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INTERPOLATION OF FAMILIES $\{L^{p(\gamma)}_{\mu(\gamma)}, \gamma \in \Gamma\}$

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Abstract

We identify the intermediate space of a complex interpolation family - in the sense of Coifman, Cwikel, Rochberg, Sagher and Weiss- of L^p spaces with change of measure, for the complex interpolation method associated to an analytic functional.

0. Introduction

Let $\{A(\gamma)\ ;\ \gamma \in \Gamma\}$ be a complex interpolation family (c.i.f.) on $\Gamma = \{|z| = 1\}$ in the sense of [**3**]. Let U be the containing space and $\mathcal{F} = \mathcal{F}(A(\cdot), \Gamma)$ the space of analytic U-valued functions associated to the family.

Let T be an analytic functional on the unit disc D and define the interpolated space $A[T]$ as

$$A[T] = \{x \in U\ ;\ \exists f \in \mathcal{F}\ ,\ T(f) = x\}$$

with the usual norm $\|x\|_{A[T]} = \inf\{\|f\|_{\mathcal{F}}\ ;\ T(f) = x\}$. We shall say that T is of finite support if T admits a representation of the type

$$T = \sum_{j=0}^{n} \sum_{l=0}^{m(j)} a_{jl}\delta^{(l)}(z_j). \tag{1}$$

The set $\{z_0, \cdots, z_n\}$ is said to be the support of T.

The two following results are easily proved.

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Proposition 1. *Let $\{A(\gamma)\ ;\ \gamma \in \Gamma\}$ and $\{B(\gamma)\ ;\gamma \in \Gamma\}$ be two c.i.f. with contaninig spaces U, V and log-intersection space $\mathcal{A}$ and $\mathcal{B}$ respectively. Let $L : \mathcal{A} \longrightarrow \cap_{\gamma\in\Gamma} B(\gamma)$ be a linear operator such that, for each $a \in \mathcal{A}$ and for almost every $\gamma \in \Gamma$,*

$$\|La\|_{B(\gamma)} \leq M(\gamma)\|a\|_{A(\gamma)}$$

where $\log\ M(\cdot) \in L^1(\Gamma)$.

Under theses conditions, if $L : U \longrightarrow V$ is continuous,

$$L : A[GT] \longrightarrow B[T]$$

with norm ≤ 1, where

$$G(z) = \exp\left(-\frac{1}{2\pi}\int_0^{2\pi} \log\ M(\gamma)\ dH_z(\gamma)\right),$$

H_z being the Herglotz kernel.

Proposition 2.

(a) *If $n > m$, $A[\delta^{(m)}(z_0)]$ is continuously embedded in $A[\delta^{(n)}(z_0)]$.*
(b) *If T is of the type (1), $A[T] \equiv \sum_{j=0}^{n} A[\delta^{(m(j))}(z_j)]$.*

Let X be a measure space and $\mu(\gamma, x) \geq 0$ a measurable function on $\Gamma \times X$ such that, for almost every $x \in X$,

$$\int_\Gamma \frac{1}{p(\gamma)} \log\ \mu(\gamma, x) dP_z(\gamma) < +\infty,$$

with $p(\gamma) \geq 1$ a measurable function on Γ and P_z the Poisson kernel.

We shall denote by $\mu(\gamma)$ the measure $\mu(\gamma, x)dx$ with dx the σ-finite measure of X, and by $L^p_{\mu(\gamma)} = L^p(\mu(\gamma))$ the corresponding L^p space.

Assume that the family $\{L^{p(\gamma)}_{\mu(\gamma)}\ ,\ \gamma \in \Gamma\}$ is a c.i.f. with containing space $\mathcal{U}$. Consider the function

$$\mu(z, x) = \exp\left(p(z)\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{p(\gamma)} \log\ \mu(\gamma, x) dH_z(\gamma)\right).$$

It is known (see [6]) that if $T = \delta(z_0)$, $[L^{p(\cdot)}_{\mu(\cdot)}][T] \equiv L^{p(z_0)}_{\mu(z_0)}$, where

$$\frac{1}{p(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{p(\gamma)} dP_z(\gamma).$$

The aim of this paper is to identify the interpolated spaces $[L^{p(\cdot)}_{\mu(\cdot)}][T]$ when T is of finite support.

1. Main results

From Proposition 2, we shall only need to identify a space $[L^{p(\cdot)}_{\mu(\cdot)}][\delta^{(n)}(z_0)]$ with $z_0 \in D$ and $n \in N$. We shall do an induction with respect to n using the following result.

Lemma 3. *Let $F : D \longrightarrow \mathcal{U}$ be an analytic function with non-tangential limit a.e. $\gamma \in \Gamma$ and such that, for almost every $x \in X$, the function $F(z,x) \in N^+(D)$. Assume that, for almost every $\gamma \in \Gamma$, $F(\gamma,\cdot) \in L^{p(\gamma)}_{\mu(\gamma)}$ and*

$$\operatorname{ess}\sup_{\gamma\in\Gamma} \|F(\gamma,\cdot)\|_{L^{p(\gamma)}_{\mu(\gamma)}} = M < +\infty.$$

Then, if $F(z_0,\cdot) = 0$, $F'(z_0,\cdot)$ is in $[L^{p(\cdot)}_{\mu(\cdot)}][\delta(z_0)] = L^{p(z_0)}_{\mu(z_0)}$.

Proof:

We shall prove it with the help of the Fundamental inequality (F.I.) of Hernández (see [**6**]).

Under the hypothesis given, we can consider the function

$$G(z,x) = \begin{cases} F(z,x)/z - z_0 & z \neq z_0 \\ F'(z_0,x) & z = z_0. \end{cases}$$

From the F.I. and the fact that the function $G(z,x)\mu(z,x)^{\alpha(z)}$, with $\alpha(z) = 1/p(z)$, is in $N^+(D)$, we have

$$\begin{aligned}
&\int_X |G(z,x)|^{p(z)}|\mu(z,x)|d\mu = \int_X |G(z,x)\mu(z,x)^{\alpha(z)}|^{p(z)} d\mu \leq \\
&\leq \int_X \exp\left(p(z)\frac{1}{2\pi}\int_0^{2\pi} \log|G(\gamma,x)\mu(\gamma,x)^{1/p(\gamma)}|dP_z(\gamma)\right) d\mu \overset{F.I.}{\leq} \\
&\leq \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{p(z)}{p(\gamma)} \log\left(\int_X |G(\gamma,x)\mu(\gamma,x)^{1/p(\gamma)}|^{p(\gamma)} d\mu\right) dP_z(\gamma)\right) = \\
&= \exp\left(p(z)\frac{1}{2\pi}\int_0^{2\pi} \frac{1}{p(\gamma)} \log\left(\int_X \left(\frac{|F(\gamma,x)|}{|e^{i\gamma} - z_0|}\right)^{p(\gamma)} \mu(\gamma,x)d\mu\right) dP_z(\gamma)\right) \leq \\
&\leq \exp\left(p(z)\frac{1}{2\pi}\int_0^{2\pi} \log\ \|\frac{F(\gamma,\cdot)}{e^{i\gamma} - z_0}\|_{L^{p(\gamma)}_{\mu(\gamma)}} dP_z(\gamma)\right) = \\
&= \exp\left(p(z)\log\frac{M}{d(z_0,\Gamma)}\right) = \left(\frac{M}{d(z_0,\Gamma)}\right)^{p(z)}.
\end{aligned}$$

Thus, the proof is finished from Fatou's Lemma. Moreover,

$$\|F'(z_0,\cdot)\|_{L^{p(z_0)}_{\mu(z_0)}} \leq \frac{M}{d(z_0,\Gamma)}. \quad \blacksquare$$

For each $f \in L^{p(z_0)}_{\mu(z_0)}$, we shall express by H_f the function

$$H_f(z,x)=\mu(z,x)^{-\alpha(z)}\mu(z_0,x)^{w(z)}\|f\|_{L^{p(z_0)}_{\mu(z_0)}}\frac{f(x)}{|f(x)|}\left(\frac{|f(x)|}{\|f\|_{L^{p(z_0)}_{\mu(z_0)}}}\right)^{w(z)p(z_0)},$$

where $\omega(z) = \alpha(z) + \tilde{\alpha}(z)$, with $\tilde{\alpha}(z)$ the conjugate function of α such that $\tilde{\alpha}(z_0) = 0$. We shall assume, in the sequel, that $\omega'(z_0) \neq 0$.

Proposition 4. *$f \in [L^{p(\cdot)}_{\mu(\cdot)}][\delta'(z_0)]$ if and only if there exist f_0 and f_1 in $L^{p(z_0)}_{\mu(z_0)}$ such that*

$$f(x) = f_0(x) + f_1(x)(\log\ |f_1(x)| + H_\mu(z_0,x)), \tag{3}$$

where

$$H_\mu(z_0,x) = \left(\mu(z,x)^{-\alpha(z)}\mu(z_0,x)^{w(z)}\right)'(z_0).$$

Moreover,

$$\|f\|_{[L^{p(\cdot)}_{\mu(\cdot)}][\delta'(z_0)]} \equiv \inf\{\|f_0 + f_1 \log\ \|f_1\|_{L^{p(z_0)}_{\mu(z_0)}}\|_{L^{p(z_0)}_{\mu(z_0)}} + \|f_1\|_{L^{p(z_0)}_{\mu(z_0)}}\ ;\ f \text{ satisfies (3)}\ \}. \tag{4}$$

Proof:

To simplify notation, we shall denote by $E(n)$ the space $[L^{p(\cdot)}_{\mu(\cdot)}][\delta^{(n)}(z_0)]$ for every $n \in \mathbf{N}$. Thus, $E(0) = L^{p(z_0)}_{\mu(z_0)}$.

Let $f \in E(1)$ and $F \in \mathcal{F}(L^{p(\cdot)}_{\mu(\cdot)},\Gamma)$ with $F'(z_0,\cdot) = f$.

Consider $A = \{x \in X\ ;\ F(z_0,x) = 0\}$. It is clear, from the previous lemma, that $f_0^*(x) = f(x)\chi_A(x) \in E(0)$ and

$$\|f_0^*\|_{E(0)} \leq \frac{\|F\|_{\mathcal{F}}}{d(z_0,\Gamma)}.$$

If $x \in A^c$, $F(z_0,x) \neq 0$ and we can consider the function $H(z,x) = H_{F(z_0,x)}\chi_{A^c}(x)$.

It is easy to see that H satisfies the hypothesis of the previous lemma but $H(z_0, \cdot) = 0$. So, the function $G(z,x) = F(z,x)\chi_{A^c}(x) - H(z,x)$ satisfies the necessary hypothesis to ensure that if $f_1 = F(z_0,x)\chi_{A^c}(x)$,

$$\begin{aligned} G'(z_0,x) &= f(x)\chi_{A^c}(x) - f_1(x)(p(z_0)w'(z_0)\log\ |f_1(x)|)+ \\ &\quad + p(z_0)w'(z_0)f_1(x)\log\ \|f_1\|_{E(0)} + H_\mu(z_0,x)f_1(x) \end{aligned}$$

is in $E(0)$ with norm $\leq 2\|F\|_{\mathcal{F}}/d(z_0,\Gamma)$.

Combinating the previous results and joining all the terms of $E(0)$ in a single function f_0, we obtain the desired results as well as one of the inequalities of (4).

Conversely, let $f = f_0 + f_1\,(H_\mu(z_0) + w'(z_0)p(z_0)\log\ |f_1|) = f_0 + g$. If we consider the function H_{f_1}, we obtain, from the previous lemma, that if $F \in \mathcal{F}(L^{p(\cdot)}_{\mu(\cdot)}, \Gamma)$ satisfies $F(z_0,x) = f_1$, then

$$\begin{aligned} f_1^*(x) &= F'(z_0,x) - H'_{f_1}(z_0,x) = \\ &= F'(z_0,x) - f_1(x)\,\Big(p(z_0)w'(z_0)\log\ |f_1(x)| - \\ &\qquad -p(z_0)w'(z_0)\log\ \|f_1\|_{L^{p(z_0)}_{\mu(z_0)}} + H_\mu(z_0,x)\Big) = \\ &= F'(z_0,x) + f_1(x)p(z_0)w'(z_0)\log\ \|f_1\|_{E(0)} - g(x) \end{aligned}$$

is in $E(0)$ and, thus, $g \in E(1)$. $E(0)$ being continuously embedded in $E(1)$ we obtain the desired algebraic equality. Moreover,

$$\begin{aligned} &\|f\|_{E(1)} = \|f_0 + g\|_{E(1)} = \\ &= \|f_0 - f_1^* + F'(z_0,x) + f_1 p(z_0)w'(z_0)\log\ \|f_1\|_{E(0)}\|_{E(1)} \leq \\ &\leq \|f_0 + f_1 p(z_0)w'(z_0)\log\ \|f_1\|_{E(0)}\|_{E(1)} + \|f_1^* - F'(z_0,\cdot)\|_{E(1)} \leq \\ &\leq C\|f_0 + f_1 p(z_0)w'(z_0)\log\ \|f_1\|_{E(0)}\|_{E(0)} + \\ &+ \frac{1}{d(z_0,\Gamma)}(\|F\|_{\mathcal{F}} + \|f_1\|_{E(0)}) + \|F\|_{\mathcal{F}}. \end{aligned}$$

Now, (4) follows easily. ■

Proposition 5. *$f \in [L^{p(\cdot)}_{\mu(\cdot)}][\delta^{(n)}(z_0)]$ if and only if there exist $f_0, \ldots, f_n$ in $L^{p(z_0)}_{\mu(z_0)}$ such that $f(x) = f_0(x) + H'_1(z_0,x) + \cdots + H^{(n)}_n(z_0,x)$, where $H_j = H_{f_j}$.*

Proof:

$E(n)$ still denotes the space $[L^{p(\cdot)}_{\mu(\cdot)}][\delta^{(n)}(z_0)]$ as in the preceeding proof.

It is already known that the result is true for $n = 0$ and $n = 1$. Assume that it is true for $n - 1$ and let us see it for $n > 1$.

Let $f \in E(n)$ and $F \in \mathcal{F}(L^{p(\cdot)}_{\mu(\cdot)}, \Gamma)$ with $F^{(n)}(z_0, \cdot) = f$. Consider the set

$$A = \{x \in X \ ; \ F(z_0, x) = 0\}$$

and assume the following

Claim. *If F satisfies the hypothesis of Lemma 3, then we get that $F^{(n)}(z_0, \cdot) \in E(n-1)$.*

It is clear then, that $(F(z, \cdot)\chi_A(\cdot))^{(n)}(z_0)$ is in $E(n-1)$ and if $f_n = F(z_0, \cdot)\chi_{A^c}$ and $H_n = H_{f_n}$, then $G_n(z, x) = F(z, x)\chi_{A^c}(x) - H_n(z, x)$ satisfies the hypothesis of the claim and therefore, $G_n^{(n)}(z_0, \cdot) \in E(n-1)$.

Consequently, if we call $g(\cdot) = (F(z, \cdot)\chi_A(\cdot))^{(n)}(z_0) + G_n^{(n)}(z_0, \cdot)$ we have, from the induction hypothesis, that there exist $f_0, \cdots, f_{n-1}$ in $E(0)$ such that

$$g(x) = f_0(x) + \sum_{j=1}^{n-1} H_j^{(j)}(z_0, x).$$

Finally, as $f(x) = g(x) + H_n^{(n)}(z_0, x)$, the desired result is obtained. The converse is quite similar.

Proof of the claim:

We know that the claim is true for $n = 1$. Let us consider the set $B = \{x \in X \ ; \ F'(z_0, x) = 0\}$. Then, from the induction hypothesis, $(F(z, \cdot)\chi_B(\cdot))^{(n)}(z_0)$ is in $E(n-2)$.

Let now $x \notin B$. One can consider the function

$$G_F(z, x) = \frac{F(z, x)}{z - z_0}\chi_{B^c}(x) - H_F(z, x)$$

where $H_F = H_{F'(z_0, \cdot)\chi_{B^c}}$.

Because G_F satisfies the hypothesis of Lemma 3, $G_F^{(n-1)}(z_0, \cdot)$ is in $E(n-2)$ and, thus, as

$$(F(z, x)\chi_{B^c})^{(n)}(z_0) = n\left(G_F^{(n-1)}(z_0, x) + H_F^{(n-1)}(z_0, x)\right)$$

and $H_F^{(n-1)}(z_0, \cdot) \in E(n-1)$, we get that $F^{(n)}(z_0, \cdot)$ is in $E(n-1)$. ∎

Corollary 6. *Let* $J(z,x) = (\mu(z_0,x)/\mu(z,x))^{1/p}$. *Then, the space* $[L^p_{\mu(\cdot)}][\delta^{(n)}(z_0)]$ *is equivalent to*

$$L^p(\mu(z_0)) + L^p(\mu(z_0)(J'(z_0,x))^{-p}) + \cdots + L^p(\mu(z_0)(J^{(n)}(z_0,x))^{-p}) \equiv$$
$$\equiv L^p(\mu(z_0)(\sum_{j=1}^{n} |J^{(j)}|(z_0,x))^{-p}).$$

Proof:
Let us denote $\mu_k = \mu(z_0)J^{(k)}(z_0,x)^{-p}$ for every $k \in \mathbf{N}$.
If $p(\gamma) = p$, $H_f(z,x) = J(z,x)f(x)$ and, as $f \in L^p(\mu_0)$,

$$H_f^{(k)}(z_0,x) = f(x)J^{(k)}(z_0,x) \in L^p(\mu_k).$$

Now we see the equivalence of the norms. Assume initially that $n = 1$ and let $f \in [L^p_{\mu(\cdot)}][\delta'(z_0)]$. Let $F \in \mathcal{F}(L^p_{\mu(\cdot)}, \Gamma)$ with $F'(z_0,x) = f(x)$ and consider $G(z,x) = F(z,x) - J(z,x)F(z_0,x)$. It is satisfied that $G(z_0,\cdot) = 0$ and, therefore, $G'(z_0,\cdot) \in L^p(\mu_0)$. Moreover,

$$\|G'(z_0,\cdot)\|_{L^p(\mu_0)} \leq \frac{1}{d(z_0,\Gamma)}(\|F\|_{\mathcal{F}} + \|F(z_0,x)\|_{L^p(\mu_0)}) \leq \frac{2\|F\|_{\mathcal{F}}}{d(z_0,\Gamma)}.$$

Thus, $F'(z_0,x) = G'(z_0,x) + J'(z_0,x)F(z_0,x) = f_0(x) + f_1(x)$ with f_0 in $L^p(\mu_0)$ and f_1 in $L^p(\mu_1)$. Moreover,

$$\|f_0\|_{L^p(\mu_0)} + \|f_1\|_{L^p(\mu_1)} \leq \frac{2\|F\|_{\mathcal{F}}}{d(z_0,\Gamma)} + \|F\|_{\mathcal{F}} = \left(1 + \frac{2}{d(z_0,\Gamma)}\right)\|F\|_{\mathcal{F}},$$

that, a fortiori, yields the equivalence of the norms. Now, assume that the result is true for $n-1$ and let $f \in [L^p_{\mu(\cdot)}][\delta^{(n)}(z_0)]$ and F in $\mathcal{F}(L^p_{\mu(\cdot)}, \Gamma)$ with $F^{(n)}(z_0,x) = f(x)$. The function $G^{(n)}(z_0,x) = f(x) - J^{(n)}(z_0,x)F(z_0,x)$ is in $[L^p_{\mu(\cdot)}][\delta^{(n-1)}(z_0)]$ and, from the induction hypothesis, there exist $f_j \in L^p(\mu_0)$ $(0 \leq j \leq n-1)$ such that

$$f(x) = f_0(x) + f_1(x)J'(z_0,x) + \cdots + f_{n-1}(x)J^{(n-1)}(z_0,x) + J^{(n)}(z_0,x)F(z_0,x).$$

Moreover,

$$\|f_0\|_{L^p(\mu_0)} + \cdots + \|f_{n-1}\|_{L^p(\mu_{n-1})} \ll \|G^{(n)}(z_0,x)\|_{[L^p_{\mu(\cdot)}][\delta^{(n-1)}(z_0)]} \ll \|F\|_{\mathcal{F}}.$$

Now, the proof is easily ended. ■

Corollary 7. *Let w_0, w_1 be two positive measurable functions on X. Then f is in $[L^{p_0}(w_0), L^{p_1}(w_1)]_{\delta'(\theta)}$ if and only if there exist f_0, f_1 in $L^p(w)$, ($1/p = (1-\theta)/p_0 + \theta/p_1$ and $w = w_0^{p(1-\theta)/p_0} w_1^{p\theta/p_1}$) such that*

$$f(x) = f_0(x) + f_1(x)(\frac{1}{p_0}\log\ w_0(x) - \frac{1}{p_1}\log\ w_1(x)) + f_1(x)\log\ |f_1(x)|.$$

Proof:

Given $0 < \theta < 1$, there exists a measurable set $\Gamma_1 \subset \Gamma$ such that $\int_{\Gamma_1} dP_{z_0}(\gamma) = \theta$. So, if we consider $A(\gamma) = L^{p_0}(w_0)$ for each $\gamma \in \Gamma \setminus \Gamma_1$ and $A(\gamma) = L^{p_1}(w_1)$ for each $\gamma \in \Gamma_1$, we have $A(\gamma) = [L^{p_0}(w_0), L^{p_1}(w_1)]_{\alpha(\gamma)}$ with $\alpha(\cdot) = \chi_{\Gamma_1}(\cdot)$.

It is known (see [**11**, 1.18.5]) that $A(\gamma) = L^{p(\gamma)}_{\mu(\gamma,x)}$, where, for each $\gamma \in \Gamma$,

$$\frac{1}{p(\gamma)} = \frac{1-\alpha(\gamma)}{p_0} + \frac{\alpha(\gamma)}{p_1} \quad \text{and}$$
$$\mu(\gamma, x) = w_0^{p(\gamma)(1-\alpha(\gamma))/p_0} w_1^{p(\gamma)\alpha(\gamma)//p_1}.$$

Moreover, α attains the values 0 and 1, and thus, as we have proved in [**2**] in quite analogy with the reiteration results of [**3**], if $T = \delta^{(n)}(z_0)$ ($n \in \mathbf{N}$) and $w'(z_0) \neq 0$, then

$$A[T] \equiv [L^{p_0}(w_0), L^{p_1}(w_1)]_S,$$

where $S(\varphi) = T(\varphi \circ w)$ and $[L^{p_0}(w_0), L^{p_1}(w_1)]_S$ is defined like in the interpolation method of [**10**]. So,

$$[L^{p_0}(w_0), L^{p_1}(w_1)]_S \equiv [L^{p_0}(w_0), L^{p_1}(w_1)]_{\delta'(\alpha(z_0))} = [L^{p_0}(w_0), L^{p_1}(w_1)]_{\delta'(\theta)}.$$

Hence, the space we want to identify is a particular case of Proposition 5. But, in this case,

$$\mu(z, x) = w_0^{p(z)(1-w(z))/p_0} w_1^{p(z)w(z)/p_1}.$$

If we call $B(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{p(\gamma)} dP_z(\gamma)$ and $\mathcal{B}(z) = B(z) + i\tilde{B}(z)$ with $\tilde{B}(z_0) = 0$, we have

$$\mu(z_0, x)^{\mathcal{B}(z)} \mu(z, x)^{-B(z)} = w_0^{((w(z)-1)+(1-\theta)p\mathcal{B}(z))/p_0} w_1^{(p\mathcal{B}(z)\theta - w(z))/p_1}.$$

Now we can apply Proposition 4 to end the proof. ∎

Remark. In view of the Corollary 6 and the above calculation, one can easily obtain that

$$[L^p(w_0), L^p(w_1)]_{\delta^{(n)}(\theta)} \equiv L^p(w_0^{1-\theta} w_1^{\theta} (1 + |\log (w_0/w_1)|^n)^{-p})$$

as it is said in [**7**].

Remark.

Let $\varphi(x,t)$ be a function that, for each $x \in M$, is an increasing function of t in $0 \leq t < \infty$, and $\varphi(x,0) = 0$. Denote by $\varphi(X)$ the class of measurable functions g on M such that there exist $\lambda > 0$ and $f \in X$ with $\|f\|_X \leq 1$ and

$$|g(x)| \leq \lambda\varphi(x, \lambda|f(x)|) \qquad \text{a.e. } x \in M.$$

Define the "norm" of g, $\|g\|_{\varphi(X)}$, as the infimun of the values λ for which such an inequality holds.

It is known (see [**1**]) that if $\varphi(x,t)$ is a concave function of t and change the previous norm by

$$\|g\| = \inf\{\lambda > 0 \ ; \ |g(x)| \leq \lambda\varphi(x, |f(x)|) \qquad \text{a.e. } x \in M\},$$

then $(\varphi(X), \|\cdot\|)$ is a Banach Lattice. In our case, we can only assure that the space $\varphi(X)$ is a Frechet Lattice.

We say that a function f is equivalent to g in R^+ if and only if there exist $a,\ b > 0$ such that

$$a\ f(x) \leq\ g(x) \leq\ b\ f(x) \qquad \text{a.e. } x \in X.$$

It is also known that $L^{p(\gamma)}_{\mu(\gamma)} = \varphi_\gamma(L^1)$, where

$$\varphi_\gamma(x,t) = \mu(\gamma, x)^{-1/p(\gamma)} t^{1/p(\gamma)}.$$

Consider the function

$$\varphi_z(x,t) = \exp\left(\frac{1}{2\pi}\int_\Gamma \log\ \varphi_\gamma(x,t)\ dH_z(\gamma)\right).$$

Then $\varphi_z(x,t) = \mu(z,x)^{-1/p(z)} t^{\omega(z)}$.

Finally we assume that, for each $1 \leq k \leq n$, the function $\varphi_k(x,t) = |\delta^{(k)}(z_0)(\varphi_z(x,t))|$ is equivalent to an increasing function that we shall continue denoting by φ_k.

Proposition 8. *If $T = \delta^{(n)}(z_0)$, the space $[L^{p(\cdot)}_{\mu(\cdot)}][T]$ is equivalent to $\sum_{k=0}^{n} \varphi_k(L^1)$.*

Proof:

Let $f \in \varphi_k(L^1)$ and let $h \in L^1$ with $\|h\|_{L^1} \leq 1$ and $\lambda > 0$ such that $|f(x)| \leq \lambda\ \varphi_k(x, \lambda|h(x)|)$.

We have

$$\varphi_k(x, \lambda|h(x)|) = |\delta^{(k)}(z_0)\left(\mu(z,x)^{-1/p(z)}(\lambda|h(x)|)^{\omega(z)}\right)|.$$

It is easy to see that the function $F(z,\cdot) = \mu(z,\cdot)^{-1/p(z)}(\lambda|h(x)|)^{\omega(z)}$ is in $\mathcal{F}(L^{p(\cdot)}_{\mu(\cdot)})$, and hence, f is in $[L^{p(\cdot)}_{\mu(\cdot)}][\delta^{(k)}(z_0)]$.

Moreover, $\|f\|_{[L^{p(\cdot)}_{\mu(\cdot)}][\delta^{(k)}(z_0)]} \leq \lambda\|F\|_{\mathcal{F}}$. So it is clear that if $(f_n)_n$ converges to zero in $\varphi_k(L^1)$, $(f_n)_n$ converges to zero in $[L^{p(\cdot)}_{\mu(\cdot)}][\delta^{(k)}(z_0)]$.

Conversely, from Proposition 5, one can obtain that if $f \in [L^{p(\cdot)}_{\mu(\cdot)}][\delta^{(n)}(z_0)]$, $f(x) = g(x) + H_n^{(n)}(z_0, x)$ where $|H_n^{(n)}(z_0,x)| \equiv \varphi_n(x, |g_n(x)|)$ with $g_n = |f_n|^{p(z_0)}\mu(z_0) \in L^1$. An induction ends the proof. ∎

2. Applications

Example 1.

If $b \in BMO$ has a norm enough small s, then $W = e^b$ and W^{-1} are weight of A_p. Furthermore. for any Calderón Zygmund integral operator (CZO), L,

$$L : L^p(W) \longrightarrow L^p(W) \quad \text{and} \quad L : L^p(W^{-1}) \longrightarrow L^p(W^{-1}).$$

(See [8]).

Proposition 9. *Under the previous hypothesis, for each $b \in BMO$,*

$$\int_{\mathbf{R}^n} |L(g(x)|b(x)|)|^p \frac{1}{(\|b\|_* + \mathrm{s}|b(x)|)^p} dx \ll \frac{1}{\mathrm{s}^p}\|g\|_p^p \qquad \forall g \in L^p.$$

Proof:

It is a trivial consequence of the fact that

$$L : [L^p(W), L^p(W^{-1})]_{\delta'(\theta)} \longrightarrow [L^p(W), L^p(W^{-1})]_{\delta'(\theta)}$$

and that for $\theta = 1/2$,

$$[L^p(W), L^p(W^{-1})]_{\delta'(\frac{1}{2})} \equiv L^p((1 + |b|)^{-p}).$$

So, if $f \in L^p((1 + |b|)^{-p})$, $\|L(f)\|_{L^p((1+|b|)^{-p})} \ll \|f\|_{L^p((1+|b|)^{-p})}$.
On the other hand, if $g \in L^p$, $L(g) \in L^p$ and

$$\|L(g)\|_{L^p((1+|b|)^{-p})} \leq \|L(g)\|_p \leq c\|g\|_p.$$

The combination of all these results ends the proof. ■

Corollary 10. *If L is a CZO,*

$$\sup_{b\in BMO} \left(\int_X |L(|b(x)|)|^p \frac{1}{(\|b\|_* + \mathrm{s}|b(x)|)^p} dx \right)^{\frac{1}{p}} \ll \frac{1}{\mathrm{s}} |X|,$$

for any Lebesgue measurable set X and $|X|$ its measure.

Example 2.

Consider $0 < \gamma < n$, $1 < p_1 < (n/\gamma)$ and $1/p_2 = 1/p_1 - \gamma/n$. If $b \in BMO$, it is proved in [**9**] that if $L_\gamma = *|x|^{\gamma - n}$ (Riesz Potentials), then

$$L_\gamma : L^{p_1}(e^b) \longrightarrow L^{p_2}(e^b) \quad \text{and}$$
$$L_\gamma : L^{p_1}(e^{-b}) \longrightarrow L^{p_2}(e^{-b}).$$

Thus, with an argument quite similar to the one of Proposition 9, we get the following result.

Proposition 11. *Under the previous conditions,*

$$\sup_{b\in BMO} \left(\int_X |L_\gamma |b(x)|\,|^{p_2} \frac{1}{(\|b\|_* + \mathrm{s}|b(x)|)^{p_2}} dx \right)^{\frac{1}{p_2}} \ll \frac{1}{\mathrm{s}} |X|^{\frac{1}{p_1}}.$$

Example 3.

Let $1 < p_1 < p_2 < \infty$ and $p = 2(p_1^{-1} + p_2^{-1})^{-1}$. If $g \in L^p(\mathbf{R}^n)$ and g^* is the Maximal function of Hardy-Littlewood, there exists α such that $(g^*)^{\pm\alpha}$ are weights in the classes A_{p_1} and A_{p_2} ([**4**, Prop. 2]). Consequently, if L is a CZO,

$$L : L^{p_1}((g^*)^{\pm\alpha}) \longrightarrow L^{p_1}((g^*)^{\pm\alpha}) \quad \text{and}$$
$$L : L^{p_2}((g^*)^{\pm\alpha}) \longrightarrow L^{p_2}((g^*)^{\pm\alpha}).$$

Proposition 12. *Under the previous conditions, for each $f \in L^p(\mathbf{R}^n)$ $(p_1 \le p \le p_2)$,*

$$\left(\int_{\mathbf{R}^n} |L(f|\log\ g^*|)|^p \frac{1}{(1+\alpha|\log\ g^*|)^p} dx\right)^{\frac{1}{p}} \ll \frac{1}{\alpha}\|f\|_p.$$

Example 4. On the Hardy-Littlewood maximal operator.

Let M be the Hardy-Littlewood maximal operator. If $0 < \alpha < 1$, then

$$f(x) = M(\|x\|^{-\alpha n})(1 + |\log\ M(\|x\|^{-\alpha n})\ |)^{-1} \in L^{1/\alpha}(\mathbf{R}^n).$$

If we take $p = 1/\alpha$ and $u = 1$, it will be a particular case of the following result.

Proposition 13. *Let $u \in A_2$ and $p > 1$. If $f(1 + |\log\ |f|\ |)^{-1} \in L^p(u^{-1})$ and $g = M(fu^{-1})u$, then $g(1 + |\log\ |g|\ |)^{-1} \in L^p(u^{-1})$.*

Proof: Let $\alpha : \Gamma \longrightarrow (0,1)$ a measurable function such that

$$\frac{1}{p} = \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{1+\alpha(\gamma)}\, d\gamma.$$

Then

(a) $u^{\alpha(\gamma)} \in A_{\alpha(\gamma)+1}$ (see [5]) and, therefore, if $p(\gamma) = 1 + \alpha(\gamma)$

$$M : L^{p(\gamma)}(u^{\alpha(\gamma)}) \longrightarrow L^{p(\gamma)}(u^{\alpha(\gamma)}).$$

(b) By interpolation

$$M : [L^{p(\cdot)}(u^{\alpha(\cdot)})][\delta'(0)] \longrightarrow [L^{p(\cdot)}(u^{\alpha(\cdot)})][\delta'(0)].$$

(c) If

$$u : [L^{p(\cdot)}(u^{\alpha(\cdot)})][\delta'(0)] \longrightarrow L_\phi(u^{-1})$$

is defined by $u(f) = uf$, then u is an isomorfism, where $L_\phi(u^{-1})$ is the Orlicz space associated to $\phi(t) = \varphi^{-1}(t)^p$, and $\varphi(t) = t(1+|\log\ t|)$. This result is a consequence of Proposition 4 with $H_\mu(0,x) = pw'(0)\log\ u$ and from the fact that $L_\phi(u^{-1})$ is the space of the measurable functions such that $f(1 + |\log\ |f|\ |)^{-1} \in L^p(u^{-1})$.

Now the proof ends from (a), (b) and (c). ■

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