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POINTWISE CONVERGENCE OF THE FOURIER TRANSFORM ON LOCALLY COMPACT ABELIAN GROUPS

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Abstract

We extend to locally compact abelian groups, Fejer's theorem on pointwise convergence of the Fourier transform. We prove that $\lim \varphi_U * f(y) = f(y)$ almost everywhere for any function f in the space $(L^p, l^\infty)(G)$ (hence in $L^p(G)$), $2 \leq p \leq \infty$, where $\{\varphi_U\}$ is Simon's generalization to locally compact abelian groups of the summability Fejer Kernel. Using this result, we extend to locally compact abelian groups a theorem of F. Holland on the Fourier transform of unbounded measures of type q.

1. Notation and Preliminary Results

Throughout, G is a locally compact abelian group, with dual group Γ, and Haar measure m. By the structure theorem, G is represented by $\mathbf{R}^a \times G_1$, where a is a nonnegative integer and G_1 is a group which contains an open compact subgroup H. The set of basic neighbourhoods of $x \varepsilon G$ is denoted by $\mathcal{N}_x(G)$. We write $C_c(G), C_0(G)$ for the spaces of functions on G that are continuous, with compact support and vanish at infinity, respectively. We consider the amalgam spaces $(L^p, l^q)(G), (C_0, l^q)(G) (1 \leq p, q \leq \infty)$ as defined in [**S**]. The Fourier transform (inverse Fourier transform) of a measure μ, is denoted by $\hat{\mu}$ $(\check{\mu})$. We let $A_c(G)$ be the set of all functions f in $C_c(G)$ such that $\hat{f} \varepsilon L^1(\Gamma)$. The characteristic function of a subset E of G is denoted by χ_E. The conjugate p' of a number p is such that $1/p + 1/p' = 1$. For each $U \varepsilon \mathcal{N}_0(G)$, A.B. Simon [**Si**] defined a function φ_U as the product of two functions α_U and β_U defined on $\mathbf{R}^a$ and on G_1, respectively.

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The function β_U is continuous, nonnegative, with $L^1(G)$-norm equal to 1, and

$$\sup_G |\beta_U(x)| = B_U \leq 2m(U)/1 - 2m(U) \qquad \text{finite.} \tag{1}$$

Hence $B_U \to 0$ as $U \to 0$.

The function α_U is defined as follows.

Let $(-\delta_1, \delta_1) \times \cdots \times (-\delta_a, \delta_a) \times U_H$ be a product neighbourhood contained in U, where $\delta_i > 0 \quad (i = 1, \ldots, a)$, and U_H is an element of $\mathcal{N}_0(G)$ included in H. For $i = 1, \ldots, a$ we set $U_i = (-\delta_i, \delta_i), N_i = 1/\delta_i$, and define the function α_{U_i} on $\mathbf{R}$ by

$$\alpha_{U_i}(t) = \frac{1 - \cos(N_i t)}{\pi N_i t^2}$$

For $t = (t_1, \ldots, t_a)$ in $\mathbf{R}^a$, the function α_U is given by $\alpha_U(t) = \Pi_{i=1}^a \alpha_{U_i}(t_i)$. Clearly α_U is continuous, nonnegative, and its $L^1(\mathbf{R}^a)$-norm is equal to 1. Each φ_U has the following properties. For a proof see [**Si**].

1.1) φ_U is continuous, nonnegative and bounded
1.2) φ_U is integrable and $||\varphi_U||_1 = 1$
1.3) $\hat{\varphi}_U \,\varepsilon\, C_c(\Gamma)$ and $||\hat{\varphi}_U||_\infty \leq 1$
1.4) $\varphi_U(x) = \int_\Gamma \hat{\varphi}(\gamma)\gamma(x)d\gamma$ by 1.3)
1.5) For $\varepsilon > 0$ and $U \,\varepsilon\, \mathcal{N}_0(G)$ given, we can find a V such that if $V' \leq V$ and $x \not\varepsilon \; U$, then $\varphi_V(x) < \varepsilon$ and $\int_{G-V} \varphi_{V'}(x)dx < \varepsilon$.
1.6) $\lim_U \hat{\varphi}_U(\gamma) = 1$.
1.7) the family $\{\varphi_U | U \,\varepsilon\, \mathcal{N}_0(G)\}$ is an approximate identity in $L^1(G)$.

We add to this list the fact that each φ_U belongs to the Wiener algebra $(C_0, l^1)(G)$ [W].

Proposition 1.1. *For each U in $\mathcal{N}_0(U)$, the function α_U belongs to* $(C_0, l^1)(\mathbf{R}^a)$.

Proof: . Since $\alpha_{U_i}(i = 1, \ldots, a)$ is an even function we have for n in $\mathbf{Z} - \{0, -1\}$ that

$$\sup_{t \varepsilon [0,1]} \alpha_{U_i}(t + n) = \sup_{t \varepsilon [0,1]} \alpha_{U_i}(t - (1 + n)) \leq \frac{2}{N_i \pi} \frac{1}{n^2}.$$

If $n \,\varepsilon\, \{0, -1\}$, then there exists a constant C_i such that

$$\sup_{t \varepsilon [0,1]} \alpha_{U_i}(t + n) \leq \frac{N_i}{\pi} C_i$$

because the limit

$$\lim_{t\to -n} \frac{1-\cos\ N_i(n+t)}{(N_i(n+t))^2}$$

exists.

Therefore for all $i = 1, \ldots, a$ and all integer n we have that

$$\sup_{t\varepsilon[0,1]} \quad |\alpha_{U_i}(t+n)| \leq ca_n$$

where

$$c = \max_{1\leq i\leq a}(2/(N_i\pi), N_iC_i/\pi)$$

and a_n is equal to $1/n^2$ if $n \,\varepsilon\, \mathbf{Z} - \{0,-1\}$, and to 1 if $n \,\varepsilon\, \{0,-1\}$.

Finally, for $i = 1, \ldots, a$ we have that

$$\begin{aligned}||\alpha_{U_i}||_{\infty 1} &= \sum_{\mathbf{Z}} \sup_{t\varepsilon[n,n+1]} |\alpha_{U_i}(t)| \\ &= \sum_{\mathbf{Z}} \sup_{t\varepsilon[0,1]} |\alpha_{U_i}(t+n)| \\ &\leq C \sum_{\mathbf{Z}} a_n < \infty.\end{aligned}$$

From the definition of the norm $|| \ ||_{\infty 1}$, it is easy to see that

$$||\alpha_U||_{\infty 1} = \Pi_{i=1}^a ||\alpha_{U_i}||_{\infty 1}. \quad \blacksquare$$

Corollary 1.2. *For each U in $\mathcal{N}_0(G)$, the function φ_U belongs to $(C_0, l^1)(G)$.*

Proof: By (1) we have for all (t,s) in G that

$$\varphi_U(t,s) = \alpha_U(t)\beta_U(s) \leq B_U\alpha_U(t),$$

hence

$$||\varphi_U||_{\infty 1} \leq B_U \sum_{n\varepsilon \mathbf{Z}^a} \quad \sup_{t+n\varepsilon[0,1]^a} |\alpha_U(t)| = B_U||\alpha_U||_{\infty 1}. \quad \blacksquare$$

For the rest of this paper φ_U, α_U, and β_U are as indicated in this section.

2. Main Theorem

In this second section we want to prove that

$$\lim_{U\to 0} \varphi_U * f(y) = f(y) \qquad \text{almost everywhere} \tag{2}$$

for all f in $(L^p, l^\infty)(G)(2 < p \leq \infty)$.

First, we prove two lemmas.

Lemma 2.1. *Let V and K be two elements of $\mathcal{N}_0(G)$ of the form*

$$V = (-\delta_1, \delta_1) \times \cdots \times (-\delta_a, \delta_a) \times V_H$$

and $K = [-\gamma_1, \gamma_1] \times \cdots \times [-\gamma_a, \gamma_a] \times K_H$, where $\delta_i > 0, \gamma_i > 0$ $(i = 1, \dots, a), V_H$ and K_H are elements of $\mathcal{N}_0(G)$ contained in H, and K_H is compact.

For $1 \leq p < \infty$, we define $\eta_i = \min(\delta_i^{2p}, \gamma_i)$ $\quad (i = 1, \dots, a)$, and we let W_H be the interior of K_H. Then the set $W = [-\eta_1, \eta_1,] \times \cdots \times [-\eta_a, \eta_a] \times W_H$ belongs to $\mathcal{N}_0(G)$ and for a fixed $y = (y_0, s_0) = (y_1, \dots, y_a, s_0)$ in G, the element $W_y = y + W$ of $N_y(G)$ has the following properties:

2.1) $W_y \subseteq y + K_H$

2.2) *If $\Pi_a = [-\eta_1 + y_1, \eta_1 + y_1] \times \cdots \times [-\eta_a + y_a, \eta_a + y_a]$, then*

$$\left[\int_{\Pi_a} \alpha_U(y_0 - x)^p dx\right]^{1/p} = O(\Pi_{i=1}^a \delta_i)$$

2.3) $\mathbf{R}^a - \Pi_a \subseteq \cup\, I_n$, *where $\{I_n\}$ is a countable family of compact subsets of $\mathbf{R}^a$, and*

$$\sum_{\mathbf{N}} \left[\int_{I_n} \alpha_U(y_0 - x)^p dx\right]^{1/p} = O(\Pi_{i=1}^a \delta_i).$$

2.4) *There exists a constant C such that $\sup_{\mathbf{N}} \mathcal{C}(I_n) \leq C$, where $\mathcal{C}(I_n)$ is the cardinality of the set*

$$\{j \in \mathbf{Z}^a \mid (j + [0,1]^a) \cap I_n \neq \phi\}.$$

Proof: Several constants will appear during the proof and since their specific value is irrelevant for our needs we just write $C_1, C_2, \dots C_q$. Part

2.1) is clear. Set $J_i = [-\eta_i + y_i, \eta_i + y_i](i = 1, \ldots, a)$. Part 2.2) follows from the continuity of α_{U_i} because

(3)
$$\left[\int_{J_i} \alpha_{U_i}(y_i - x)^p dx\right]^{1/p} = \left[\int_{-\eta_i}^{\eta_i} \alpha_{U_i}(x)^p dx\right]^{1/p} \leq C_1 N_i \eta_i^{1/p} \leq C_2 \delta_i.$$

Now, for each $i = 1, \ldots, a$, let $L(n,i)$ and $R(n,i)(n\varepsilon\mathbf{N})$ be the intervals

$$[-n - 1 - \eta_i + y_i, -n - \eta_i + y_i] \text{ and } [n + \eta_i + y_i, n + 1 + \eta_i + y_i]$$

respectively. Then

$$\mathbf{R} - J_i = (-\infty, -\eta_i + y_i) \cup (\eta_i + y_i, \infty) \subseteq \bigcup_{\mathbf{N}} L(i,n) \cup \bigcup_{\mathbf{N}} R(i,n),$$

and

$$\int_{L(n,i)} \alpha_{U_i}(y_i - x)^p dx \leq C_3 \delta_i^p a_n$$

where

$$a_n = \frac{1}{(n_i + n)^{2p-1}} - \frac{1}{(n_i + n + 1)^{2p-1}}.$$

Since $\sum a_n^{1/p}$ converges we conclude that

$$\sum_{\mathbf{N}} \left[\int_{L(n,i)} \alpha_{U_i}(y_i - x)^p dx\right]^{1/p} \leq C_4 \delta_i.$$

Similarly

$$\sum_{\mathbf{N}} \left[\int_{R(n,i)} \alpha_{U_i}(y_i - x)^p dx\right]^{1/p} = C_5 \delta_i.$$

Clearly $\sup_{\mathbf{N}} C\ (L(n,i))$ and $\sup_{\mathbf{N}} C\ (R(n,i))$ are less than or equal to 2, hence for $i = 1, \ldots, a$, the set $\mathbf{R} - J_i$ is equal to $\cup I_n$, where each I_n is compact, $\sup C\ (I_n) \leq 2$ and

$$\sum_{\mathbf{N}} \left[\int_{I_n} \alpha_{U_i}(y_i - x)^p dx\right]^{1/p} = O(\delta_i). \tag{4}$$

Since $\mathbf{R} = (\mathbf{R} - J_a) \cup J_a$, and J_a is compact, by (3) and (4) we see that $\mathbf{R} = \cup K_n$, with each K_n compact, $\sup C(K_n) \leq C_6$, and

$$\sum_{\mathbf{N}} \left[\int_{K_n} \alpha_{U_a}(y_a - x)^p dx\right]^{1/p} = O(\delta_a). \tag{5}$$

We prove properties 2.3) and 2.4) by induction on a. The case $a = 1$ follows from (3). Suppose that 2.3) and 2.4) hold for $a-1$. That is, $\mathbf{R}^{a-1} - \Pi a - 1 \subseteq \cup I_n$, each I_n a compact subset of $\mathbf{R}^{a-1}$, $\sup C(I_n) \leq C_7$, and

$$\sum_{\mathbf{N}} \left[\int_{I_n} \Pi_{i=1}^{a-1} \alpha_{U_i}(y_i - x_i)^p dx \right]^{1/p} = O(\Pi_{i=1}^{a-1} \delta_i). \tag{6}$$

By (4) with $i = a$, we have that $\mathbf{R} - J_a \leq \cup I_j$, each I_j a compact subset of $\mathbf{R}, \sup \mathcal{C}(I_j) \leq 2$ and

$$\sum_{\mathbf{N}} \left[\int_{I_j} \alpha_{U_a}(y_a - x)^p dx \right]^{1/p} = O(\delta_a). \tag{7}$$

Then

$$\begin{aligned} \mathbf{R}^a - \Pi a &= (\mathbf{R}^{a-1} \times \mathbf{R}) - (\Pi(a-1) \times Ja) \\ &= (\mathbf{R}^{a-1} - \Pi(a-1)) \times (\mathbf{R} \cup \Pi(a-1)) \times (\mathbf{R} - Ja) \\ &\leq \bigcup_{n,m} (I_n \times K_m) \bigcup_{\mathbf{N}} (\Pi(a-1) \times Ij). \end{aligned}$$

The sets $I_n \times K_m$ and $\Pi(a-1) \times Ij$ are compact subsets of $\mathbf{R}$, for all n, m, j. Hence $\sup \mathcal{C}(I_n \times K_m) \leq C_8$ and $\sup \mathcal{C}(\Pi(a-1) \times Ij) \leq C_9$. Therefore 2.4) holds with $C = \max(C_8, C_9)$. Finally, by (5) and (6) we have that

$$\begin{aligned} &\sum_{n,m} \left[\int_{I_n \times K_m} \alpha_U(y_0 - x)^p dx \right]^{1/p} = \\ &= \sum_{\mathbf{N}} \left[\int_{I_n} \Pi_{i=1}^{a-1} \alpha_{U_i}(y_i - x_i)^p dx \right]^{1/p} \sum_{\mathbf{N}} \left[\int_{K_m} \alpha_{U_a}(y_a - x)^p dx \right]^{1/p} = \\ &\qquad = O(\Pi_{i=1}^{a} \delta_i). \end{aligned}$$

We conclude from (3) and (7) that

$$\begin{aligned} &\sum_{\mathbf{N}} \left[\int_{\Pi a - 1 \times Ij} \alpha_U(y_0 - x)^p dx \right]^{1/p} = \\ &= \Pi_{i=1}^{a-1} \left[\int_{I_i} \alpha_{U_i}(y_i - x)^p dx \right]^{1/p} \sum_{\mathbf{N}} \left[\int_{I_j} \alpha_{U_a}(y_a - x)^p dx \right]^{1/p} = \\ &\qquad = O(\Pi_{i=1}^{a} \delta_i). \end{aligned}$$

Lemma 2.2. *For each V_y in $\mathcal{N}_y(G)$ $(y \varepsilon G)$*

$$\lim_{u \to 0} \int_{G-V_y} \varphi_U(y-x) f(x) dx = 0$$

for all f in $(L^p, l^\infty)(G)(1 < p \leq \infty)$.

Proof: Let $y = (y_1, \dots, y_a, s_0) = (y_0, s_0)$ be an element of $\mathbf{R}^a \times G_1$. We choose two elements V and K of $\mathcal{N}_0(G)$ with the same form as in Lemma 2.2, such that $y + K \subseteq V_y$ and $V \subseteq U$.

Following the notation of Lemma 2.2, we set $\eta_i = \min(\delta_i^{2p}, \gamma_i)(i = 1, \dots, a)$, and W_H the interior of K_H. Then the set $W = [-\eta_1, \eta_1] \times \cdots \times [-\eta_a, \eta_a] \times W_H$ satisfies the properties listed in Lemma 2.2. Hence by property 2.1) it is enough to prove that

$$\lim_{U \to 0} \int_{G-W_y} \varphi_U(y-x) f(x) dx = 0.$$

Since

$$G - W_y = (\mathbf{R}^a - \Pi_a) \times G_1 \cup \Pi_a \times (G_1 - (s_0 + W_H)),$$

we have by the definition of the function φ_U, that

$$\varphi_U(y-x) = \alpha_U(y_0 - t)\, \beta_U\, (s_0 - s) = 0$$

if $s_0 - s \not\in H$, and $x = (t, s)$ in G. Hence

$$\begin{aligned} \int_{G-W_y} \varphi_U(y-x) f(x) dx &= \int_{(\mathbf{R}^a - \Pi a) \times (s_0 + H)} \varphi_U(y-x) f(x) dx \\ (8) \qquad &+ \int_{\Pi a \times (s_0 + (H - W_g))} \varphi_U(y-x) f(x) dx. \end{aligned}$$

Let $\{I_n\}$ be the countable family of sets given by property 2.3). For each I_n we have by the Hölder inequality and (1)

$$\int_{I_n \times (s_0 + H)} | \varphi_U(y-x) f(x) | dx \leq ||f \chi_{I_n \times (s_0 + H)}||_p B_U \leq$$

$$\leq \left[\int_{I_n} \alpha_U(y_0 - x)^{p'} dx \right]^{1/p'}$$

By property 2.4) $\sup_{\mathbf{N}} |S(I_n \times (s_0 + H))| \leq C$, where C is a constant, and $|S(I_n \times (s_0 + H))|$ is the number of K_α's (as defined in [S]) such that $I_n \times (s_0 + H) \cap K_\alpha \neq \phi$. This implies that for all $n \,\varepsilon\, \mathbf{N}$

$$||f\chi_{I_n \times (s_0+H)}||_p \leq |S(I_n \times (s_0 + H))| \, ||f||_{p\infty} \leq C||f||_{p\infty}.$$

Thus, we conclude from 2.2) that

$$(9) \quad \int_{(\mathbf{R}^a - \Pi_a)\times G_1} \varphi_U(y-x) \mid f(x) \mid dx \leq$$
$$\leq C||f||_\infty B_U \sum_{\mathbf{N}} \left[\int_{I_n} \alpha_U(y_0 - x)^{p'} dx\right]^{1/p'} =$$
$$= O(\Pi_{i=1}^a \delta_i B_U).$$

Applying Hölder's inequality we get

$$\int_{\Pi a \times (s_0 + (H - W_H))} \varphi_U(y-x)|f(x)|dx \leq$$
$$\leq B_U||f||_{p\infty}|S(\Pi a \times (s_0 + (H - W_H))| \left[\int_{\Pi a} \alpha_U(y_0 - x)^{p'} dx\right]^{p'}.$$

Note that $\Pi a \times (s_0 + (H - W_H))$ is compact (H is compact and $H - W_H$ is closed), and because $B_U \to 0 \quad \text{as } U \to 0$

Now, since $\Pi a \to y \quad \text{as } U \to 0 \quad \text{and } s_0 + (H - W_H) \leq s_0 + H$ is independent of U, we have that $|S(\Pi a \times (s_0 + (H - W_H)| \to 0$ as $U \to 0$. Therefore by property 2.2)

$$(10) \qquad \int_{\Pi a \times (s_0 + (H - W_H))} \varphi_U(y-x)|f(x)|dx \to 0 \quad \text{as} \quad U \to 0.$$

The result follows from (8), (9), and (10).

Theorem 2.3. *For all f in $(L^p, l^\infty)(G), 2 \leq p \leq \infty$,*

$$\lim_{U \to 0} \int_G \varphi_U(y-x) f(x) dx = f(y)$$

almost everywhere.

Proof: Let V_y be in N_y compact. We have to show, by Lemma 2.2, that

$$\lim_{V \to 0} \int_{V_y} \varphi_U(y-x) f(x) dx$$

converges to $f(x)$ almost everywhere. Since the function f belongs to $(L^p, l^\infty) \subseteq (L^2, l^\infty)$, the function $g = f\chi_{V_y}$ belongs to $L^2(G)$, and by Corollary 1.2, 1.1), and 1.4) each φ_U also belongs to $L^2(G)$. Hence by the Parseval identity, we have that

$$\begin{aligned}\int_{V_y} \varphi_U(y-x)f(x)dx &= \int_G \varphi_U(y-x)g(x)dx \\ &= \int_\Gamma \hat{\varphi}_U(\hat{x})\hat{g}(-\hat{x})\overline{[y,\hat{x}]}d\hat{x}\end{aligned}$$

By the Lebesque Dominated Convergence theorem (see properties 1.3 and 1.6) we have that

$$\begin{aligned}\lim_{U\to 0}\int_{V_y} \varphi_U(y-x)f(x)dx &= \lim_{U\to 0}\int_\Gamma \hat{\varphi}_U(\hat{x})\hat{g}(-\hat{x})\overline{[y,\hat{x}]}d\hat{x} \\ &= \int_\Gamma \hat{g}(-\hat{x})\overline{[y,\hat{x}]}d\hat{x} = g(y)\end{aligned}$$

almost anywhere.

3. Fourier Transform of Unbounded Measures

The space $M_q(G)(1 \leq p < \infty)$ of unbounded measures of type q [**S**], consists of Radon measures μ with finite norm $\|\mu\|_q$ given by $\left[\sum_J |\mu|(K_\alpha)^q\right]^{1/q}$. If $G = \mathbf{R}$, then the family $\{K_\alpha\}$ can be taken as $\{[n, n+1] | n\varepsilon\mathbf{Z}\}$.

In this section we generalize to locally compact abelian groups, the following theorem due to F. Holland [**H**].

Theorem 3.1. *Let $1 \leq q \leq 2$ and $\mu \,\varepsilon\, M_q(\mathbf{R})$. Then as $N \to \infty$*

$$\frac{1}{\sqrt{2\pi}}\int_{-N}^{N} e^{-ixt}d\mu(t)$$

converges in the norm of $(L^{q'}, l^\infty)$ to a function $\hat{\mu}$ and

$$\int h(x)\hat{\mu}(x)dx = \int \hat{h}(x)d\mu(x) \quad (h\varepsilon(L^q, l^1)(\mathbf{R})).$$

Further

$$\sqrt{2\pi}\hat{\mu}(x) = (C.1)\int e^{-ixt}d\mu(t)$$

almost everywhere.

$(C.1)$ means that the integral on the right is summable by the Cesáro method of order 1 to the value $\sqrt{2\pi}\hat{\mu}(x)$.

It is easy to see, that for any measure μ in $M_q \quad (1 \leq q \leq 2)$, there is a net μ_α of bounded measures such that $\lim ||\mu_\alpha - \mu||_q = 0$, and therefore by [**S. Theorem 4.2**] $lim||\hat{\mu}_\alpha - \hat{\mu}||_{q\infty} = 0$. This generalizes the first part of the theorem.

Theorem 3.2. *Let μ be an element of $M_q \quad (1 \leq q \leq 2)$*

i) $\int_\Gamma \overline{f(\gamma)}\hat{\mu}(\gamma)d\gamma = \int_G \overline{\check{f}(x)}d\mu(x)$ *for all f in $(L^q, l^1)(\Gamma)$.*

ii) *(C.1)* $\int_G \overline{\gamma(x)}d\mu(x) := \lim_{U\to 0} \int_G \check{\varphi}_U(x)\overline{\gamma(x)}d\mu(x) = \hat{\mu}(\gamma)$ *almost everywhere.*

(C.1) means that the integral on the right is summable by the Cesáro method of order 1 *to the value $\hat{\mu}(\gamma)$.*

Proof: Let μ_α be the net of bounded measures related to μ, as mentioned above.

Since (L^q, l^1) is a subspace of L^1 [**S**,(3,4)], we have by the Extended Parseval Formula [**S. Lemma 4.1**] that for any f in $(L^q, l^1)(\Gamma)$,

$$\int_\Gamma \overline{f(\gamma)}\hat{\mu}_\alpha(\gamma)d\gamma = \int_G \overline{\check{f}(x)}d\mu_\alpha(x) \tag{11}$$

By the Hölder inequality

$$\int_\Gamma |f(\gamma)| \; |\hat{\mu}_\alpha(\gamma) - \hat{\mu}(\gamma)|d\gamma \leq$$

$$\leq \sum_J \left[\int_{K_\beta} |f(\gamma)|^q\right]^{1/q} \left[\int_{K_\beta} |\hat{\mu}_\alpha(\gamma) - \hat{\mu}(\gamma)|^{q'}d\gamma\right]^{1/q'} \leq$$

$$\leq ||f||_{q1}||\hat{\mu}_\alpha - \hat{\mu}||_{q'\infty}.$$

Similarly

$$\int_G |\check{f}(x)| \quad d|\mu_\alpha - \mu|(x) \leq ||\check{f}||_{\infty q'}||\mu_\alpha - \mu||_q.$$

Therefore the left side of (ii) converges to $\int_\Gamma \overline{f(\gamma)}d\mu(\gamma)$, and the right side to $\int_G \check{f}(x)d\mu(x)$. This proves i).

By proposition 1.1 and [**S**,(3.1)], each φ_U belongs to $(L^q, l^1)(\Gamma)$, so from i)

$$\int_\Gamma \varphi_U(y-\gamma)\hat{\mu}(\gamma)d\gamma = \int_G y(x)\overline{\check{\varphi}_U(x)}d\mu(x).$$

Hence, part ii) follows from Theorem 2.3.

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