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AN INDESTRUCTIBLE BLASCHKE PRODUCT IN THE LITTLE BLOCH SPACE

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Abstract

The little Bloch space, $\mathcal{B}_0$, is the space of all holomorphic functions f on the unit disk such that $\lim_{|z|\to 1} |f'(z)|(1-|z|^2) = 0$. Finite Blaschke products are clearly in $\mathcal{B}_0$, but examples of infinite products in $\mathcal{B}_0$ are more difficult to obtain (there are now several constructions due to Sarason, Stephenson and the author, among others). Stephenson has asked whether $\mathcal{B}_0$ contains an infinite, indestructible Blaschke product, i.e., a Blaschke product B so that $(B(z)-a)/(1-\bar{a}B(z))$, is also a Blaschke product for every $a \in D$. In this paper we give an afirmative answer to his question by constructing such a Blaschke product. We also answer a question of Carmona and Cufí by constructing a VMO function, f, so that $\|f\|_\infty = 1$ and whose range set, $R(f,a) = \{w : \text{ there exists } z_n \to a, f(z_n) = w\}$, equals the open unit disk for every $a \in \mathbf{T}$.

1. Introduction

Let $\mathbf{D} = \{|z| < 1\}$ denote the unit disk. The little Bloch space, $\mathcal{B}_0$, is the space of holomorphic functions f on $\mathbf{D}$ such that

$$\lim_{|z|\to 1} |f'(z)|(1-|z|^2) = 0.$$

Basic facts about $\mathcal{B}_0$ can be found in **[2]**. A Blaschke product is a holomorphic function of the form

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$$B(z) = \prod_n \frac{z_n - z}{1 - \bar{z}_n z} \frac{|z_n|}{z_n},$$

where $\sum(1 - |z_n|) < \infty$. Finite Blaschke products are clearly in $\mathcal{B}_0$, but examples of infinite products in $\mathcal{B}_0$ are not so obvious. Such functions do exist, as is shown in [**1**], [**8**], [**10**]. A more explicit example, as well as a characterization of such products in terms of the zero sequence $\{z_n\}$, has been given in [**3**]. That result answers several questions about $\mathcal{B}_0$, but does not resolve the following question from [**10**]: does $\mathcal{B}_0$ contain an infinite, indestructible Blaschke product, i.e., a Blaschke product B so that

$$B_a(z) = \frac{B(z) - a}{1 - \bar{a}B(z)},$$

is also a Blaschke product for every $a \in D$?

The question arises because of Frostman's theorem. An inner function F is a holomorphic function on $\mathbf{D}$ with boundary values of absolute value 1 a.e. on $\mathbf{T}$. Any such function can be written as a product

$$F(z) = B(z)S(z) = \left(\prod_{n=1}^{\infty} \frac{z_n - z}{1 - \bar{z}_n z} \frac{|z_n|}{z_n}\right)\left(\exp\left(-\int \frac{e^{i\theta} + z}{e^{i\theta} - z} d\mu(\theta)\right)\right),$$

of a Blaschke product and a singular inner function (μ is a finite, positive measure singular to $d\theta$). Frostman's theorem states that for any inner function F on $\mathbf{D}$,

$$F_a(z) = \frac{F(z) - a}{1 - \bar{a}F(z)},$$

is a Blaschke product for every $a \in \mathbf{D} \setminus E$ where E is an exceptional set of zero logarithmic capacity [**5**, Theorem II.6.4]. The constructions of Blaschke products in [**8**], [**10**] first build an inner function in $\mathcal{B}_0$ and then apply Frostman's theorem. Stephenson asked if this was unavoidable, e.g., does $\mathcal{B}_0$ contain any indestructible Blaschke products? The example in [**3**] is built by constructing the zero set, so does not use Frostman's theorem. Furthermore, a variant of Stephenson's construction gives a Blaschke product without using Frostman's theorem (see next section). In this note we expand on this observation to give a "cut and paste" construction of an indestructible Blaschke product in $\mathcal{B}_0$.

One could also try to produce such an example by finding a sufficient condition on the zeros for the product to be indestructible and which includes some sequences satisfying the $\mathcal{B}_0$ condition from [**3**]. One such sufficient condition for indestructibility is that the sequence be thin, i.e.,

$$\lim_{n\to\infty} \left|\prod_{k\neq n} \frac{z_k - z_n}{1 - \bar{z}_n z_k}\right| = 1.$$

However, this condition is incompatible with the $\mathcal{B}_0$ condition. An even more ambitious problem is to characterize indestructibility in terms of the zero-set. In [**6**], Morse has constructed a destructible Blaschke product which becomes indestructible when a single point is deleted from its zero-set. This indicates any characterization of indestructibility in terms of the zero set would be very delicate (and probably very difficult). A related problem has been solved, however. In some sense, finite products of interpolating Blaschke products are the "conformally invariant" class of Blaschke products. In [**7**] Nicolau has given a zero-set characterization of those Blaschke products B so that B_a is a finite product of interpolating Blaschke products for every a in the disk. Thus he has solved the conformally invariant version of the problem of characterizing indestructibility.

Our construction gives a Blaschke product whose singular set (the accumulation set of the zeros) has measure zero. If we could construct an example whose singular set was the entire circle, this function would also have the property that its range set, $R(f,a) = \{w : \text{ there exists } z_n \to a, f(z_n) = w\}$, equals the whole disk for every $a \in \mathbf{T}$. Carmona and Cufí had asked in [**4**] if there was a function in $H^\infty \cap \mathcal{B}_0$ with this property. I believe the construction can be modified to give such an example, but rather than do this, I will sketch the construction of a function $f \in H^\infty \cap \mathrm{VMO}$ with $\|f\|_\infty = 1$ and $R(f,a) = \mathbf{D}$ for every $a \in \mathbf{T}$. Since $\mathrm{VMO} \subset \mathcal{B}_0$, this is an even stronger result (again answering a question of Carmona and Cufí).

I thank Arturo Nicolau for bringing this question to my attention and our discussions on it. I also thank the referee for his helpful comments. His suggestions have clarified the exposition in several places.

2. The $\mathcal{B}_0$ construction

The idea is quite simple; we will build a simply connected Riemann surface by taking copies of the unit disk with slits and "gluing" different copies along the slits. A simple example of this idea is to take infinitely many copies of $\mathbf{D}\backslash[\frac{1}{2}, 1)$, and identifying the "top" edge of one copy with the "bottom" edge of the next. See Figure 1.

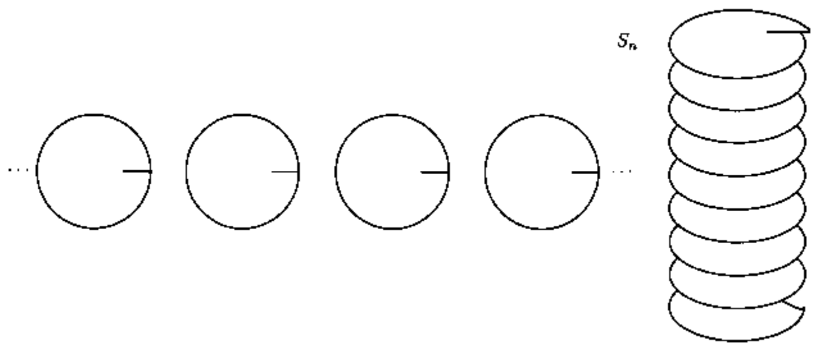

Figure 1. A single example

Let S_0 be the initial sheet, which we also refer to as the "zeroth sheet". This sheet contains a point corresponding to zero in the unit disk and we refer to this point as "0" on the surface. Let S_n be nth stage of this construction (the union of copies $-n$ to n) and $S = \cup_n S_n$ the limiting surface. For each of these surfaces the point "0" refers to the point 0 on S_0. In the rest of this paper we shall assume that any Riemann mapping of the unit disk to a constructed surface like S_n or S maps 0 in the disk to the point 0 on the surface. Whenever we talk about harmonic on the surface it is the push forward of normalized Lebesgue measure on the circle under such a Riemann mapping, i.e., harmonic measure will always be with respect to the point 0 on the zeroth sheet.

S is simply connected so there is a Riemann mapping $\Phi : \mathbf{D} \to S$ and there is an obvious holomorphic projection $P : S \to \mathbf{D}$. We claim that $F = P \circ \Phi$ must be an inner function because all the harmonic measure for S lives on the part of the boundary above the the unit circle. To prove this we consider S_n and show that the harmonic measure of the two radial slits in its boundary are $O(\frac{1}{n})$. To do this we map S_n to a half infinite strip $W = \{(x, y) : -\infty < x < 0, -(2n+1)\pi < y < (2n+1)\pi\}$ by the mapping

$$z \to \log(\frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}),$$

which has a well defined branch on S_n. The point 0 on the surface is mapped to $-1/2$ and the radial edges are mapped to the horizontal edges of the strip. Standard estimates (e.g., map the strip to a halfplane via $\sin(z/i)$ and use the Poisson integral) show that the harmonic measure of the horizontal edges of the strip with respect to the point $-1/2$ are approximately $1/n$. By conformal invariance of harmonic measure the claim about S_n is proved. Thus the circular part of ∂S_n has measure $\geq 1 - C/n$. Taking $n \to \infty$ we see that the circular part of ∂S has full measure, i.e, $|F| = 1$ a.e. on the unit circle.

In fact, F must be Blaschke product. To see this, recall that a function f in the unit ball of $H^\infty(\mathbf{D})$ is a Blaschke product iff the least harmonic majorant of $\log|f|$ is 0 (e.g., [**5**, Theorem II.2.4]). This says that F is a Blaschke product iff 0 is the least harmonic majorant of $\log|P(z)|$ on S. Let u be the least harmonic majorant of $\log|P(z)|$ on S. Then u restricted to S_n is harmonic and has boundary values 0 on $P^{-1}(\mathbf{T})$ and $\geq \log\frac{1}{2}$ on the two radial slits in ∂S_n. Thus $0 \geq u(0) \geq \frac{C}{n}\log\frac{1}{2}$. Since this holds for any n, $u(0) = 0$ (recall that the "0" in $u(0)$ refers to the designated point on the zeroth sheet). Since u is nonpositive this implies $u \equiv 0$ and so F is a Blaschke product.

Let $a \in \mathbf{D}$ and $\tau_a(z) = (z - a)/(1 - \bar{a}z)$. An argument like the one above shows shows that if $a \neq \frac{1}{2}$ then $F_a = \tau_a \circ F$ is a Blaschke product. However, since no point of S covers the point $\{\frac{1}{2}\}$, $F_{\frac{1}{2}}$ is never zero, so must be a singular inner function (in fact, since F is continuous except for one boundary point, up to rotations it must be $\exp(\lambda\frac{1-z}{1+z})$, for some $\lambda > 0$, i.e., the singular inner function corresponding to a positive point mass).

To build an indestructible Blaschke product we will have to vary the construction, adding sheets which cover the omitted points of earlier generations, and in particular, so that the least harmonic majorant of $\log|\tau_a(P(z))|$ on S is 0 for any choice of $a \in \mathbf{D}$. This says that not only is each point covered infinitely often, but there is some sense in which it is "frequently" covered. To get F into the little Bloch space imposes another constraint: given any $\epsilon > 0$ only finitely many of the sheets we attach may contain a disk of radius ϵ. This arises because of a geometric characterization of $\mathcal{B}_0$ due to Stegenga and Stephenson [**9**]. For f analytic on $\mathbf{D}$ and $a \in \mathbf{D}, r > 0$ they define $\Omega_a(r)$ to be the component of $f^{-1}(D(f(a), r))$ containing a, $\Gamma_a(r) = \partial\Omega_a(r) \cap \mathbf{T}$ and $r_f(a) = \sup\{r : \Gamma_a(r) = \emptyset\}$. Then $f \in \mathcal{B}_0$ iff $r_f(a) = o(1)$ as $|a| \to 1$. In particular, if the Riemann surface of f is obtained by identifying copies of $\mathbf{D}$ along slits, then the endpoints of any such slit are in the ideal boundary of the surface. Therefore f will be in the little Bloch space if for every $\epsilon > 0$, these endpoints of pasted edges are ϵ-dense in $\mathbf{D}$ for all but finitely many sheets.

We will inductively construct a sequence of positive numbers $\{\epsilon_n\}$ tending to zero, a sequence of finite point sets $\{E_n\}$, a collection of radial line segments T_n, and two sequences of integers $\{g_n\}, \{h_n\}$ tending to infinity. The sets $\{E_n\}, \{T_n\}$ will satisfy

(1) $T_n \subset T_{n+1}$, $E_n \subset E_{n+1}$, $E_n \subset T_n$.
(2) The endpoint of each segment in T_n is in E_n.
(3) $\epsilon_n/\epsilon_{n+1}$ is an even integer.
(4) Adjacent points of E_n on a segment of T_n are at distance ϵ_n from each other.
(5) $\sup_{z\in\mathbf{D}} \operatorname{dist}(z, E_n) \leq 10\epsilon_n$.

See Figure 2. An "edge" I of T_n denotes a subinterval on T_n connecting two adjacent points of E_n, i.e., a component of $T_n \backslash E_n$. We let e_n denote the number of edges in T_n. In the construction below each such edge will be treated as two separate pieces of the boundary of the domain $R_n = \mathbf{D}\backslash T_n$ corresponding to its two sides. One side will be pasted to a sheet of previous generation, the other pasted to one or more sheets in the next higher generation.

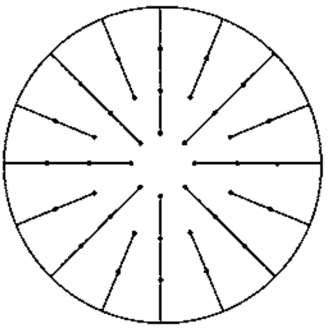

Figure 2. E_n, T_n, R_n

Given an edge I in the boundary of R_n we can either attach another copy of R_n, or divide the edge into $m = \epsilon_n/\epsilon_{n+1}$ edges in T_{n+1} (since $E_n \subset E_{n+1}$) and attach m copies of R_{n+1}. Given a sequence of integers $\{g_i\}$ we could build a Riemann surface as follows. Start with one copy of R_1 and attach $2e_1$ copies of of R_1 along (both sides of) each edge of T_1. Call this S_1. Then attach more copies of R_1 along each edge in ∂S_1 to obtain S_2 and continuing for g_1 generations, obtaining a nested sequence of surfaces $S_1 \subset S_2 \subset \cdots \subset S_{g_1}$. The term "generations" refers to the fact that to connect the point 0 in the zeroth sheet S_0 to any of the unpasted edges of S_k a path must pass though at least $k+1$ different sheets (i.e., copies of R_1) belonging to S_0, $S_1 \backslash S_0$, ... , $S_k \backslash S_{k-1}$.

We have obtained S_{g_1} by pasting together identical sheets, i.e., copies of R_1. To get the next surface, S_{g_1+1}, we attach to each unpasted edge of S_{g_1} ϵ_1/ϵ_2 copies of the sheet R_2. We obtain S_{g_1+2} by pasting a copy of R_2 to each unpasted edge of S_{g_1+1}. We continue in this way for g_2 generations, obtaining a surface $S_{g_1+g_2}$.

The next surface $S_{g_1+g_2+1}$, is constructed by attaching copies of R_3 to the unpasted edges of $S_{g_1+g_2}$. Thus given the sequence of integers $\{g_k\}$ (which tells us for how many generations to attach copies of R_k) and continuing in the obvious manner, we obtain an increasing, nested sequence of simply connected surfaces, $\{S_n\}$. Then $S = \cup_n S_n$, is a simply connected connected Riemann surface. If $\Phi : \mathbf{D} \to S$ is the Riemann map (mapping 0 to 0 on S_0), and $P : S \to \mathbf{D}$ the projection then $F = P \circ \Phi$ is a holomorphic function on the unit disk which we claim is an infinite Blaschke product in $\mathcal{B}_0$, if the parameters are chosen correctly.

This is essential Stephenson's construction in [**10**]. The fact that $F \in \mathcal{B}_0$ follows from the characterization of Stegenga and Stephenson mentioned earlier. If the sequence $\{g_i\}$ grows quickly enough, Stephenson shows the mapping F is an inner function. If $\text{dist}(0, T_n) \geq \epsilon_n$ then F is actually a Blaschke product (again if $g_n \nearrow \infty$ fast enough). To prove this, consider the least harmonic majorant u of $\log |P(z)|$ restricted to $S_N = S_{g_1+\cdots+g_n}$ The boundary breaks into two pieces $\partial S_N = \partial_1 S_N \cup \partial_2 S_N$ corresponding respectively to $P^{-1}(\mathbf{T})$ and the radial edges. Then u has boundary values 0 on $\partial_1 S_N$ and $u \geq \log \epsilon_n$ on $\partial_2 S_N$. The set $\partial_2 S_N$ can be made to have as small harmonic measure as we wish by taking g_n large enough, so we may take

$$0 \geq u \geq \omega(\partial_2 S_N) \log \epsilon_n \geq -\frac{1}{n}$$

if g_n is large enough (recall that as before, harmonic measure refers to the harmonic measure with respect to the point 0 on the zeroth sheet

S_0). Thus F is a Blaschke product, but it cannot be indestructible since it only takes values in each E_n finitely often. As Stephenson points out, this example shows the exceptional set in Frostman's theorem may be dense in the unit disk.

To make F indestructible, we modify the construction slightly. Associated to each E_n define another set F_n by replacing each $z \in E_n$ by a point $w \in E_{n+1}$ with $|z - w| = \frac{1}{2}\epsilon_n$ and such that w is on same radius as z. The sets F_n satisfy approximately the same density conditions as the E_n (with ϵ_n replaced by $2\epsilon_n$). Our idea is to modify the construction by alternating the use of the sets E_n and F_n in the construction. Since $E_n \cap F_n = \emptyset$ this means our surface will cover the whole disk and since $\max(\text{dist}(z, E_n), \text{dist}(z, F_n)) \geq \epsilon_n/4$ for every point z in the disk, we should be able to prove our function is indestructible by estimating harmonic measure either on the "E_n-sheets" or "F_n-sheets" (depending on whether z is far from E_n or far from F_n). However since $E_n \cap F_n = \emptyset$, we need some further modifications to to able to attach an "F_n-sheet" to an "E_n-sheet".

This is how we attach a F_n-sheet to an E_n-sheet. Consider a component interval I of T_n with endpoints in E_n. Let $\tilde{T}_n$ be the analogue of T_n for the set F_n and let $\tilde{R}_n = \mathbf{D} \backslash \tilde{T}_n$. Assume (without loss of generality) that F_n has been chosen so $T_n \subset \tilde{T}_n$. Let $\{a_0, a_1, \ldots, a_n\} = I \cap E_{n+1}$, listed in order (e.g., a_0, a_n are the endpoints of I). Let $F_{n,j} = F_n \cup \{a_j, a_{j+1}\}$. Along each interval (a_j, a_{j+1}) attach a copy of $\tilde{R}_n$. To this sheet we attach copies of $\tilde{R}_n$ along all component intervals of $\tilde{T}_n \backslash F_j$. We continue in this way, attaching copies of $\tilde{R}_n$ along intervals of $\tilde{T}_n \backslash F_n$, except for those sheets reached by either looping around a_j or around a_{j+1}, in which case we are forced to attach copies of $\tilde{R}_n$ along intervals of the form $\tilde{T}_n \backslash F_n \cup \{a_j\}$ (or $\tilde{T}_n \backslash F_n \cup \{a_{j+1}\}$). Some of these identifications are illustrated in Figure 3.

More precisely, Figure 3 shows regions on four sheets, labeled I, II, III, IV. Sheet I is pasted to sheet II along the edge $[a_j, a_{j+1}]$. Sheet II is pasted to sheet III along edge $[q, a_j]$ and to sheet IV along the edge $[p, q]$, where p, q are points of F_n adjacent to a_j. The solid and doted curves illustrate paths from sheet I to sheets III and IV respectively which (must) pass through sheet III. Notice that the point $A \in E_n$ in the ideal boundary of sheet I is covered when sheets II and IV are pasted along $[p, q]$. Similarly $a_0 \in E_n$ is covered when II is pasted to III along $[q, a_j]$ (assuming $j \neq 0$; otherwise it would be covered by some sheet attached to sheet IV).

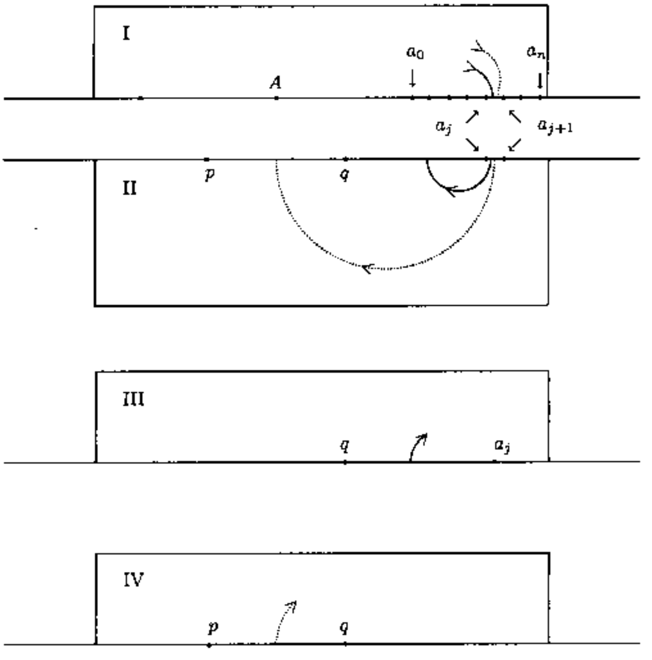

Figure 3. Modifications to cover E_n

Suppose we have already constructed a surface S_n whose boundary consists of arcs covering $\mathbf{T}$ or edges of T_n. To each component interval I of $T_n \backslash E_n$ we attach copies of $\tilde{R}_n$ as described above. Do this for g_{n+1} generations. The resulting sheets cover E_n (the only sheets which do not cover every point of E_n are those attached along subintervals of the form (a_0, a_1) or (a_{n-1}, a_n) in the construction above). We call the resulting surface $\tilde{S}_n$. To the boundary of $\tilde{S}_n$ attach copies of $R_{n+1} = \mathbf{D} \backslash T_{n+1}$ along component intervals of $T_{n+1} \backslash E_{n+1}$ for h_{n+1} generations (this poses no difficulties since E_n, F_n and all points of the form a_j in the previous stage of construction were in E_{n+1}; thus every radial interval in the boundary of $\tilde{S}_n$ has endpoints in E_{n+1}). The resulting surface is called S_{n+1} and satisfies the induction hypothesis. The union over n of these (nested) surfaces is denoted S and we obtain the desired function by mapping the disk to S and then projecting back to the disk. All that remains is to choose the sequences $\{\epsilon_n\}$, $\{g_n\}$ and $\{h_n\}$ so that the harmonic measure estimates hold. We will first choose g_n, then ϵ_{n+1} and then h_{n+1}.

Let u be the least harmonic majorant of $\log |\tau_a \circ P(z)|$. We want to show $u \equiv 0$. If $\text{dist}(a, E_n) \geq \frac{1}{4}\epsilon_n$ then we estimate u on S_n by an argument similar to the one used before. More precisely, let W be S_n minus the components of $P^{-1}(\mathbf{D} \backslash D(a, \frac{1}{8}\epsilon_n))$ which hit ∂S_n (recall that $D(a, \frac{1}{8}\epsilon_n) \cap E_n = \emptyset$ so they can only hit ∂S_n in radial edges if at all). Divide the boundary up into two pieces $\partial W = \partial_1 W \cup \partial_2 W$ according to whether the points lie over $\mathbf{T}$ or $T_n \cup \partial D(a, \frac{1}{8}\epsilon_n)$. The function u is zero on $\partial_1 W$ and $u \geq \log \frac{1}{8}\epsilon_n$ on $\partial_2 W$. Moreover we can make the harmonic measure of $\partial_2 W$ as small as we wish by taking g_n large enough. We choose g_n so that

$$0 \geq \omega(\partial_2 W) \log \frac{1}{8}\epsilon_n \geq -\frac{1}{n}.$$

Then $0 \geq u(0) \geq -\frac{1}{n}$ in this case.

If $\text{dist}(a, E_n) \leq \frac{1}{4}\epsilon_n$ then we do the computation on $\tilde{S}_n$. In this case let W be $\tilde{S}_n$ minus the components of $P^{-1}(D(a, \frac{1}{8}\epsilon_{n+1}))$ which hit $\partial \tilde{S}_n$. Divide the boundary into three pieces $\partial W = \partial_1 W \cup \partial_2 W \cup \partial_3 W$, according to whether the points lie over $\mathbf{T}$, T_n or $\partial D(a, \frac{1}{8}\epsilon_n)$. As before $u = 0$ on $\partial_1 W$. On the rest of the boundary $0 \geq u \geq \log(\epsilon_{n+1}/8)$. The harmonic measure of $\partial_3 W \cap S_n$ can be made as small as desired by taking ϵ_{n+1} small enough (in fact the measure decays like $C(\epsilon_{n+1})^\alpha$ for some α depending on S_n (it depends on the number of sheets in S_n)). The boundary piece $\partial_3 W \backslash S_n$ can only be reached by leaving S_n through at most two of the intervals in $T_n \backslash E_{n+1}$ (i.e., intervals of the form (a_j, a_{j+1}) described above). These intervals also have harmonic measure in S_n dominated by $C(\epsilon_{n+1})^\alpha$. Therefore we can choose ϵ_{n+1} so that

$$0 \geq \log(\frac{1}{8}\epsilon_{n+1})\omega(\partial_3 W) \geq -1/n.$$

The harmonic measure of $\partial_2 W$ can be made as small as we wish by taking h_{n+1} large enough, i.e., choose h_{n+1} so that

$$0 \geq \log(\frac{1}{8}\epsilon_{n+1})\omega(\partial_2 W) \geq -1/n.$$

Thus $u(0) = 0$ and so $u \equiv 0$ on S. Hence F is an indestructible Blaschke product, as desired.

3. The VMO Construction

In this section we will construct an $f \in \text{VMO}$ with $\|f\|_\infty = 1$ so that $R(f, a) = \mathbf{D}$ for every $a \in \mathbf{T}$. The construction is a "cut and paste" argument like the one in the previous section. As before we obtain the surface by identifying copies of the unit disk along radial slits, although the details are somewhat different.

First we will describe how to construct a function $f \in H^\infty \cap \mathrm{VMO}$ whose range sets $R(f,a)$ are dense in $\mathbf{D}$ for every $a \in \mathbf{T}$. Then we will then indicate how to modify the construction to obtain a function whose range sets equal $\mathbf{D}$ for every a. Recall that a dyadic subinterval of $[0,1]$ is an interval of the form $[j2^{-n},(j+1)2^{-n}]$ for some $n \geq 0$ and $0 \leq j < 2^n$. A dyadic subinterval of a line segment $[a,b]$ denotes an image of such an interval under the affine mapping from $[0,1]$ to $[a,b]$.

For each $n = 1,2,3,\ldots$ we construct a collection of radial line segments T_n with one endpoint on the unit circle. Each line segment in T_n is subdivided into a countable number of subintervals $I_{n,j}$. The collection of all endpoints of some such $I_{n,j}$ is denoted by E_n. It is a simple exercise to construct such sets so that

(1) $T_n \subset T_{n+1}$, $E_n \subset E_{n+1}$.
(2) The radial segments in T_n have arguments which are rational multiples of 2π.
(3) Every dyadic subinterval of $I_{n,j}$ is also of the form $I_{m,k}$ for some $m > n$.
(4) $\mathrm{diam}(I_{n,j}) \leq \mathrm{dist}(I_{n,j}, \mathbf{T})$.
(5) For every $z \in \mathbf{D}$, $\mathrm{dist}(z, E_n) \leq 2^{-n}(1-|z|)$.
(6) For every $z \in E_n$, $\mathrm{dist}(z, E_n \backslash z) \geq 2^{-n}(1-|z|)/100$.

Details are left to the reader (see Figure 4).

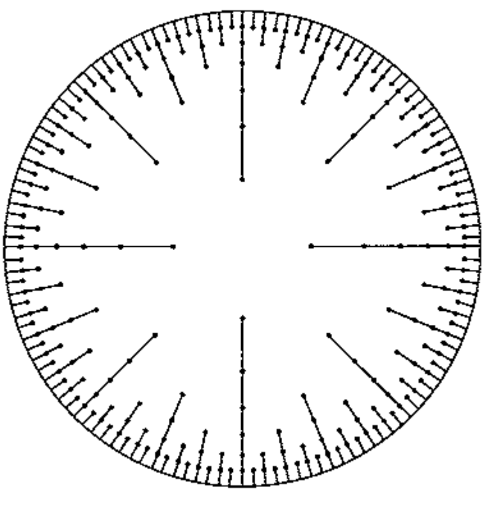

Figure 4. T_n and E_n

We will also need a Cantor set $C \subset [-1,1]$ of positive length such that

(7) If $z \notin \{|y| \leq (1-|x|)/10\}$ (a diamond shaped neighborhood of C) then the harmonic measure with respect to z satisfies $\omega(z, C, \mathbf{R}^2 \backslash [-1,1]) > 1 - 10^{-4}$. (In other words the set C looks like a solid line segment from far away).

(8) If $z \in [-1,1] \backslash C$ and I is the interval of $[-1,1] \backslash C$ containing z then

$$\omega(z, D(z, 10\text{dist}(z,C)) \cap C, D(z, 10\text{dist}(z,C) \backslash C) \geq 1 - 10^{-4}.$$

(9) All the component intervals of $[-1,1] \backslash C$ are dyadic.

The construction is standard and left to the reader.

To each interval $I_{n,j} \subset T_n$ associate the Cantor set $C_{n,j} \subset I_{n,j}$ which is the affine image of C under the obvious map from $[-1,1]$ to $I_{n,j}$. The component intervals of $I_{n,j} \backslash C_{n,j}$ will be denoted $J_{n,j,k}$ (which is as many subscripts as I dare use). Each of these is a dyadic subinterval of $I_{n,j}$ and hence is equal to some $I_{m,s}$ for some m and s. We build our surface by identifying copies of the disk along such segments.

More precisely, let $R_n = \mathbf{D} \backslash T_n$, i.e., R_n is the disk with countable many radial slits removed. To start the construction let $S_1 = R_1$. For each interval of the form $J_{n,j,k}$ in the boundary of S_1 find the m for which $J_{n,j,k} = I_{m,s}$ and attach a copy of R_m to S_2 along this common segment (as in the previous section we think of different sides of a boundary slit as being different boundary points; thus each $J_{n,j,k}$ corresponds to two boundary intervals). Do this for every $J_{n,j,k}$ gives S_2. We obtain S_3 in the same way, by attaching the appropriate R_n to each of the boundary intervals $J_{n,j,k}$ for each copy of R_n in $S_2 \backslash S_1$. In an obvious manner we obtain an increasing sequence $S_1 \subset S_2 \subset \ldots$ of simply connected Riemann surfaces. The union, S, is the desired surface. Let f be the function obtained by composing the covering map $\mathbf{D} \to S$ (say one that maps $0 \in \mathbf{D}$ to $0 \in S_1$) with the projection map from S to $\mathbf{D}$. Clearly f a holomorphic function with $\|f\|_\infty = 1$.

For a point $w \in S$ let $r(w)$ denote the radius of the largest disk centered at w contained in S. Properties (7) and (8) of the Cantor set C imply that there is an absolute constant M so that for any $\epsilon > 0$

$$\omega(w, \partial S \cap D(w, Mr(w)), S) \geq \frac{99}{100}.$$

(Since R_n is simply connected, a Brownian path starting at w hits $\partial R_n \cap D(w, Mr(w))$ with high probability (Beurling's theorem says with probability $\geq 1 - CM^{-1/2}$). If it never leaves R_n then it certainly hits $\partial S \cap D(w, Mr(w))$. If it does leave R_n then estimates (7) and (8) say that it hits a third sheet with probability less than 10^{-4}. Thus it hits $\partial S \cap D(w, Mr(w))$ with high probability.)

In terms of f this means

$$\omega(z, \{x \in \mathbf{T} : |f(x) - f(z)| > Mr(f(z))\}, \mathbf{D}) \leq \frac{1}{100}.$$

Since the set of points where $r(f(z))$ is larger than some fixed number is a compact subset of the disk we see that $1 - |z| < \delta = \delta(\epsilon)$ implies

$$|\{x \in I_z : |f(x) - f(z)| > \epsilon\} \leq \frac{1}{10}|I_z|,$$

where I_z denotes the interval centered at $z/|z|$ of length $1 - |z|$. This condition is well known to imply $f \in$ VMO (e.g. [**5**, Exercise VI.4]).

Thus we have constructed a $f \in$ VMO with $\|f\|_\infty = 1$. We claim that the range set $R(f, a)$ is dense in $\mathbf{D}$ for every boundary point a. To see this, note that each edge we paste along corresponds to a cross cut in the unit disk and that the diameter of the cross cut is comparable to its harmonic measure with respect to zero, which equals the harmonic measure of the edge in the surface with respect to the point 0 in the zeroth sheet. Thus a path which crosses infinitely many sheets corresponds to a path in the disk crossing infinitely many such cross cuts whose harmonic measures (and hence diameters) tend to zero, and thus defines a boundary point of the disk. The range set at that point will be dense in the disk since each sheet is open and dense and is covered by the part of the disk separated from the origin by one of the cross cuts and hence covered by a neighborhood of the boundary point. Since a countable intersection of dense open sets is dense, the range set at that boundary point is dense (in fact, a dense G_δ). The set of boundary points corresponding to paths which cross infinitely many sheets is dense (if a path approaches the boundary through one sheet we can always change it arbitrarily close to the boundary so that it crosses infinitely many sheets)). Therefore the range set is dense for every boundary point (since a countable intersection of dense G_δ's is dense).

However, we do not have $R(f, a) = \mathbf{D}$ because the copies of the Cantor set are omitted. To take care of this problem we can modify the construction slightly. Instead of attaching a copy of R_m to R_n as above, attach a "perturbed" copy $\tilde{R}_m$ of R_m. We define $\tilde{R}_m = \mathbf{D} \backslash \tilde{T}_m$, where $\tilde{T}_m, \tilde{E}_m$ have the same properties as before except that T_m and $\tilde{T}_m$ only intersect along one radial segment; the one where we want to attach $\tilde{R}_m$ to R_n. For example, $\tilde{T}_m$ could be obtained from T_m by simply changing the argument of each radial segment by some homeomorphism of the circle which only fixes the one desired angle and maps the other rationals (e.g., the arguments of the other radial segments of T_m) onto irrational values. Such a map is easy to construct, e.g., $\theta \to \theta + \lambda\theta(\theta - 2\pi)$ with a small irrational λ will fix 0 and map other rational multiples of 2π to irrational ones.

When we attach such a modified sheet $\tilde{R}_m$ to R_n the union $R_n \cup \tilde{R}_m$ covers the whole disk except for a subset of the radial segment $T_n \cap \tilde{T}_m$. This exceptional set can be covered when we attach a normal" sheet R_p to $\tilde{R}_m$ along any radial segment, except one with the same argument as the radial segment where we attached $\tilde{R}_m$ to R_n. Thus by alternating "normal" and "perturbed" versions of R_n in the construction we obtain a VMO function with $\|f\|_\infty = 1$ and $R(f,a) = \mathbf{D}$ for every $a \in \partial\mathbf{D}$. (One sheet does not cover the disk; however a given sheet and all sheets separated by it from the zeroth sheet do cover the whole disk. Thus the cross cut argument above shows the range set is the whole disk at every boundary point corresponding to a path crossing infinitely many cross cuts. Since such points are dense on the boundary, it is easy to see $R(f,a) = \mathbf{D}$ for every a.)

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