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ON HARMONIC VECTOR FIELDS

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Abstract

A tangent bundle to a Riemannian manifold carries various metrics induced by a Riemannian tensor. We consider harmonic vector fields with respect to some of these metrics. We give a simple proof that a vector field on a compact manifold is harmonic with respect to the Sasaki metric on TM if and only if it is parallel. We also consider the metrics II and $I + II$ on a tangent bundle (cf. [**YI**]) and harmonic vector fields generated by them.

1. Preliminaries

1.1. Let (M, g) be a smooth manifold. We denote by $N := TM$ the tangent bundle. Then there is given the canonical projection $\pi : N \to M$. By $\Gamma(T^*M)$ we shall denote the set of 1-forms on M. Then there exists a natural map

$$\mathbf{i} : \Gamma(T^*M) \longrightarrow C^\infty(N)$$

such that $(\mathbf{i}\theta)(v) := \theta(v)$ for each $\theta \in \Gamma(T^*M)$ and $v \in N$.

Suppose that $X \in \chi(M)$ is a vector field on M. Then there is defined *the vertical lift* X^V *of* X *to* N. The vector field X^V has the property that

$$X^V(\mathbf{i}\theta) = \theta(X) \circ \pi$$

for all 1-forms θ on M. Moreover it is well-known that the above equality determines uniquely the vertical lift of X.

Observation 1.1. The vertical lift of a vector field depends pointwisely on vectors.

Observation 1.2. Suppose that for a given $x \in M$ we have that X, $v \in T_xM$. Then the vertical lift of X to N at v is a vector which is tangent at zero to a curve.

$$t \longrightarrow v + tX.$$

Observation 1.3. The following map

$$T_{\pi(v)}M \ni X \longrightarrow X^V \in T_v N^V$$

is an isomorphism for all $v \in N$ (cf. 2.2. for the definition of TN^V).

Let $X \in \chi(M)$. Then there is defined *the complete lif X^c of X to N*. The field X^c is uniquely determined by the following property: for all $f \in C^\infty(M)$ we have that

$$X^c(\mathrm{i}df) = \mathrm{i}d(X(f)).$$

Observation 1.4. If φ_t is a local flow of X on M then

$$v \longrightarrow d\varphi_t(v)$$

is a local flow of X^c on N (cf. [**CDL**]).

2.2. Let (M, g) be a pseudo-Riemannian structure on the manifold M. Then the tensor g determines the Levi-Civita connection ∇ on M.

The connection ∇ induces a 1-form

$$\omega^\nabla \in T^*N \otimes_N \pi^{-1}TM$$

defined uniquely by the following equations

$$\omega^\nabla(dY(X)) = \nabla_X Y$$
$$\omega^\nabla(X^V) = X$$

for all vector fields $X, Y \in \chi(M)$. By $\pi^{-1}TM$ we denote the pull-back bundle of TM along the projection $\pi : N \to M$. In the above formula $dY(X)$ denotes the differential of Y evaluated at X. More precisely, if $(U, (x^1, \dots, x^m))$ is a chart on M and $(N|_{U'}, (x^1, \dots, x^m, y^1, \dots, y^m))$ is the induced chart on N then

$$dY(X) = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i} + \sum_{i=1}^{m} \sum_{\alpha=1}^{m} X^i \frac{\partial Y^\alpha}{\partial x^i} \frac{\partial}{\partial y^\alpha}$$

where $X = \sum_{i=1}^{m} X^i \frac{\partial}{\partial x^i}$ and $Y = \sum_{j=1}^{m} Y^j \frac{\partial}{\partial y^j}$. Then we define the vertical and horizontal subspaces of the bundle $TN \to N$ in the following way:

$$TN^V = \{X^V \in TN |\, X \in TM\} = \ker d\pi$$
$$(TN)^H = \{Z \in TN |\, \omega^\nabla(Z) = 0\}.$$

It is well-know that $(TN)^V$ and $(TN)^H$ are smooth subbundles of $TN \to N$ and that we have the following direct sum of vector bundles:

$$TN = (TN)^V \oplus (TN)^H.$$

This decomposition implies that

$$d\pi : (TN)^H \longrightarrow TM$$

is an isomorphism on the fibres.

Suppose that $X, v \in TM$ then *the horizontal lift of X to T_vN* is a vector $X^H \in T_vN$ such that

$$\begin{aligned} &(1). \quad X^H \text{ is horizontal;} \\ &(2). \quad d\pi(X^H) = X. \end{aligned}$$

It is clear that the conditions above define uniquely the horizontal lift of a vector. In a natural way the horizontal lift is extended to vector fields.

Observation 1.5. From the construction of the horizontal subbundle it follows that

$$\begin{aligned} (TN)^H_v &= \{dY(X) \in TN | \forall X \in T_{\pi(v)}M \\ &\qquad \text{and } \forall Y \in \chi(M) \text{ such that } \nabla Y_x = 0\} \\ &= imdY_x \text{ where } Y \in \chi(M) \text{ and } \nabla Y_x = 0. \end{aligned}$$

The direct sum decomposition of T_vN and the above identifications allow us to define maps p^H and p^V such that

$$\begin{aligned} p^H &: T_vN \longrightarrow T_{\pi(v)}M \\ p^V &: T_vN \longrightarrow T_{\pi(v)}M \end{aligned}$$

where p^H is just $d_v\pi$ and p^V is a composition of a projection from T_vN onto T_vN^V with the identification of this space with $T_{\pi(v)}M$ (cf. Observation 1.3). The maps p^H and p^V serve for construction of three symmetric bilinear forms

$$\begin{aligned} I &:= g(p^H(*), p^H(*)) \\ II &:= g(p^H(*), p^V(*)) + g(p^V(*), p^H(*)) \\ III &:= g(p^V(*), p^V(*)) \end{aligned}$$

We may repeat the construction of these forms point by point to obtain global forms on N. The forms II, $I + II$, $I + III$, $II + III$ appear to be Riemannian or pseudo-Riemannian metrics on N. These metrics are studied in [**YI**]; an interesting exposition of this subject may be found also in [**Ia**].

There are the following relations between the lifts defined above:

Proposition 1.6. *Let $X, Y \in \chi(M)$ and $p \in M$. We also denote $v = X_p$. Then the following identities hold*

$$\text{(i)} \qquad Y_v^c = Y_v^H + (\nabla_X Y)_v^V$$

$$\text{(ii)} \qquad dX(Y_p) = Y_v^H + (\nabla_Y X)_v^V$$

$$\text{(iii)} \qquad dX(Y_p) = Y_v^c + [Y, X]_v^V$$

(cf. also [**YI**]*).*

Proof: (ii) let $dX(Y)_v = A + B$ where $A \in T_vN^H$ and $B \in T_vN^V$. Then

$$Y_p = d\pi \circ dX(Y) = d\pi(A + B) = d\pi(A).$$

Since $d\pi$ restricted to T_pN^H is an isomorphism then we get that $A = Y_v^H$. On the other hand we have that

$$(\nabla_Y X)_p = \omega_v^\nabla(dX(Y_p)) = \omega_v^\nabla(A + B) = \omega_v^\nabla(B).$$

Since ω_v^∇ is an isomorphism when restricted to T_vN^V then it follows that $(\nabla_Y X)_v^V = B$. Hence (ii) follows.

We shall demonstrate (iii) using a chart $(U, (x_1, \dots, x_m))$ such that $p \in U \subset M$. Then we may express X and Y as the linear combinations of the standard basis

$$X = \sum_{i=1}^m X^i \frac{\partial}{\partial x^i}, \qquad Y = \sum_{i=1}^m Y^i \frac{\partial}{\partial x^i}$$

where X^i, Y^j are C^∞-functions on U. The local coordinates on M determine in a natural way the local coordinates $(N|_U, (x_1, \dots, x_m, y_1, \dots, y_1))$. It easy to check that in this new local coordinates the following equalities hold:

$$dX(Y_p) = \left(Y_p^1, \dots, Y_p^m, \sum_{\alpha=1}^m \frac{\partial X^1}{\partial x^\alpha}|_p \cdot Y_p^\alpha, \dots, \sum_{\alpha=1}^m \frac{\partial X^m}{\partial x^\alpha}|_p \cdot Y_p^\alpha \right)$$

$$Y_v^c = \left(Y_p^1, \dots, Y_p^m, \sum_{\alpha=1}^m X_p^\alpha \cdot \frac{\partial Y^1}{\partial x^\alpha}|_p, \dots, \sum_{\alpha=1}^m X_p^\alpha \cdot \frac{\partial Y^m}{\partial x^\alpha}|_p \right)$$

(for the second equality cf. [**YI**, p. 15]). Then it follows that

$$dX(Y_p) - Y_v^c = [Y, X]_v^V$$

and then (iii) follows.

Equality (i) is a consequence of (ii) and (iii). In fact, from (iii) we get that

$$Y_v^c = dX(Y_p) - [Y, X]_v^V.$$

Then from (ii) and from the fact that ∇ is torsionless we get that

$$\begin{aligned} Y_v^c &= Y_v^H + (\nabla_Y X)_v^V - [Y, X]_v^V \\ &= Y_v^H + (\nabla_Y X + [X, Y])_v^V \\ &= Y_v^H + (\nabla_X Y)_v^V. \end{aligned}$$

This ends the proof of (i) and of the proposition. ■

1.3. If $\phi : (M_1, g_1) \to (M_2, g_2)$ is a smooth map between two pseudo-Riemannian manifolds then the *tension field of* ϕ is defined as

$$r(\phi) = \mathrm{trace}_{g_1} \nabla\, d\phi.$$

Then ϕ is called *harmonic* if the tension field vanishes. The equivalent definition of harmonicity of ϕ is that ϕ is a stationary point of the energy functional

$$E(\phi) = \frac{1}{2} \int_{M_1} \mathrm{trace}_{g_1}(\phi^* g_2) \nu_{g_1}.$$

By ν_{g_1} we denote the canonical measure on M_1 induced by g_1. If M_1 is not compact then the energy may be defined on its compact subsets. Then ϕ appears harmonic iff such energies defined on compact subsets are stationary with respect to the compactly supported variations. The function

$$e(\phi) = \frac{1}{2} \mathrm{trace}_{g_1}(\phi^* g_2)$$

is called *energy density* of ϕ (cf. [**K**]). For more details about harmonic maps and techniques used in that theory cf. [$\mathbf{EL}_1$], [$\mathbf{EL}_2$].

2. Energy densities

In this part of the paper we shall consider the properties of energy densities associated with different symmetric tensors.

Let (M, g) be a pseudo-Riemannian manifold and let $X \in \chi(M)$. We fix $p \in M$ and suppose that $X_p = v$. By $E_1, \ldots, E_m$ we denote a local orthonormal frame. Then the energy density associated with X :

$(M,g) \to (N,I)$ is the following:

$$\begin{aligned} 2e_I(X)_p &= \operatorname{trace}_g I\left(dX(*), dX(*)\right)_v \\ &= \sum_{i=1}^{m} I\left(dX(E_i), dX(E_i)\right)_v \cdot g(E_i, E_i)_p \\ &= \sum_{i=1}^{m} I\left(E_i^H + (\nabla_{E_i} X)^V, E_i^H + (\nabla_{E_i} X)^V\right)_v \cdot g(E_i, E_i)_p \\ &= \sum_{i=1}^{m} g(E_i, E_i)_p^2 \\ &= m \end{aligned}$$

Similarily we calculate that

$$\begin{aligned} 2e_{II}(X)_p &= \operatorname{trace}_g II\left(dX(*), dX(*)\right)_v \\ &= \sum_{i=1}^{m} II\left(dX(E_i), dX(E_i)\right)_v \cdot g(E_i, E_i)_p \\ &= 2\sum_{i=1}^{m} g(E_i, \nabla_{E_i} X)_p \cdot g(E_i, E_i)_p \end{aligned}$$

and

$$2e_{III}(X)_p = \sum_{i=1}^{m} g(\nabla_{E_i} X, \nabla_{E_i} X)_p \cdot g(E_i, E_i)_p.$$

It follows that for a given real number $t \in R$

$$\begin{aligned} e_I(tX) &= \text{constant} = \frac{m}{2} \\ e_{II}(tX) &= t e_{II}(X) \\ e_{III}(tX) &= t^2 e_{III}(X) \end{aligned}$$

If M is compact then there are defined the energies

$$\begin{aligned} E_I(X) &= \int_M e_I(X)\nu_g \\ E_{II}(X) &= \int_M e_{II}(X)\nu_g \\ E_{III}(X) &= \int_M e_{III}(X)\nu_g. \end{aligned}$$

Since I, III are degenerated metrics on N the quantities defined above are not classical energies. However they have the following properties:

$$E_I(tX) = \frac{m}{2}\,\mathrm{vol}(M)$$
$$E_{II}(tX) = tE_{II}(X)$$
$$E_{III}(tX) = t^2 E_{III}(X).$$

3. The metric $I + III$ (the Sasaki metric)

Harmonic vector fields $X : (M, g) \to (TM, I + III)$ were investigated by Ishihara (cf. [**I**]). The tension field obtained by Ishihara is the following:

$$\tau(X) = (\mathrm{trace}_g\, R(\nabla_* X, X)*)^H + ((\mathrm{trace}_g\, \nabla^2 X)^V$$

cf. also [**CS**]. The vector field $\mathrm{trace}_g \nabla^2 X$ is called the *rough Laplacian* and is denoted by ΔX. In the case when M is compact it was proved that X is harmonic iff it is parallel (cf. [**I**]). In the proof there is used Bochner's theorem ([**Y**, p. 39]). We give below very simple proof of the theorem of Ishihara for compact manifolds.

Theorem 3.1. *Let (M, g) be a compact Riemannian manifold and $X \in \chi(M)$. Then X is harmonic with respect to Sasaki metric on TM if and only if X is parallel.*

Proof: Suppose that X is harmonic and let consider the following variation of X

$$M \times \mathbf{R} \ni (x, t) \longrightarrow tX_x \in TM.$$

Since X is a critical point of the energy functional we have that:

$$\begin{aligned} 0 &= \frac{d}{dt} E_{I+III}(tX)|_{t=1} \\ &= \frac{d}{dt} E_{III}(tX)|_{t=1} \\ &= \frac{d}{dt} t^2 E_{III}(X)|_{t=1} \\ &= E_{III}(X). \end{aligned}$$

This implies that

$$0 = e_{III}(X) = \mathrm{trace}_g\, g(\nabla_* X, \nabla_* X)$$

hence $\nabla X = 0$.

If X is parallel then it is clear that the tension field of X vanishes hence the vector field is harmonic. ∎

4. Metrics II, $I+II$

Let (M,g) be a pseudo-Riemannian manifold. The Levi-Civita connections on $N=TM$ defined by the metrics II and $I+II$ are the same. In fact this is the complete lift of the Levi-Civita connection ∇ to N (cf. Proposition 6.6, p. 45, Proposition 3.1, p. 149 [**YI**]). The complete lift of ∇ we shall denote by ∇^c. This connection is characterized in the following way: if X, $Y\in\chi(M)$ then

$$\begin{aligned}\nabla^c_{X^V}Y^V&=0\\ \nabla^c_{X^V}Y^c&=(\nabla_XY)^V\\ \nabla^c_{X^c}Y^V&=(\nabla_XY)^V\\ \nabla^c_{X^c}Y^c&=(\nabla_XY)^c.\end{aligned}$$

Since the Levi-Civita connections of metrics II and $I+II$ coincide so do harmonic maps defined by these metrics. We shall calculate the tension field of a vector field $X\in\chi(M)$. Let $(E_1,\dots,E_m)$ be a local orthonormal frame around the point $p\in M$ and let $\varepsilon_i:=g(E_i,E_i)_p$ for $i=1,\dots,m$. Then applying Proposition 1.6 we get

$$\begin{aligned}\tau(X)_p&=\operatorname{trace}_g(\nabla^c dX)_p\\ &=\sum_{i=1}^m\varepsilon_i\nabla^c_{E_i}dX(E_i)|_v-\varepsilon_i dX(\nabla_{E_i}E_i)|_v\\ &=\sum_{i=1}^m\varepsilon_i\Big(\nabla^c_{E_i^c+[E_i,X]^V}(E_i^c+[E_i,X]^V)\\ &\quad-(\nabla_{E_i}E_i)^c-[\nabla_{E_i}E_i,X]^V\Big)_v\\ &=\sum_{i=1}^m\varepsilon_i\left(\nabla_{E_i}[E_i,X]+\nabla_{[E_i,X]}E_i-[\nabla_{E_i}E_i,X]\right)^V_v.\end{aligned}$$

We would like to remark that in the second equation above we consider the covariant derivative ∇^c along $X:M\to N$. Hence we are interested only in the values of the vector fields on the image of X. This justifies the application of Proposition 1.6. Since the connection ∇ is torsionless we have that for all $i=1,\dots,m$

$$\begin{aligned}\nabla_{E_i}[E_i,X]&=\nabla_{E_i}\nabla_{E_i}X-\nabla_{E_i}\nabla_XE_i\\ [\nabla_{E_i}E_i,X]&=\nabla_{(\nabla_{E_i},E_i)}X-\nabla_X\nabla_{E_i}E_i.\end{aligned}$$

We apply these formulas to compute $\tau(X)_p$.

$$\begin{aligned}
\tau(X)_p &= \left(\sum_{i=1}^{m} \varepsilon_i \nabla_{E_i}\nabla_{E_i}X - \varepsilon_i \nabla_{E_i}\nabla_X E_i\right)^V_v + \left(\sum_{i=1}^{m} \varepsilon_i \nabla_{[E_i,X]}E_i\right)^V_v \\
&\quad - \left(\sum_{i=1}^{m} \varepsilon_i \nabla_{(\nabla_{E_i}E_i)}X - \varepsilon_i \nabla_X \nabla_{E_i}E_i\right)^V_v \\
&= \left(\sum_{i=1}^{m} \varepsilon_i \nabla_{E_i}\nabla_{E_i}X - \varepsilon_i \nabla_{\nabla_{E_i}E_i}X\right)^V_v \\
&\quad + \left(\sum_{i=1}^{m} \varepsilon_i \nabla_X\nabla_{E_i}E_i - \varepsilon_i \nabla_{E_i}\nabla_X E_i + \varepsilon_i \nabla_{[E_i,X]}E_i\right)^V_v \\
&= \left(\operatorname{trace}_g \nabla^2 X + \sum_{i=1}^{m} \varepsilon_i R(X,E_i)E_i\right)^V_v \\
&= (\Delta X + \operatorname{trace}_g R(X,*)*)^V_v .
\end{aligned}$$

In the above formula R denotes the curvature tensor of ∇. Then applying Observation 1.3 we get the following proposition.

Proposition 4.1. *Let (M,g) be a pseudo-Riemannian manifold and let TM be aquipped with the metric II or $I+II$ then a vector field $X \in \chi(M)$ is harmonic with respect to these metrics if and only if*

$$\Delta X + \operatorname{trace}_g R(X,*)* = 0.$$

Let us observe that for any $Y \in \chi(M)$ we have that

$$\begin{aligned}
g(\operatorname{trace}_g R(X,*)*, Y) &= \operatorname{trace}_g g(R(*,X)Y,*) \\
&= \mathcal{R}(X,Y)
\end{aligned}$$

where $\mathcal{R}$ denotes the Ricci tensor of (M,g). Hence we have that

Corollary 4.2. *If (M,g) is a pseudo-Riemannian manifold and TM is equipped with one of the metrics II or $I+II$ then a vector field X is harmonic iff*

$$g(\Delta X,*) + \mathcal{R}(X,*) = 0.$$

Corollary 4.3. *If (M,g) is a compact pseudo-Riemannian manifold and $X \in \chi(M)$ is harmonic with respect to the metrics II or $I+II$ then $E_{II}(X) = 0$.*

Proof: Let us consider the variation $(x,t) \to tX$ then

$$0 = \frac{d}{dt}E_{II}(tX)|_{t=1} = \frac{d}{dt}tE_{II}(X)|_{t=1} = E_{II}(X). \quad \blacksquare$$

Corollary 4.4. *If (M,g) is a pseudo-Riemannian manifold and X is a Killing vector field then X is harmonic with respect to the metrics II or $I+II$.*

Proof: If X is a Killing vector field then

$$\operatorname{div} X = g(\Delta X, *) + \mathcal{R}(X,*) = 0,$$

converse is true if M is compact (cf. [**P**]). Hence our corollary follows. $\blacksquare$

Corollary 4.5. *If (M,g) is a Riemannian manifold with Ricci tensor negatively semi-defined (i.e. for each $V \in \xi(M)$ $\mathcal{R}(V,V) \leq 0$). Then a vector field X is harmonic with respect to the metric II or $I+II$ if and only if X is parallel. Moreover, if $\mathcal{R}$ is negatively defined (i.e. $\mathcal{R}(V,V) = 0$ iff $V = 0$) then zero sections are the only harmonic vector fields.*

Proof: We shall apply methods used in [**P**]. Suppose that X is harmonic. We have the following Bochner's formula valid for all vector fields:

$$2g(\Delta X, X) + 2\operatorname{trace}_g(\nabla X, \nabla X) + \Delta g(X,X) = 0.$$

Since

$$g(\Delta X, X) = -\mathcal{R}(V,V) \geq 0$$

we get that

$$2\operatorname{trace}_g(\nabla X, \nabla X) + \Delta g(X,X) \leq 0$$

then $\Delta g(X,X) \leq 0$. On a compact manifold this implies that $\Delta g(X,X) = 0$ and then $\operatorname{trace}_g(\nabla X, \nabla X) = 0$, so $\nabla X = 0$. We have also obtained that $\mathcal{R}(X,X) = 0$. Hence if $\mathcal{R}$ is negatively defined then $X = 0$. $\blacksquare$

Notation. If $\varphi : M_1 \to M_2$ is a diffeomorphism then by

$$\bar{\varphi} : \chi(M_1) \longrightarrow \chi(M_2)$$

we denote the isomorphism of the modules of vector fields such that $\bar{\varphi}(X) := d\varphi \circ X \circ \varphi^{-1}$. The operator $\bar{\varphi}$ extends on the tensors of an arbitrary type (cf. [**KN**, p. 28]).

Observation 4.6. Let (M, g) be a pseudo-Riemannian manifold and let Y be a Killing vector field and X a harmonic vector field. Then $[Y, X]$ is harmonic.

Proof: Let φ_t be a local flow of Y. Since φ_t are local isometries then we have that

$$\tilde{\varphi}_t \Delta X = \Delta \tilde{\varphi}_t X \text{ and } \tilde{\varphi}_t \operatorname{trace}_g R(X, *)* = \operatorname{trace}_g R(\tilde{\varphi}_t X, *)*$$

These equations and harmonicity of X imply that

$$\begin{aligned} 0 &= -\frac{d}{dt} \tilde{\varphi}_t (\Delta X + \operatorname{trace}_g R(X, *)*)|_{t=0} \\ &= -\frac{d}{dt} \left(\Delta \tilde{\varphi}_t X + \operatorname{trace}_g R(\tilde{\varphi}_t X, *)*\right)|_{t=0} \\ &= \Delta [Y, X] + \operatorname{trace}_g R([Y, X], *) * . \end{aligned}$$

Hence from Proposition 4.1 it follows that $[Y, X]$ is harmonic. It is clear that $[X, Y]$ is also harmonic since the multiplication by -1 is an isometry of N. ■

Example 4.7. If M is a Riemann surface of genus greater than one then its universal covering of M is a hyperbolic plane with the constant curvature equal to -1. There is a group of deck transformations acting isometrically on M. One can project the Riemannian structure from its universal covering to M. In this way it is possible to construct a Riemannian metric on M with curvature -1. Then Ricci curvature of M is negatively defined and hence the only harmonic vector fields are zero sections. The same construction of a compact Riemannian manifold with constant negative curvature can be done for each dimension.

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