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GROUP RINGS WITH FC-NILPOTENT UNIT GROUPS

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Abstract

Let $U(RG)$ be the unit group of the group ring RG. Groups G such that $U(RG)$ is FC-nilpotent are determined, where R is the ring of integers $\mathbb{Z}$ or a field K of characteristic zero.

Let R be a commutative ring with identity, G be a group. $U(RG)$ the group of units of the group ring RG and $\Delta(G)$ the FC-subgroup of G. Define,

$$\Delta_{k+1}(G)/\Delta_k(G) = \Delta(G/\Delta_k(G)) \text{ for } k \geq 1 \text{ and } \Delta_1(G) = \Delta(G).$$

A group G is said to be FC-nilpotent if $\Delta_n(G) = G$ for some n.

In the present note, we determine groups G such that $U(RG)$ is FC-nilpotent, where R is either the ring of integers $\mathbb{Z}$ or a field K of characteristic zero.

The main results are

Theorem 1. *If $U(\mathbb{Z}G)$ is FC-nilpotent, then G is FC-nilpotent and $T(G)$ the set of torsion elements of G, is an abelian or a Hamiltonian 2-group with every subgroup of $T(G)$ normal in G.*

Conversely if G is FC-nilpotent with $T(G)$ satisfying the above conditions and $G/T(G)$ is right ordered, then $U(\mathbb{Z}G)$ is FC-nilpotent.

Theorem 2. *Let G be a finitely generated group and K a field of characteristic zero. If $U(KG)$ is FC-nilpotent, then G is FC-nilpotent and $T(G)$ is an abelian subgroup of G with every idempotent of $KT(G)$ central in KG.*

Conversely if G satisfies the above conditions and $G/T(G)$ is right ordered, then $U(KG)$ is FC-nilpotent.

Proof of Theorem 1: Let $U(\mathbb{Z}G)$ be FC-nilpotent, $t \in T(G)$ and $x \in G$. Then $H = \langle x, t\rangle$ is finitely generated FC-nilpotent and so H is nilpotent by finite [**1**]. Also $U(\mathbb{Z}H)$ cannot have free noncyclic subgroups because it is FC-nilpotent. Thus, by [**3**], $\langle t\rangle$ is a normal subgroup of H. Hence $T(G)$ is either an abelian or a Hamiltonian group with every subgroup of $T(G)$ normal in G.

If $T(G)$ is nonabelian and has an element of odd order, then again by [**3**], $U(\mathbb{Z}T(G))$ will have a non cyclic free subgroup. Thus $T(G)$ is either an abelian or a Hamiltonian 2-group.

Conversely, by [**3**, Theorem 2], $U(\mathbb{Z}G) = U(\mathbb{Z}T(G))G$. If $T(G)$ is a Hamiltonian 2-group, then $U(\mathbb{Z}G) = \pm G$ [**5**, II.2.5] and $U(\mathbb{Z}G)$ is FC-nilpotent.

If $T(G)$ is abelian, $\alpha \in U(\mathbb{Z}T(G))$ and $\gamma = \beta x$, where $\beta \in U(\mathbb{Z}T(G))$, $x \in G$, a unit of $\mathbb{Z}G$, then

$$\gamma^{-1}\alpha\gamma = x^{-1}\beta^{-1}\alpha\beta x = x^{-1}\alpha x.$$

Now supp $(\alpha) \subseteq T(G)$ and $\langle t \rangle$ is normal in G for every $t \in T(G)$, implies that $\alpha \in \Delta(U(\mathbb{Z}(G))$. Hence $U(\mathbb{Z}T(G)) \subseteq \Delta(U(\mathbb{Z}G))$ and $T(G) \subseteq \Delta(U(\mathbb{Z}G)) \cap G = S(G)$.

Also, if $g \in S(G)$ and $\gamma = \beta x$; $\beta \in U(\mathbb{Z}T(G))$, $x \in G$, then

$$g^{\beta x} = g^x(g,\beta)^x \in U(\mathbb{Z}T(G))S(G).$$

Thus $U(\mathbb{Z}T(G))S(G)$ is a normal subgroup of $U(ZG)$. Now

$$U(\mathbb{Z}G)/U(\mathbb{Z}T(G))S(G) = U(\mathbb{Z}T(G))G/U(\mathbb{Z}T(G))S(G) \cong G/S(G)$$

Further, $G/S(G)$ is FC-nilpotent and $S(G) \subseteq \Delta(G)$. So $U(\mathbb{Z}G)/U(\mathbb{Z}T(G))S(G)$ is FC-nilpotent. But, $U(\mathbb{Z}T(G))S(G) \subseteq \Delta(U(\mathbb{Z}G))$. Hence, $U(\mathbb{Z}G)$ is FC-nilpotent. ∎

Proof of Theorem 2: If $U(KG)$ is FC-nilpotent, then $U(\mathbb{Z}G)$ is also FC-nilpotent. Thus by Theorem 1, G is FC-nilpotent and $T(G)$ is a subgroup which is either abelian or a Hamiltonian 2-group with every subgroup of $T(G)$ normal in G.

If $T(G)$ is non abelian, then $K_8 \subseteq T(G)$ and thus

$$\mathbb{Q}K_8 \cong \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q} \oplus S,$$

where S is a Quaternion algebra over rationals. By [**2**], $U(S)$ has a free non cyclic subgroup and so $U(\mathbb{Q}K_8)$ is not FC-nilpotent. Thus $T(G)$ is abelian with every subgroup of $T(G)$ normal in G.

Now as G is finitely generated FC-nilpotent, it satisfies maximal condition on subgroups and thus $T(G)$ is finite abelian. By [**5**, VI.3.12] every idempotent of $KT(G)$ is central in KG as $GL_n(K)$, $n > 1$ has a free non cyclic subgroup.

Conversely, if G is finitely generated FC-nilpotent, then $T(G)$ is finite. Since $T(G)$ is finite abelian therefore $KT(G) = \oplus \sum_{i=1}^{r} F_i$, a direct sum of fields.

Further, as every idempotent of $KT(G)$ is central in KG, so $KG = KT(G) * G/T(G) = \oplus \sum_{i=1}^{r} F_i * G/T(G)$.

Thus $U(KG) = Dr_{i=1}^{r} U(F_i * G/T(G))$. Now $G/T(G)$ is right ordered, by [**5**, VI.1.6] $U(F_i * G/T(G))$ has only trivial units. Thus

$$U(F_i * G/T(G))/U(F_i) = U(F_i)(G/T(G))/U(F_i) \cong G/T(G).$$

Hence to prove that $U(F_i * G/T(G))$ is FC-nilpotent, it is sufficient to prove that $U(F_i) \subseteq \Delta((U(F_i * G/T(G)))$.

Let $\alpha \in U(F_i)$ and $\beta x \in U(F_i * G/T(G))$, where $\beta \in U(F_i)$ and x is an element of a transversal of $T(G)$ in G. Then $(\beta x)^{-1}\alpha(\beta x) = x^{-1}\beta^{-1}\alpha\beta x = x^{-1}\alpha x$. Now as $\alpha \in U(F_i) \subseteq U(KT(G))$ and every subgroup of $T(G)$ is normal in G. So α has finitely many conjugates in $U(F_i * G/T(G))$. Thus $U(F_i) \subseteq \Delta(U(F_i * G/T(G)))$. This further gives that $U(KG) = Dr_{i=1}^{r} U(F_i * G/T(G))$ is FC-nilpotent. ■

Remark. If $G = \langle x, y | x^{-1}y^2x = y^{-2}, y^{-1}x^2y = x^{-2}\rangle$, then by [4, p. 606] G is torsion free but not right ordered. However, G is FC-nilpotent because it is abelian by finite. Hence G is torsion free FC-nilpotent group which is not right ordered.

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