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GALOIS H-OBJECTS WITH A NORMAL BASIS IN CLOSED CATEGORIES. A COHOMOLOGICAL INTERPRETATION

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Abstract

In this paper, for a cocommutative Hopf algebra **H** in a symmetric closed category $\mathcal{C}$ with basic object K, we get an isomorphism between the group of isomorphism classes of Galois **H**-objects with a normal basis and the second cohomology group $H^2(\mathbf{H}, K)$ of **H** with coefficients in K. Using this result, we obtain a direct sum decomposition for the Brauer group of **H**-module Azumaya monoids with inner action:

$$\mathrm{BM}_{\mathrm{inn}}(\mathcal{C}, \mathbf{H}) \cong \mathrm{B}(\mathcal{C}) \oplus H^2(\mathbf{H}, K)$$

In particular, if $\mathcal{C}$ is the symmetric closed category of K-modules with K a field, $H^2(H, K)$ is the second cohomology group introduced by Sweedler in **[21]**. Moreover, if H is a finitely generated projective, commutative and cocommutative Hopf algebra over a commutative ring with unit K, then the above decomposition theorem is the one obtained by Beattie **[5]** for the Brauer group of H-module algebras.

Preliminary

A monoidal category $(\mathcal{C}, \otimes, K)$ consists of a category $\mathcal{C}$ with a bifunctor $-\otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and a basic object K, and with natural isomorphisms:

$$a_{ABC} : A \otimes (B \otimes C) \cong (A \otimes B) \otimes C$$
$$l_A : K \otimes A \cong A$$
$$r_A : A \otimes K \cong A$$

such that

$$(A \otimes a_{BCD}) \circ a_{A(B\otimes C)D} \circ (a_{ABC} \otimes D) = a_{AB(C\otimes D)} \circ a_{(A\otimes B)CD}$$
$$(A \otimes l_B) \circ a_{AKB} = r_A \otimes B$$

If there is a natural isomorphism $\tau_B^A : A \otimes B \cong B \otimes A$ such that $\tau_A^B \circ \tau_B^A = A \otimes B$, $\tau_{B\otimes C}^A = (B \otimes \tau_C^A) \circ (\tau_B^A \otimes C)$, then $\mathcal{C}$ is called a symmetric monoidal category.

A closed category is a symmetric monoidal category in which each functor $- \otimes A : \mathcal{C} \to \mathcal{C}$ has a specified right adjoint $[A, -] : \mathcal{C} \to \mathcal{C}$ ([**12**], [**18**]).

Examples:

1) The category of sets and mappings.
2) The category of R-modules over a commutative ring R.
3) The category of chain complexes of R-modules and morphisms of degree 0, with R a commutative ring.
4) The category of sheaves of θ-modules over a topological space, with θ a sheaf of commutative rings.
5) The category of coherent sheaves of modules over a scheme.
6) The category of all R-graded modules with morphisms of degree 0 (R is a commutative graded ring).
7) (R, σ)-Mod, with R a commutative ring and σ an idempotent kernel functor in R-Mod.

In what follows, $\mathcal{C}$ denotes a symmetric closed category with equalizers, co-equalizers and projective basic objec K. We denote by α_M and β_M the unit and the co-unit, respectively, of the $\mathcal{C}$-adjuntion $M \otimes - \dashv [M, -] : \mathcal{C} \to \mathcal{C}$ which exists for each object M of $\mathcal{C}$.

1. An object M of $\mathcal{C}$ is called profinite in $\mathcal{C}$ if the morphism $[M, \beta_M(K) \otimes M] \circ \alpha_M(\hat{M} \otimes M) : \hat{M} \otimes M \to [M, M] = E(M)$ is an isomorphism, where $\hat{M} = [M, K]$. If, moreover, the factorization of $\beta_M(K) : M \otimes \hat{M} \to K$ through the co-equalizer of the morphisms $\beta_M(M) \otimes \hat{M}$ and $M \otimes ([M, \beta_M(K) \circ (\beta_M(M) \otimes \hat{M})] \circ \alpha_M(E(M) \otimes \hat{M})) : M \otimes E(M) \otimes \hat{M} \to M \otimes \hat{M}$ is an isomorphism, we say that M is a progenerator in $\mathcal{C}$.

2. A monoid in $\mathcal{C}$ is a triple $\mathbf{A} = (A, \eta_A, \mu_A)$ where A is an object in $\mathcal{C}$ and $\mu_A : A \otimes A \to A$, $\eta_A : K \to A$ are morphisms in $\mathcal{C}$ such that $\mu_A \circ (A \otimes \eta_A) = A = \mu_A \circ (\eta_A \otimes A)$, $\mu_A \circ (\mu_A \otimes A) = \mu_A \circ (A \otimes \mu_A)$. If $\mu_A \circ \tau_A^A = \mu_A$, then we will say that $\mathbf{A}$ is a commutative monoid. Given two monoids $\mathbf{A} = (A, \eta_A, \mu_A)$ and $\mathbf{B} = (B, \eta_B, \mu_B)$ in $\mathcal{C}$, $f : A \to B$ is a monoid morphism if $\mu_B \circ (f \otimes f) = f \circ \mu_A$ and $f \circ \eta_A = \eta_B$.

A comonoid (cocommutative), $\mathbf{D} = (D, \varepsilon_D, \delta_D)$ is an object D in $\mathcal{C}$ together with two morphisms $\varepsilon_D : D \to K$, $\delta_D : D \to D \otimes D$, such that $(\delta_D \otimes D) \circ \delta_D = (D \otimes \delta_D) \circ \delta_D$ and $(\varepsilon_D \otimes D) \circ \delta_D = 1_D = (D \otimes \varepsilon_D) \circ \delta_D (\tau_D^D \circ \delta_D = \delta_D)$. If $\mathbf{D} = (D, \varepsilon_D, \delta_D)$ and $\mathbf{E} = (E, \varepsilon_E, \delta_E)$ are comonoids, $f : D \to E$ is a comonoid morphism if $(f \otimes f) \circ \delta_D = \delta_E \circ f$ and $\varepsilon_E \circ f = \varepsilon_D$.

3. For a monoid $\mathbf{A} = (A, \eta_A, \mu_A)$ and a comonoid $\mathbf{D} = (D, \varepsilon_D, \delta_D)$ in $\mathcal{C}$, we denote by $\mathrm{Reg}(D, A)$ the group of invertible elements in $\mathcal{C}(D, A)$ (morphisms in $\mathcal{C}$ from D to A) with the operation "convolution" given by: $f * g = \mu_A \circ (f \otimes g) \circ \delta_D$. The unit element is $\varepsilon_D \otimes \eta_A$.

Observe that $\mathrm{Reg}(D, A)$ is an abelian group when $\mathbf{D}$ is cocommutative and $\mathbf{A}$ is commutative.

4. Definition. Let $\Pi = (C, \eta_C, \mu_C)$ be a monoid and $\mathbf{C} = (C, \varepsilon_C, \delta_C)$ a comonoid in $\mathcal{C}$ and let $\lambda : C \to C$ be a morphism. Then $\mathbf{H} = (\mathbf{C} = (C, \varepsilon_C, \delta_C), \Pi = (C, \eta_C, \mu_C), \tau^C, \lambda)$ is a Hopf algebra in $\mathcal{C}$ with respect to the comonoid $\mathbf{C}$ if ε_C and δ_C are monoid morphisms (equivalently, η_C and μ_C are comonoid morphisms) and λ is the inverse of $1_C : C \to C$ in $\mathrm{Reg}(C, C)$.

We say that $\mathbf{H}$ is a finite Hopf algebra if C is profinite in $\mathcal{C}$.

5. Definition. $(\mathbf{A}, \varphi_A) = (A, \eta_A, \mu_A; \varphi_A)$ is a left $\mathbf{H}$-module monoid if:

i) $\mathbf{A} = (A, \eta_A, \mu_A)$ is a monoid in $\mathcal{C}$.
ii) (A, φ_A) is a left $\mathbf{H}$-module ($\varphi_A \circ (C \otimes \varphi_A) = \varphi_A \circ (\mu_C \otimes A)$, $\varphi_A \circ (\eta_C \otimes A) = A$).
iii) η_A, μ_A are morphisms of left $\mathbf{H}$-modules ($\varphi_A \circ (C \otimes \eta_A) = \eta_A \otimes \varepsilon_C$ and $\varphi_A \circ (C \otimes \mu_A) = \mu_A \circ \varphi_{A \otimes A}$, where $\varphi_{A \otimes A} = (\varphi_A \otimes \varphi_A) \circ (C \otimes \tau_A^C \otimes A) \circ (\delta_C \otimes A \otimes A)$).

We say that the action φ_A of $\mathbf{H}$ in $\mathbf{A}$ is inner if there exists a morphism f in $\mathrm{Reg}(C, A)$ such that $\varphi_A = \mu_A \circ (A \otimes (\mu_A \circ \tau_A^A)) \circ (f \otimes f^{-1} \otimes A) \circ (\delta_C \otimes A) : C \otimes A \to A$, where f^{-1} is the convolution inverse of f.

6. Definition. If $\mathbf{H}$ is a cocommutative Hopf algebra and $(\mathbf{A}, \varphi_A)$ is a commutative $\mathbf{H}$-module monoid, then, we say that a morphism σ in $\mathrm{Reg}(C \otimes C, A)$ is a 2-cocycle if $\partial_1(\sigma) * \partial_3(\sigma) = \partial_2(\sigma) * \partial_4(\sigma)$, where $\partial_1(\sigma) = \varphi_A \circ (C \otimes \sigma)$, $\partial_2(\sigma) = \sigma \circ (\mu_C \otimes C)$, $\partial_3(\sigma) = \sigma \circ (C \otimes \mu_C)$ and $\partial_4(\sigma) = \sigma \otimes \varepsilon_C$.

Two 2-cocycles σ and γ are said to be cohomologous, written $\sigma \sim \gamma$, if there exists a morphism $v \in \mathrm{Reg}(C, A)$ such that $\sigma * \partial_2(v) = \partial_1(v) * \partial_3(v) * \gamma$, where $\partial_1(v) = \varphi_A \circ (C \otimes v)$, $\partial_2(v) = v \circ \mu_C$ and $\partial_3(v) = v \otimes \varepsilon_C$.

Trivially, "$\sim$" is an equivalence relation.

The set of equivalence classes shall be called the second cohomology group of the cocommutative Hopf algebra $\mathbf{H}$ with the coefficients in the left $\mathbf{H}$-module monoid $(\mathbf{A}, \varphi_A)$, and will be denoted by $H^2(\mathbf{H}, \mathbf{A})$.

If σ is a 2-cocycle in $\mathrm{Reg}(C \otimes C, A)$, then the morphism $\bar{\sigma} = \sigma * \partial_1(\pi) * \partial_2(\pi^{-1}) * \partial_3(\pi)$ is a 2-cocycle in $\mathrm{Reg}(C \otimes C, A)$ cohomologous with σ such that $\bar{\sigma} \circ (\eta_C \otimes C) = \varepsilon_C \otimes \eta_A = \bar{\sigma} \circ (C \otimes \eta_C)$, where $\pi = \sigma^{-1} \circ (C \otimes \eta_C)$ is a morphism in $\mathrm{Reg}(C, A)$ with inverse $\pi^{-1} = \sigma \circ (C \otimes \eta_C)$. Moreover, if

γ is cohomologous with σ, then there exists a morphism $\bar{\vartheta} \in \operatorname{Reg}(C, A)$ such that $\bar{\vartheta} \circ \eta_C = \eta_A$ and $\bar{\sigma} * \partial_2(\bar{\vartheta}) = \partial_1(\bar{\vartheta}) * \partial_3(\bar{\vartheta}) * \bar{\gamma}$.

Remark. Let $\mathcal{C}$ the category of K-modules over a field. In this case, $H^2(\mathbf{H}, \mathbf{A})$ is the second cohomology group of the Sweedler's complex $\{\operatorname{Reg}(\overset{q}{\otimes}C, A); \Delta_q\}_{q \geq 0}$

$$\operatorname{Reg}(K, A) \xrightarrow{\Delta_0} \operatorname{Reg}(C, A) \xrightarrow{\Delta_1} \dots \xrightarrow{\Delta_{q-1}} \operatorname{Reg}(\overset{q}{\otimes}C, A) \xrightarrow{\Delta_q} \operatorname{Reg}(\overset{q+1}{\otimes}C, A) \xrightarrow{\Delta_{q+1}} \dots$$

where $\Delta_q := \partial_1 * \partial_2^{-1} * \dots * \partial_{q+2}^{(-1)^{q+1}}$ and for each $f \in \operatorname{Reg}(\overset{q}{\otimes}C, A)$,

$$\begin{aligned} &\partial_1(f) = \varphi_A \circ (C \otimes f) \\ &\vdots \\ &\partial_i(f) = f \circ (C \otimes \overset{i-2}{\dots} \otimes C \otimes \mu_C \otimes C \otimes \overset{q-i+1}{\dots} \otimes C) \\ &\vdots \\ &\partial_{q+2}(f) = f \otimes \varepsilon_C \end{aligned}$$

([**21**]).

7. Definition. $(\mathbf{B}, \rho_B) = (B, \eta_B, \mu_B; \rho_B)$ is a right $\mathbf{H}$-comodule monoid if:

i) $\mathbf{B} = (B, \eta_B, \mu_B)$ is a monoid in $\mathcal{C}$
ii) (B, ρ_B) is a right $\mathbf{H}$-comodule ($(\rho_B \otimes C) \circ \rho_B = (B \otimes \delta_C) \circ \rho_B$; $(B \otimes \varepsilon_C) \circ \rho_B = B$).
iii) $\rho_B : B \to B \otimes C$ is a monoid morphism from (B, η_B, μ_B) to the product monoid $\mathrm{B\Pi} = (B \otimes C, \eta_B \otimes \eta_C, (\mu_B \otimes \mu_C) \circ (B \otimes \tau_B^C \otimes C))$ (that is, $\rho_B \circ \eta_B = \eta_B \otimes \eta_C$ and $\rho_B \circ \mu_B = (\mu_B \otimes \mu_C) \circ (B \otimes \tau_B^C \otimes C) \circ (\rho_B \otimes \rho_B)$).

From now on we assume that $\mathbf{H}$ is a finite cocommutative and commutative Hopf algebra.

8. Definition. A right $\mathbf{H}$-comodule monoid $(\mathbf{B}, \rho_B)$ is said to be a Galois $\mathbf{H}$-object if and only if:

i) The morphism $\gamma_B := (\mu_B \otimes C) \circ (B \otimes \rho_B) : B \otimes B \to B \otimes C$ is an isomorphism.
ii) B is a progenerator in $\mathcal{C}$.

For example, in the case of (R, σ)-Mod, a commutative $\mathbf{H}$-comodule monoid is a couple $(\mathbf{B}, \rho_B)$, where B is a commutative (R, σ)-algebra and $\rho_B : B \to B \perp H := Q_\sigma(B \otimes_R H)$ is a morphism of algebras and it defines a right $\mathbf{H}$-comodule structure over B.

$(\mathbf{B}, \rho_B)$ is a Galois $\mathbf{H}$-object if and only if B is a (R, σ)-progenerator and the mapping $\rho_B^{\hat{H}} : B \underset{\sigma}{\#} \hat{H} \to \mathrm{Hom}(B, B)$ arising from the left $B \underset{\sigma}{\#} \hat{H}$-module structure on B is an isomorphism ([**15**, (1.3.17)]).

If a Galois $\mathbf{H}$-object is isomorphic to $\mathbf{H}$ as an $\mathbf{H}$-comodule then we say that it has a normal basis.

If $\mathbf{B}_1$ and $\mathbf{B}_2$ are Galois $\mathbf{H}$-objects, $f : B_1 \to B_2$ is a morphism of Galois $\mathbf{H}$-objects if it is a morphism of $\mathbf{H}$-comodules ($\rho_{B_2} \circ f = (f \otimes C) \circ \rho_{B_1}$) and of monoids.

If $(\mathbf{A}, \rho_A)$ and $(\mathbf{B}, \rho_B)$ are $\mathbf{H}$-comodule monoids, then $\mathbf{A} \circ \mathbf{B}$, defined by the following equalizer diagram

$$A \circ B \overset{i_{AB}}{\to} A \otimes B \underset{\partial^2_{AB}}{\overset{\partial^1_{AB}}{\rightrightarrows}} A \otimes B \otimes C$$

where

$$\partial^1_{AB} = (A \otimes \tau_B^C) \circ (\rho_A \otimes B), \text{ and}$$
$$\partial^2_{AB} = A \otimes \rho_B$$

is an $\mathbf{H}$-comodule monoid to be denoted by $(\mathbf{A} \circ \mathbf{B}, \rho_{AB})$.

If moreover $(\mathbf{A}, \rho_A)$ and $(\mathbf{B}, \rho_B)$ are Galois $\mathbf{H}$-objects, then $(\mathbf{A} \circ \mathbf{B}, \rho_{AB})$ is also a Galois $\mathbf{H}$-object, where ρ_{AB} is the factorization of the morphism $\partial^1_{AB} \circ i_{AB}$ (or $\partial^2_{AB} \circ i_{AB}$) through the equalizer $i_{AB} \otimes C$.

The set of isomorphism classes of Galois $\mathbf{H}$-objects (with a normal basis), with the operation induced by the one given above, is an abelian group to be denoted by $\mathbf{Gal}_C(\mathbf{H})(\mathbf{N}_C(\mathbf{H}))$. The unit element is the class of (Π, δ_C) and the opposite of $[(\mathbf{B}, \rho_B)]$ is $[(\mathbf{B}^{\mathrm{op}}, (B \otimes \lambda) \circ \rho_B)]$ where $\mathbf{B}^{\mathrm{op}} = (B, \eta_B, \mu_B \circ \tau_B^B)$.

Remark. In the case of a finitely generated projective, commutative and cocommutative Hopf algebra $\mathbf{H}$ over a commutative ring R, $\mathbf{Gal}_C(\mathbf{H})$ is the group of Galois $\mathbf{H}$-objects in the sense of S. Chase and M. Sweedler in [**9**].

9. Proposition. *If* $[(\mathbf{B}, \rho_B)] \in \mathbf{N}_C(\mathbf{H})$, *then, there is a 2-cocycle* σ *in* $\mathrm{Reg}(C \otimes C, K)$ *satisfying* $\sigma \circ (\eta_C \otimes C) = \varepsilon_C = \sigma \circ (C \otimes \eta_C)$.

Proof:

Let $(\mathbf{B}, \rho_B)$ a Galois $\mathbf{H}$-object with a normal basis. Then we have an isomorphism $\gamma_B : B \otimes B \to B \otimes C$ and an $\mathbf{H}$-comodule isomorphism $r : C \to B$. Therefore the morphism of $\mathbf{H}$-comodules $f = (\varepsilon_C \otimes B) \circ (r^{-1} \otimes r) \circ (\eta_B \otimes C) : C \to B$ is in $\mathrm{Reg}(C, B)$ with inverse

$$f^{-1} = \mu_B \circ (B \otimes \varepsilon_C \otimes B) \circ (B \otimes r^{-1} \otimes r) \circ (\gamma_B^{-1} \otimes \eta_C) \circ (\eta_B \otimes C)$$

and satisfying $f \circ \eta_C = \eta_B$.

Indeed:

$$
\begin{aligned}
f * f^{-1} &= \mu_B \circ (\varepsilon_C \otimes B \otimes \varepsilon_C \otimes B) \circ (r^{-1} \otimes B \otimes r^{-1} \otimes r) \circ \\
&\circ (\eta_B \otimes \gamma_B^{-1} \otimes \eta_C) \circ (r \otimes C) \circ \delta_C = \\
&= \mu_B \circ (\varepsilon_C \otimes B \otimes \varepsilon_C \otimes B) \circ (r^{-1} \otimes B \otimes r^{-1} \otimes r) \circ \\
&\circ (B \otimes [\gamma_B^{-1} \circ \gamma_B] \otimes C) \circ (\eta_B \otimes \eta_B \otimes B \otimes \eta_C) \circ r = \\
&= (\varepsilon_C \otimes B \otimes \varepsilon_C) \circ (r^{-1} \otimes r \otimes C) \circ ([\rho_B \circ \eta_B] \otimes C) = \\
&= \varepsilon_C \otimes \eta_B \\
f^{-1} * f &= \mu_B \circ (B \otimes [\varepsilon_C \circ r^{-1}] \otimes B) \circ (\gamma_B^{-1} \otimes r) \circ (\eta_B \otimes \delta_C) = \\
&= \mu_B \circ (B \otimes [\varepsilon_C \circ r^{-1}] \otimes r) \circ (B \otimes \rho_B) \circ \gamma_B^{-1} \circ (\eta_B \otimes C) = \\
&= \mu_B \circ (B \otimes \varepsilon_C \otimes r) \circ (B \otimes \delta_C) \circ (B \otimes r^{-1}) \circ \gamma_B^{-1} \circ (\eta_B \otimes C) = \\
&= (B \otimes \varepsilon_C) \circ \gamma_B \circ \gamma_B^{-1} \circ (\eta_B \otimes C) = \\
&= \varepsilon_C \otimes \eta_B
\end{aligned}
$$

because r is an $\mathbf{H}$-comodule isomorphism and the equalities:

$$
\begin{aligned}
&(\mu_B \otimes B) \circ (B \otimes \gamma_B^{-1}) = \gamma_B^{-1} \circ (\mu_B \otimes C) \\
&(\varepsilon_C \otimes B) \circ (r^{-1} \otimes r) \circ (\eta_B \otimes \eta_C) = (\varepsilon_C \otimes B) \circ (r^{-1} \otimes r) \circ \rho_B \circ \eta_B = \eta_B \\
&(\gamma_B^{-1} \otimes C) \circ (B \otimes \delta_C) = (B \otimes \rho_B) \circ \gamma_B^{-1}
\end{aligned}
$$

Trivially, f is a morphism of $\mathbf{H}$-comodules and $f \circ \eta_C = \eta_B$.

The morphism $\sigma_f = (\mu_B \circ (f \otimes f)) * (f^{-1} \otimes \mu_C) : C \otimes C \to B$ factors through the equalizer

$$
K \xrightarrow{\eta_B} B \underset{B \otimes \eta_C}{\overset{\rho_B}{\rightrightarrows}} B \otimes C
$$

because $\mathbf{H}$ is a Hopf algebra, $(\mathbf{B}, \rho_B)$ an $\mathbf{H}$-comodule monoid, f a morphism of $\mathbf{H}$-comodules and the equality $\rho_B \circ f^{-1} = (f^{-1} \otimes \lambda) \circ \tau_C^C \circ \delta_C$ ([**14**, (2.3)]).

Moreover, the factorization, $\bar{\sigma}_f$, of σ_f is in $\mathrm{Reg}(C \otimes C, K)$ with inverse the factorization of the morphism $\sigma_f^{-1} = (f \circ \mu_C) * (\mu_B \circ \tau_B^B \circ (f^{-1} \otimes f^{-1}))$: $C \otimes C \to B$ through the equalizer η_B.

The morphism $\bar{\sigma}_f : C \otimes C \to K$ is a 2-cocycle. Indeed:

$$\begin{aligned}
&\eta_B \circ (\partial_1(\bar{\sigma}_f) * \partial_3(\bar{\sigma}_f)) = \\
&= (f * f^{-1}) \circ (\bar{\sigma}_f \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) \circ \\
&\circ (C \otimes \bar{\sigma}_f \otimes \mu_C) \circ (C \otimes C \otimes \tau_C^C \otimes C) \circ (C \otimes \delta_C \otimes \delta_C) = \\
&= \mu_B \circ (B \otimes f^{-1}) \circ \rho_B \circ f \circ (\bar{\sigma}_f \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) \circ \\
&\circ (C \otimes \bar{\sigma}_f \otimes \mu_C) \circ (C \otimes C \otimes \tau_C^C \otimes C) \circ (C \otimes \delta_C \otimes \delta_C) = \\
&= \mu_B \circ (B \otimes f^{-1}) \circ \rho_B \circ \mu_B \circ (f \otimes f) \circ (C \otimes \bar{\sigma}_f \otimes \mu_C) \circ \\
&\circ (C \otimes C \otimes \tau_C^C \otimes C) \circ (C \otimes \delta_C \otimes \delta_C) = \\
&= \mu_B \circ (B \otimes f^{-1}) \circ \rho_B \circ \mu_B \circ (\mu_B \otimes C) \circ (f \otimes f \otimes f) = \\
&= \mu_B \circ (B \otimes f^{-1}) \circ \rho_B \circ f \circ (\bar{\sigma}_f \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) \circ \\
&\circ (\bar{\sigma}_f \otimes \mu_C \otimes C) \circ (C \otimes \tau_C^C \otimes C \otimes C) \circ (\delta_C \otimes \delta_C \otimes C) = \\
&= (f * f^{-1}) \circ (\bar{\sigma}_f \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) \circ \\
&\circ (\bar{\sigma}_f \otimes \mu_C \otimes C) \circ (C \otimes \tau_C^C \otimes C \otimes C) \circ (\delta_C \otimes \delta_C \otimes C) = \\
&= \eta_B \circ (\partial_2(\bar{\sigma}_f) * \partial_4(\bar{\sigma}_f))
\end{aligned}$$

because f is a morphism of **H**-comodules, **H** a cocommutative Hopf algebra and the equality

$$f \circ (\bar{\sigma}_f \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \mu_B \circ (f \otimes f)$$

and then, since η_B is a monomorphism, $\bar{\sigma}_f$ is a 2-cocycle.

Trivially, $\bar{\sigma}_f \circ (\eta_C \otimes C) = \varepsilon_C = \bar{\sigma}_f \circ (C \otimes \eta_C)$. ■

Remark. If $[(\mathbf{B}_1, \rho_{B_1})] = [(\mathbf{B}_2, \rho_{B_2})] \in \mathbf{N}_C(\mathbf{H})$ then there is an isomorphism of **H**-comodule monoids $h : B_1 \to B_2$. Clearly, $\overline{\sigma_{f_1}} = \overline{\sigma_{h \circ f_1}}$. (Notice that $h \circ f_1 \in \operatorname{Reg}(C, B_2)$ with inverse $h \circ f_1^{-1}$). Moreover, $\overline{\sigma_{f_1}} \sim \overline{\sigma_{f_2}}$.

Indeed:

The morphism $e = (h \circ f_1) * f_2^{-1} : C \to B_2$ factors through the equalizer η_{B_2}:

$$\begin{aligned}
\rho_{B_2} \circ e = (\mu_{B_2} \otimes \mu_C) \circ ((h \circ f_1) \otimes \tau_{B_2}^C \otimes \lambda) \circ (\delta_C \otimes f_2^{-1} \otimes C) \circ \\
\circ (C \otimes (\tau_C^C \circ \delta_C)) \circ \delta_C = (B_2 \otimes \eta_C) \circ e
\end{aligned}$$

and then, there exists a morphism $\bar{e} : C \to K$ such that $\eta_{B_2} \circ \bar{e} = e$. Clearly, $\bar{e} \circ \eta_C = K$. Moreover, $\bar{e}$ is in $\operatorname{Reg}(C, K)$ with inverse $\bar{e}^{-1}$, the factorization of $e^{-1} = f_2 * (h \circ f_1^{-1})$ throught the equalizer η_{B_2}.

We also have that:

$$\eta_{B_2}\circ(\overline{\sigma_{f_1}}*\partial_2(\bar{e}))=$$
$$=\mu_{B_2}\circ(\eta_{B_2}\otimes\eta_{B_2})\circ(\overline{\sigma_{h\circ f}}*\partial_2(\bar{e}))=$$
$$=\mu_{B_2}\circ(\mu_{B_2}\otimes\mu_{B_2})\circ[(h\circ f_1)\otimes(h\circ f_1)\otimes((h\circ f_1^{-1})*(h\circ f_1))\otimes$$
$$\otimes f_2^{-1}]\circ(C\otimes C\otimes\delta_C)\circ(C\otimes C\otimes\mu_C)\circ(C\otimes\tau_C^C\otimes C)\circ(\delta_C\otimes\delta_C)=$$
$$=\mu_{B_2}\circ(\mu_{B_2}\otimes B_2)\circ(\mu_{B_2}\otimes(\mu_{B_2}\circ\tau_{B_2}^{B_2})\otimes\mu_{B_2})\circ$$
$$\circ((h\circ f_1)\otimes(h\circ f_1)\otimes(f_2^{-1}*f_2)\otimes f_2^{-1}\otimes f_2\otimes f_2^{-1})\circ$$
$$\circ(C\otimes C\otimes C\otimes\delta_C\otimes\mu_C)\circ(C\otimes C\otimes C\otimes\tau_C^C\otimes C)\circ$$
$$\circ(C\otimes C\otimes\delta_C\otimes\delta_C)\circ(C\otimes\tau_C^C\otimes C)\circ(\delta_C\otimes\delta_C)=$$
$$=\mu_{B_2}\circ(\mu_{B_2}\otimes B_2)\circ((\eta_{B_2}\circ\bar{e})\otimes(\eta_{B_2}\circ\bar{e})\otimes(\eta_{B_2}\circ\overline{\sigma_{f_2}}))\circ$$
$$\circ(C\otimes\tau_C^C\otimes C)\circ(\delta_C\otimes\delta_C)=$$
$$=\eta_{B_2}\circ(\partial_1(\bar{e})*\partial_3(\bar{e})*\overline{\sigma_{f_2}})$$

and then, since η_{B_2} is a monomorphism, $\overline{\sigma_{f_1}}\sim\overline{\sigma_{f_2}}$.

10. Proposition. *If σ is a 2-cocycle in $\mathrm{Reg}(C\otimes C,K)$ such that $\sigma\circ(\eta_C\otimes C)=\varepsilon_C=\sigma\circ(C\otimes\eta_C)$, then $(\mathbf{C}_\sigma=(C,\eta_C,\mu_{C_\sigma}=(\sigma\otimes\mu_C)\circ(C\otimes\tau_C^C\otimes C)\circ(\delta_C\otimes\delta_C));\delta_C)$ is a Galois $\mathbf{H}$-object with a normal basis.*

Proof:

Trivially, $(\mathbf{C}_\sigma,\delta_C)$ is an $\mathbf{H}$-comodule monoid.

The morphism $\gamma_{C_\sigma}=(\mu_{C_\sigma}\otimes C)\circ(C\otimes\delta_C):C\otimes C\to C\otimes C$ is an isomorphism with inverse

$$\gamma_{C_\sigma}^{-1}=(\mu_C\otimes C)\circ(C\otimes\sigma^{-1}\otimes\lambda\otimes C)\circ(C\otimes\mu_C\otimes C\otimes\delta_C)\circ(\delta_C\otimes\lambda\otimes C\otimes C)\circ$$
$$\circ(C\otimes\delta_C\otimes C)\circ(C\otimes\delta_C)$$

Indeed:

$$\gamma_{C_\sigma}^{-1}\circ\gamma_{C_\sigma}=$$
$$=(\mu_C\otimes C)\circ(\mu_C\otimes\sigma^{-1}\otimes\lambda\otimes C)\circ(C\otimes C\otimes\mu_C\otimes C\otimes\delta_C)\circ$$
$$\circ(C\otimes C\otimes\mu_C\otimes\lambda\otimes C\otimes C)\circ(C\otimes\tau_C^C\otimes C\otimes\delta_C\otimes C)\circ$$
$$\circ(\sigma\otimes\delta_C\otimes\delta_C\otimes\delta_C)\circ(C\otimes\tau_C^C\otimes C\otimes C)\circ(\delta_C\otimes\delta_C\otimes C)\circ(C\otimes\delta_C)=$$
$$=(\mu_C\otimes C)\circ(\mu_C\otimes\sigma^{-1}\otimes\lambda\otimes C)\circ(C\otimes C\otimes\mu_C\otimes\delta_C\otimes C)\circ$$
$$\circ(C\otimes\tau_C^C\otimes[\mu_C\circ(C\otimes\lambda)\circ\delta_C]\otimes\delta_C)\circ(C\otimes C\otimes\delta_C\otimes C)\circ$$
$$\circ(\sigma\otimes\delta_C\otimes\delta_C)\circ(C\otimes\tau_C^C\otimes C)\circ(\delta_C\otimes\delta_C)=$$

$$= (\mu_C \otimes C) \circ (\mu_C \otimes \sigma^{-1} \otimes \lambda \otimes C) \circ (C \otimes \tau_C^C \otimes [\tau_C^C \circ \delta_C] \otimes C) \circ$$
$$\circ (\sigma \otimes \delta_C \otimes C \otimes \delta_C) \circ (C \otimes \tau_C^C \otimes \delta_C) \circ (\delta_C \otimes \delta_C) =$$
$$= (\mu_C \otimes C) \circ (C \otimes [\mu_C \circ (C \otimes \lambda) \circ \delta_C] \otimes C) \circ$$
$$\circ (\sigma \otimes C \otimes C \otimes \sigma^{-1} \otimes C) \circ (C \otimes C \otimes C \otimes \tau_C^C \otimes C \otimes C) \circ$$
$$\circ (C \otimes C \otimes \delta_C \otimes \delta_C \otimes C) \circ (C \otimes \tau_C^C \otimes \delta_C) \circ (\delta_C \otimes \delta_C) =$$
$$= (\sigma \otimes C \otimes \sigma^{-1} \otimes C) \circ (C \otimes C \otimes [\tau_C^C \circ \delta_C] \otimes \delta_C) \circ$$
$$\circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) =$$
$$= ([\sigma * \sigma^{-1}] \otimes C \otimes C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) =$$
$$= C \otimes C$$

$$\gamma_{C_\sigma} \circ \gamma_{C_\sigma}^{-1} =$$
$$= (\sigma \otimes \mu_C \otimes C) \circ (\mu_C \otimes \tau_C^C \otimes C \otimes C) \circ (C \otimes C \otimes \mu_C \otimes \delta_C \otimes C) \circ$$
$$\circ (C \otimes \tau_C^C \otimes C \otimes \delta_C) \circ (\delta_C \otimes \sigma^{-1} \otimes \delta_C \otimes C) \circ (C \otimes \mu_C \otimes C \otimes \lambda \otimes C) \circ$$
$$\circ (C \otimes C \otimes \lambda \otimes C \otimes \delta_C) \circ (\delta_C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) =$$
$$= (\sigma \otimes \mu_C \otimes C) \circ (C \otimes \tau_C^C \otimes C \otimes C) \circ (\mu_C \otimes \mu_C \otimes [\tau_C^C \circ \delta_C] \otimes C) \circ$$
$$\circ (C \otimes \tau_C^C \otimes C \otimes \delta_C) \circ (\delta_C \otimes \sigma^{-1} \otimes [(\lambda \otimes \lambda) \circ \tau_C^C \circ \delta_C] \otimes C) \circ$$
$$\circ (C \otimes \mu_C \otimes C \otimes C \otimes C) \circ (\delta_C \otimes \lambda \otimes C \otimes \delta_C) \circ$$
$$\circ (C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) =$$
$$= (\sigma \otimes C \otimes C) \circ (C \otimes \tau_C^C \otimes C) \circ (\mu_C \otimes \mu_C \otimes \delta_C) \circ$$
$$\circ (C \otimes \tau_C^C \otimes [\mu_C \circ (\lambda \otimes C) \circ \delta_C] \otimes C) \circ (\delta_C \otimes \sigma^{-1} \otimes \lambda \otimes \delta_C) \circ$$
$$\circ (C \otimes \mu_C \otimes C \otimes \delta_C) \circ (\delta_C \otimes \lambda \otimes C \otimes C) \circ (C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) =$$
$$= (\sigma \otimes C \otimes C) \circ (\mu_C \otimes \tau_C^C \otimes C) \circ (C \otimes \tau_C^C \otimes \delta_C) \circ$$
$$\circ ([\tau_C^C \circ \delta_C] \otimes \sigma^{-1} \otimes \lambda \otimes C) \circ (C \otimes \mu_C \otimes C \otimes C \otimes C) \circ$$
$$\circ (C \otimes C \otimes \lambda \otimes C \otimes C \otimes C) \circ (\delta_C \otimes \delta_C \otimes C \otimes C) \circ$$
$$\circ (C \otimes [\tau_C^C \circ \delta_C] \otimes C) \circ (C \otimes \delta_C) =$$
$$= (C \otimes \sigma \otimes C) \circ (C \otimes C \otimes \sigma^{-1} \otimes \delta_C) \circ (C \otimes [\delta_C \circ \mu_C] \otimes C \otimes C) \circ$$
$$\circ (C \otimes C \otimes \lambda \otimes C \otimes C) \circ (\delta_C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) =$$
$$= (C \otimes \sigma \otimes \sigma^{-1} \otimes C) \circ (C \otimes C \otimes \tau_C^C \otimes C \otimes C) \circ$$
$$\circ (C \otimes \delta_C \otimes \delta_C \otimes C) \circ (C \otimes \mu_C \otimes \delta_C) \circ (\delta_C \otimes \lambda \otimes C) \circ (C \otimes \delta_C) =$$
$$= (C \otimes [\sigma * \sigma^{-1}] \otimes C) \circ (C \otimes \mu_C \otimes \delta_C) \circ (\delta_C \otimes \lambda \otimes C) \circ (C \otimes \delta_C) =$$
$$= C \otimes C$$

and thus (C_σ, δ_C) is a Galois **H**-object with a normal basis. ■

Remark. If σ and γ are two 2-cocycles such in Proposition 10 and cohomologous, then $(\ddot{e} \otimes C) \circ \delta_C : (\mathbf{C}_\sigma, \delta_C) \to (\mathbf{C}_\gamma, \delta_C)$ is an isomorphism of $\mathbf{H}$-comodule monoids where $\bar{e}$ is the morphism which exists because $\sigma \sim \gamma$, and thus, $[(\mathbf{C}_\sigma, \delta_C)] = [(\mathbf{C}_\gamma, \delta_C)]$ in $\mathbf{N}_C(\mathbf{H})$.

11. Theorem. *There is an isomorphism of abelian groups* $F : \mathbf{N}_C(\mathbf{H}) \to H^2(\mathbf{H}, K)$ *defined by* $F([(\mathbf{B}, \rho_B)]) = [\bar{\sigma}_f]$ *with inverse* $G([\sigma]) = [(\mathbf{C}_\sigma, \delta_C)]$.

Proof:

The morphisms F and G are well defined by Propositions 9 and 10. Moreover, the morphism of $\mathbf{H}$-comodules $f : (\mathbf{C}_{\bar{\sigma}_f} = (C, \eta_C, \mu_{C_{\sigma_f}}); \delta_C) \to (\mathbf{B} = (B, \eta_B, \mu_B); \rho_B)$ is a morphism of monoids and therefore it is an isomorphism ([**13**, (4.3.9)]) and $[(\mathbf{B}, \rho_B)] = [(C_{\bar{\sigma}_f}, \delta_C)]$ in $\mathbf{N}_C(\mathbf{H})$.

If $[\sigma] \in H^2(\mathbf{H}, K)$ then the 2-cocycle γ defined from C_σ (Proposition 9) equals to σ:

$$
\begin{aligned}
\eta_C \circ \gamma = {} & \mu_{C_\sigma} \circ (\mu_{C_\sigma} \otimes [(C \otimes \varepsilon_C) \circ \gamma_{C_\sigma}^{-1} \circ (\eta_C \otimes C)]) \circ (C \otimes C \otimes \mu_C) \circ \\
& \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
& = (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) \circ (\sigma \otimes \mu_C \otimes \sigma^{-1} \otimes C) \circ \\
& \circ (C \otimes \tau_C^C \otimes C \otimes \lambda \otimes C \otimes \lambda) \circ (\delta_C \otimes \delta_C \otimes \delta_C \otimes C) \circ \\
& \circ (C \otimes C \otimes [\delta_C \circ \mu_C]) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
& = (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \sigma^{-1} \otimes \delta_C) \circ \\
& \circ (C \otimes \lambda \otimes C \otimes \lambda) \circ (C \otimes \delta_C \otimes C) \circ (C \otimes \delta_C) \circ \delta_C \circ \\
& \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
& = (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \sigma^{-1} \otimes C \otimes C) \circ \\
& \circ (C \otimes \lambda \otimes C \otimes [(\lambda \otimes \lambda) \circ \tau_C^C \circ \delta_C]) \circ (C \otimes \delta_C \otimes C) \circ \\
& \circ (C \otimes [\tau_C^C \circ \delta_C]) \circ \delta_C \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
& = (\sigma \otimes C \otimes \sigma^{-1}) \circ (C \otimes \tau_C^C \otimes \lambda \otimes C) \circ \\
& \circ (C \otimes [\mu_C \circ (C \otimes \lambda) \circ \delta_C] \otimes \lambda \otimes \delta_C) \circ (C \otimes C \otimes \delta_C) \circ (C \otimes \delta_C) \circ \\
& \circ \delta_C \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
& = (\sigma \otimes \sigma^{-1} \otimes \eta_C) \circ (C \otimes \lambda \otimes \lambda \otimes C) \circ (\delta_C \otimes \delta_C) \circ \delta_C \circ \\
& \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
& = ([\partial_1(\sigma^{-1}) * \partial_4(\sigma)] \otimes \eta_C) \circ (C \otimes \lambda \otimes C) \circ (C \otimes \delta_C) \circ \delta_C \circ \\
& \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
& = ([\partial_2(\sigma^{-1}) * \partial_3(\sigma)] \otimes \eta_C) \circ (C \otimes \lambda \otimes C) \circ (C \otimes \delta_C) \circ \delta_C \circ \\
& \circ (\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) = \\
& = (\sigma^{-1} \otimes \sigma \otimes \eta_C) \circ (\mu_C \otimes \tau_C^C \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes \tau_C^C \otimes C) \circ
\end{aligned}
$$

$$\circ(\delta_C \otimes [(\lambda \otimes \lambda) \circ \tau_C^C \circ \delta_C] \otimes \delta_C) \circ (C \otimes \delta_C) \circ \delta_C \circ$$
$$\circ(\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) =$$
$$= (\sigma \otimes \sigma^{-1} \otimes \eta_C) \circ (C \otimes \tau_C^C \otimes C) \circ$$
$$\circ(C \otimes [\mu_C \circ (C \otimes \lambda) \circ \delta_C] \otimes [\mu_C \circ (\lambda \otimes C) \circ$$
$$\circ \delta_C] \otimes C) \circ (\delta_C \otimes \delta_C) \circ \delta_C \circ$$
$$\circ(\sigma \otimes \mu_C) \circ (C \otimes \tau_C^C \otimes C) \circ (\delta_C \otimes \delta_C) =$$
$$= \eta_C \circ \sigma$$

because σ is a 2-cocyle and $\mathbf{H}$ a cocommutative Hopf algebra.

If $[(\mathbf{B}, \rho_B)]$, $[(\mathbf{B}', \rho_{B'})] \in \mathbf{N}_{\mathcal{C}}(\mathbf{H})$ then the morphism $(f \otimes g) \circ \delta_C : C \to B \otimes B'$ factors through the equalizer $i_{BB'}$, where f and g are the morphisms defined in Proposition 9 for $(\mathbf{B}, \rho_B)$ and $(\mathbf{B}', \rho_{B'})$ respectively. Let $h : C \to B \circ B'$ be this factorization. The morphism h is in $\mathrm{Reg}(C, B \circ B')$ with inverse the factorization of the morphism $(f^{-1} \otimes g^{-1}) \circ \delta_C$ through the equalizer $i_{BB'}$, and satisfying $h \circ \eta_C = \eta_{BB'}$. Moreover, $\bar{\sigma}_f * \bar{\sigma}_g = \bar{\sigma}_h$. ■

Remark. If $\mathcal{C} = K$-Mod (K a field), this result has been obtained by Sweedler for the $\mathbf{H}$-module algebra $(K, \varepsilon_H \otimes K)$ in **[21]**.

12. Definition. A monoid $\mathbf{A} = (A, \eta_A, \mu_A)$ is said to be Azumaya if and only if:

i) A is a progenerator in $\mathcal{C}$.
ii) The morphism of monoids $\chi_A : A \otimes A \to [A, A]$; $\chi_A := [A, \mu_A \circ (A \otimes \mu_A) \circ (\tau_A^A \otimes A)] \circ \alpha_A(A \otimes A)$ is an isomorphism.

13. Definition. On the set of $\mathbf{H}$-module monoid isomorphism classes of $\mathbf{H}$-module Azumaya monoids we define the following equivalence relation:

$$(\mathbf{A}, \varphi_A) \sim (\mathbf{B}, \varphi_B) \Longleftrightarrow \mathbf{AE}(M)^{\mathrm{op}} \cong \mathbf{BE}(N)^{\mathrm{op}}$$

for some progenerators $\mathbf{H}$-modules (M, φ_M) and (N, φ_N).

The set of equivalence classes of $\mathbf{H}$-module Azumaya monoids forms a group under the operation induced by the tensor product, $(\mathbf{AB}, \varphi_{A\otimes B} = (\varphi_A \otimes \varphi_B) \circ (C \otimes \tau_A^C \otimes B) \circ (\delta_C \otimes A \otimes B))$. The unit element is the class of the $\mathbf{H}$-module Azumaya monoid:

$$(E(M)^{\mathrm{op}}, \varphi_{E(M)} = [M, \varphi_M \circ (C \otimes \beta_M(M)) \circ (\tau_C^M \otimes [M, M]) \circ$$
$$\circ(\varphi_M \otimes C \otimes [M, M]) \circ (\tau_C^M \otimes C \otimes [M, M]) \circ (M \otimes \tau_C^C \otimes [M, M]) \circ$$
$$\circ(M \otimes C \otimes \lambda \otimes [M, M]) \circ (M \otimes \delta_C \otimes [M, M])] \circ \alpha_M(C \otimes [M, M]))$$

for some progenerator $\mathbf{H}$-module (M, φ_M); and the opposite of (A, φ_A) is $(A^{\mathrm{op}}, \varphi_A)$. This group is denoted by $\mathbf{BM}(\mathcal{C}, \mathbf{H})$.

If $\mathbf{1} = (1, 1, \tau^K, 1)$ is the trivial Hopf algebra in $\mathcal{C}$, then we define the Brauer group of Azumaya monoids in $\mathcal{C}$ as $\mathbf{BM}(\mathcal{C}, 1)$ and we will denote it by $\mathbf{B}(\mathcal{C})$.

Examples.

1) If $\mathcal{C}$ is the category of modules over a commutative ring R, then $B(\mathcal{C})$ is the Brauer group of R defined by Auslander and Goldman in [**4**].
2) If $\mathcal{C}$ is the category of sheaves of θ-modules, $B(\mathcal{C})$ is the Brauer group defined by Auslander in [**3**].
3) If $\mathcal{C}$ is the category of (R, σ)-Mod with σ an idempotent noetherian kernel functor in R-Mod, López and Villanueva obtain, in [**17**], a homomorphism $Br(R, \sigma) \to B((R, \sigma)\text{-Mod})$ which is an isomorphism if R is a noetherian ring, where $Br(R, \sigma)$ is the relative Brauer group introduced by Oystaeyen and Verschoren in [**22**].

14. Definition. We denote by $\mathbf{BM}_{\text{inn}}(\mathcal{C}, \mathbf{H})$ the subgroup of $\mathbf{BM}(\mathcal{C}, \mathbf{H})$ built up with the equivalence classes that can be represented by an $\mathbf{H}$-module Azumaya monoid with inner action.

([**2**, 16]).

15. Theorem. $\mathbf{BM}_{\text{inn}}(\mathcal{C}, \mathbf{H}) \cong \mathbf{B}(\mathcal{C}) \oplus H^2(\mathbf{H}, K)$.

Proof:

The sequence

$$1 \longrightarrow \mathbf{B}(\mathcal{C}) \xrightarrow{\ i\ } \mathbf{BM}_{\text{inn}}(\mathcal{C}, \mathbf{H}) \xrightarrow{\ \Pi\ } \mathbf{N}_{\mathcal{C}}(\mathbf{H}) \longrightarrow 1$$

is split exact, where the morphism i is given by $i[(A)] = [(A, \varepsilon_C \otimes A)]$ and the morphism Π is given by $\Pi[(A, \varphi_A)] := (\Pi(\mathbf{A}), \rho_{\Pi(A)})$ with $\Pi(\mathbf{A}) := \text{Ig}(m, n)$

$$\Pi(A) \xrightarrow{\ j\ } A \otimes C \underset{n}{\overset{m}{\rightrightarrows}} [A, A \otimes C]$$

where

$$m = [A, \mu_A \otimes C] \circ \alpha_A(A \otimes C)$$
$$n = [A, (\mu_A \otimes C) \circ (A \otimes (\varphi_A \circ \tau_C^A) \otimes C) \circ (\tau_A^A \otimes \delta_C)] \circ \alpha_A(A \otimes C)$$

([**2**, 17]).

If H is a finitely generated projective, commutative and cocommutative Hopf algebra over a commutative ring K, this result generalizes the one obtained by Beattie in [**5**], and if the action of H over A is inner, the description of $\Pi(A)$ is due to Beattie and Ulbricht ([**6**]). ■

References

1. ALONSO ÁLVAREZ, J. N., Extensiones Cleft en categorías cerradas. Interpretación cohomológica, *Alxebra* **58** (1992), Depto. Alxebra, Santiago de Compostela.
2. ALONSO ÁLVAREZ, J. N. AND FERNÁNDEZ VILABOA, J. M., Inner actions and Galois H-objects in a symmetric closed category, Depto. Alxebra, Santiago de Compostela (Preprint).
3. AUSLANDER, M., The Brauer group of a ringed space, *Journal of Algebra* **4** (1966), 220–273.
4. AUSLANDER, M. AND GOLDMAN, O., The Brauer group of a commutative ring, *Transactions A.M.S.* **97** (1960), 367–409.
5. BEATTIE, M., A direct sum decomposition for the Brauer group of H-module algebras, *Journal of Algebra* **43** (1976), 686–693.
6. BEATTIE, M. AND ULBRICHT, K. H., A Skolem-Noether theorem for Hopf algebra actions, *Comm. in Algebra* **18** (1990), 3713-3724.
7. BLATTNER, R. J. AND MONTGOMERY, S., Crossed products and Galois extensions of Hopf algebras, *Pacific J. of Math.* **137** (1989), 37–54.
8. CAENEPEEL, S., A note on inner actions of Hopf algebras, *Proc. A.M.S.* **113, no. 1** (1991).
9. CHASE, S. U. AND SWEEDLER, M. E., "*Hopf algebras and Galois theory*," Lect. Notes in Math. **97**, 1969.
10. DOI, Y., Equivalent crossed products for a Hopf algebra, *Communications in Algebra* **17** (1989), 3053–3085.
11. EARLY, T. E. AND KREIMER, H. F., Galois algebras and Harrison cohomology, *Journal of Algebra* **58** (1979), 136–147.
12. EILENBERG, S. AND KELLY, G. M., Closed categories, Proc. of the Conference on Categorical Algebra, La Jolla (1966), 421–562.
13. FERNÁNDEZ-VILABOA, J. M., Grupos de Brauer y de Galois de un álgebra de Hopf en una categoría cerrada, *Alxebra* **42** (1985), Depto. Alxebra, Santiago de Compostela.
14. FERNÁNDEZ-VILABOA, J. M. AND LÓPEZ-LÓPEZ, M. P., Cleft comodule triples in a symmetric closed category, *Bull. Soc. Math. Belgique, ser. B* **40** (1988), 283–297.
15. JEREMÍAS-LÓPEZ, A., El grupo fundamental relativo. Teoría de Galois y localización, *Alxebra* **56** (1991), Depto. Alxebra, Santiago de Compostela.
16. KREIMER, H. F. AND TAKEUCHI, M., Hopf algebras and Galois extensions of an algebra, *Indiana Univ. Math. J.* **30** (1981), 675–692.

17. LÓPEZ-LÓPEZ, M. P. AND VILLANUEVA-NOVOA, E., The Brauer group of the category (R, σ)-Mod. (An alternative to the theory of relative invariants), Proc. of the first Belgian-Spanish week on Algebra and Geometry (1988), Rijksuniversitair Centrum Antwerpen Groenenborgelaan 171, B-2020 Antwerpen, Belgium, 75–93.
18. MAC LANE, S., "*Categories for the working mathematicien*," G.T.M. **5**, Springer, 1971.
19. PACHUASHVILI, B., Cohomologies and extensions in monoidal categories, *Journal of Pure and Applied Algebra* **72** (1991), 109–147.
20. PAREIGIS, B., "*Non-additive ring and module theory IV: The Brauer group of a symmetric monoidal category*," Lecture Notes in Math. **549**, 1976, pp. 122–133.
21. SWEEDLER, M. E., Cohomology of algebras over Hopf algebras, *Transactions of the A.M.S.* **133** (1968), 205–239.
22. VAN OYSTAEYEN, F. AND VERSCHOREN, A., "*Relative invariants of rings. The commutative theory*," Lect. Notes in Pure and App. Math. **79**, Marcel Dekker, New York, 1983.
23. YOKOGAWA, K., The cohomological aspects of Hopf-Galois extensions over a commutative ring, *Osaka J. Math.* **18** (1981), 75–93.

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