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GLOBAL DIMENSION IN NOETHERIAN RINGS AND RINGS WITH GABRIEL AND KRULL DIMENSION

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Abstract

In this paper we compute the global dimension of Noetherian rings and rings with Gabriel and Krull dimension by taking a subclass of cyclic modules determined by the Gabriel filtration in the lattice of hereditary torsion theories.

Introduction

In this paper we exhibit a nice subclass of cyclic modules to compute the global dimension of a ring (see [**9**], [**12**], [**13**], [**15**]) whose origins are in [**3**] and [**11**]. In the first part, the left global dimension of a noetherian ring, R, is computed in terms of the injective dimensions of the following subclass of R-mod. If $\tau_{-1} < \tau_0 < \cdots < \tau_\beta$ is the Gabriel filtration in the lattice of hereditary torsion theories of R, i.e. R-tors [**2**], then the subclass consist of all the cyclic τ_μ-cocritical left R-modules whose injective dimension equals the injective dimension of every one of its submodules, with μ ranging over all the ordinals less than β. Also, we obtain some of the classical results for noetherian rings as consequences of our results. In the second part we note that all our results can be dualized.

Throughout this paper, R will denote an associative ring with 1 and R-mod the category of all unitary left R-modules. Torsion classes and torsion theories will always be hereditary; all terminology concerning torsion theories is quoted from [**2**]. Given a nonzero $M \in R$-mod, $\mathrm{Id}(M)$ and $\mathrm{Pd}(M)$ denote respectively, the injective and projective dimensions of M, setting $\mathrm{Id}(0) = \mathrm{Pd}(0) = -\infty$. The left global dimension of R will be denoted by $lgl\dim(R)$, and the Gabriel dimension $G\dim(R)$. For further details on each of these dimensions we refer respectively to [**13**] and [**2**].

1. Injective dimension

The Strong Injective Dimension of a left R-module, M is defined as $\mathrm{Sid}(M) = \sup\{\mathrm{Id}(M')|0 \to M' \to M \text{ is exact}\}$. Following [**11**], given $n \in \mathbb{N}$, we will denote by $\mathcal{L}_n$ the class of left R-modules M, with $\mathrm{Sid}(M) \leq n$. We define $\mathrm{Sid}(M) = \infty$ when for all $n \in \mathbb{N}$, there exists a submodule $M' \subseteq M$ such that $\mathrm{Id}(M') \geq n$ (with the convention $n < \infty$). Observe that if there exist $M' \subseteq M$ with $\mathrm{Id}(M') = \infty$, then $\mathrm{Sid}(M) = \infty$. We remark [**11**] that if R is a left noetherian ring then the classes $\mathcal{L}_n$ $(n = 0, 1, \dots)$ are torsion classes and [**11**] if ${}_RR$ is the ring R taken as left R-module the $\mathrm{Sid}({}_RR) = lgl\dim(R)$.

Note that there is a chain of torsion classes $\mathcal{L}_0 \subseteq \mathcal{L}_1 \subseteq \cdots \subseteq \mathcal{L}_n \subseteq \dots$ We add $\mathcal{L}_{-\infty} = \{0\}$ and $\mathcal{L}_\infty = R$-mod. Let σ_n be the torsion theory corresponding to $\mathcal{L}_n$. Note that $M \in R$-mod is σ_n-torsionfree if and only if for all submodules $0 \neq M' \subseteq M$ there exist a submodule $N \subseteq M'$ such that $\mathrm{Id}(N) > n$. Within the above chain there exists a strictly increasing subchain; that is, if $n_0 = -\infty$, then $\mathcal{L}_{n_0} \subset \cdots \subset \mathcal{L}_{n_j} \subset \dots$ where the length of the subchain is at most ω.

1.1 Examples.

(i) In $\mathbb{Z}$ we have $\mathcal{L}_{-\infty} = \mathcal{L}_0 \subset \mathcal{L}_1 = \mathbb{Z}$-mod.

(ii) Let K be a field. For $R = \begin{pmatrix} K & K[X,Y] \\ 0 & K[X,Y] \end{pmatrix}$ we have $\mathcal{L}_{-\infty} \subset \mathcal{L}_0 \subset \mathcal{L}_1 \subset \mathcal{L}_2 = R$-mod.

(iii) Let R be a commutative noetherian regular local ring, with J maximal ideal. Suppose that $\mathrm{Id}(R/J) = n$, then we have in this case $\mathcal{L}_{-\infty} = \mathcal{L}_0 = \cdots = \mathcal{L}_{n-1} \subset \mathcal{L}_n = R$-mod.

(iv) Let R be a left artinian left local ring [**2**]. Then R-mod has a chain as in (iii).

(v) In [**3**], there are examples where all the inclusions in the chain are proper.

(vi) In any left artinian ring which has at least two simple left R-modules with different injective dimensions (as $\mathbb{Z}_{12}$) the Gabriel filtration has less terms than the subchain.

1.2 Lemma. *Let R be a left noetherian ring and $\mathcal{L}_{n_j}$ $(j = -\infty, 0, 1, \dots)$ a term in the subchain such that $\mathcal{L}_{n_j} \neq R$-mod. Then there exists a σ_{n_j}-cocritical left R-module M, such that $n_j < Id(M) = Id(M')$ for all submodules $0 \neq M' \subseteq M$.*

Proof: Since R is noetherian and $\mathcal{L}_{n_j} \neq R$-mod then there exist a σ_{n_j}-cocritical left R-module M. (a) If $\mathrm{Id}(M) > n_j$ (including ∞) then in each exact sequence $0 \to M' \to M \to M'' \to 0$ we have $\mathrm{Id}(M'') \leq n_j$ and $\mathrm{Id}(M) > n_j$, and so $\mathrm{Id}(M') = \mathrm{Id}(M)$; hence M is the required object. (b) If $\mathrm{Id}(M) \leq n_j$ then since $\mathrm{Sid}(M) > n_j$ then there exists a submodule $0 \neq M' \subset M$ such that $\mathrm{Id}(M') > n_j$ (including ∞) and since M' is also σ_{n_j}-cocritical we are again in case (a), and M' is now the required object. ■

1.3 Proposition. *Let R be a left noetherian ring, $\mathcal{L}_{n_j}$ $(j = 0, 1, \dots)$ $\mathcal{L}_{n_j} \neq R$-mod and $m = \min\{Id(C) | C$ is cyclic σ_{n_j}-cocritical with $Id(C) > n_j\}$. Then $m = n_{j+1}$.*

Proof: By hypothesis and Lemma 1.2 it is clear that $\mathcal{L}_m$ always exists and that $\mathcal{L}_{n_j} \subset \mathcal{L}_m$. Suppose that there exists $k \in \mathbb{N}$ such that $\mathcal{L}_{n_j} \subset \mathcal{L}_k \subseteq \mathcal{L}_m$. Since R is noetherian [**2**] there exists a σ_{n_j}-cocritical σ_k-torsion left R-module M, and hence $n_j < \mathrm{Sid}(M) \leq k$. So, by the part (a) in the proof of Lemma 1.2, there exists a cyclic submodule $C \subseteq M$ such that $\mathrm{Id}(C) = \mathrm{Sid}(M)$. Since C is also σ_{n_j}-cocritical and $\mathrm{Id}(C) > n_j$. Then, by the definition of m we must have $\mathrm{Id}(C) \geq m$. Hence $k \geq m$ and thus $k = m$.

Note that, in particular, if $m = \min\{\mathrm{Id}(S) | S \in R\text{-mod is simple}\}$ then $m = n_1$. ■

1.4 Observation. Since every subchain has at least two terms, it is natural to analize the step $\sigma_{n_j} < \sigma_{n_{j+1}}$. In each of these steps there exists a σ_{n_j}-cocritical cyclic left R-module C, such that $\mathrm{Id}(C) = \mathrm{Id}(C') = \mathrm{Sid}(C) \geq n_{j+1}$ for all submodules $0 \neq C' \subseteq C$.

From here, Theorem C of B. Osofsky in [**5**] follows immediately. In the next theorem, we will see that we can to extract a nice subclass of the class of cyclic left R-modules, to compute the left global dimension.

1.5 Theorem. *Let R be a left noetherian ring, such that $G\dim(R) = \beta$. Then $lgl\dim(R) = \sup\{Id(C) | C$ is cyclic, $Id(C) = Id(C')$, for all $0 \neq C' \subseteq C$ and τ_μ-cocritical, with $\mu < \beta\}$.*

Proof: Let $\tau_{-1} < \tau_0 < \cdots < \tau_\beta$ be the Gabriel filtration in R-tors and let $lgl\dim(R) = n_k$ (or ∞). For any given $j < k$ we have a step $\sigma_{n_j} < \sigma_{n_{j+1}}$ and by [**2**], there exists an ordinal $\alpha \leq \beta$ which is least with the property that $\tau_\alpha \not\leq \sigma_{n_j}$. Note that α is a successor. Then by Observation 1.4 and the fact that $\tau_\alpha \not\leq \sigma_{n_j}$ there exists a $\tau_{\alpha-1}$-cocritical σ_{n_j}-cocritical left R-module C, such that $\mathrm{Id}(C) = \mathrm{Id}(C') \geq n_{j+1}$ for all submodules $0 \neq C' \subseteq C$. Setting $\mu = \alpha - 1$ we have the result in view that the choice of j was arbitrary. ■

The next corollary is of particular importance inasmuch as there exist an abundance of examples where the subchain is finite.

1.6 Corollary. *Let R be a left noetherian ring such that $G\dim(R) = \beta$. Suppose that we have finitely many terms in the subchain. Then $lgl\dim(R) = \sup\{Id(C) | C$ is cyclic, $Id(C) = Id(C')$ for all $0 \neq C' \subseteq C$ and τ_μ-cocritical where $\mu < \beta$ is fixed$\}$.*

Proof: Consider the last step in the subchain, $\sigma_{n_{j-1}} < \sigma_{n_j} = \chi$. Then there exists $\alpha \leq \beta$ such that $\tau_\alpha \not\leq \sigma_{n_{j-1}}$. Here, $\mu = \alpha - 1$. ■

1.7 Observation. Let R be a commutative noetherian ring. We take in this case the original definition of Krull dimension over the prime ideals of R. Suppose that now, for all $S \in R$-mod simple we have that $S \in \mathcal{L}_n$ for some $n \in \mathbb{N}$ (fixed). Let J be any maximal ideal of R, then R/J is a simple left R-module and $\mathrm{Supp}(R/J) = J$; furthermore, $R/J \in \mathcal{L}_n$. Then, by [**11**] the local ring R_J is regular with $K\dim(R_J) \leq n$. By [**13**] we have $\mathrm{Sid}_{R_J}(R_J) = lgl\dim(R_J) \leq n$, for all J. By [**13**] we have $lgl\dim(R) \leq n$, hence $R \in \mathcal{L}_n$. That is, we have just proved that for any commutative noetherian ring, if $\sigma_{n_j} \neq \chi$ then $\tau_0 \not\leq \sigma_{n_j}$ and by Theorem 1.5 we have the well-known result: $lgl\dim(R) = \sup\{\mathrm{Id}(S) | S$ is simple R-module$\}$.

2. Projective dimension

When we compute the left global dimension as the supremum of the projective dimensions of cocritical and critical left R-modules we have analogous results to the above; furthermore, we can relax the noetherian condition on the ring R. The Strong Projective Dimension [**11**] is defined as $\mathrm{Spd}(M) = \sup\{\mathrm{Pd}(N) | M \to N \to 0$ is exact$\}$. We have for all $n \in \mathbb{N}$, the classes $\mathcal{U}_n = \{M \in R\text{-mod} \,|\mathrm{Spd}(M) \leq n\}$. In this case [**11**] $\mathcal{U}_n$ are torsion classes for an arbitrary ring R, and $lgl\dim(R) = \mathrm{Spd}(_RR)$. We denote by ρn the torsion theory corresponding to $\mathcal{U}_n$. Again, we have a chain $\mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \cdots \subseteq \mathcal{U}_n \subseteq \ldots$ and adding $\mathcal{U}_{-\infty}$ and $\mathcal{U}_\infty$ we can take a strictly ascending subchain $\mathcal{U}_{n_0} \subset \cdots \subset \mathcal{U}_{n_j} \subset \ldots$ with $n_0 = -\infty$. From here, we can do the dualization in a similar way to the first part and we can remove the noetherian condition. So, we will write only the principal result. For ρ_{n_j}-cocritical left R-modules C, such that $\mathrm{Pd}(C) \geq n_{j+1}$, the consequence $\mathrm{Pd}(C) = \mathrm{Pd}(C')$ for all submodules $0 \neq C' \subseteq C$ will be removed in view that all ρ_{n_j}-cocritical satisfy it.

2.1 Theorem. *Let R be a ring with Gabriel dimension, suppose that $G\dim(R) = \beta$. Then*

$$lgl\dim(R) = \sup\{Pd(C)|C \text{ is cyclic and } \tau_\mu\text{-cocritical, with } \mu < \beta\}.$$

2.2 Examples. (i) In a non-noetherian ring R, with Gabriel dimension, the classes $\mathcal{L}_n$ are not in general torsion classes, but they are Serre subcategories (see [**5**]). Even if subchains can be found, the results that we have seen do not hold. For example, let S be $(\mathbb{Z}_2)^{\mathbb{N}}$ and $R \subset S$ the subring generated by $(\mathbb{Z}_2)^{(\mathbb{N})}$ together with $1 \in S$. Then R is a commutative boolean semiartinian hereditary V-ring, having as chain $\mathcal{L}_{-\infty} \subset \mathcal{L}_0 \subset \mathcal{L}_1 = R$-mod. Note that the torsion class generated by $\mathcal{L}_0$ is the same that $\mathcal{L}_1$.

(ii) Let R be a ring with Gabriel dimension and nonsingular as left R-module ($Z({}_RR) = 0$). Then if R is not semi simple we have

$$lgl\dim(R) = \sup\{\mathrm{Pd}(C)|C \text{ is cyclic singular}\}.$$

Proof: We shall prove that every C of Theorem 2.1 admits another singular module D such that $\mathrm{Pd}(D) \geq \mathrm{Pd}(C)$. By [**4**], in every nonsingular ring, cyclic uniform modules are either singular or nonsingular. Since it is clear when C is singular, we assume that C is nonsingular. So take a left ideal I of R such that $R/I \cong C$. Because I is not large in R, there is a left ideal $0 \neq J$ of R with $I \oplus J$ large in R. By taking $D = R/I \oplus J$ we have $\mathrm{Pd}(D) = 1 + \mathrm{Pd}(I \ominus J) \geq 1 + \mathrm{Pd}(I) \geq \mathrm{Pd}(C)$ and R/I is singular.

(iv) If R is left semiartinian, $lgl\dim(R) = \sup\{\mathrm{Pd}(S)|S \text{ is simple}\}$ (see [**8**]). Semi perfect rings are semiartinian, for instance.

(v) Finally we refer to injective and projective dimension in left Fully Bounded Noetherian (FBN) rings without any other assumption (like the commonly used right coherence of [**12**], [**15**]). In this rings, the (two sided) prime ideals often have not hard descriptions and we can see how our classes work.

Let R be a left FBN ring and take $\mathcal{L}_{n_j} \neq R$-mod ($\mathcal{U}_{n_k} \neq R$-mod). By the results above, there exists a cyclic σ_{n_j}-cocritical (ρ_{n_j}-cocritical) left R-module C. Take [**14**] the associated prime ideal $\mathrm{ass}(C)$ and note that by [**14**] if $x \in C$ is such that $R \cdot x = C$ then $\mathrm{ass}(C) \subseteq l(x)$ (the left annihilator of x) and hence, $R/\mathrm{ass}(C) \notin \mathcal{L}_{n_j}$ ($R/\mathrm{ass}(C) \notin \mathcal{U}_{n_k}$) and by [**14**] we have $R/\mathrm{ass}(C)$ is σ_{n_j}-torsionfree (ρ_{n_k}-torsionfree). Since R is left FBN then, the injective hulls $E(R/\mathrm{ass}(C)) \cong E(C)$ and hence there exists a copy of C, say C again, $C \subseteq E(R/\mathrm{ass}(C))$. Take $K = C \cap R/\mathrm{ass}(C)$ and note that since $K \subseteq C$ then K has the properties of modules in Theorems 1.5 and 2.2. Since $R/\mathrm{ass}(C)$ is a left order in a simple ring [7], [14] then for any $x \in K$ we have $(R/\mathrm{ass}(C)) \cdot x \cong R/\mathrm{ass}(C)$ as left $R/\mathrm{ass}(C)$-modules and hence as left R-modules. This implies that $\mathrm{Id}(R/\mathrm{ass}(C)) = \mathrm{Id}(C)$ $(\mathrm{Pd}(R/\mathrm{ass}(C) = \mathrm{Pd}(C))$. So we have that if R is a left FBN ring then

$$lgl\dim(R) = \sup\{\mathrm{Id}(R/I) | I \in \mathrm{Spec}(R)\} = \sup\{\mathrm{Pd}(R/I) | I \in \mathrm{Spec}(R)\}. \blacksquare$$

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References

1. M.F. ATIYAH AND I.G. MACDONALD, "*Introduction to commutative algebra*," Addison-Wesley, Reading, Mass., 1969.
2. J.S. GOLAN, "*Torsion theories*," Longman Scientific & Technical - John Wiley & Sons, 1986.
3. J.S. GOLAN AND Z. PAPP, Cocritically nice rings and Boyle's conjecture, *Comm. Algebra* **8(18)** (1980), 1775–1798.
4. K.R. GOODEARL, "*Ring Theory*," Marcel Dekker, New York, 1976.
5. R. GORDON AND J.C. ROBSON, Krull dimension, *Mem. Amer. Math. Soc.* **133** (1973).
6. J.J. KOKER, Homological dimension of rings with Krull and Gabriel dimension, Ph.D. Thesis, University of Wisconsin-Milwaukee, 1990.
7. J.C. MC CONNELL AND J.C. ROBSON, "*Noncommutative noetherian rings*," John Wiley & Sons, Chichester New York Brisbane Toronto Singapore, 1987.
8. C. NĂSTĂSESCU, Dimension globale des anneaux semi-artiniens, *C.R. Acad. Sci. Paris, Sér. A-B* **268** (1969), A685–A688.
9. B.L. OSOFSKY, Global dimension of valuation rings, *Trans. Amer. Math. Soc.* **127** (1967), 136–149.
10. I. PALMÉR AND J.-E. ROOS, Formules explicites pour la dimension homologique des anneaux de matrices generalisées, *C.R. Acad. Sci. Paris, Sér. A-B* **273** (1971), A1026–A1029.
11. Z. PAPP, On the strong injective (projective) dimension of modules, *Arch. Math.* **25** (1974), 354–360.
12. J. RAINWATER, Global dimension of fully bounded noetherian rings, *Comm. Algebra* **15(10)** (1987), 2146–2156.
13. J. ROTMAN, "*An introduction to Homological Algebra*," Orlando, Academic Press, 1979.

14. BO STENSTRÖM, "*Rings of quotients*," Springer-Verlag, Berlin, Heidelberg, New York, 1975.

15. M. TEPLY, Global dimensions of right coherent rings with left Krull dimension, *Bull. Austral. Math. Soc.* **39, no 2** (1989), 215–223.

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