Publicacions Matemàtiques, Vol 36 (1992), 761–772.

NON-OBSTRUCTED SUBCANONICAL SPACE CURVES

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Abstract

Recall that a closed subscheme $X \subset \mathbf{P}$ is non-obstructed if the corresponding point x of the Hilbert scheme $\underline{Hilb}^n_{p(t)}$ is non-singular. A geometric characterization of non-obstructedness is not known even for smooth space curves. The goal of this work is to prove that subcanonical k-Buchsbaum, $k \leq 2$, space curves are non-obstructed. As a main tool, we use Serre's correspondence between subcanonical curves and vector bundles.

Introduction

In 1960 ([**G**]), A. Grothendieck proved that there is a k-projective scheme $\underline{Hilb}^n_{p(t)}$ which parametrizes from the functorial point of view all closed subschemes of $\mathbf{P}^n$ with given Hilbert polynomial $p(t) \in \mathbf{Q}[t]$; though so far very few of these schemes has been studied in detail and there are no general results about these schemes concerning connected components, dimension, smoothness, rationality, topological invariants,... From now on, we will say that a closed subscheme $X \subset \mathbf{P}^n$ is non-obstructed if the corresponding point x of the Hilbert scheme $\underline{Hilb}^n_{p(t)}$ is non-singular; otherwise, we will say that X is obstructed. A geometrical characterization of non-obstructedness is not known even for smooth space curves and several examples of obstructed smooth space curves has been given, for instance, in [**M**], [**S**], [**EF**], [**K1**], [**K2**], [**K3**], [**E1**], and [**BKM**].

In this paper, we will prove the non-obstructedness of subcanonical 2-Buchsbaum space curves (Cf. Theorem 2.5). As a corollary we will get that 2-Buchsbaum quasi-complete space curves (Cf. Definition 2.6)

*Partially suported by DGICYT PB88-0224.

not contained in a surface of degree < 9 are also non-obstructed (Cf. Proposition 2.7).

Recall that a curve $C \subset \mathbf{P}^3$ is subcanonical if the canonical sheaf ω_C of C is isomorphic to $\mathcal{O}_C(a)$ for some integer a; and a curve $C \subset \mathbf{P}^3$ is k-Buchsbaum if and only if $k = \min\{t \mid m^t M(C) = 0\}$ where $m = (X_0, X_1, X_2, X_3)$ and $M(C)$ is the Hartshorne-Rao module of C. Note that a curve $C \subset \mathbf{P}^3$ is arithmetically Cohen-Macaulay (Resp. arithmetically Buchsbaum) if and only if is 0-Buchsbaum (Resp. 1-Buchsbaum). So, the notion of k-Buchsbaum can be viewed as a natural extension of the notions arithmetically Cohen-Macaulay and arithmetically Buchsbaum. Moreover, every curve $C \subset \mathbf{P}^3$ is k-Buchsbaum for some integer k.

A classical theorem of Gherardelli says that a smooth irreducible subcanonical curve $C \subset \mathbf{P}^3$ is arithmetically Cohen-Macaulay if and only if it is complete intersection (For a weaker characterization of complete intersection space curves see [**CV**]). In [**EF**], Ellia-Fiorentini prove that an integral subcanonical curve $C \subset \mathbf{P}^3$ is arithmetically Buchsbaum if and only if C is the zero scheme of a section of $N(t)$, $t \geq 1$, where N is the null correlation bundle. Therefore, subcanonical k-Buchsbaum space curves, $0 \leq k \leq 1$, are non-obstructed. The aim of this paper is to extend this knowledge to subcanonical, 2-Buchsbaum space curves with the hope of finding a clue which could facilitate the study of arbitrary subcanonical space curves.

In section 1, we establish some preliminary results. In section 2, we prove the main results of this paper. We see that any subcanonical, 2-Buchsbaum space curve is in the even liaison class of three disjoint lines. From this we can show that any subcanonical, 2-Buchsbaum space curve is non-obstructed, it has maximal rank and we give a resolution of its ideal sheaf. In section 3, we conclude by studying some examples and adding some remarks.

Notations

Throughout this paper we work over an algebraically closed field $\mathbf{k}$ of characteristic zero. We set $S = \mathbf{k}[X_0, \ldots, X_3]$, $m = (X_0, \ldots, X_3) \subset S$ and $\mathbf{P}^3 = \operatorname{Proj}(S)$. By a curve we mean a closed, locally Cohen-Macaulay, one-dimensional subscheme $X \subset \mathbf{P}^3$. For a coherent sheaf F on X, $F(n)$ as usual will be $F \otimes \mathcal{O}_X(n)$ and we let $h^i F(n) = \dim_{\mathbf{k}} H^i(X, F(n))$.

Given a curve $C \subset \mathbf{P}^3$, we denote $d =$ degree of C, $p_a =$ arithmetic genus of C, $s = \min\{t \mid H^0 I_C(t) \neq 0\}$, $e = \max\{t \mid H^1 \mathcal{O}_C(t) \neq 0\}$

and $c = \max\{t \mid H^1 I_C(t) \neq 0\}$ ($c = -\infty$, if C is arithmetically Cohen-Macaulay). For a curve C in $\mathbf{P}^3$, the Hartshorne-Rao module $M(C) = \oplus_n H^1 I_C(n)$ is a graded S-module of finite length. Recall that a curve C in $\mathbf{P}^3$ is said to be a-subcanonical if the canonical sheaf ω_C of C is isomorphic to $\mathcal{O}_C(a)$ and a curve C in $\mathbf{P}^3$ is said to have maximal rank if the restriction map $H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(n)) \to H^0(C, \mathcal{O}_C(n))$ is of maximal rank, for every integer n.

For a coherent sheaf E on $\mathbf{P}^3$ we denote $H^1_*(E)$ the graded S-module $\oplus_n H^1(\mathbf{P}^3, E(n))$. A rank 2 vector bundle E on $\mathbf{P}^3$ is said to be stable if $H^0 E_{\text{norm}} = 0$ where E_{norm} denotes the twist of E which has first Chern class equal 0 or -1. Our main reference for this subject is [**H**].

1. Generalities

In the present section we recall the definitions and basic properties needed later.

Definition 1.1. ([**E**], [**MM**]) Let $C \subset \mathbf{P}^3$ be a curve. We say that C is k-Buchsbaum if and only if $k = \min\{t \mid m^t M(C) = 0\}$.

See [**MM**] for general results on k-Buchsbaum curves.

Remarks 1.1.1. (a) C is 0-Buchsbaum (Resp. 1-Buchsbaum) if and only if C is arithmetically Cohen-Macaulay (Resp. arithmetically Buchsbaum).

(b) For any curve $C \subset \mathbf{P}^3$ there is an integer k such that C is k-Buchsbaum.

(c) Let $C, D \subset \mathbf{P}^3$ be two curves in the same liaison class. Then, C is k-Buchsbaum if and only if D is k-Buchsbaum.

Definition 1.2. ([**E**], [**MM**]) A rank 2 vector bundle E on $\mathbf{P}^3$ is said to be k-Buchsbaum if and only if $k = \min\{t \mid m^t H^1_* E = 0\}$.

Remark 1.2.1. If E is a rank 2, k-Buchsbaum vector bundle on $\mathbf{P}^3$, then the zero set of a section of $E(n)$ is a k-Buchsbaum, subcanonical curve. Conversely, any subcanonical, k-Buchsbaum curve corresponds to a rank 2, k-Buchsbaum vector bundle on $\mathbf{P}^3$.

From [**E**], we get the following characterization for rank 2, k-Buchsbaum, $k \leq 2$, vector bundles on $\mathbf{P}^3$.

Proposition 1.3. *Let E be a normalized, rank 2, k-Buchsbaum vector bundle on $\mathbf{P}^3$. Then:*

$k = 0$ if and only if E is direct sum of line bundles,

$k = 1$ if and only if E is the a correlation bundle,

$k = 2$ if and only if E is stable, $c_1 E = 0$ and $c_2 E = 2$.

Proposition 1.4. *Let E be a rank 2 stable vector bundle on $\mathbf{P}^3$ with $c_1E = 0$, $c_2E = 2$. Then:*

(1) *$h^3E(t) = 0$ for $t \geq -4$, $h^2E(t) = 0$ for $t \neq -3, -4$ and $h^2E(t) = 2$ for $t = -3, -4$, $h^1E(t) = 0$ for $t \neq -1, 0$ and $h^1E(t) = 2$ for $t = 0, -1$, $h^0E(t) = 0$ for $t \leq 0$ and $h^0E(t) = \chi E(t)$ for $t > 0$, and $E(2)$ is generated by its global sections,*

(2) *There is a section $s \in H^0E(1)$ whose zero set is three disjoint lines,*

(3) *E has a locally free resolution of the following kind:*

$$0 \longrightarrow 2\mathcal{O}(-4) \longrightarrow 6\mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-1) \oplus 4\mathcal{O}(-2) \longrightarrow E \longrightarrow 0.$$

Furthermore, the moduli space $M(0,2)$ of rank 2 stable vector bundles on $\mathbf{P}^3$ with $c_1 = 0$, $c_2 = 2$ is an irreducible, smooth variety of dimension 13.

Proof: I will only proof (3); for the other results see [**H**]. By (2), there is a section $s \in H^0E(1)$ which gives us an exact sequence:

$$(1) \qquad 0 \longrightarrow \mathcal{O} \longrightarrow E(1) \longrightarrow I_Y(2) \longrightarrow 0$$

where Y is the disjoint union of three lines.

By [**I**, Proposition 7.2.2], I_Y has a resolution of the following kind:

$$(2) \quad 0 \longrightarrow 2\mathcal{O}(-5) \longrightarrow 6\mathcal{O}(-4) \longrightarrow \mathcal{O}(-2) \oplus 4\mathcal{O}(-3) \longrightarrow I_Y \longrightarrow 0.$$

Set $K := \operatorname{Ker}(4\mathcal{O}(-3) \oplus \mathcal{O}(-2) \to I_Y)$. Then, the exact sequence (2) breaks up and gives us the exact sequences:

$$(3) \qquad 0 \longrightarrow 2\mathcal{O}(-5) \longrightarrow 6\mathcal{O}(-4) \longrightarrow K \longrightarrow 0$$

$$(4) \qquad 0 \longrightarrow K \longrightarrow \mathcal{O}(-2) \oplus 4\mathcal{O}(-3) \longrightarrow I_Y \longrightarrow 0.$$

The exact sequences (1) and (4) give us the following diagram:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K(1) & = & K(1) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & 2\mathcal{O}(-1)\oplus 4\mathcal{O}(-2) & \longrightarrow & \mathcal{O}(-1)\oplus 4\mathcal{O}(-2) & \longrightarrow & \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}(-1) & \longrightarrow & E & \longrightarrow & I_Y(1) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Finally, the exact sequence (3) together with the exact sequence:

$$0 \longrightarrow K(1) \longrightarrow 2\mathcal{O}(-1) \oplus 4\mathcal{O}(-2) \longrightarrow E \longrightarrow 0$$

gives us the resolution of E :

$$0 \longrightarrow 2\mathcal{O}(-4) \longrightarrow 6\mathcal{O}(-3) \longrightarrow 2\mathcal{O}(-1) \oplus 4\mathcal{O}(-2) \longrightarrow E \longrightarrow 0. \quad \blacksquare$$

2. Subcanonical, 2-Buchsbaum space curves

In this section, we give a complete description of subcanonical, 2-Buchsbaum space curves.

Theorem 2.1. *Let $C \subset \mathbf{P}^3$ be an integral, a-subcanonical curve. Then, C is 2-Buchsbaum if and only if C is the zero scheme of a section of a rank 2 stable vector bundle E on $\mathbf{P}^3$ with $c_1E = 0$ and $c_2E = 2$.*

Proof: Let E be a rank 2 stable vector bundle on $\mathbf{P}^3$ with $c_1E = 0$ and $c_2E = 2$. By Proposition 1.3, E is 2-Buchsbaum. In particular, any curve associated to E is 2-Buchsbaum.

Conversely, let $C \subset \mathbf{P}^3$ be an integral, 2-Buchsbaum, a-subcanonical curve. Set $b = a/2$ if a is even, and $b = (a+1)/2$ if a is odd. A general section $0 \neq s \in H^0\omega_C \simeq H^0\mathcal{O}_C(a)$ gives us an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow E(b+2) \longrightarrow I_C(a+4) \longrightarrow 0$$

where E is a rank 2 vector bundle on $\mathbf{P}^3$. Since a 2-Buchsbaum curve gives rise to a 2-Buchsbaum sheaf, applying Proposition 1.3, we get that E is stable, $c_1E = 0$ and $c_2E = 2$. $\blacksquare$

Proposition 2.2. *Let E be a rank 2 stable vector bundle on $\mathbf{P}^3$ with $c_1 = 0$, $c_2 = 2$ and let $C \subset \mathbf{P}^3$ be the zero scheme of a section of $E(t)$, $t \geq 1$. Then:*

(1) $\omega_C \simeq \mathcal{O}_C(2t-4)$, $e(C) = 2t-4$, $s(C) = t+1$, $\deg(C) = t^2+2$, $p_a = t^3 - 2t^2 + 2t - 3$,

(2) $M(C)_{t-1} \simeq M(C)_t \simeq \mathrm{k}^2$, $M(C)_n = 0$ *for* $n \neq t, t-1$,

(3) *C has maximal rank and belongs to the even liaison class of three disjoint lines,*

(4) *The ideal sheaf I_C of C has a locally free resolution of the following kind:*

$$0 \longrightarrow 2\mathcal{O}(-5) \longrightarrow 6\mathcal{O}(-4) \longrightarrow 4\mathcal{O}(-3) \oplus \mathcal{O}(-2) \longrightarrow I_C \longrightarrow 0$$

if $t = 1$, and

$$0 \longrightarrow 2\mathcal{O}(-4-t) \longrightarrow 6\mathcal{O}(-3-t) \oplus \mathcal{O}(-2t) \longrightarrow \\ \longrightarrow 4\mathcal{O}(-2-t) \oplus 2\mathcal{O}(-1-t) \longrightarrow I_C \longrightarrow 0$$

if $t > 1$.

Proof: (1)-(3) follows from Proposition 1.4 and the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow E(t) \longrightarrow I_C(2t) \longrightarrow 0.$$

(4) If $t = 1$, then C is the disjoint union of three lines and the result follows from [**I**, Proposition 7.2.2].

Assume $t > 1$ and consider the locally free resolution of E given in Proposition 1.4:

$$(1) \quad 0 \longrightarrow 2\mathcal{O}(-4) \longrightarrow 6\mathcal{O}(-3) \longrightarrow 4\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow E \longrightarrow 0.$$

Set $K := \mathrm{Ker}(4\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \to E)$. Then, the exact sequence (1) breaks up and gives us the exact sequences:

$$(2) \qquad 0 \longrightarrow 2\mathcal{O}(-4) \longrightarrow 6\mathcal{O}(-3) \longrightarrow K \longrightarrow 0$$

$$(3) \qquad 0 \longrightarrow K \longrightarrow 4\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow E \longrightarrow 0.$$

The exact sequence (3) together with the exact sequence:

$$0 \longrightarrow \mathcal{O}(-2t) \longrightarrow E(-t) \longrightarrow I_C \longrightarrow 0$$

gives us the following diagram:

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & K \oplus \mathcal{O}(-2t) & \longrightarrow & \mathcal{O}(-2t) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K & \longrightarrow & 2\mathcal{O}(-1-t) \oplus 4\mathcal{O}(-2-t) & \longrightarrow & E(-t) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & I_C & = & I_C & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Finally, using the exact sequence (2) and the exact sequence:

$$0 \longrightarrow K \oplus \mathcal{O}(-2t) \longrightarrow 2\mathcal{O}(-1-t) \oplus 4\mathcal{O}(-2-t) \longrightarrow I_C \longrightarrow 0$$

we get the locally free resolution of I_C:

$$0 \longrightarrow 2\mathcal{O}(-4-t) \longrightarrow 6\mathcal{O}(-3-t) \oplus \mathcal{O}(-2t) \longrightarrow$$
$$\longrightarrow 4\mathcal{O}(-2-t) \oplus 2\mathcal{O}(-1-t) \longrightarrow I_C \longrightarrow 0. \blacksquare$$

Lemma 2.3. *Let E be a rank 2 stable vector bundle on $\mathbf{P}^3$ with $c_1E = 0$, $c_2E = 2$. Then, $h^2E \otimes E = h^3E \otimes E = 0$.*

Proof: By proposition 1.4, E has a locally free resolution of the following kind:

(1) $\quad 0 \longrightarrow 2\mathcal{O}(-4) \longrightarrow 6\mathcal{O}(-3) \longrightarrow 4\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow E \longrightarrow 0.$

Set $K := \mathrm{Ker}(4\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \to E)$. Then, the exact sequence (1) breaks up and gives us the exact sequences:

(2) $\quad 0 \longrightarrow 2\mathcal{O}(-4) \longrightarrow 6\mathcal{O}(-3) \longrightarrow K \longrightarrow 0$

(3) $\quad 0 \longrightarrow K \longrightarrow 4\mathcal{O}(-2) \oplus 2\mathcal{O}(-1) \longrightarrow E \longrightarrow 0.$

Finally, tensoring (2) and (3) by E and taking cohomology we get $h^2E \otimes E = h^3E \otimes E = 0$. $\blacksquare$

Let $\mathcal{F}(t)$ be the family of curves zero schemes of sections of $E(t)$, $t \geq 1$, where E is a rank 2 stable vector bundle on $\mathbf{P}^3$ with $c_1E = 0$ and $c_2E = 2$.

Proposition 2.4. *(1) For all $t \geq 1$, $\mathcal{F}(t)$ is an irreducible family of dimension 12 if $t = 1$ and $2t^2 + 10 + (t^3 + 5t)/3$ if $t > 1$.*

(2) For $t \neq 3, 4$, the closure of $\mathcal{F}(t)$ in $\operatorname{Hilb} \mathbf{P}^3$ is an irreducible component generically smooth.

(3) $\mathcal{F}(3)$ (Resp. $\mathcal{F}(4)$) is contained in a unique component of $\operatorname{Hilb} \mathbf{P}^3$, generically smooth of dimension 44 (Resp. 72).

Proof: (1) The irreducibility of $\mathcal{F}(t)$ follows from the irreducibility of $M(0,2)$. On the other hand, $\dim \mathcal{F}(t) = \dim M(0,2) + h^0 E(t) - h^0 \mathcal{O}_C$ for a general E in $M(0,2)$ and a general C in $\mathcal{F}(t)$. So, the dimension of $\mathcal{F}(t)$ is 12 if t=1 and $2t^2 + 10t + (t^3 + 5t)/3$ if $t > 1$.

(2) Assume $t \neq 3, 4$. Let C be a curve in $\mathcal{F}(t)$ and let N_C be its normal bundle. Since the Zariski tangent space of $\operatorname{Hilb} \mathbf{P}^3$ at the point corresponding to C is isomorphic to $H^0 N_C$ and $h^0 N_C = 4 \deg(C) + h^1 N_C$, it is enough to compute $h^1 N_C$ and to check that $\dim \mathcal{F}(t) = h^0 N_C$. But, C is the zero scheme of a section of $E(t)$, hence $N_C \simeq E(t) \otimes \mathcal{O}_C$ and we have $h^1 N_C = h^1(E(t) \otimes \mathcal{O}_C) = h^2(E(t) \otimes I_C)$. On the other hand, tensoring by $E(t)$ the exact sequence:

$$0 \longrightarrow \mathcal{O}(-2t) \longrightarrow E(-t) \longrightarrow I_C \longrightarrow 0,$$

taking cohomology and using lemma 2.3, we get:

$$h^2(E(t) \otimes I_C) = \begin{cases} 0 & \text{if} \quad t = 1, 2 \\ 2 - 2t^2 + (t^3 + 5t)/3 & \text{if} \quad t > 4. \end{cases}$$

Putting altogether we have: for $t = 1, 2$, $\dim \mathcal{F}(t) = h^0 N_C = 4 \deg(C)$ and $h^1 N_C = 0$ and for $t > 4$, $\dim \mathcal{F}(t) = h^0 N_C$; which gives what we want.

(3) Assume $t = 3, 4$. Let C be a curve of $\mathcal{F}(3)$ (Resp. $\mathcal{F}(4)$) and let N_C its normal bundle. As before, we compute $h^1 N_C$ and we get: $h^1 N_C = 0$ and $\dim \mathcal{F}(t) < 4 \deg(C) = h^0 N_C = 44$ (Resp. 72). So, $\mathcal{F}(3)$ (Resp. $\mathcal{F}(4)$) is contained in a unique irreducible component of $\operatorname{Hilb} \mathbf{P}^3$, generically smooth of dimension 44 (Resp. 72). ■

Theorem 2.5. *Let $C \subset \mathbf{P}^3$ be an irreducible, subcanonical curve. If C is k-Buchsbaum and $k \leq 2$, then C is non-obstructed (i.e. the corresponding point of the Hilbert scheme is smooth).*

Proof: The case $k = 2$ follows from Proposition 2.4, the case $k = 1$ from [**EF**, Proposition 3.5] and the case $k = 0$ from [**El**]. ■

Definition 2.6. A curve $C \subset \mathbf{P}^3$ is said to be quasi-complete intersection (briefly q.c.i.) if there exists a surjection $\oplus_{i=1}^3 \mathcal{O}_{\mathbf{P}^3}(-a_i) \to I_C \to 0$ for some integers $a_1 \le a_2 \le a_3$.

Remark 2.6.1. This definition is equivalent to saying that there are homogeneous elements $f_1, f_2, f_3 \in I(X)$ of degrees a_1, a_2, a_3, respectively, such that $I(X)/(f_1, f_2, f_3)$ is a graded S-module of finite length.

Remark 2.6.2. Let X be a curve in $\mathbf{P}^3$. If X is the zero scheme of a section of a rank 2 vector bundle on $\mathbf{P}^3$ and Z is linked to X then Z is q.c.i. Conversely, if Z is q.c.i. of three surfaces of degrees $a_1 \le a_2 \le a_3$ and X is linked to Z by means of two surfaces of degrees a_i and a_j, respectively, then X is the zero scheme of a section of a rank 2 vector bundle on $\mathbf{P}^3$.

Proposition 2.7. *Let $Y \subset \mathbf{P}^3$ be an integral 2-Buchsbaum curve q.c.i. of three surfaces of degrees $9 \le a_1 \le a_2 \le a_3$. Then, Y is non-obstructed.*

Remark. See [**Mi**], for the case of integral, 1-Buchsbaum, quasi-complete intersection space curves.

Proof: By definition there exists a surjection $\oplus_{i=1}^3 \mathcal{O}_{\mathbf{P}^3}(-a_i) \to I_Y \to 0$. Set $F := \mathrm{Ker}(\oplus_{i=1}^3 \mathcal{O}_{\mathbf{P}^3}(-a_i) \to I_Y)$ and $E := F((a_1 + a_2 + a_3)/2)$. By Proposition 1.3, E is a 2-Buchsbaum stable rank 2 vector bundle on $\mathbf{P}^3$ with Chern classes $c_1 = 0$, $c_2 = 2$. Now, we link Y to an irreducible curve X by means of two surfaces of degrees a_1 and a_2, respectively. By [**PS**, Proposition 2.5], the ideal sheaf I_X of X has a locally free resolution of the following type:

$$0 \longrightarrow \mathcal{O}(a_3 - a_2 - a_1) \longrightarrow E((a_3 - a_2 - a_1)/2) \longrightarrow I_X \longrightarrow 0,$$

and by propositions 2.2 and 2.4 X is a non-obstructed, maximal rank space curve. Moreover, the hypothesis on a_i's and Proposition 1.4 imply that $H^1 I_X(a_2 - 4) = H^1 I_X(a_1 - 4) = 0$. Thus, Y is non-obstructed [**K2**, Corollary 3.10]. ■

3. Comments and Questions

If we try to generalize the results of section 2 to higher values of k, we immediately encounter difficulties of various kinds, to be pointed out presently.

First of all, note that all subcanonical, 0-Buchsbaum curves are in the liaison class of a line; all subcanonical, 1-Buchsbaum curves are in the liaison class of the disjoint union of two lines; and all subcanonical, 2-Buchsbaum curves are in the liaison class of the disjoint union of three lines. Thus, it is natural to ask if all subcanonical, k-Buchsbaum curves ($k > 2$) are in the liaison class of $\gamma = \gamma(k)$ lines or, at least, if all subcanonical, k-Buchsbaum curves ($k > 2$) are in the same liaison class. The answer, in general, is no. For instance:

Example 3.1. Take Y_1 the disjoint union of 4 lines, Y_2 the disjoint union of 5 lines, Y_3 a general, irreducible, smooth, elliptic curve of degree 7 and Y_4 a general, irreducible, smooth, elliptic curve of degree 8. Y_1, Y_2, Y_3 and Y_4 are subcanonical and it is not difficult to see that they are 3-Buchsbaum. However, computing their Hartshorne-Rao modules we easily get that they belong to four different liaison classes.

In section 2, we prove that subcanonical, k-Buchsbaum curves, $k \leq 2$, are non-obstructed. We wonder if the hypothesis $k \leq 2$ can be avoid. To be more precise, we suggest the following problems:

3.2.1. To characterize non-obstructed subcanonical space curves, and
3.2.2. To characterize non-obstructed quasi-complete intersection space curves.

In particular,

3.3.1. Are subcanonical space curves of maximal rank non-obstructed?
3.3.2. Are quais-complete intersection space curves of maximal rank non-obstructed?

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Rebut el 10 de Setembre de 1991