

## OSCILLATIONS OF THE SOLUTIONS OF NONLINEAR HYPERBOLIC EQUATIONS OF NEUTRAL TYPE

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### Abstract

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In this paper nonlinear hyperbolic equations of neutral type of the form

$$\frac{\partial^2}{\partial t^2} [u(x, t) + \lambda(t)u(x, t - \tau)] - [\Delta u(x, t) + \mu(t)\Delta u(x, t - \sigma)] + c(x, t, u) = f(x, t), \quad (x, t) \in \Omega \times (0, \infty) \equiv G,$$

are considered, where  $\tau, \sigma = \text{const} > 0$ , with boundary conditions

$$\frac{\partial u}{\partial n} + \gamma(x, t)u = g(x, t), \quad (x, t) \in \partial\Omega \times [0, \infty)$$

or

$$u = 0, \quad (x, t) \in \partial\Omega \times [0, \infty).$$

Under certain constraints on the coefficients of the equation and the boundary conditions, sufficient conditions for oscillation of the solutions of the problems considered are obtained.

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### 1. Introduction

In the last few years results related to the oscillatory properties and asymptotic behaviour of the solutions of some classes of hyperbolic equations were published. We shall mention especially the work of K. Kreith, T. Kusano and N. Yoshida [4] in which sufficient conditions for oscillation of the solutions of the nonlinear hyperbolic equation

$$u_{tt} - \Delta u + c(x, t, u) = f(x, t)$$

considered in a cylindrical domain are obtained. Oscillatory properties of the solutions of hyperbolic differential equations with a deviating argument were investigated in the works of D. Georgiou, K. Kreith [2], D. Georgiou [3]. Hyperbolic differential equations with *maxima* were investigated in the work of D. Mishev [6] and some conditions for oscillation of the solutions of hyperbolic equations of neutral type were obtained by D. Mishev and D. Bainov in [7], [8].

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## 2. Preliminary notes

In the present paper sufficient conditions for oscillation of the solutions of nonlinear hyperbolic equations of neutral type of the form

$$(1) \quad \frac{\partial^2}{\partial t^2} [u(x, t) + \lambda(t)u(x, t - \tau)] - [\Delta u(x, t) + \mu(t)\Delta u(x, t - \sigma)] + c(x, t, u) = f(x, t), \quad (x, t) \in \Omega \times (0, \infty) \equiv G,$$

are obtained, where  $\tau, \sigma = \text{const} > 0$ ,  $\Delta u(x, t) = \sum_{i=1}^n u_{x_i x_i}(x, t)$  and  $\Omega$  is a bounded domain in  $\mathbb{R}^n$  with a piecewise smooth boundary.

Consider boundary conditions of the form

$$(2) \quad \frac{\partial u}{\partial n} + \gamma(x, t)u = g(x, t), \quad (x, t) \in \partial\Omega \times [0, \infty)$$

$$(3) \quad u = 0, \quad (x, t) \in \partial\Omega \times [0, \infty)$$

We shall say that conditions (H) are satisfied if the following conditions hold:

$$\text{H1. } \lambda(t) \in C^2([0, \infty); [0, \infty)),$$

$$\mu(t) \in C([0, \infty); \mathbb{R}),$$

$$\text{H2. } c(x, t, u) \in C(G \times \mathbb{R}; \mathbb{R}),$$

$$\text{H3. } c(x, t, -u) = -c(x, t, u), \quad (x, t, u) \in G \times (0, \infty),$$

$$\text{H4. } c(x, t, u) \geq p(t) \cdot h(u), \quad (x, t, u) \in G \times (0, \infty),$$

where  $p(t)$  is a continuous and positive function in the interval  $(0, \infty)$  and  $h(u)$  is a continuous, positive and convex function in the same interval  $(0, \infty)$ .

$$\text{H5. } f(x, t) \in C(G; \mathbb{R})$$

$$\text{H6. } g(x, t) \in C(\partial\Omega \times [0, \infty); \mathbb{R})$$

$$\text{H7. } \gamma(x, t) \in C(\partial\Omega \times [0, \infty); [0, \infty)).$$

**Definition 1.** The solution  $u(x, t) \in C^2(G) \cap C^1(\bar{G})$  of problem (1), (2) ((1), (3)) is said to oscillate in the domain  $G$  if for any positive number  $\mu$  there exists a point  $(x_0, t_0) \in \Omega \times [\mu, \infty)$ , such that the equality  $u(x_0, t_0) = 0$  holds.

In the subsequent theorems sufficient conditions for oscillation of the solutions of problems (1), (2) and (1), (3) in the domain  $G$  are obtained. We shall note that in the work of K. Kreith, T. Kusano, N. Yoshida [4] conditions are obtained for the oscillation of the solutions only of problem (1), (2) in the case when  $\lambda(t) \equiv 0$ ,  $\mu(t) \equiv 0$  and  $\gamma(x, t) \equiv 0$ .

Introduce the following notation:

$$(4) \quad F(t) = \frac{1}{|\Omega|} \cdot \int_{\Omega} f(x, t) dx, \quad t > 0,$$

$$(5) \quad G(t) = \frac{1}{|\Omega|} \cdot \int_{\partial\Omega} g(x, t) ds, \quad t > 0,$$

where  $|\Omega| = \int_{\Omega} dx$ .

With any solution  $u(x, t) \in C^2(G) \cap C^1(\bar{G})$  of problem (1), (2) we associate the function

$$(5) \quad v(t) = \frac{1}{|\Omega|} \cdot \int_{\Omega} u(x, t) dx, \quad t > 0.$$

**Lemma 1.** *Let conditions (H) hold and let  $u(x, t)$  be a positive solution of problem (1), (2) in the domain  $G$ . Then the function  $v(t)$  defined by (5) satisfies the differential inequality of neutral type*

$$(6) \quad \frac{d^2}{dt^2} [v(t) + \lambda(t)v(t - \tau)] + p(t)h(v(t)) \leq G(t) + \mu(t)G(t - \sigma) + F(t),$$

$t \geq t_0$ , where  $t_0$  is a sufficiently large positive number.

*Proof:* Let  $u(x, t)$  be a positive solution in the domain  $G$  of problem (1), (2) and  $t_0 = \max\{\tau, \sigma\}$ . Then  $u(x, t - \tau) > 0$  and  $u(x, t - \sigma) > 0$  for  $(x, t) \in \Omega \times [t_0, \infty)$ . We integrate both sides of equation (1) with respect to  $x$  over the domain  $\Omega$  and obtain for  $t \geq t_0$ :

$$(7) \quad \frac{d^2}{dt^2} \left[ \int_{\Omega} u(x, t) dx + \lambda(t) \int_{\Omega} u(x, t - \tau) dx \right] - \left[ \int_{\Omega} \Delta u(x, t) dx + \mu(t) \int_{\Omega} \Delta u(x, t - \sigma) dx \right] + \int_{\Omega} c(x, t, u) dx = \int_{\Omega} f(x, t) dx.$$

From Green's formula and condition H7 it follows that

$$(8) \quad \int_{\Omega} \Delta u(x, t) dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} ds = \int_{\partial\Omega} [g(x, t) - \gamma(x, t)u] ds \leq \int_{\partial\Omega} g(x, t) ds$$

$$(9) \quad \int_{\Omega} \Delta u(x, t - \sigma) dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}(x, t - \sigma) ds = \\ = \int_{\partial\Omega} [g(x, t - \sigma) - \gamma(x, t - \sigma) \cdot u(x, t - \sigma)] ds \leq \int_{\partial\Omega} g(x, t - \sigma) ds$$



From Lemma 1 it follows that the function  $\frac{1}{|\Omega|} \int_{\Omega} (-u(x, t)) dx$  is a positive solution of inequality (12) for  $t \geq t_0 + \mu$  which also contradicts the assumption of the theorem. Thus Theorem 1 is proved. ■

Now we shall investigate the oscillatory properties of the solutions of problem (1), (3). Consider in the domain  $\Omega$  the following Dirichlet problem:

$$\begin{cases} \Delta U + \alpha U = 0 \text{ in } \Omega \\ U|_{\partial\Omega} = 0 \end{cases}$$

where  $\alpha = \text{const}$ . It is well known [1] that the smallest eigenvalue  $\alpha_0$  is positive and the corresponding eigenfunction  $\varphi(x)$  can be chosen to satisfy the inequality  $\varphi(x) > 0$  for  $x \in \Omega$ .

With any solution  $u(x, t) \in C^2(G) \cap C^1(\bar{G})$  of problem (1), (3) we associate the function

$$(13) \quad w(t) = \int_{\Omega} u(x, t)\varphi(x) dx \cdot \left( \int_{\Omega} \varphi(x) dx \right)^{-1}, \quad t > 0$$

We shall note that a similar averaging was first used by N. Yoshida in the work [10].

**Lemma 2.** *Let conditions H1-H6 hold and let  $u(x, t)$  be a positive solution in the domain  $G$  of problem (1), (3). Then the function  $w(t)$  defined by (13) satisfies the differential inequality of neutral type*

$$(14) \quad \frac{d^2}{dt^2} [w(t) + \lambda(t)w(t - \tau)] + \alpha_0 w(t) + \alpha_0 \mu(t) \cdot w(t - \sigma) + p(t) \cdot h(w(t)) \leq \int_{\Omega} f(x, t)\varphi(x) dx \cdot \left( \int_{\Omega} \varphi(x) dx \right)^{-1}, \quad t \geq t_0,$$

where  $t_0$  is a sufficiently large positive number.

*Proof:* Let  $u(x, t)$  be a positive solution in the domain  $G$  of problem (1), (3) and  $t_0 = \max\{\tau, \sigma\}$ . Then  $u(x, t - \tau) > 0$  and  $u(x, t - \sigma) > 0$  for  $(x, t) \in \Omega \times (t_0, \infty)$ . Multiply both sides of equation (1) by the eigenfunction  $\varphi(x)$  of the Dirichlet problem and integrate with respect to  $x$  over the domain  $\Omega$ . For  $t \geq t_0$  we obtain

$$(15) \quad \frac{d^2}{dt^2} \left[ \int_{\Omega} u(x, t)\varphi(x) dx + \lambda(t) \int_{\Omega} u(x, t - \tau)\varphi(x) dx \right] - \left[ \int_{\Omega} \Delta u(x, t)\varphi(x) dx + \mu(t) \int_{\Omega} \Delta u(x, t - \sigma)\varphi(x) dx \right] + \int_{\Omega} c(x, t, u)\varphi(x) dx = \int_{\Omega} f(x, t)\varphi(x) dx.$$

From Green's formula it follows that

$$(16) \quad \int_{\Omega} \Delta u(x, t) \varphi(x) dx = \int_{\Omega} u(x, t) \Delta \varphi(x) dx = \\ = -\alpha_0 \cdot \int_{\Omega} u(x, t) \varphi(x) dx = -\alpha_0 w(t) \cdot \int_{\Omega} \varphi(x) dx$$

$$(17) \quad \int_{\Omega} \Delta u(x, t - \sigma) \varphi(x) dx = \int_{\Omega} u(x, t - \sigma) \Delta \varphi(x) dx = \\ = -\alpha_0 \int_{\Omega} u(x, t - \sigma) \varphi(x) dx = -\alpha_0 w(t - \sigma) \cdot \int_{\Omega} \varphi(x) dx,$$

where  $\alpha_0$  is the smallest eigenvalue. Moreover, from condition H4 and Jensen's inequality it follows that

$$(18) \quad \int_{\Omega} c(x, t, u) \varphi(x) dx \geq p(t) \int_{\Omega} h(u) \varphi(x) dx \geq \\ \geq p(t) \cdot h \left( \int_{\Omega} u(x, t) \varphi(x) dx \cdot \left( \int_{\Omega} \varphi(x) dx \right)^{-1} \right) \cdot \int_{\Omega} \varphi(x) dx = \\ = p(t) \cdot h(w(t)) \cdot \int_{\Omega} \varphi(x) dx$$

Using (16)-(18) and condition H1, from (15) we obtain

$$\frac{d^2}{dt^2} [w(t) + \lambda(t)w(t - \tau)] \leq -\alpha_0 [w(t) + \mu(t)w(t - \sigma)] - \\ - p(t) \cdot h(w(t)) + \int_{\Omega} f(x, t) \varphi(x) dx \cdot \left( \int_{\Omega} \varphi(x) dx \right)^{-1},$$

which completes the proof of Lemma 2. ■

Introduce the notation

$$(19) \quad F_1(t) = \int_{\Omega} f(x, t) \varphi(x) dx \cdot \left( \int_{\Omega} \varphi(x) dx \right)^{-1}, \quad t > 0$$

Analogously to Theorem 1 the following theorem is proved.

**Theorem 2.** *Let conditions H1-H5 hold and let the differential inequalities of neutral type*

$$(20) \quad \frac{d^2}{dt^2} [w(t) + \lambda(t)w(t - \tau)] + \alpha_0 [w(t) + \mu(t)w(t - \sigma)] + \\ + p(t) \cdot h(w(t)) \leq F_1(t), \quad t \geq t_0,$$

$$(21) \quad \frac{d^2}{dt^2}[w(t) + \lambda(t)w(t - \tau)] + \alpha_0[w(t) + \mu(t)w(t - \sigma)] + p(t) \cdot h(w(t)) \leq -F_1(t), \quad t \geq t_0$$

have no eventually positive solutions. Then each solution  $u(x, t)$  of problem (1), (3) oscillates in the domain  $G$ .

From the theorems proved above it follows that the finding of sufficient conditions for oscillation of the solutions of equation (1) in the domain  $G$  is reduced to the investigation of the oscillatory properties of differential inequalities of neutral type of the form

$$(22) \quad \frac{d^2}{dt^2}[x(t) + \lambda(t)x(t - \tau)] + q_0(t)x(t) + q(t)x(t - \sigma) + p(t) \cdot h(x(t)) \leq H(t), \quad t \geq t_0$$

We shall say that condition (A) are satisfied if the following conditions hold:

- A1.  $\lambda(t) \in C^2([t_0, \infty); [0, \infty))$ ,
- A2.  $q_0(t), q(t) \in C([t_0, \infty); [0, \infty))$ ,
- A3.  $p(t) \in C([t_0, \infty); [0, \infty))$ ,
- A4.  $h(u) \in C(\mathbb{R}; \mathbb{R}), h(u) > 0$  for  $u > 0$ ,
- A5.  $H(t) \in C([t_0, \infty); \mathbb{R})$ .

**Theorem 3.** *Let conditions (A) hold as well as the condition*

$$(23) \quad \liminf_{t \rightarrow \infty} \frac{1}{t - t_1} \int_{t_1}^t (t - s) \cdot H(s) ds = -\infty$$

for  $t_1 \geq t_0$ . Then the differential inequality (22) has no eventually positive solutions.

*Proof.* Suppose that this is not true and let  $x(t)$  be a positive solution of inequality (22) defined in the interval  $[t_1, \infty)$ , where  $t_1 \geq t_0$ . Then in virtue of conditions A2-A4 we obtain for  $t \geq t_2$  ( $t_2 \geq t_1 + \max\{\sigma, \tau\}$ )

$$\frac{d^2}{dt^2}[x(t) + \lambda(t)x(t - \tau)] \leq H(t) - q_0(t)x(t) - q(t)x(t - \sigma) - p(t) \cdot h(x(t)) \leq H(t).$$

We integrate twice the above inequality over the interval  $[t_2, t]$ ,  $t > t_2$  and obtain

$$x(t) + \lambda(t)x(t - \tau) \leq C_1 + C_2(t - t_2) + \int_{t_2}^t \left[ \int_{t_2}^\rho H(s) ds \right] d\rho,$$

where  $C_1, C_2 = \text{const.}$  Since

$$\int_{t_2}^t \left[ \int_{t_2}^{\rho} H(s) ds \right] d\rho = \int_{t_2}^t (t-s)H(s) ds,$$

dividing both sides of last inequality by  $t - t_2 > 0$ , we obtain

$$(24) \quad \frac{x(t) + \lambda(t)x(t-\tau)}{t-t_2} \leq \frac{C_1}{t-t_2} + C_2 + \frac{1}{t-t_2} \int_{t_2}^t (t-s)H(s) ds$$

Then for  $t \rightarrow \infty$  from (24), making use of condition (23), we obtain that

$$(25) \quad \liminf_{t \rightarrow \infty} \frac{x(t) + \lambda(t)x(t-\tau)}{t-t_2} = -\infty$$

On the other hand, using condition A1 and the fact that  $x(t) > 0$ ,  $x(t-\tau) > 0$  for  $t \geq t_2$ , we obtain that

$$\liminf_{t \rightarrow \infty} \frac{1}{t-t_0} [x(t) + \lambda(t)x(t-\tau)] \geq 0,$$

which contradicts equality (25).

This completes the proof of Theorem 3. ■

The following sufficient condition for oscillation of the solutions of problem (1), (2) is a corollary of Theorem 1 and Theorem 3.

**Theorem 4.** *Let conditions (H) hold as well as the conditions*

$$(26) \quad \liminf_{t \rightarrow \infty} \int_{t_0}^t \left(1 - \frac{s}{t}\right) (G(s) + \mu(s)G(s-\sigma) + F(s)) ds = -\infty,$$

$$(27) \quad \limsup_{t \rightarrow \infty} \int_{t_0}^t \left(1 - \frac{s}{t}\right) (G(s) + \mu(s)G(s-\sigma) + F(s)) ds = +\infty$$

for any sufficiently large number  $t_0$ , where the functions  $G(t)$  and  $F(t)$  are defined by (4). Then each solution  $u(x, t)$  of problem (1), (2) oscillates in the domain  $G$ .

The following sufficient condition for oscillation of the solutions of problem (1), (3) is a corollary of Theorem 2 and Theorem 3.



**Theorem 5.** *Let conditions H1-H5 hold as well as the conditions*

$$(28) \quad \mu(t) \geq 0 \text{ for } t \geq 0$$

$$(29) \quad \liminf_{t \rightarrow \infty} \int_{t_0}^t \left(1 - \frac{s}{t}\right) F_1(s) ds = -\infty$$

$$(30) \quad \limsup_{t \rightarrow \infty} \int_{t_0}^t \left(1 - \frac{s}{t}\right) F_1(s) ds = +\infty$$

for any sufficiently large number  $t_0$ , where the function  $F_1(t)$  is defined by (19). Then each solution  $u(x, t)$  of problem (1), (3) oscillates in the domain  $G$ .

**Example 1.** Consider the equation

$$(31) \quad u_{tt} + u_{tt}(x, t - \pi) - u_{xx} + u = 2e^t \cos x (\sin t + \cos t - e^{-\pi} \cdot \cos t),$$

$$(x, t) \in \left(0, \frac{\pi}{2}\right) \times (0, \infty) \equiv G,$$

and the boundary conditions

$$(32) \quad -u_x(0, t) = 0, \quad u_x\left(\frac{\pi}{2}, t\right) = -e^t \cdot \sin t, \quad t \geq 0$$

A straightforward verification shows that the functions

$$c(x, t, u) = u, \quad f(x, t) = 2e^t \cdot \cos x \cdot (\sin t + \cos t - e^{-\pi} \cos t),$$

$$g(0, t) = 0, \quad g\left(\frac{\pi}{2}, t\right) = -e^t \cdot \sin t, \quad \lambda(t) \equiv 1, \quad \mu(t) \equiv 0,$$

$$\gamma(x, t) \equiv 0$$

satisfy conditions (H). Moreover, from (4) we obtain that

$$G(t) = -\frac{2}{\pi} e^t \cdot \sin t, \quad t > 0,$$

$$F(t) = \frac{4}{\pi} e^t (\sin t + \cos t - e^{-\pi} \cdot \cos t), \quad t > 0.$$

By straightforward calculations we find that

$$I(t) = \int_{t_0}^t \left(1 - \frac{s}{t}\right) (G(s) + \mu(s)G(s - \sigma) + F(s)) ds =$$

$$= e^t \cdot (t\pi)^{-1} \cdot (2 \sin t - 2e^{-\pi} \sin t - \cos t) + C,$$

where  $C$  is independent of  $t$ . Hence

$$\liminf_{t \rightarrow \infty} I(t) = -\infty, \quad \limsup_{t \rightarrow \infty} I(t) = +\infty,$$

i.e. conditions (26), (27) of Theorem 4 hold as well. Then from Theorem 4 it follows that each solution of problem (31), (32) oscillates in the domain  $G = (0, \frac{\pi}{2}) \times (0, \infty)$ . For instance, the function  $u(x, t) = e^t \sin t \cos x$  is such a solution.

**Example 2.** Consider the equation

$$(33) \quad u_{tt} + u_{tt}(x, t - \pi) - [u_{xx} + u_{xx}(x, t - \pi)] + u = f(x, t), \\ (x, t) \in (0, \pi) \times (0, \infty) \equiv G,$$

where  $f(x, t) = e^t \cdot \sin x(2e^{-\pi} \sin t - 2 \sin t + e^{-\pi} \cos t)$  and the boundary conditions

$$(34) \quad u(0, t) = u(\pi, t) = 0, \quad t \geq 0.$$

It is immediately verified that the functions

$$c(x, t, u) = u, \quad f(x, t), \quad \lambda(t) = \mu(t) = 1$$

satisfy conditions H1-H5. Moreover, the smallest eigenvalue of the Sturm-Liouville problem

$$U'' + \alpha U = 0, \quad U(0) = U(\pi) = 0$$

is  $\alpha_0 = 1$  and the corresponding eigenfunction is  $\varphi(x) = \sqrt{\frac{2}{\pi}} \sin x > 0$ ,  $x \in (0, \pi)$ . Then from (19) we find that

$$F_1(t) = \int_0^\pi f(x, t) \sqrt{\frac{2}{\pi}} \sin x \, dx \cdot \left( \int_0^\pi \sqrt{\frac{2}{\pi}} \sin x \, dx \right)^{-1} = \\ = \frac{\pi}{4} e^t (2e^{-\pi} \sin t - 2 \sin t + e^{-\pi} \cos t).$$

By straightforward calculations we obtain that

$$I_1(t) = \int_{t_0}^t \left( 1 - \frac{s}{t} \right) F_1(s) \, ds = \frac{\pi}{4} \cdot \frac{e^t}{t} \left( \cos t - e^{-\pi} \cos t + \frac{1}{2} e^{-\pi} \sin t \right) + C,$$

where  $C$  is independent of  $t$ . Hence  $\liminf_{t \rightarrow \infty} I_1(t) = -\infty$  and  $\limsup_{t \rightarrow \infty} I_1(t) = +\infty$ , i.e. conditions (29), (30) of Theorem 5 hold as well. Then from

Theorem 5 it follows that each solution of problem (33), (34) oscillates in the domain  $G = (0, \pi) \times (0, \infty)$ . For instance, the function  $u(x, t) = e^t \sin x \cdot \cos t$  is such a solution.

In the subsequent theorems we shall restrict our attention to some particular cases of equation (1) for which new sufficient conditions for oscillation of the solutions are obtained.

**Case A.**

Assume that  $\lambda(t) \equiv 0$ . We shall use the following result of T. Kusano and M. Naito [5] concerning differential inequalities of the form

$$(35) \quad (q(t)(p(t) \cdot x)')' + h(t, x) \leq r(t), \quad t \geq t_0$$

We shall say that conditions (B) are satisfied if the following conditions hold:

- B1.  $p(t), q(t) \in C([t_0, \infty); (0, \infty)), \int_{t_0}^{\infty} (q(t))^{-1} dt = \infty$ .
- B2.  $h(t, x) \in C([t_0, \infty) \times (0, \infty); (0, \infty)), h(t, x)$  is a monotone increasing function of its second argument  $x$ .
- B3.  $r(t) \in C([t_0, \infty); \mathbb{R})$ .

**Theorem 6 [5].** *Let conditions (B) hold and let the differential inequality*

$$(36) \quad (q(t)(p(t)x)')' + h(t, x) \leq 0$$

*have no eventually positive solutions. Moreover, let a function  $\theta(t) \in C^2([t_0, \infty); \mathbb{R})$  exist with the following properties:*

- (37)  $\theta(t)$  takes both positive and negative values  
for arbitrarily large values of  $t$ .

$$(38) \quad (q(t)(p(t) \cdot \theta(t))')' = r(t), \quad t \geq t_0$$

$$(39) \quad \liminf_{t \rightarrow \infty} [p(t) \cdot \theta(t)] = 0$$

*Then the differential inequality (35) has no eventually positive solutions.*

The following sufficient condition for oscillation of the solutions of problem (1), (2) in the case when  $\lambda(t) \equiv 0$  is a corollary of Theorem 1 and Theorem 6.

**Theorem 7.** Let the following conditions be fulfilled:

1. Conditions (H) hold.
2. The function  $h(u)$  is monotone increasing in the interval  $(0, \infty)$ .
3. The differential inequality

$$x''(t) + p(t) \cdot h(x(t)) \leq 0, \quad t \geq t_0$$

has no eventually positive solutions.

4. There exists a function  $\theta(t) \in C^2([t_0, \infty); \mathbb{R})$  with the following properties:
  - a)  $\theta(t)$  takes both positive and negative values for arbitrarily large values of  $t$ .
  - b)  $[\theta(t)]'' = G(t) + \mu(t)G(t - \sigma) + F(t), \quad t \geq t_1$
  - c)  $\lim_{t \rightarrow \infty} \theta(t) = 0$ .

Then each solution  $u(x, t)$  of problem (1), (2) oscillates in the domain  $G$ .

**Example 3.** Consider the equation

$$(40) \quad u_{tt}(x, t) - u_{xx}(x, t) - u_{xx}(x, t - \pi) + 2u = f(x, t),$$

$$(x, t) \in \left(0, \frac{\pi}{2}\right) \times (0, \infty),$$

where  $f(x, t) = e^{-t} \cdot \cos x \cdot (3 \sin t - 2 \cos t - e^\pi \sin t)$  and the boundary condition

$$(41) \quad -u_x(0, t) = 0, \quad u_x\left(\frac{\pi}{2}, t\right) = -e^{-t} \sin t, \quad t > 0.$$

It is immediately verified that the functions

$$c(x, t, u) = 2u, \quad f(x, t), \quad \lambda(t) \equiv 0, \quad \mu(t) \equiv 1,$$

$$\gamma(x, t) \equiv 0, \quad g(0, t) = 0, \quad g\left(\frac{\pi}{2}, t\right) = -e^{-t} \cdot \sin t$$

satisfy conditions (H). Moreover, from (4) we obtain

$$G(t) = -\frac{2}{\pi} e^{-t} \sin t, \quad t > 0$$

$$F(t) = \frac{2}{\pi} e^{-t} \cdot (3 \sin t - 2 \cos t - e^\pi \sin t), \quad t > 0.$$

Then

$$I_2(t) = \int_{t_0}^t \left(1 - \frac{s}{t}\right) (G(s) + \mu(s)G(s - \sigma) + F(s)) ds =$$

$$= \int_{t_0}^t \left(1 - \frac{s}{t}\right) \cdot \frac{2}{\pi} e^{-s} \cdot (3 \sin s - 2 \cos s - e^\pi \sin s) ds,$$

which immediately implies that  $\lim_{t \rightarrow \infty} I_2(t) < \infty$ . Hence conditions (26) and (27) of Theorem 4 are not satisfied. It is easy to check that the differential inequality  $x'' + 2x \leq 0$  has no eventually positive solutions. Let  $\theta(t) = \frac{e^{-t}}{\pi} [2 \sin t + (3 - e^\pi) \cos t]$ . Then for  $n \in \mathbb{Z}$  we obtain

$$\theta\left(\frac{\pi}{2} + 2n\pi\right) > 0, \quad \theta\left(\frac{3\pi}{2} + 2n\pi\right) < 0.$$

Moreover,  $[\theta(t)]'' = \frac{2}{\pi} e^{-t} \cdot (3 \sin t - 2 \cos t - e^\pi \sin t) = G(t) + G(t - \pi) + F(t)$  and  $\lim_{t \rightarrow \infty} \theta(t) = 0$ . Hence the function  $\theta(t)$  satisfies condition 4 of Theorem 7. Then by Theorem 7 each solution  $u(x, t)$  of problem (40), (41) oscillates in the domain  $G = (0, \frac{\pi}{2}) \times (0, \infty)$ . For instance, the function  $u(x, t) = e^{-t} \sin t \cos x$  is such a solution.

We shall note that a result analogous to that of Theorem 7 can be obtained for problem (1), (3) as well.

**Case B.**

Assume that  $f(x, t) \equiv 0, g(x, t) \equiv 0$ . In this case the finding of sufficient conditions for oscillation of the solutions of equation (1) in the domain  $G$  is reduced to the investigation of the oscillatory properties of differential inequalities of neutral type of the form

$$(42) \quad \frac{d^2}{dt^2} [x(t) + \lambda(t)x(t - \tau)] + q_0(t)x(t) + q(t)x(t - \sigma) + p(t) \cdot h(x(t)) \leq 0, \quad t \geq t_0,$$

$$(43) \quad \frac{d^2}{dt^2} [x(t) + \lambda(t)x(t - \tau)] + q_0(t)x(t) + q(t)x(t - \sigma) + p(t) \cdot h(x(t)) \geq 0, \quad t \geq t_0.$$

Together with (42) and (43) we shall consider the nonlinear differential equation of neutral type

$$(44) \quad \frac{d^2}{dt^2} [x(t) + \lambda(t)x(t - \tau)] + q_0(t)x(t) + q(t)x(t - \sigma) + p(t) \cdot h(x(t)) = 0, \quad t \geq t_0.$$

We shall say that conditions (C) are satisfied if the following conditions hold:

- C1.  $\lambda(t) \in C^2([t_0, \infty); \mathbb{R})$ ,  
 $0 < \lambda_1 \leq \lambda(t) \leq \lambda_2$  for  $t \geq t_0, \lambda_1, \lambda_2 = \text{const.}$
- C2.  $q_0(t), q(t) \in C([t_0, \infty); [0, \infty))$ ,
- C3.  $p(t) \in C([t_0, \infty); (0; \infty))$ ,
- C4.  $h(u) \in C(\mathbb{R}; \mathbb{R}), h(-u) = -h(u)$ ,

$h(u)$  is a positive and monotone increasing function in the interval  $(0, \infty)$ .

**Theorem 8.** *Let the following conditions be satisfied:*

1. *Conditions (C) hold.*
2. *For any closed and measurable set  $E \subset [t_0, \infty)$  for which  $\text{meas}(E \cap [t, t + 2\tau]) \geq \tau$ ,  $t \in [t_0, \infty)$ , the following condition holds*

$$(45) \quad \int_E p(t) dt = \infty$$

*Then:*

- (i) *the differential inequality (42) has no eventually positive solutions;*
- (ii) *the differential inequality (43) has no eventually negative solutions;*
- (iii) *all solutions of the differential equation (44) oscillate.*

*Proof:*

(i) Let  $x(t)$  be an eventually positive solution of the differential inequality (42). Then there exists a number  $t_1 \geq t_0$  such that  $x(t) > 0$ ,  $x(t - \tau) > 0$  and  $x(t - \sigma) > 0$  for  $t \geq t_1$ . From conditions C2-C4 and (42) it follows that

$$(46) \quad \begin{aligned} \frac{d^2}{dt^2} [x(t) + \lambda(t)x(t - \tau)] &\leq -q_0(t)x(t) - q(t)x(t - \sigma) - \\ &- p(t) \cdot h(x(t)) \leq -p(t) \cdot h(x(t)) < 0, \quad t \geq t_1. \end{aligned}$$

Hence the function  $\frac{d}{dt}[x(t) + \lambda(t)x(t - \tau)]$  is monotone decreasing in the interval  $[t_1, \infty)$ . Suppose that there exists a number  $t_2 \geq t_1$  such that  $\frac{d}{dt}[x(t_2) + \lambda(t_2)x(t_2 - \tau)] = -c < 0$ . Then for any point  $t \geq t_2$  the following inequality holds

$$\frac{d}{dt}[x(t) + \lambda(t)x(t - \tau)] \leq \frac{d}{dt}[x(t_2) + \lambda(t_2)x(t_2 - \tau)] = -c$$

Integrate last inequality over the interval  $[t_2, t]$ ,  $t > t_2$  and obtain

$$x(t) + \lambda(t)x(t - \tau) \leq x(t_2) + \lambda(t_2)x(t_2 - \tau) - c(t - t_2).$$

Hence  $\limsup_{t \rightarrow \infty} [x(t) + \lambda(t)x(t - \tau)] \leq 0$  which contradicts the assumption that  $x(t)$  is an eventually positive solution. Hence

$$(47) \quad \frac{d}{dt}[x(t) + \lambda(t) \cdot x(t - \tau)] \geq 0, \quad t \geq t_1$$

whence we obtain that  $x(t) + \lambda(t)x(t - \tau) \geq c_1 > 0$  for  $t \geq t_1$ . From Lemma 1 [9], [11] it follows that there exists a closed and measurable set  $E \subset [t_1, \infty)$  and a constant  $c_2 > 0$  such that  $x(t) \geq c_2$  for  $t \in E$  and  $\text{meas}(E \cap [t, t + 2\tau]) \geq \tau$  for  $t \geq t_1$ . Then from condition C4 it follows that

$$h(x(t)) \geq h(c_2) = c_3 > 0 \text{ for } t \in E.$$

Integrate both sides of inequality (46) over the interval  $[t_1, t]$ ,  $t > t_1$  and using (47), we obtain

$$\begin{aligned} C_3 \int_{E \cap [t_1, t]} p(s) ds &\leq \int_{t_1}^t p(s)h(x(s)) ds \leq \frac{d}{dt}[x(t_1) + \lambda(t_1)x(t_1 - \tau)] - \\ &- \frac{d}{dt}[x(t) + \lambda(t)x(t - \tau)] \leq \frac{d}{dt}[x(t_1) + \lambda(t_1)x(t_1 - \tau)] = C_4. \end{aligned}$$

For  $t \rightarrow \infty$  from the above inequality it follows that  $\int_E p(t) dt < \infty$ , which contradicts condition (45). Thus assertion (i) of Theorem 8 is proved.

(ii) The proof follows immediately from the fact that if  $x(t)$  is an eventually negative solution of the differential inequality (43), then  $-x(t)$  is an eventually positive solution of the differential inequality (42).

(iii) The proof follows immediately from assertions (i) and (ii). ■

The following sufficient condition for oscillation of the solutions of problem (1), (2) or (1), (3) in the case when  $f(x, t) \equiv 0$  and  $g(x, t) \equiv 0$  is a corollary of Theorem 1, Theorem 2 and Theorem 8.

**Theorem 9.** *Let the following conditions hold:*

1. *Conditions (H) are fulfilled.*
2.  $0 < \lambda_1 \leq \lambda(t) < \lambda_2$ ,  $t \geq t_0$ ,  $\lambda_1, \lambda_2 = \text{const.}$
3.  $h(-u) = -h(u)$ ,  $u \in \mathbb{R}$ ;  $h(u)$  is a monotone increasing function in the interval  $(0, \infty)$ .
4. *For any closed and measurable set  $E \subset [t_0, \infty)$  for which  $\text{meas}(E \cap [t, t + 2\tau]) \geq \tau$ ,  $t \in [t_0, \infty)$  the following condition holds*

$$(48) \quad \int_E p(t) dt = \infty$$

*Then each solution  $u(x, t)$  of problem (1), (2) or (1), (3) oscillates in the domain  $G$ .*

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