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MAXIMAL QUOTIENT RINGS AND ESSENTIAL RIGHT IDEALS IN GROUP RINGS OF LOCALLY FINITE GROUPS

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Dedicated to the memory of Pere Menal

Abstract

Let k be a commutative field. Let G be a locally finite group without elements of order p in case $\operatorname{char} k = p > 0$. In this paper it is proved that the type I_∞ part of the maximal right quotient ring of the group algebgra kG is zero.

1. Introduction

Let k be a commutative field, G a group, and suppose that the group ring kG is regular in the sense of von Neumann. By [**7**, p.69], this means precisely that G is locally finite with no elements of order equal to the characteristic of k. Then the maximal right quotient ring $Q^r(kG)$ of kG is a regular right self-injective ring [**3**, Corollaries 1.2 and 1.24], and as such, is uniquely a direct product of rings of types $I_f, I_\infty, II_f, II_\infty$, and III. We refer to [**3**] for general background on regular rings. The main theorem of this paper is the following.

Theorem. *With the above notation, the type I_∞ part of $Q^r(kG)$ is zero .*

The first author has obtained this result under various supplementary hypotheses [**2**], and in particular when G is Δ-hypercentral. These results will be used in the proof of the general case. He has also shown that the type I_f part of $Q^r(kG)$ is non-zero if and only if $|G : \Delta(G)| < \infty$ and $\Delta(G)'$ is finite, where as usual, $\Delta(G)$ is the subgroup of G consisting of the elements with finitely many conjugates [**1**].

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In 1983, Menal proposed the study of the maximal quotient ring of regular group algebras to the first author. One of the problems proposed was the characterization of the type I part of $Q^r(kG)$ for kG regular. Goursaud and Valette [4] had a partial result, namely they had characterized this part when k has positive characteristic or contains all roots of unity. Finally, with [1, Theorem 2.3] and the Theorem of this paper, the problem has been solved.

The remainder of the paper began as an attempt to answer the following question, raised for example in [**6**].

Question. *Let G be a locally finite group, k a field and H an infinite subgroup of G. Is it true that if J is an essential right ideal of kH, then the right ideal JkG of kG generated by J is essential in kG?*

We are only able to answer this question in a very special case, namely the case when J is the augmentation ideal of kH. However the method we use involves attaching a certain numerical invariant to each right (or left) ideal of kG, and possibly this may be of some interest in its own right. It is closely related to the invariant d discussed in [**5**], but is not quite the same. Our hope was that this invariant would distinguish between essential right ideals and the rest, but we give an example indicating the contrary. A result giving a positive answer to the question when kG is regular and H has finite index is proved in [**2**, Lemma 1.4] . The proof there is rather indirect and it would be interesting to have a direct proof in that case.

Definition 1.1. Let G be a locally finite group, k be a field, and J be a right ideal of kG. For each finite subgroup F of G, let $\alpha(J,F) = \dim(J \cap kF)/|F|$. Let $\alpha(J) = \sup \alpha(J,F)$, where F ranges over all finite subgroups of G.

Of course, the same definition can be made for left ideals of kG. Clearly $\alpha(J) \leq 1$. The main properties of $\alpha(J)$ are the following.

Lemma 1.2. *Let G, k be as above and let H be a subgroup of G.*

(i) *If J is a right ideal of kH, then $\alpha(J) \leq \alpha(JkG)$.*
(ii) *If $\alpha(J) = 1$, then J is an essential right ideal of kG.*
(iii) *If $\omega(kG)$ denotes the augmentation ideal of kG and G is infinite, then $\alpha(\omega(kG)) = 1$.*
(iv) *If J and L are right ideals of kG and J is isomorphic to a submodule of L, then $\alpha(J) \leq \alpha(L)$.*

These facts clearly imply the following.

Corollary 1.3. *With the above notation, if H is an infinite subgroup of G, then $\omega(kH)kG$ is an essential right ideal of kG.*

This was obtained somewhat less generally in [**6**, p.250] (see below). It is unfortunate that the converse of part (ii) of Lemma 1.2 is false. In fact, we have the following.

Example 1.4. Let G be any countably infinite periodic abelian group. Then for any $\epsilon > 0$, there is an essential ideal J of $\mathbb{C}G$ such that $\alpha(J) < \epsilon$.

2. Properties of $\alpha(J)$

In this section, we shall prove Lemma 1.2, after mentioning some other basic facts. Throughout, G denotes a locally finite group, and k a field.

Lemma 2.1. *Let J be a right ideal of kG, and let F_1, F_2 be finite subgroups of G with $F_1 \leq F_2$. Then $\alpha(J, F_1) \leq \alpha(J, F_2)$.*

Proof: Clearly, $(J \cap kF_1)kF_2 \leq J \cap kF_2$. Therefore,

$$\dim(J \cap kF_1)|F_2|/|F_1| \leq \dim J \cap kF_2.$$

Dividing by $|F_2|$ gives the result. ∎

The following is a useful consequence.

Lemma 2.2. *Suppose that G is countable, and let $G_1 \leq G_2 \leq \ldots$ be a tower of finite subgroups of G such that $\cup_{i=1}^{\infty} G_i = G$. Let J be a right ideal of kG and $\alpha_i = \dim(J \cap kG_i)/|G_i|$. Then $\alpha(J) = \lim_{i\to\infty} \alpha_i$.*

Proof: Let $\beta = \lim_{i\to\infty} \alpha_i$. Clearly, $\beta \leq \alpha(J)$. On the other hand, if F is any finite subgroup of G, then $F \leq G_i$ for some $i \geq 1$, and then Lemma 2.1 shows that $\alpha(J, F) \leq \alpha_i \leq \beta$. Hence $\alpha(J) \leq \beta$, and the two are equal. ∎

Proof of Lemma 1.2: (i) Let F be a finite subgroup of H. Then clearly $J \cap kF \leq JkG \cap kF$. Hence $\dim J \cap kF/|F| \leq \dim JkG \cap kF/|F| \leq \alpha(JkG)$. Since F is an arbitrary finite subgroup of H, this gives $\alpha(J) \leq \alpha(JkG)$.

(ii) Let J be a right ideal of kG with $\alpha(J) = 1$, and suppose if possible that L is a non-zero right ideal of kG with $J \cap L = 0$. Fix a finite subgroup E of G such that $L \cap kE \neq 0$, and let F be any finite subgroup of G containing E. Then $(L \cap kE)kF \cap (J \cap kF) = 0$, and so $\dim(L \cap kE)|F|/|E| + \dim(J \cap kF) \leq |F|$. Hence $\alpha(J, F) \leq 1 - \alpha(L, E) < 1$.

Now if F_1 is any finite subgroup of G and we take $F = \langle E, F_1\rangle$, then we deduce from Lemma 2.1 that $\alpha(J, F_1) \leq 1 - \alpha(L, E) < 1$, whence $\alpha(J) < 1$, a contradiction.

(iii) This is trivial.

(iv) Let F_1 be any finite subgroup of G, and let $\phi : J \to L$ be a right G-monomorphism. Then ϕ maps $J \cap kF_1$ into $L \cap kF_2$, for some finite subgroup F_2 of G. Let $F = \langle F_1, F_2\rangle$. Then ϕ embeds $(J \cap kF_1)kF$ into $L \cap kF$. Therefore, $\dim(J \cap kF_1)|F|/|F_1| \leq \dim(L \cap kF)$. Therefore $\alpha(J, F_1) \leq \alpha(L, F) \leq \alpha(L)$, and since F_1 is arbitrary, the result follows. ■

We note that if G is a finite elementary abelian 2-group of order 2^n and k is as field of characteristic 2, then $\xi^2 = 0$ for all $\xi \in \omega(kG)$. Hence $kG\xi$ annihilates ξ, and so $\dim \xi kG \leq 2^{n-1}$. It follows from this that if H is an infinite elementary abelian 2-group, then $\alpha(J) \leq 1/2$ for each principal ideal $J \leq \omega(kG)$. This may be compared with [**6**, p.250].

3. The Example

Write $G = \cup_{i=1}^{\infty} G_i$, where the G_i form a strictly increasing tower of finite subgroups of G. We construct the ideal J to satisfy the following condition:

(∗). *For each i and primitive idempotent $e \in \mathbb{C}G_i$, there exists $\alpha \in \mathbb{C}G$ such that $0 \neq e\alpha \in J$.*

Since each non-zero ideal of $\mathbb{C}G$ contains such an element e, we see that (∗) implies that J is essential in $\mathbb{C}G$.

Now we construct J as the union of a tower $J_1 \leq J_2 \leq \dots$, where J_i is an ideal of $\mathbb{C}G_i$. We begin with any minimal ideal of $\mathbb{C}G_1$ as J_1. Thus,

$$\dim J_1 = 1. \tag{1}$$

We also let $e_1, e_2, \dots$ be a sequence formed by taking first the (finitely many) primitive idempotents in $\mathbb{C}G_1$, then those in $\mathbb{C}G_2$, and so on.

Suppose we have J_i, a proper ideal of $\mathbb{C}G_i$. Then there is a primitive idempotent e of $\mathbb{C}G_i$ not in J_i, and if we write $\overline{J_i} = J_i \mathbb{C}G_{i+1}$, we have the direct sum $e\mathbb{C}G_{i+1} \oplus \overline{J_i}$ of ideals of $\mathbb{C}G_{i+1}$. Choose the first j such that we have a direct sum $e_j \mathbb{C}G_{i+1} \oplus \overline{J_i}$. Thus

For each $l < j$, there exists $\alpha \in \mathbb{C}G_{i+1}$ such that $0 \neq e_l\alpha \in \overline{J_i}$.

It will also be clear from the next step that $j \geq i-1$. Let f be a primitive idempotent in $e_j\mathbb{C}G_{i+1}$, and put

$$J_{i+1} = \overline{J_i} \oplus \mathbb{C}f,$$

an ideal of $\mathbb{C}G_{i+1}$. Clearly, $J_i \leq J_{i+1}$, and

$$\dim J_{i+1} = 1 + (\dim J_i)|G_{i+1}|/|G_i| \tag{2}$$

Also, we claim that

$$J_{i+1} \cap \mathbb{C}G_i = J_i, \tag{3}$$

from which it follows in particular that J_{i+1} is a proper ideal of $\mathbb{C}G_{i+1}$. To verify the above claim, it suffices to note that if $J_{i+1} \cap \mathbb{C}G_i = L_i > J_i$, then $\dim J_{i+1} \geq (\dim L_i)|G_{i+1}|/|G_i| \geq \dim J_{i+1} + |G_{i+1}|/|G_i| - 1$, a contradiction, since the sequence (G_i) is strictly increasing. Putting $J = \cup_{i=1}^{\infty} J_i$, we have $(*)$, and so J is essential in $\mathbb{C}G$.

Let $\alpha_i = \dim J \cap \mathbb{C}G_i/|G_i|$. Then from (2) and (3), we have $\alpha_{i+1} = \alpha_i + 1/|G_{i+1}|$, so from (1),

$$\alpha_i = \sum_{j=1}^{i} \frac{1}{|G_j|}.$$

Now $|G_{i+1}| \geq 2|G_i|$, so

$$\alpha_i \leq \frac{1}{|G_1|}(1 + \frac{1}{2} + \ldots + \frac{1}{2^{i-1}}) \leq \frac{2}{|G_1|}.$$

Choosing G_1 suitably, we obtain what we want.

4. Proof of the Theorem

The theorem follows from [**2**, Theorem 1.3] and Lemma 4.1 below.

Lemma 4.1. *Suppose that kG is regular and the type I part of $Q^r(kG)$ is non-zero. Then G is Δ-hypercentral.*

Before beginning the proof we recall some notation and terminology. Let $\Delta(G) = \{g \in G : |G : C_G(g)| < \infty\}$. We define the transfinite upper Δ -series of G by the rules

$$\Delta_0(G) = 1,$$
$$\Delta_{\rho+1}(G)/\Delta_\rho(G) = \Delta(G/\Delta_\rho(G)),$$
$$\Delta_\beta(G) = \cup_{\alpha<\beta}\Delta_\alpha(G),$$

for ordinals ρ and limit ordinals β. The last term in this series is denoted by $\Delta_\infty(G)$ and called the Δ-hypercentre of G. We say that G is Δ-hypercentral, if $G = \Delta_\infty(G)$.

We also write $\pi(G)$ for the set of primes p such that G has an element of order p.

We use implicitly the following fact, which is well known. Since we have not found an explicit reference in the literature, we give a proof for completeness. Recall that an idempotent e in a regular ring R is called *abelian*, if every idempotent in eRe is central.

Lemma 4.2. *Let R be a regular ring with maximal right quotient ring Q. Then R contains a non-zero abelian idempotent if and only if Q does.*

Proof: Let e be an abelian idempotent in Q. Then $eQ \cap R \neq 0$, so $eQ \cap R$ contains a non-zero idempotent f. Now the map φ: $fQf \to feQfe$ defined by $\varphi(x) = xe = exe$ is a ring isomorphism with invers $\varphi^{-1}(y) = yf$. Since $feQfe \subseteq eQe$, it follows that fQf is abelian. Therefore so is fRf.

Conversely, let e be an abelian idempotent in R. We claim that eRe is essential as a right eRe-submodule of eQe, whence [**3**, Theorem 3.2 and Corollary 7.4] show that eQe is abelian. Now $eR = eRe \oplus eR(1-e)$, and since the second summand can contain no non-zero idempotent, it can contain no non-zero right ideal of R. Let x be a non-zero element of eQe. Then $xR \cap R$ is a non-zero right ideal of R contained in eR, and hence it is not contained in $eR(1-e)$. Therefore there exists an element $r \in R$ such that $0 \neq xre \in eRe$, and clearly we can replace r by ere here, as required. ■

Proof of Lemma 4.1: Since the type I part of $Q^r(kG)$ is non-zero, kG contains a non-zero abelian idempotent e. Let H be the subgroup generated by the support of e, and $G_1 = \langle \Delta_\infty(G), H \rangle$. Then e is an abelian idempotent of kG_1, and so the type I part of $Q = Q^r(kG_1)$ is non-zero. Since G_1 is Δ-hypercentral, [**2**, Theorem 1.3] tells us that the type I_∞ part of Q is zero, and therefore its type I_f part must be non-zero. By [**1**, Theorem 2.3], $|G_1 : \Delta(G_1)| < \infty$ and $\Delta(G_1)'$ is finite. Further, if M is the smallest normal subgroup of G_1 such that G_1/M is abelian-by-finite, then the type I part of Q is $(\hat{M}/|M|)Q$, by the first part of the proof of Theorem 2.3 in [1]. Now M is also the smallest normal subgroup of $\Delta_\infty(G)$ such that $\Delta_\infty(G)/M$ is abelian-by-finite, and as such, it is normal in G. Since $e \in (\hat{M}/|M|)kG \cong k[G/M]$, we see that we may assume that $\Delta_\infty(G)$ is abelian-by-finite. By [**7**, Lemma 12.2.2], $\Delta_\infty(G)$ has a characteristic abelian subgroup A of finite index.

Now A is the direct product $A = \prod A_p$ of its primary components. Let $\pi = \pi(H)$, and $A_{\pi'} = \prod_{p \notin \pi} A_p$. Then $A_{\pi'} \subset G$. Consider $\overline{G} = G/A_{\pi'}$. By [**3**, Lemma 7.6], the image of e in $k\overline{G}$ is an abelian idempotent, and clearly it is non-zero. Therefore the type I part of $Q^r(k\overline{G})$ is non-zero. Further, $\Delta_\infty(\overline{G}) = \Delta_\infty(G)/A_{\pi'}$. Therefore, we may assume that $A_{\pi'} = 1$, and so $\pi(\Delta_\infty(G)) = \sigma$ is finite.

Let p be a prime such that $\alpha(ekG) > \frac{1}{p}$, and $p \notin \pi \cup \sigma$. We shall see that $p \notin \pi(G)$. Suppose on the contrary that G contains an element g of order p. Let $\tilde{g} = 1 + g + \ldots + g^{p-1}$. Since the powers of g lie in distinct cosets of G_1, we see that $\tilde{g}e \neq 0$. Since kG is regular, there exists $\beta \in kG$ such that

$$\tilde{g}e = \tilde{g}e\beta\tilde{g}e. \tag{4}$$

By squaring it, we see that $e\beta\tilde{g}e$ is an idempotent in $ekGe$. By [**2**, Lemma 2.1], we see that $e\beta\tilde{g}e \in kG_1$. Using the fact that we have a direct sum $\sum_{i=0}^{p-1} g^i kG_1$, we deduce from (4) that $e = e\beta\tilde{g}e$. It follows from this that the map $f : ekG \to \tilde{g}e\beta kG$ defined by $f(\gamma) = \tilde{g}\gamma$ is an isomorphism, ($f^{-1}(\delta) = e\beta\delta$), and so by Lemma 1.2 (iv), $\alpha(ekG) \leq \alpha(\tilde{g}kG)$. But an easy calculation shows that $\alpha(\tilde{g}kG) = \frac{1}{p}$. This contradicts the choice of p. Hence $\pi(G)$ is finite. By [**1**, Proposition 1.2], we find that G is hypercentral, and the proof is complete. ■

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