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# ON STRONGLY NONLINEAR ELLIPTIC EQUATIONS WITH WEAK COERCITIVITY CONDITION

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*Abstract*

We prove the existence and uniqueness of weak solutions of boundary value problems in an unbounded domain $\Omega \subset \mathbb{R}^n$ for strongly nonlinear $2m$ order elliptic differential equations.

In this paper it will be proved existence and uniqueness of solutions of boundary value problems for the equation

$$(0.1) \qquad \sum_{|\alpha|=m} (-1)^m D^\alpha [f_\alpha(x, D^\alpha u)] + \sum_{|\alpha|\leqq m-1} (-1)^{|\alpha|} D^\alpha [g_\alpha(x, u, \dots, D^\beta u, \dots)] = F \text{ in } \Omega$$

where $\Omega$ is an unbounded domain in $\mathbb{R}^n$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \sum_{j=1}^{n} \alpha_j$, $D_j = \frac{\partial}{\partial x_j}$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, $|\beta| \leq m$.

Function $f_\alpha$ satisfies the Carathéodory conditions such that $\zeta_\alpha \mapsto f_\alpha(x, \zeta_\alpha)$ is strictly monotone increasing, $f_\alpha(x, 0) = 0$ and $f_\alpha$, $g_\alpha$ satisfy the "weak" coercitivity condition

$$(0.2) \qquad \sum_{|\alpha|=m} f_\alpha(x, \zeta_\alpha)\zeta_\alpha + \sum_{|\alpha|\leqq m-1} g_\alpha(x, \zeta)\zeta_\alpha \geqq c_0 \sum_{|\alpha|=m} |\zeta_\alpha|^p$$

with some constants $p > 1$, $c_0 > 0$. Functions $g_\alpha$ have some polynomial growth in $D^\beta u$, but on $f_\alpha$ no growth restriction is imposed in $D^\alpha u$.

Similar result has been proved in [1] for the equation

$$\sum_{|\alpha|\leqq m} (-1)^{|\alpha|} D^\alpha [g_\alpha(D^\alpha u)] = F$$

in a bounded $\Omega$ if the condition

$$g_\alpha(\zeta_\alpha)\zeta_\alpha \geqq c_0|\zeta_\alpha|^p - c_1, \qquad |\alpha| \leqq m$$

is fulfilled with some constants $p > 1$, $c_j > 0$. The proof of the existence theorem is based on a method called by F.E. Browder "elliptic super-regularization" (see [1] - [3]). Our results can be extended to equations of the form

$$\sum_{|\alpha|=m} (-1)^m D^\alpha[f_\alpha(x, u, \dots, D^\beta u, \dots] + \\ + \sum_{|\alpha|\leqq m-1} (-1)^{|\alpha|} D^\alpha[g_\alpha(x, u, \dots, D^\beta u, \dots] = F$$

where $|\beta| \leqq m$ (see [2] - [5]).

It is to be mentioned that [6] is connected with our result where D. Fortunato has considered equation $Lu + f(x, u) = 0$; by $L$ is denoted a second order linear elliptic operator with weak coercitivity conditions in an unbounded domain. Similarly to our consideration, in [6] the solution $u$ must satisfy the "asymptotic condition" $\int_\Omega |\text{grad } u|^2\, dx < +\infty$.

## 1. The existence theorem

Let $\Omega \subset \mathbb{R}^n$ be an unbounded domain with bounded boundary $\partial\Omega$, having the uniform $C^m$-regularity property and $\Omega_r = \Omega \cap B_r$ where $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ (see [7]). Denote by $W_p^m(\Omega)$ the usual Sobolev space of real valued functions $u$ whose distributional derivatives belong to $L^p(\Omega)$. The norm on $W_p^m(\Omega)$ is

$$\|u\| = \left\{ \sum_{|\alpha|\leqq m} \int_\Omega |D^\alpha u|^p\, dx \right\}^{1/p}.$$

By $W_{p,\text{loc}}^m(\bar\Omega)$ will be denoted the set of functions $f$ such that $\varphi f \in W_p^m(\Omega)$ for all $\varphi \in C_0^\infty(\mathbb{R}^n)$, i.e. for all infinitely differentiable functions $\varphi$ with compact support.

Denote by $\tilde W_{p,0}^m(\Omega)$ the set of functions $u \in W_{p,\text{loc}}^m(\bar\Omega)$ satisfying the conditions: $D^\alpha u \in L^p(\Omega)$ if $|\alpha| = m$ and the trace of $D^\beta u$ on $\partial\Omega$ equals to 0 if $|\beta| \leqq m - 1$. The norm in $\tilde W_{p,0}^m(\Omega)$ is defined by

$$\|u\|_{\tilde W_{p,0}^m(\Omega)} = \left\{ \sum_{|\alpha|=m} \int_\Omega |D^\alpha u|^p\, dx \right\}^{1/p}.$$

It is not difficult to show that $\tilde{W}^m_{p,0}(\Omega)$ is a reflexive Banach space. Let $V$ be a closed linear subspace of $\tilde{W}^m_{p,0}(\Omega)$.

Let $N$ be the number of multiindices $\beta = (\beta_1, \ldots, \beta_n)$ satisfying $|\beta| \leqq m$. Assume that

I. Functions $f_\alpha : \Omega \times \mathbb{R} \to \mathbb{R}\,(|\alpha| = m)$ satisfy the Carathéodory conditions, i.e. $f_\alpha(x, \zeta_\alpha)$ is measurable in $x$ for each fixed $\zeta_\alpha \in \mathbb{R}$ and it is continuous in $\zeta_\alpha$ for almost all $x \in \Omega$.

II. $f_\alpha(x, \zeta_\alpha)$ is strictly monotone increasing with respect to $\zeta_\alpha$, $f_\alpha(x, 0) = 0$.

III. For any $s > 0$ there is a function $f_{\alpha,s}$ such that $f_{\alpha,s} \in L^1(\Omega_r)$ for each $r > 0$ and

$$|f_\alpha(x, \zeta_\alpha| \leqq f_{\alpha,s}(x) \text{ if } |\zeta_\alpha| \leqq s.$$

Further, there exist constants $c_1$, $c_2 > 0$ and a function $f^*_\alpha \in L^1(\Omega)$ such that for a.e. $x \in \Omega$

$$|f_\alpha(x, \zeta_\alpha)| \leqq f^*_\alpha(x) + c_1|\zeta_\alpha|^{p-1} \text{ if } |\zeta_\alpha| \leqq c_2$$

with some $p > 1$.

IV. There exists a constant $c_3 > 0$ such that for all $\zeta_\alpha \in \mathbb{R}$, a.e. $x \in \Omega$

$$|f_\alpha(x, \zeta_\alpha)| \geqq c_3|\zeta_\alpha|^{p-1}.$$

V. Functions $g_\alpha : \Omega \times \mathbb{R}^N \to \mathbb{R}\,(|\alpha| \leqq m-1)$ satisfy the Carathéodory conditions.

VI. There exists a bounded domain $\Omega' \subset \Omega$ such that $g_\alpha(x, \zeta) = 0$ for all $\zeta \in \mathbb{R}^N$, a.e. $x \in \Omega \backslash \Omega'$; further,

$$\sum_{|\alpha| \leqq m-1} g_\alpha(x, \zeta)\zeta_\alpha \geqq 0.$$

VII. There exist constants $\rho_{|\alpha|}$, functions $\Phi_\alpha \in L^{p/\rho_\alpha}(\Omega')$ and a continuous function $C_\alpha$ such that

$$p - 1 \leqq \rho_{|\alpha|} < p - 1 + \frac{(m - |\alpha|)p}{n}, \qquad \rho_{|\alpha|} \leqq p$$

and for all $\zeta \in \mathbb{R}^N$, a.e. $x \in \Omega'$

$$|g_\alpha(x, \zeta)| \leqq C_\alpha(\zeta')\,[\Phi_\alpha(x) + |\zeta''|^{\rho_{|\alpha|}}]$$

where $\zeta = (\zeta', \zeta'')$ and $\zeta'$ consists of those $\zeta_\gamma$ for which $|\gamma| < m - n/p$.

**Remark 1.** Function $f_\alpha$ satisfies conditions I - IV e.g. in the following special case:

$$f_\alpha(x,\zeta_\alpha) = \chi_\alpha(x)\varphi_\alpha(\zeta_\alpha) + \Psi_\alpha(\zeta_\alpha)$$

where $\chi_\alpha \in L^1(\Omega)$, $\chi_\alpha \geqq 0$; $\varphi_\alpha$, $\Psi_\alpha$ are continuous functions, $\varphi_\alpha$ is monotone increasing, $\Psi_\alpha$ is strictly monotone increasing, $\varphi_\alpha(0) = 0$, $\Psi_\alpha(0) = 0$ and

$$c|\zeta_\alpha|^{p-1} \leqq |\Psi_\alpha(\zeta_\alpha)| (\zeta_\alpha \in \mathbb{R}),\ |\Psi_\alpha(\zeta_\alpha)| \leqq \tilde{c}|\zeta_\alpha|^{p-1} \text{ if } |\zeta_\alpha| < 1$$

by $c$, $\tilde{c}$ are denoted positive constants.

**Theorem 1.** *Assume that conditions I - VII are fulfilled. Then for any $G \in V^*$ (i.e. for linear continuous functional over $V$) with compact support there is $u \in V$ such that*

$$f_\alpha(x, D^\alpha u) D^\alpha u \in L^1(\Omega), \tag{1.1}$$

(1.2)

$$|f_\alpha(x, D^\alpha u)| \leqq f_\alpha^{(1)} + f_\alpha^{(2)} \text{ where } f_\alpha^{(1)} \in L^1(\Omega),\ f_\alpha^{(2)} \in L^q(\Omega),\ \frac{1}{p} + \frac{1}{q} = 1,$$

$$\sum_{|\alpha|=m} \int_\Omega f_\alpha(x, D^\alpha u) D^\alpha v\, dx + \\ + \sum_{|\alpha| \leqq m-1} \int_{\Omega'} g_\alpha(x, u, \ldots, D^\beta u, \ldots) D^\alpha v\, dx = \langle G, v \rangle \tag{1.3}$$

*for all $v \in C_0^\infty(\mathbb{R}^n)$ with $v|_\Omega \in V$.*

This theorem will be a simple consequence of Theorem 2 formulated below.

Let $V_r$ be the closure in $W_p^m(\Omega_r)$ of

$$\{\varphi|_{\Omega_r} : \varphi \in C_0^\infty(B_r) \cap V\}.$$

Then $V_r$ is a closed linear subspace of $W_p^m(\Omega_r)$ and –extending function $u \in V_r$ as 0 to $\Omega \backslash \Omega_r$– the extensions belong to $V$. Let $s > \max\{n, p\}$ then by Sobolev's imbedding theorem $W_s^{m+1}(\Omega_r)$ is continuously and also compactly imbedded into $W_p^m(\Omega_r)$ and $C_B^m(\Omega_r)$ (see e.g. [7]) where $C_B^m(\Omega_r)$ denotes the set of $m$ times continuously differentiable functions

$u$ with finite norm $\|u\| = \sum_{|\alpha|\leqq m} \sup_{\Omega_r} |D^\alpha u|$. Denote by $\overset{\circ}{W}{}_s^{m+1}(\Omega_r)$ the closure in $W_s^{m+1}(\Omega_r)$ of

$$\{\varphi|_{\Omega_r} : \varphi \in C_0^\infty(B_r)\}.$$

Then –extending $u \in \overset{\circ}{W}{}_s^{m+1}(\Omega_r)$ as 0 to $\Omega\backslash\Omega_r$– the extension belongs to $W_s^{m+1}(\Omega)$. Further, let

$$W_r = \overset{\circ}{W}{}_s^{m+1}(\Omega_r) \cap V_r$$

with the norm of $W_s^{m+1}(\Omega_r)$. Then $W_r$ is a closed linear subspace of $W_s^{m+1}(\Omega_r)$. Functions $u \in W_r$ will be extended to $\Omega\backslash\Omega_r$ as 0.

For any $u$, $v \in W_r$ define

$$\langle S_r(u), v\rangle = \sum_{|\alpha|\leqq m+1} \int_{\Omega_r} |D^\alpha u|^{s-2}(D^\alpha u)(D^\alpha v)\,dx,$$

$$\langle T_r(u), v\rangle = \sum_{|\alpha|\leqq m} \int_{\Omega_r} f_\alpha(x, D^\alpha u) D^\alpha v\,dx,$$

$$\langle Q_r(u), v\rangle = \sum_{|\alpha|\leqq m-1} \int_{\Omega'} g_\alpha(x, u, \dots, D^\beta u, \dots) D^\alpha v\,dx.$$

By Hölder's inequality, Sobolev's imbedding theorem, assumptions I, III, V, VII $S_r$, $T_r$, $Q_r : W_r \to W_r^*$ are bounded nonlinear operators i.e. they map bounded sets of $W_r$ onto bounded sets of $W_r^*$.

**Theorem 2.** *Assume that conditions I - VII are fulfilled, $G \subset V^*$ has compact support and $\lim_{l\to\infty} r_l = +\infty$. Then for sufficiently large $l$ there exists at least one soluton $u_l \in W_{r_l}$ of*

$$\text{(1.4)}\quad \frac{1}{l}\langle S_{r_l}(u_l), v\rangle + \langle T_{r_l}(u_l), v\rangle + \langle Q_{r_l}(u_l), v\rangle = \langle G, v\rangle \text{ for all } v \in W_{r_l}.$$

*Further, there is a subsequence $(u_l')$ of $(u_l)$ which is weakly converging in $V$ to a function $u \in V$ satisfying (1.1) - (1.3). If (1.1) - (1.3) may have at most one solution then also $(u_l)$ converges weakly to $u$.*

*Proof:* Clearly, $\frac{1}{l}S_{r_l}$ is a pseudomonotone operator. Since $W_{r_l}$ is compactly imbedded into $C_B^m(\Omega_{r_l})$ thus by use of assumptions I, III, V, VII it is easy to show that also $(\frac{1}{l}S_{r_l} + T_{r_l} + Q_{r_l}) : W_{r_l} \to W^*_{r_l}$ is pseudomonotone. Assumptions II, IV, VI imply that for each $u \in W_{r_l}$

(1.5)
$$\left\langle \left(\frac{1}{l}S_{r_l} + T_{r_l} + Q_{r_l}\right)(u), u \right\rangle \geqq \frac{1}{l}\|u\|^s_{W_{r_l}} + c_3 \sum_{|\alpha|=m} \int_{\Omega_{r_l}} |D^\alpha u|^p \, dx,$$

hence $\frac{1}{l}S_{r_l} + T_{r_l} + Q_{r_l}$ is coercive. So by the theory of pseudomonotone operators (see e.g. [8]) there is at least one solution $u_l \in W_{r_l}$ of (1.4).

Since $G$ has compact support (contained in $\Omega_{\tilde{r}}$) thus

(1.6)
$$|\langle G, u\rangle| \leqq \|G\|_{V^*} \|u\|_{W^m_p(\Omega_{\tilde{r}})} \leqq c\|G\|_{V^*} \left\{ \sum_{|\alpha|=m} \int_{\Omega_{\tilde{r}}} |D^\alpha u|^p \, dx \right\}^{1/p} \leqq$$
$$\leqq c\|G\|_{V^*} \left\{ \sum_{|\alpha|=m} \int_{\Omega_{r_l}} |D^\alpha u|^p \, dx \right\}^{1/p}$$

for sufficiently large $l$. (The norm in $W^m_p(\Omega_{\tilde{r}})$ is equivalent with $\{ \sum_{|\alpha|=m} \int_{\Omega_{\tilde{r}}} |D^\alpha u|^p \, dx\}^{1/p}$ for functions satisfying $D^\beta u|_\Gamma = 0$ if $|\beta| \leqq m-1$.)

From (1.4) - (1.6), $p > 1$ it follows that

$$\frac{1}{l}\|u_l\|^s_{W_{r_l}} \text{ is bounded and} \tag{1.7}$$

$$\|u_l\|_V \text{ is bounded.} \tag{1.8}$$

Equality (1.4), VI and (1.8) imply that

$$\sum_{|\alpha| \leqq m} \int_{\Omega_{r_l}} f_\alpha(x, D^\alpha u_l) \, dx \text{ is bounded.} \tag{1.9}$$

By Hölder's inequality, for any fixed $j$, $v \in W_{r_j}$

$$\left|\frac{1}{l}\langle S_{r_l}(u_l), v\rangle\right| \leqq \frac{1}{l}\|u_l\|^{s-1}_{W_{r_l}} \|v\|_{W_{r_l}} \text{ if } l \geqq j$$

and so by (1.7)

$$\lim_{l\to\infty} \frac{1}{l}\langle S_{r_l}(u_l), v\rangle = 0. \tag{1.10}$$

From (1.8) it follows that there are a subsequence $(u_l')$ of $(u_l)$ and $u \in V$ such that

$$(u_l') \longrightarrow u \text{ weakly in } V \tag{1.11}$$

and

$$(D^\gamma u_l') \longrightarrow D^\gamma u \text{ a.e. in } \Omega \text{ for } |\gamma| \leqq m-1 \tag{1.12}$$

because by compact imbedding theorems it may be supposed that for any fixed $r > 0$

$$(D^\gamma u_l') \longrightarrow D^\gamma u \text{ in } L^p(\Omega_r),\ |\gamma| \leqq m-1 \tag{1.13}$$

and by VII

$$(D^\gamma u_l') \longrightarrow D^\gamma u \text{ in } L^{q_{|\gamma|}}(\Omega'),\ |\gamma| \leqq m-1 \tag{1.14}$$

where $q_{|\gamma|}$ is defined by

$$\frac{1}{p/\rho_{|\gamma|}} + \frac{1}{q_{|\gamma|}} = 1.$$

**Lemma 1.** *For all $\alpha$ and each fixed $r > 0$ the integrals*

$$\int_{\Omega_r} |f_\alpha(x, D^\alpha u_l')|\, dx$$

*are uniformly bounded and the functions $f_\alpha(x, D^\alpha u_l')$ are uniformly equi-integrable in $\Omega_r$.*

*Proof:* From II it follows that for any $\zeta_\alpha$, $\tilde{\zeta}_\alpha$

$$f_\alpha(x, \zeta_\alpha)\tilde{\zeta}_\alpha \leqq f_\alpha(x, \zeta_\alpha)\zeta_\alpha + f_\alpha(x, \tilde{\zeta}_\alpha)\tilde{\zeta}_\alpha.$$

Applying this inequality to $\tilde{\zeta}_\alpha = \rho \operatorname{sgn} f_\alpha(x, \zeta_\alpha)$ with arbitrary fixed number $\rho > 0$ we obtain

$$\rho[\operatorname{sgn} f_\alpha(x, \zeta_\alpha)] f_\alpha(x, \zeta_\alpha) \leqq f_\alpha(x, \zeta_\alpha)\zeta_\alpha + f_\alpha(x, \tilde{\zeta}_\alpha)\rho \operatorname{sgn} f_\alpha(x, \zeta_\alpha)$$

where $|\tilde{\zeta}_\alpha| = \rho$. Thus by III we have

$$|f_\alpha(x, \zeta_\alpha)| \leqq \frac{f_\alpha(x, \zeta_\alpha)\zeta_\alpha}{\rho} + f_{\alpha,\rho}(x).$$

Combining this estimation with (1.9) we obtain Lemma 1. ∎

By using the methods of [1], [2], [9] we obtain

**Lemma 2.** *There is a subsequence* $(u_{l_k})$ *of* $(u_l')$ *such that*

$$(D^\alpha u_{l_k}) \longrightarrow D^\alpha u \text{ a.e. in } \Omega \text{ if } |\alpha| = m.$$

*(See* [**4**, Lemma 4]*)*.

Consider a fixed $v \in C_0^\infty(\mathbb{R}^n)$ such that $v|_\Omega \in V$ and apply (1.4) to this $v$ and $l = l_k$. Then passing to the limit in (1.4), in virtue of I, V, (1.10) - (1.14), Lemma 1, Lemma 2, Vitali's theorem and Hölder's inequality we obtain (1.3). (1.1) is a consequence of (1.9), II and Fatou's lemma. Since by III

$$|f_\alpha(x,\zeta_\alpha)| \leqq \sup_{|\zeta_\alpha| \leqq c_2} |f_\alpha(x,\zeta_\alpha)| + \frac{1}{c_2}|f_\alpha(x,\zeta_\alpha)\zeta_\alpha| \leqq$$

$$\leqq f_\alpha^*(x) + c_1|\zeta_\alpha|^{p-1} + \frac{1}{c_2}|f_\alpha(x,\zeta_\alpha)\zeta_\alpha|$$

thus (1.1) implies (1.2).

## 2. The uniqueness theorem

In addtion to I - VII it will be assumed that the following conditions are fulfilled:

VIII. There is a constant $c_4$ such that for all $\zeta_\alpha \in \mathbb{R}$, $|\alpha| = m$, a.e. $x \in \Omega$

$$|f_\alpha(x,\zeta_\alpha)| \leqq c_4 |f_\alpha(x, -\zeta_\alpha|.$$

IX. For each $\zeta$, $\tilde{\zeta} \in \mathbb{R}^N$, a.e. $x \in \Omega$

$$\sum_{|\alpha| \leqq m-1} [g_\alpha(x,\zeta) - g_\alpha(x,\tilde{\zeta})](\zeta_\alpha - \tilde{\zeta}_\alpha) \geqq 0.$$

IX. For each $\zeta$, $\tilde{\zeta} \in \mathbb{R}^N$, a.e. $x \in \Omega$

$$\sum_{|\alpha| \leqq m-1} [g_\alpha(x,\zeta) - g_\alpha(x,\tilde{\zeta})](\zeta_\alpha - \tilde{\zeta}_\alpha) \geqq 0.$$

X. $\Omega$ is a starlike domain in the following sense: there exist $x_0 \in \mathbb{R}^n$ and $\delta > 0$ such that $1 < \lambda < 1 + \delta$ implies $\bar{\Omega}_\lambda \subset \Omega$ where

$$\Omega_\lambda = \{x_0 + \lambda(x - x_0) : x \in \Omega\}.$$

XI. There exist numbers $\varepsilon_1$, $\varepsilon_2$, $c_5 > 0$ and a function $k \in L^q(\Omega)$ such that for all $\zeta \in \mathbb{R}^N$, a.e. $x$, $x' \in \Omega$

$$|f_\alpha(x,\zeta_\alpha)| \leqq c_5 |f_\alpha(x',\zeta_\alpha)| + k(x)$$

if $|x - x'| \leqq \varepsilon_1$ or if $x' = x_0 + \frac{1}{\lambda}(x - x_0)$ where $0 < \lambda - 1 < \varepsilon_2$, $x_0$ is defined in $X$.

**Theorem 3.** *If conditions I - XI are fulfilled then problem (1.1) - (1.3) has a unique solution $u \in V$.*

**Remark 2.** Functions $f_\alpha$ satisfy the conditions of Theorem 3 e.g. in the following special case:

$$f_\alpha(x, \zeta_\alpha) = h_\alpha^{(1)}(\zeta_\alpha)\chi_\alpha(x) + h_\alpha^{(2)}(\zeta_\alpha)$$

where $h_\alpha^{(j)}$ are continuous, (for $j = 2$ strictly) monotone increasing functions, $h_\alpha^{(j)}(0) = 0$. Further, with suitable positive constants $c_1^* - c_3^*$ we have

$$|h_\alpha^{(j)}(-\zeta_\alpha)| \leqq c_1^* |h_\alpha^{(j)}(\zeta_\alpha)|,\ c_2^* |\zeta_\alpha|^{p-1} \leqq |h_\alpha^{(2)}(\zeta_\alpha)|;$$
$$\text{for } |\zeta_\alpha| < 1 \quad |h_\alpha^{(2)}(\zeta_\alpha)| \leqq c_3^* |\zeta_\alpha|^{p-1}.$$

$\chi_\alpha \equiv 0$ or $\chi_\alpha > 0$, $\chi_\alpha \in L^1(\Omega)$ and with some positive constants $\varepsilon_1$, $\varepsilon_2$, $c_5$

$\chi_\alpha(x) \leqq c_5 \chi_\alpha(x')$ if $|x - x'| < \varepsilon_1$ or $x' = x_0 + \frac{1}{\lambda}(x - x_0)$ where $0 < \lambda - 1 < \varepsilon_2$. $\chi_\alpha$ satisfies the above conditions e.g. if $x_0 = 0$, $\chi_\alpha$ is continuous, positive and out of some $B_a$ $\chi_\alpha(x) = \chi_\alpha^1(|x|)$ where $\chi_\alpha^1$ is monotone decreasing and its derivative is bounded.

In the proof of Theorem 3 we need

**Lemma 3.** *For each $\zeta_\alpha$, $\tilde{\zeta}_\alpha$, a.e. $x \in \Omega$*

$$|f_\alpha(x, \zeta_\alpha)\tilde{\zeta}_\alpha| \leqq c_4 [f_\alpha(x, \zeta_\alpha)\zeta_\alpha + f_\alpha(x, \tilde{\zeta}_\alpha)\zeta_\alpha].$$

*Proof:* Define $\tilde{\zeta}'_\alpha = |\tilde{\zeta}_\alpha|(\text{sgn } \zeta_\alpha)$ then II implies

$$f_\alpha(x, \zeta_\alpha)\tilde{\zeta}'_\alpha + f_\alpha(x, \tilde{\zeta}'_\alpha)\zeta_\alpha \leqq f_\alpha(x, \zeta_\alpha)\zeta_\alpha + f_\alpha(x, \tilde{\zeta}'_\alpha)\tilde{\zeta}'_\alpha$$

whence by $f_\alpha(x, \zeta_\alpha)\zeta_\alpha \geqq 0$, $f_\alpha(x, \tilde{\zeta}'_\alpha)\tilde{\zeta}'_\alpha \geqq 0$

$$f_\alpha(x, \zeta_\alpha)\tilde{\zeta}'_\alpha \leqq f_\alpha(x, \zeta_\alpha)\zeta_\alpha + f_\alpha(x, \tilde{\zeta}'_\alpha)\tilde{\zeta}'_\alpha.$$

Thus in virtue of $f_\alpha(x, \zeta_\alpha)\zeta_\alpha \geqq 0$, VIII we have

$$|f_\alpha(x, \zeta_\alpha)\tilde{\zeta}_\alpha| = f_\alpha(x, \zeta_\alpha)\tilde{\zeta}'_\alpha \leqq$$
$$\leqq f_\alpha(x, \zeta_\alpha)\zeta_\alpha + f_\alpha(x, \tilde{\zeta}'_\alpha)\tilde{\zeta}'_\alpha \leqq f_\alpha(x, \zeta_\alpha)\zeta_\alpha + c_4 f_\alpha(x, \tilde{\zeta}_\alpha)\tilde{\zeta}_\alpha. \blacksquare$$

*The Proof of Theorem 3:* Assume that $u = u'$ and $u = u''$ satisfy (1.1) - (1.3). We shall show that (1.3) is fulfilled with $v = u'$, $v = u''$. This will imply $u' = u''$ a.e. since then

$$\sum_{|\alpha|=m} \int_\Omega [f_\alpha(x, D^\alpha u') - f_\alpha(x, D^\alpha u'')](D^\alpha u' - D^\alpha u'')\, dx +$$
$$+ \sum_{|\alpha| \leqq m-1} \int_{\Omega'} [g_\alpha(x, u', \ldots, D^\beta u', \ldots) - g_\alpha(x, u'', \ldots, D^\beta u'', \ldots)]$$
$$(D^\alpha u' - D^\alpha u'')\, dx = 0$$

and so by II, IX $D^\alpha u' = D^\alpha u''$ a.e. in $\Omega$ if $|\alpha| = m$ which implies $u' = u''$ a.e. as $u', u'' \in \tilde{W}^m_{p,0}(\Omega)$.

Let $\lambda_j$ be a sequence of numbers such that $\lim(\lambda_j) = 1$ and $1 - \delta < \lambda_j < 1, \quad j = 1, 2, \ldots$ . Define functions $v_j$ in $\mathbb{R}^n$ by

$$v_j(x) = \begin{cases} u'\left(x_0 + \dfrac{1}{\lambda_j}(x - x_0)\right) & \text{if } x \in \Omega_{\lambda_j} \\ 0 \text{ otherwise} \end{cases} \tag{2.1}$$

and consider the convolution $v_j * \eta_\varepsilon$ where $\varepsilon$ is a positive number and $\eta_\varepsilon \in C_0^\infty(\mathbb{R}^n)$ is such that $\eta_\varepsilon \geqq 0$, $\eta_\varepsilon(x) = 0$ for $|x| > \varepsilon$ and $\int \eta_\varepsilon\, dx = 1$. Then $v_j * \eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ and by Hölder's inequality for $|\alpha| = m$

$$D^\alpha(v_j * \eta_\varepsilon) = D^\alpha v_j * \eta_\varepsilon \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \tag{2.2}$$

since the trace of $D^\beta v_j$ on $\partial\Omega_{\lambda_j}$ is 0 if $|\beta| \leqq m - 1$.

By using an idea of V. Komornik, we show that (1.3) holds with $u = u''$, $v = v_j * \eta_\varepsilon$ if $\varepsilon > 0$ is sufficiently small.

Let $w = v_j * \eta_\varepsilon$. Further, consider a fixed function $\varphi \in C_0^\infty(\mathbb{R}^n)$ such that $0 \leqq \varphi \leqq 1$, $\varphi(x) = 0$ if $|x| \geqq 1$, $\varphi(x) = 1$ if $|x| \leqq 1/2$ and define $w_k$ by

$$w_k(x) = \varphi\left(\frac{x}{k}\right) w(x).$$

Then

$$D^\alpha w_k(x) = \sum_{\gamma \leqq \alpha} c_\gamma \frac{1}{k^{|\gamma|}} D^\gamma \varphi\left(\frac{x}{k}\right) D^{\alpha-\gamma} w(x) \tag{2.3}$$

whence

$$\|D^\alpha w_k\|_{L^\infty(\mathbb{R}^n)} \leqq \sum_{\gamma\leqq\alpha} \frac{d_\gamma}{k^{|\gamma|}} \sup_{B_k} |D^{\alpha-\gamma} w|, \tag{2.4}$$

$$\|D^\alpha w_k\|_{L^p(\mathbb{R}^n)} \leqq \sum_{\gamma\leqq\alpha} \frac{d_\gamma}{k^{|\gamma|}} \|D^{\alpha-\gamma} w\|_{L^p(B_k)}. \tag{2.5}$$

In order to estimate the right hand sides of (2.4), (2.5) we prove estimations

$$\|f\|_{L^\infty(B_k)} \leqq \text{const } k^l \sum_{|\beta|=l} \|D^\beta f\|_{L^\infty(B_k)}, \tag{2.6}$$

$$\|f\|_{L^p(B_k)} \leqq \text{const } k^l \sum_{|\beta|=l} \|D^\beta f\|_{L^p(B_k)}, \tag{2.7}$$

if $f(x) = 0$ in a neighbourhood of 0. Indeed, we have

$$f(x) = \int_0^{|x|} \frac{x}{|x|} Df\left(t\frac{x}{|x|}\right) dt$$

and so

$$\|f\|_{L^\infty(B_k)} \leqq k\|Df\|_{L^\infty(B_k)}. \tag{2.8}$$

Further,

$$|f(x)| \leqq \int_0^{|x|} \left|Df\left(t\frac{x}{|x|}\right)\right| dt \leqq |x|^{\frac{1}{q}} \left(\int_0^{|x|} \left|Df\left(t\frac{x}{|x|}\right)^p\right| dt\right)^{1/p}$$

and, consequently, by using the notation $S_r = \{x \in \mathbb{R}^n : |x| = r\}$

$$\|f\|^p_{L^p(B_k)} \leqq \int_0^k \left[\int_{S_r} |x|^{\frac{p}{q}} \left\{\int_0^{|x|} \left|Df\left(t\frac{x}{|x|}\right)\right|^p dt\right\} d\sigma_x\right] dr \leqq \tag{2.9}$$

$$\leqq \int_0^k r^{\frac{p}{q}}\, dr \|Df\|^p_{L^p(B_k)} = \frac{1}{p} k^{\frac{p}{q}+1} \|Df\|^p_{L^p(B_k)}, \; \|f\|_{L^p(B_k)} \leqq$$

$$\leqq \left(\frac{1}{p}\right)^{\frac{1}{p}} k \|Df\|_{L^p(B_k)}.$$

Applying (2.8) resp. (2.9) successively we obtain (2.6) resp. (2.7).

Clearly, without loss of generality, we may assume that $0 \in \partial\Omega$ and so for sufficiently small $\varepsilon > 0$ $\quad w = v_j * \eta_\varepsilon$ is 0 in a neighbourhood of 0.

Thus we may estimate the right hand sides of (2.4), (2.5) by (2.6) resp. (2.7) and so (2.2) implies that

$$\|D^\alpha w_k\|_{L^\infty(\mathbb{R}^n)},\ \|D^\alpha w_k\|_{L^p(\mathbb{R}^n)}$$

are bounded $k = 1, 2, \ldots$ . Further, by the definition of $w_k$

$$w_k = w \text{ in } B_{\frac{k}{2}}.$$

Therefore, applying (1.3) to $u = u''$, $v = w_k$, by using Vitali's theorem we obtain as $k \to \infty$ that (1.3) holds with $u = u''$, $v = v_j * \eta_\varepsilon$.

Now, we shall prove that (1.3) is valid also with $u = u''$, $v = v_j$. Let $\varepsilon_k > 0$ be such that $\lim(\varepsilon_k) = 0$. Then for each fixed $r \geqq r_0$

$$\lim_{k\to\infty} \|v_j * \eta_{\varepsilon_k} - v_k\|_{W_p^m}(\Omega_r) = 0$$

(see e.g. [7]), consequently, for a suitable subsequence $(\varepsilon_k')$ of $(\varepsilon_k)$

$$D^\alpha(v_j * \eta_{\varepsilon_k'}) \longrightarrow D^\alpha v_j \qquad (|\alpha| \leqq m) \tag{2.10}$$

a.e. in $\Omega_r$. Applying this statement to $r = r_0, r_0 + 1, r_0 + 2, \ldots$ we may extract a subsequence $(\varepsilon_k'')$ such that (2.10) holds a.e. in $\Omega$.

Now we prove that for a fixed $j$, $|\alpha| = m$ the sequence of functions

$$f_\alpha(x, D^\alpha u'')D^\alpha(v_j * \eta_{\varepsilon_k''}), \qquad k = 1, 2, \ldots \tag{2.11}$$

is equiintegrable in $\Omega$. According to (2.1) $v_j(y) = u'(\Phi_j(y))$ where $\Phi_j(y) = x_0 + \frac{1}{\lambda_j}(y - x_0)$ (out of $\Omega$ $u'$ is considered to be 0). Consequently, with some positive constant $c_6 > 0$ we obtain

$$|D^\alpha(v_j * \eta_{\varepsilon_k''})(x)| = \left|\int_{\mathbb{R}^n} D^\alpha v_j(y)\eta_{\varepsilon_k''}(x - y)\, dy\right| \leqq$$
$$\leqq c_6 \int_{\mathbb{R}^n} |D^\alpha u'(\Phi_j(y))|\eta_{\varepsilon_k''}(x - y)\, dy.$$

Therefore, by using Lemma 3, XI and $\int_{\mathbb{R}^n} \eta_{\varepsilon_k''} = 1$, functions (2.11) can be estimated for sufficiently large $k$ in the following way:

$$
\begin{aligned}
&|f_\alpha(x, D^\alpha u''(x)) D^\alpha (v_j * \eta_{\varepsilon''_k})(x)| \leqq \\
&\leqq c_6 \int_{\mathbb{R}^n} |f_\alpha(x, D^\alpha u''(x)) D^\alpha u'(\Phi_j(y))| \eta_{\varepsilon''_k}(x-y)\, dy \leqq \\
&\leqq c_4 c_6 \int_{\mathbb{R}^n} f_\alpha(x, D^\alpha u''(x)) D^\alpha u''(x) \eta_{\varepsilon''_k}(x-y)\, dy + \\
&+ c_4 c_6 \int_{\mathbb{R}^n} f_\alpha(x, D^\alpha u'(\Phi_j(y))) D^\alpha u'(\Phi_j(y)) \eta_{\varepsilon''_k}(x-y)\, dy \leqq \\
&\leqq c_4 c_6 f_\alpha(x, D^\alpha u''(x)) D^\alpha u''(x) + \\
&+ c_4 c_5^2 c_6 \int_{\mathbb{R}^n} f_\alpha(\Phi_j(y), D^\alpha u'(\Phi_j(y))) D^\alpha u'(\Phi_j(y)) \eta_{\varepsilon''_k}(x-y)\, dy + \\
&+ 2 c_4 c_6 k(x) \int_{\mathbb{R}^n} |D^\alpha u'(\Phi_j(y))| \eta_{\varepsilon''_k}(x-y)\, dy.
\end{aligned}
$$

In the last sum the first term is Lebesgue integrable in $\Omega$, the second and third terms are equiintegrable in $\Omega\,(k = 1, 2, \dots)$ since for some $\Omega_0 \supset \bar{\Omega}$

$$
\begin{aligned}
y \mapsto f_\alpha(\Phi_j(y), D^\alpha u'(\Phi_j(y)) D^\alpha u'(\Phi_j(y)) \in L^1(\Omega_0), \\
D^\alpha u'(\Phi_j(y)) \in L^p(\Omega_0),\ k \in L^q(\Omega).
\end{aligned}
$$

Thus the sequence of functions (2.11) is equiintegrable in $\Omega$ and so by (2.10) and Vitali's theorem we find

$$
\lim_{k\to\infty} \int_\Omega f_\alpha(x, D^\alpha u'') D^\alpha (v_j * \eta_{\varepsilon''_k})\, dx = \int_\Omega f_\alpha(x, D^\alpha u'') D^\alpha v_j\, dx. \tag{2.12}
$$

By using (2.10), VI, VII, Sobolev's imbedding theorem, Hölder's inequality and Vitali's theorem it is not difficult to show that for $|\alpha| \leqq m-1$

$$
\begin{aligned}
\lim_{k\to\infty} \int_{\Omega'} g_\alpha(x, u'', \dots, D^\beta u'', \dots) D^\alpha (v_j * \eta_{\varepsilon''_k})\, dx = \\
= \int_{\Omega'} g_\alpha(x, u'', \dots, D^\beta u'', \dots) D^\alpha v_j\, dx.
\end{aligned}
\tag{2.13}
$$

Finally, $\|v_j * \eta_{\varepsilon''_k}\|_V \leqq \|v_j\|_V$, thus it may be supposed: we have chosen subsequence $(\varepsilon''_k)$ of $(\varepsilon'_k)$ such that

$$
(v_j * \eta_{\varepsilon''_k}) \longrightarrow v_j \text{ weakly in } V. \tag{2.14}
$$

Since (1.3) holds with $u = u''$, $v = v_j * \eta_{\varepsilon''_k}$, thus from (2.12) - (2.14) we obtain as $k \to \infty$ that (1.3) holds with $u = u''$, $v = v_j$. Consequently, similarly to the above arguments, we obtain as $j \to \infty$ that (1.3) is valid for $u = u''$, $v = u'$. Analogously can be considered cases $u = u''$, $v = u''$; $u = u'$, $v = u'$ resp. $u''$. ∎

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