Publicacions Matemàtiques, Vol 35 (1991), 363–374.

# ON UNIVERSAL COMPOSITIONS OF MAPS

ZVONKO ČERIN*

*Abstract*

In this paper we shall introduce notions of $\mathcal{F}$-universality and $\mathcal{F}$-e-universality for maps between compact Hausdorff spaces and explore the behaviour of these properties under the operation of composition of maps. We consider both the quest for conditions on maps $f$ and $g$ which would imply that their composition $g \circ f$ is either $\mathcal{F}$-universal or $\mathcal{F}$-e-universal and the quest for consequences on $f$ and $g$ when the composition $g \circ f$ is either $\mathcal{F}$-universal or $\mathcal{F}$-e-universal. In our approach $\mathcal{F}$ is an arbitrary class of maps. For a special choice of $\mathcal{F}$, the notion of $\mathcal{F}$-universality reduces to Holsztyński's notion of universality while $\mathcal{F}$-e-universality reduces to Sanjurjo's notion of proximate universality.

Throughout the paper, unless stated otherwise, by a space we mean a compact Hausdorff space and by a map we mean a continuous function between spaces. We shall always consider maps $f : X \to Y$ and $g : Y \to Z$ and their composition $g \circ f : X \to Z$. The letters $M$, $N$, and $P$ will be reserved for spaces containing $X$, $Y$, and $Z$ as closed subsets, respectively.

We shall use $\mathcal{F}$ and $\mathcal{G}$ to denote arbitrary classes of maps. The fact that a map $f$ belongs to $\mathcal{F}$ will be expressed by saying that $f$ is an $\mathcal{F}$-*map*. Let $\mathcal{F}_A^Z$ denote the class of all $\mathcal{F}$-maps $a : A \to B$ with $B$ contained in $Z$. Let $\mathcal{A}$ be the class of all maps.

For maps $a : X \to Y$ and $b : X \to Y$ between spaces $X$ and $Y$ and an open cover $\sigma$ of $Y$ we let $a \approx b$, $a \overset{\sigma}{\approx} b$, and $a \overset{\sigma}{=} b$ mean that $a(x) = b(x)$ for some $x \in X$, $a(x) \overset{\sigma}{=} b(x)$ (i. e., that some member of $\sigma$ contains both $a(x)$ and $b(x)$) for some $x \in X$, and $a(x) \overset{\sigma}{=} b(x)$ for every $x \in X$, respectively.

A map $f : X \to Y$ is $\mathcal{F}$-*universal* provided $f \approx a$ for every $\mathcal{F}_X^Y$-map $a$. Observe that a map is $\mathcal{A}$-universal iff it is universal in the sense of Holsztyński [**5**].

In order to define $\mathcal{F}$-e-universal maps we shall use Borsuk's method from [**2**]. In this approach we do not require exact coincidence and we allow that maps slip outside of compacta into neighborhoods in an ambient space.

*Dedicated to Chelo

Let $\tilde{X}$ denote the collection of all open covers of a space $X$. Let $n(A, X)$ and $kn(A, X)$ stand for the collection of all open and of all compact neighborhoods in $X$ of a subset $A$ in $X$, respectively. Let $\tilde{X}_A$ denote all collections of open subsets of $X$ which cover $A$ and let $i_{A,X}$ be the inclusion of $A$ into $X$.

A map $f : X \to Y$ is $\mathcal{F}$-e-*universal in* $N$ provided for every $\sigma \in \tilde{N}_Y$ there is a $U \in n(Y, N)$ such that $f \overset{\sigma}{\approx} a$ for every $a \in \mathcal{F}_X^U$. Observe that a map of metric compacta is $\mathcal{A}$-e-universal in the Hilbert cube $Q$ iff it is proximately universal in the sense of Sanjurjo [**10**, Theorem 5].

It was already noticed by Holsztyński in [**7**] that the composition of universal maps need not be universal. Some partial results in the identification of sufficient conditions on maps $f : X \to Y$ and $g : Y \to Z$ which imply that the composition $g \circ f : X \to Z$ is (proximately) universal are included in [**6**], [**7**], [**1**], and [**10**].

In the present paper we establish theorems which give answers to the above problem for $\mathcal{F}$-universality and $\mathcal{F}$-e-universality in $N$ and also to a question to find conditions which imply that either $f$ or $g$ is $\mathcal{F}$-universal and $\mathcal{F}$-e-universal in $N$ and in $P$ when the composition $g \circ f$ is $\mathcal{F}$-universal and $\mathcal{F}$-e-universal in $P$, respectively.

Our first result shows that in the definition of universalities with respect to a class $\mathcal{F}$ we can always pass on to a larger class.

A map $f$ is an $[\mathcal{F}]$-*map* provided for every $\sigma \in \tilde{Y}$ there is an $a \in \mathcal{F}_X^Y$ with $f \overset{\sigma}{=} a$. It is an $[\mathcal{F}, N)$-*map* provided for every $\sigma \in \tilde{N}_Y$ and every $V \in n(Y, N)$ there is an $a \in \mathcal{F}_X^V$ with $f \overset{\sigma}{=} a$. Similarly, $f$ is an $(\mathcal{F}, M, N)$-*map* provided for every $\sigma \in \tilde{N}_Y$ and every $V \in n(Y, N)$ there is a $U \in kn(X, M)$ and an $a \in \mathcal{F}_U^V$ with $f \overset{\sigma}{=} a|_X$.

**Theorem 1.**

(a) *A map* $f$ *is* $\mathcal{F}$*-universal iff it is* $[\mathcal{F}]$*-universal.*
(b) *A map* $f$ *is* $\mathcal{F}$*-e-universal in* $N$ *iff it is* $[\mathcal{F}, N)$*–e-universal in* $N$.

*Proof:* (b). Let $\xi \in \tilde{N}_Y$. Let $\xi^*$ and $\xi^{**}$ denote the set of all members $\sigma$ of $\tilde{N}_Y$ such that the star of $\sigma$ and the double-star of $\sigma$ refines $\xi$, respectively. Let $\sigma \in \xi^*$. By assumption, there is a $V \in n(Y, N)$ such that $f \overset{\sigma}{\approx} h$ for every $h \in \mathcal{F}_X^V$.

Consider a $k \in [\mathcal{F}, N)_X^V$. Choose an $h \in \mathcal{F}_X^V$ with $h \overset{\sigma}{=} k$. Since $f \overset{\sigma}{\approx} h$, we get $f \overset{\xi}{\approx} k$. ■

The following theorem resembles Theorem (3.3) in [7] and Theorem 9 in [**10**].

For a map $f : X \to Y$ and a space $Z$, let $\sigma_Z(f) : X \times Z \to Z \times Y$ be a map defined by $\sigma_Z(f)(x, z) = (z, f(x))$ for every $(x, z) \in X \times Z$. Let $\sigma_Z(\mathcal{F}) = \{\sigma_Z(f) : f \in \mathcal{F}\}$.

**Theorem 2.**

(a) *The composition* $g \circ f$ *is* $\mathcal{F}$*-universal iff the product* $f \times g$ *is* $\sigma_Y(\mathcal{F})$*-universal.*

(b) *The composition* $g \circ f$ *is* $\mathcal{F}$*-e-universal in* $P$ *iff the product* $f \times g$ *is* $\sigma_Y(\mathcal{F})$*-e-universal in* $Y \times P$.

*Proof:* (b). Let $g \circ f$ be $\mathcal{F}$-e-universal in $P$. Let $\xi \in \widetilde{Y \times P}_{Y \times Z}$. Choose an $\eta \in \tilde{P}_Z$ such that $b \overset{\eta}{=} c$ in $P$ implies $(a,b) \overset{\xi}{=} (a,c)$ in $Y \times P$ for every $a \in Y$. By assumption, there is a $V \in n(Z,P)$ with $h \overset{\eta}{\approx} g \circ f$ for every $h \in \mathcal{F}_X^V$. Let $U = Y \times V \in n(Y \times Z, Y \times P)$.

Consider a $k \in \sigma_Y(\mathcal{F})_{X \times Y}^U$. Pick an $h \in \mathcal{F}_X^V$ such that $k = \sigma_Y(h)$. The way in which $V$ was selected implies the existence of an $x \in X$ with $h(x) \overset{\eta}{=} g \circ f(x)$. Then

$$k(x, f(x)) = \sigma_Y(h)(x, f(x)) = (f(x), h(x))$$

and $(f(x), h(x)) \overset{\xi}{=} (f(x), g \circ f(x)) = (f \times g)(x, f(x))$. Hence, $k \overset{\xi}{\approx} f \times g$.

Conversely, suppose that $f \times g$ is $\sigma_Y(\mathcal{F})$-e-universal in $Y \times P$. Let $\eta \in \tilde{P}_Z$. Let $\mu \in \eta^*$. Let $\theta = g^{-1}(\mu)$. Let $\xi = \{E \times M : E \in \theta, M \in \mu\} \in \widetilde{Y \times P}_{Y \times Z}$. By assumption, there is a $U \in n(Y \times Z, Y \times P)$ such that $k \overset{\xi}{\approx} f \times g$ for every $k \in \sigma_Y(\mathcal{F})_{X \times Y}^U$. Choose a $V \in n(Z,P)$ with $Y \times V \subset U$.

Consider an $h \in \mathcal{F}_X^V$. Since $\sigma_Y(h)$ is in $\sigma_Y(\mathcal{F})_{X \times Y}^U$, there is an $(x,y) \in X \times Y$ with $(y, h(x)) \overset{\xi}{=} (f(x), g(y))$. In other words, $y \overset{\theta}{=} f(x)$ and $h(x) \overset{\mu}{=} g(y)$. Hence, $h \overset{\eta}{\approx} g \circ f$. ∎

The first half of the above theorem clearly includes Theorem (3.3) in [**7**]. In order to see that it also generalizes Theorem 9 in [**10**], we need the following theorem.

Recall [**9**] that a space $X$ is an *approximate polyhedron* provided for every $\sigma \in \tilde{X}$ there is a polyhedron $K$ and maps $u : X \to K$ and $d : K \to X$ with $d \circ u \overset{\sigma}{=} id_X$, where $id_X$ is the identity map on $X$. One can easily see that a space is an approximate polyhedron iff it is an approximate absolute neighborhood retract (in the sense of Clapp) for the class of all compact Hausdorff spaces [**4**].

A class $\mathcal{F}$ of maps is *solid* provided $f \circ g \in \mathcal{F}$ for every $f \in \mathcal{F}$ and every map $g$ such that $f \circ g$ can be defined. Similarly, $\mathcal{F}$ is a *legal* class provided $f \circ g \in \mathcal{F}$ for every $g \in \mathcal{F}$ and every map $f$ such that $f \circ g$ can be defined. Clearly, the class $\mathcal{A}$ is both solid and legal.

**Theorem 3.** *Let* $\mathcal{F}$ *be a legal class of maps. If* $f$ *is* $\mathcal{F}$*-e-universal in some approximate polyhedron* $R$*, then* $f$ *is* $\mathcal{F}$*-e-universal in every space* $N$.

*Proof:* Let $\sigma \in \tilde{N}_Y$. Let $T$ be a space obtained by glueing $R$ and $N$ along $Y$. Since $N$ is a closed subset of $T$, there is an $\eta \in \tilde{T}_Y$ such that $\eta|_N$ refines

$\sigma$. Let $\xi \in \eta^*$. By assumption, there is a $V \in n(Y,R)$ with $f \overset{\xi}{\approx} h$ for every $h \in \mathcal{F}_X^V$. Select refinements $\pi$ of $\xi|_R$ and $\rho$ of $\xi$ such that the star $st(Y,\pi)$ of $Y$ with respect to the collection $\pi$ is a subset of $V$ and that $\rho|_R$ refines $\pi$. Let $\tau \in \rho^*$. Since $R$ is an approximate polyhedron, there is an $S \in n(R,T)$ and a map $r : S \to R$ such that $y \overset{\tau}{=} r(y)$ for every $y \in Y$. Hence, there is a $U \in n(Y,N)$ such that $U \subset S \bigcap st(Y,\tau)$ and $u \overset{\tau}{=} r(u)$ for every $u \in U$.

Consider an $h \in \mathcal{F}_X^U$. Let $k = r \circ h$. Since $\mathcal{F}$ is legal, $k \in \mathcal{F}$. But, $k(X) \subset st(Y,\rho|_R) \subset st(Y,\pi) \subset V$. Hence, $k \overset{\xi}{\approx} f$. It follows that $f \overset{\eta}{\approx} h$ because $k \overset{\tau}{\approx} h$. Since $f(X) \bigcup h(X) \subset N$, we get $f \overset{\sigma}{\approx} h$. ■

The next three results are related to Proposition (3.9) in [1] and the first half of Theorem 8 in [10].

A map $g$ is an $X[\mathcal{F},\mathcal{G}]$-e-*retraction* provided for every $\sigma \in \tilde{Z}$ and every $a \in \mathcal{F}_X^Z$ there is a $b \in \mathcal{G}_X^Y$ with $a \overset{\sigma}{=} g \circ b$.

A map $g$ is an $X(\mathcal{F},\mathcal{G}]$-e-*retraction in* $P$ provided for every $\sigma \in \tilde{P}_Z$ there is a $W \in n(Z,P)$ such that for every $a \in \mathcal{F}_X^W$ there is a $b \in \mathcal{G}_X^Y$ with $a \overset{\sigma}{=} g \circ b$.

A map $g$ is an $X(\mathcal{F},\mathcal{G})$-e-*retraction in* $(N,P)$ provided for every $\sigma \in \tilde{P}_Z$ there is a $U \in n(Y,N)$ and a map $G : U \to P$ such that $g \overset{\sigma}{=} G|_Y$ and for every $V \in n(Y,U)$ there is a $W \in n(Z,P)$ so that for every $a \in \mathcal{F}_X^W$ there is a $b \in \mathcal{G}_X^V$ with $a \overset{\sigma}{=} G \circ b$.

Observe that g is a $Z[\mathcal{A},\mathcal{A}]$-e-retraction iff it is ARI (approximately right invertible) [8]. Also, $g$ is a $Z(\mathcal{A},\mathcal{A})$-e-retraction in $(Q,Q)$ iff it is a weakly refinable map (see [10, Theorem 6]).

**Theorem 4.**

(a) *If $f$ is $\mathcal{G}$-universal and $g$ is an $X[\mathcal{F},\mathcal{G}]$-e-retraction, then $g \circ f$ is $\mathcal{F}$-universal.*
(b) *If $f$ is $\mathcal{G}$-universal and $g$ is an $X(\mathcal{F},\mathcal{G}]$-e-retraction in $P$, then $g \circ f$ is $\mathcal{F}$-e-universal in $P$.*
(c) *If $f$ is $\mathcal{G}$-e-universal in $N$ and $g$ is an $X(\mathcal{F},\mathcal{G})$-e-retraction in $(N,P)$, then $g \circ f$ is $\mathcal{F}$-e-universal in $P$.*

*Proof:* (c). Let $\sigma \in \tilde{P}_Z$. Let $\eta \in \sigma^*$. Since $g$ is an $X(\mathcal{F},\mathcal{G})$-e-retraction in $(N,P)$, there is a $U \in n(Y,N)$ and a map $G : U \to P$ such that $g \overset{\eta}{=} G|_Y$ and for every $V \in n(Y,U)$ there is a $W \in n(Z,P)$ so that for every $a \in \mathcal{F}_X^W$ there is a $b \in \mathcal{G}_X^V$ with $a \overset{\eta}{=} G \circ b$. Let $\theta = G^{-1}(\eta) \in \tilde{N}_Y$.

Since $f$ is $\mathcal{G}$-e-universal in $N$, there is an $S \in n(Y,N)$ with the property that $h \overset{\theta}{\approx} f$ for every $h \in \mathcal{G}_X^S$. Let $V = S \bigcap U \in n(Y,U)$. By assumption, there is a $W \in n(Z,P)$ such that for every $a \in \mathcal{F}_X^W$ there is a $b \in \mathcal{G}_X^V$ with $a \overset{\eta}{=} G \circ b$.

Consider a $k \in \mathcal{F}_X^W$. Choose an $h \in \mathcal{G}_X^V$ with $k \overset{\eta}{=} G \circ h$. The way in which $V$ was selected implies $h \overset{\theta}{=} f$. Hence, $G \circ h \overset{\eta}{\approx} G \circ f \overset{\eta}{=} g \circ f$. Finally, we get

$k \overset{\sigma}{\approx} g \circ f$. ■

A map $g$ is $ARI[\mathcal{F}]$ provided for every $\sigma \in \tilde{Z}$ there is an $s \in \mathcal{F}_Z^Y$ with $g \circ s \overset{\sigma}{=} id_Z$.

A map $g$ is $ARI[\mathcal{F})$ *in* $(N, P)$ provided for every $\sigma \in \tilde{P}_Z$ there is a $U \in n(Y, N)$ and a map $G : U \to P$ such that $g \overset{\sigma}{=} G|_Y$ and for every $V \in n(Y, U)$ there is an $s \in \mathcal{F}_Z^V$ with $G \circ s \overset{\sigma}{=} i_{Z,P}$.

A map $g$ is $ARI(\mathcal{F})$ *in* $(N, P)$ provided for every $\sigma \in \tilde{P}_Z$ there is a $U \in n(Y, N)$, a map $G : U \to P$, and a $W \in kn(Z, P)$ such that $g \overset{\sigma}{=} G|_Y$ and for every $V \in n(Y, U)$ there is an $s \in \mathcal{F}_W^V$ with $G \circ s \overset{\sigma}{=} i_{W,P}$.

Clearly, $ARI[\mathcal{A})$ and $ARI(\mathcal{A})$ in $(Q, Q)$ maps agree with weakly refinable maps, while $ARI[\mathcal{A}]$ maps agree with ARI maps.

**Theorem 5.** *Let $\mathcal{F}$ be a solid class of maps.*

(a) *If $f$ is $\mathcal{F}$-universal and $g$ is $ARI[\mathcal{F}]$, then $g \circ f$ is $\mathcal{A}$-universal.*
(b) *If $f$ is $\mathcal{F}$-e-universal in $N$ and $g$ is $ARI[\mathcal{F})$ in $(N, P)$, then $g \circ f$ is $\mathcal{A}$-universal.*
(c) *If $f$ is $\mathcal{F}$-e-universal in $N$ and $g$ is $ARI(\mathcal{F})$ in $(N, P)$, then $g \circ f$ is $\mathcal{A}$-e-universal in $P$.*

*Proof:* (c). Let $\sigma \in \tilde{P}_Z$. Let $\xi \in \sigma^*$. Since $g$ is $ARI(\mathcal{F})$ in $(N, P)$, there is a $V \in n(Y, N)$ and a map $G : V \to P$ such that $g \overset{\xi}{=} G|_Y$ and for every $W \in n(Y, V)$ there is a $U \in kn(Z, P)$ and an $s \in \mathcal{F}_U^W$ with $G \circ s \overset{\xi}{=} i_{U,P}$.

Let $\theta = G^{-1}(\xi)$. Select a $W \in n(Y, V)$ such that $f \overset{\theta}{\approx} h$ for every $h \in \mathcal{F}_X^W$. By assumption, there is a $U \in kn(Z, P)$ and an $s \in \mathcal{F}_U^W$ with $G \circ s \overset{\xi}{=} i_{U,P}$.

Let $k : X \to intU$ be a map. Observe that $s \circ k \in \mathcal{F}_X^W$ because $\mathcal{F}$ is a solid class of maps. Hence, $f \overset{\theta}{\approx} s \circ k$, $g \circ f \overset{\xi}{=} G \circ f \overset{\xi}{\approx} G \circ s \circ k \overset{\xi}{=} k$, and $k \overset{\sigma}{\approx} g \circ f$. ■

In the statement of the next theorem we shall need the following notions. They could be regarded as dual to the notions of nearly extendable maps [**2**] (or, equivalently, e-movable maps [**3**]).

A map $f$ is $[\mathcal{F}, \mathcal{G}]$-e-*liftable in* $Z$ provided for every $\sigma \in \tilde{Z}$ and every $a \in \mathcal{F}_Y^Z$ there is a $b \in \mathcal{G}_X^Z$ with $b \overset{\sigma}{=} a \circ f$.

A map $f$ is $(\mathcal{F}, \mathcal{G}]$-e-*liftable in* $(P, Z)$ provided for every $\sigma \in \tilde{P}_Z$ there is a $W \in n(Z, P)$ such that for every $a \in \mathcal{F}_Y^W$ there is a $b \in \mathcal{G}_X^Z$ with $b \overset{\sigma}{=} a \circ f$.

A map $f$ is $[\mathcal{F}, \mathcal{G})$-e-*liftable in* $(P, Z)$ provided for every $\sigma \in \tilde{P}_Z$, every $U \in n(Z, P)$, and every $a \in \mathcal{F}_Y^Z$ there is a $b \in \mathcal{G}_X^U$ with $b \overset{\sigma}{=} a \circ f$.

A map $f$ is $(\mathcal{F}, \mathcal{G})$-e-*liftable in* $(P, Z)$ provided for every $\sigma \in \tilde{P}_Z$ and every $V \in n(Z, P)$ there is a $W \in n(Z, P)$ such that for every $a \in \mathcal{F}_Y^W$ there is a $b \in \mathcal{G}_X^V$ with $b \overset{\sigma}{=} a \circ f$.

Let $\lambda_Z[\mathcal{F},\mathcal{G}]$ denote the class of all $[\mathcal{F},\mathcal{G}]$-e-liftable in $Z$ maps. The notations $\lambda_{(P,Z)}(\mathcal{F},\mathcal{G}]$, $\lambda_{(P,Z)}[\mathcal{F},\mathcal{G})$, and $\lambda_{(P,Z)}(\mathcal{F},\mathcal{G})$ have analogous meanings.

A map $g$ is *extendable in* $(N,P)$ provided for every $\sigma \in \tilde{P}_Z$ there is a $U \in n(Y,N)$ and a map $G : U \to P$ such that $g \overset{\sigma}{=} G|_Y$.

**Theorem 6.** *Let $\mathcal{F}$ be a solid class of maps.*

(a) *If $f$ is $\mathcal{G}$-universal and $g$ is $ARI[\mathcal{F}]$, then $g \circ f$ is $\lambda_Y[\mathcal{F},\mathcal{G}]$-universal.*
(b) *If $f$ is $\mathcal{G}$-e-universal in $N$ and $g$ is both $ARI[\mathcal{F}]$ and extendable in $(N,P)$, then $g \circ f$ is $\lambda_{(N,Y)}[\mathcal{F},\mathcal{G})$-universal.*
(c) *If $f$ is $\mathcal{G}$-universal and $g$ is $ARI[\mathcal{F})$ in $(N,P)$, then $g \circ f$ is $\lambda_{(N,Y)}(\mathcal{F},\mathcal{G}]$-universal.*
(d) *If $f$ is $\mathcal{G}$-e-universal in $N$ and $g$ is $ARI[\mathcal{F})$ in $(N,P)$, then $g \circ f$ is $\lambda_{(N,Y)}(\mathcal{F},\mathcal{G})$-universal.*
(e) *If $f$ is $\mathcal{G}$-e-universal in $N$ and $g$ is $ARI(\mathcal{F})$ in $(N,P)$, then $g \circ f$ is $\lambda_{(N,Y)}(\mathcal{F},\mathcal{G})$-e-universal in $P$.*

*Proof:* (*d*). Let a $\sigma \in \tilde{Z}$ and a $\lambda_{(N,Y)}(\mathcal{F},\mathcal{G})$-map $k : X \to A$ with $A$ contained in $Z$ be given. Pick an $\eta \in \tilde{P}$ such that the restriction $\eta|_Z$ of $\eta$ to $Z$ refines $\sigma$. Let $\xi \in \eta^{**}$. Since $g$ is $ARI[\mathcal{F})$ in $(N,P)$, there is a $U \in n(Y,N)$ and a map $G : U \to P$ such that $g \overset{\xi}{=} G|_Y$ and for every $V \in n(Y,U)$ there is an $s \in \mathcal{F}_Z^V$ with $G \circ s \overset{\xi}{=} i_{Z,P}$. Let $\theta = G^{-1}(\xi) \in \tilde{N}_Y$.

By assumption, there is a $V \in n(Y,U)$ such that $h \overset{\theta}{\approx} f$ for every $h \in \mathcal{G}_X^V$. Since $k$ is $(\mathcal{F},\mathcal{G})$-e-liftable in $(N,P)$, there is a $W \in n(Y,U)$ with the property that that for every $a \in \mathcal{F}_A^W$ there is a $b \in \mathcal{G}_X^V$ with $b \overset{\theta}{=} a \circ k$. By assumption on $g$, there is an $s \in \mathcal{F}_Z^W$ with $G \circ s \overset{\xi}{=} i_{Z,P}$. Choose a $t \in \mathcal{G}_X^V$ with $t \overset{\theta}{=} (s|_A) \circ k$.

The way in which $V$ was selected implies $f \overset{\theta}{\approx} t$. Combining the above relations, we get $g \circ f \overset{\xi}{=} G \circ f \overset{\xi}{\approx} G \circ t \overset{\xi}{=} G \circ (s|_A) \circ k \overset{\xi}{=} i_{A,P} \circ k = k$. Hence, $k \overset{\sigma}{\approx} g \circ f$.

(*e*). Let $\sigma \in \tilde{P}_Z$. Let $\eta \in \sigma^{**}$. Since $g$ is $ARI(\mathcal{F})$ in $(N,P)$, there is a $U \in n(Y,N)$, a map $G : U \to P$, and a $W \in kn(Z,P)$ such that $g \overset{\eta}{=} G|_Y$ and for every $V \in n(Y,U)$ there is an $s \in \mathcal{F}_W^V$ with $G \circ s \overset{\eta}{=} i_{W,P}$. Let $\theta = G^{-1}(\eta) \in \tilde{N}_Y$.

Let $k : X \to A$ be a $\lambda_{(N,Y)}(\mathcal{F},\mathcal{G})$-map and assume that $A$ is a subset of $W$. Since $f$ is $\mathcal{G}$-e-universal in $N$, there is a $V \in n(Y,U)$ such that $h \overset{\theta}{\approx} f$ for every $h \in \mathcal{G}_X^V$. Since $k$ is $(\mathcal{F},\mathcal{G})$-e-liftable in $(N,P)$, there is an $R \in n(Y,U)$ such that for every $a \in \mathcal{F}_A^R$ there is a $b \in \mathcal{G}_X^V$ with $b \overset{\eta}{=} a \circ k$. Select an $s \in \mathcal{F}_W^R$ and a $t \in \mathcal{G}_X^V$ with $G \circ s \overset{\eta}{=} i_{W,P}$ and $t \overset{\theta}{=} (s|_A) \circ k$. Observe that $t \overset{\theta}{\approx} f$. Hence, $k = i_{W,P} \circ k \overset{\eta}{=} G \circ (s|_A) \circ k$ and $G \circ (s|_A) \circ k \overset{\eta}{=} G \circ t \overset{\eta}{\approx} G \circ f \overset{\eta}{=} g \circ f$. It follows that $k \overset{\sigma}{\approx} g \circ f$. ■

A map $f$ is an $[\mathcal{F}, \mathcal{G}]$-e-*progression in* $Z$ provided for every $\sigma \in \tilde{Z}$ and every $a \in \mathcal{F}_X^Z$ there is a $b \in \mathcal{G}_Y^Z$ with $a \overset{\sigma}{=} b \circ f$.

A map $f$ is an $(\mathcal{F}, \mathcal{G}]$-e-*progression in* $(P, Z)$ provided for every $\sigma \in \tilde{P}_Z$ there is a $V \in n(Z, P)$ such that for every $a \in \mathcal{F}_X^V$ there is a $b \in \mathcal{G}_Y^Z$ with $a \overset{\sigma}{=} b \circ f$.

A map $f$ is an $(\mathcal{F}, \mathcal{G})$-e-*progression in* $(P, Z)$ provided for every $\sigma \in \tilde{P}_Z$ and every $U \in n(Z, P)$ there is a $V \in n(Z, P)$ such that for every $a \in \mathcal{F}_X^V$ there is a $b \in \mathcal{G}_Y^U$ with $a \overset{\sigma}{=} b \circ f$.

Observe that an ALI (approximately left invertible) map $f$ (i. e., a map such that for every $\sigma \in \tilde{X}$ there is a map $s : Y \to X$ with $s \circ f \overset{\sigma}{=} id_X$) is an $[\mathcal{A}, \mathcal{A}]$-e-progression in every space $Z$. Similarly, an ALI in $M$ map $f$ (i. e., a map such that for every $\sigma \in \tilde{M}_X$ and every $U \in n(X, M)$ there is a map $s : Y \to U$ with $s \circ f \overset{\sigma}{=} i_{X,U}$) will be an $(\mathcal{A}, \mathcal{A})$-e-progression in every pair $(P, Z)$ where $P$ is an absolute neighborhhod retract. It will be an $(\mathcal{A}, \mathcal{A}]$-e-progression provided, in addition, $Z$ is an approximate polyhedron.

**Theorem 7.**

(a) *If $f$ is a surjective $[\mathcal{F}, \mathcal{G}]$-e-progression and $g$ is $\mathcal{G}$-universal, then $g \circ f$ is $\mathcal{F}$-universal.*

(b) *If $f$ is a surjective $(\mathcal{F}, \mathcal{G}]$-e-progression in $(P, Z)$ and $g$ is $\mathcal{G}$-universal, then $g \circ f$ is $\mathcal{F}$-e-universal in $P$.*

(c) *If $f$ is a surjective $(\mathcal{F}, \mathcal{G})$-e-progression in $(P, Z)$ and $g$ is $\mathcal{G}$-e-universal in $P$, then $g \circ f$ is $\mathcal{F}$-e-universal in $P$.*

*Proof:* (c). Let $\sigma \in \tilde{P}_Z$. Let $\eta \in \sigma^*$. Since $g$ is $\mathcal{G}$-e-universal in $P$, there is a $U \in n(Z, P)$ such that $g \overset{\eta}{\approx} h$ for every $h \in \mathcal{G}_Y^U$. Choose a $V \in n(Z, P)$ such that for every $a \in \mathcal{F}_X^V$ there is a $b \in \mathcal{G}_Y^U$ with $a \overset{\eta}{=} b \circ f$.

Let $k \in \mathcal{F}_X^V$. Select an $h \in \mathcal{G}_Y^U$ with $k \overset{\eta}{=} h \circ f$. By assumption, $g \overset{\eta}{\approx} h$. But, since $f$ is onto, the last relation implies $g \circ f \overset{\eta}{\approx} h \circ f$. Hence, $k \overset{\sigma}{\approx} g \circ f$. ■

The (b) part of the next theorem improves the (b) part of Theorem 8 in [10]. We replace Sanjurjo's assumption that a map $r$ is refinable with a weaker assumption (for example, that it is weakly refinable).

**Theorem 8.** *Let $\mathcal{F}$ be a legal class of maps.*

(a) *If $f$ is $ARI[\mathcal{F}]$ and $g$ is $\mathcal{F}$-universal, then $g \circ f$ is $\mathcal{A}$-universal.*

(b) *If $f$ is $ARI[\mathcal{F})$ in $(M, N)$, $g$ is $\mathcal{F}$-e-universal in $P$, and $P$ is an approximate polyhedron, then $g \circ f$ is $\mathcal{A}$-e-universal in $P$.*

*Proof:* (b). Let $\sigma \in \tilde{P}_Z$. Let $\xi \in \sigma^{**}$ and $\eta \in \xi^*$. Since $g$ is $\mathcal{F}$-e-universal in $P$, there is a $U \in n(Z, P)$ such that $g \overset{\eta}{\approx} h$ for every $h \in \mathcal{F}_Y^U$. We claim that $k \overset{\sigma}{\approx} g \circ f$ for every map $k : X \to U$.

Indeed, let $k : X \to U$ be a map. Since $P$ is an approximate polyhedron, there is a $W \in n(X, M)$, a $V \in n(Y, N)$, and maps $K : W \to U$ and $G : V \to P$ such that $k \overset{\xi}{=} K|_X$ and $g \overset{\xi}{=} G|_Y$. Let $\theta = G^{-1}(\eta)$. Since $f$ is $ARI[\mathcal{F})$ in $(M, N)$, there is an $R \in n(X, W)$ and a map $F : R \to V$ such that $f \overset{\theta}{=} F|_X$ and for every $T \in n(X, R)$ there is an $s \in \mathcal{F}_Y^T$ with $F \circ s \overset{\theta}{=} i_{Y,N}$.

Let $T \in n(X, R)$. Choose an $s \in \mathcal{F}_Y^T$ with $F \circ s \overset{\theta}{=} i_{Y,N}$. Since the class $\mathcal{F}$ is legal, the composition $K \circ s : Y \to U$ is from $\mathcal{F}_Y^U$. By assumption, $g \overset{\eta}{\approx} K \circ s$. On the other hand, from $F \circ s \overset{\theta}{=} i_{Y,N}$ we get $g \overset{\eta}{=} G \circ F \circ s$. Hence, we have $K \circ s \overset{\xi}{\approx} G \circ F \circ s$ and $K|_T \overset{\xi}{\approx} G \circ F|_T$. Since $T$ was arbitrary, $K|_X \overset{\xi}{\approx} G \circ F|_X$. But, $G \circ F|_X \overset{\eta}{=} G \circ f \overset{\xi}{=} g \circ f$, and $k \overset{\xi}{=} K|_X$ so that $k \overset{\sigma}{\approx} g \circ f$. ■

A map $f$ is $Z[\mathcal{F}, \mathcal{G}]$-e-*movable* provided for every $\sigma \in \tilde{Y}$ and every $a \in \mathcal{F}_Z^X$ there is a $b \in \mathcal{G}_Z^Y$ with $b \overset{\sigma}{=} f \circ a$.

A map $f$ is $Z(\mathcal{F}, \mathcal{G}]$-e-*movable* in $(M, N)$ provided for every $\sigma \in \tilde{N}_Y$ there is a $U \in n(X, M)$ and a map $F : U \to N$ such that $f \overset{\sigma}{=} F|_X$ and for every $a \in \mathcal{F}_Z^U$ there is a $b \in \mathcal{G}_Z^Y$ with $b \overset{\sigma}{=} F \circ a$.

A map $f$ is $Z(\mathcal{F}, \mathcal{G})$-e-*movable* in $(M, N)$ provided for every $\sigma \in \tilde{N}_Y$ there is a $U \in n(X, M)$ and a map $F : U \to N$ such that $f \overset{\sigma}{=} F|_X$ and for every $a \in \mathcal{F}_Z^U$ and every $V \in n(Y, N)$ there is a $b \in \mathcal{G}_Z^V$ with $b \overset{\sigma}{=} F \circ a$.

Observe that a map $f$ is $Z(\mathcal{A}, \mathcal{A}]$-e-movable in $(Q, Q)$ iff $f$ is internally $Z$-e-movable [**3**]. Similarly, a map $f$ is $Z(\mathcal{A}, \mathcal{A})$-e-movable in $(Q, Q)$ iff $f$ is $Z$-e-movable [**3**].

Let $\mu Z[\mathcal{F}, \mathcal{G}]$ denote the class of all $Z[\mathcal{F}, \mathcal{G}]$-e-movable maps. The following notations $\mu_{(M,N)} Z(\mathcal{F}, \mathcal{G}]$ and $\mu_{(M,N)} Z(\mathcal{F}, \mathcal{G})$ have analogous meanings.

**Theorem 9.**

(a) *If $f$ is $ARI[\mathcal{F}]$ and $g$ is $\mathcal{G}$-universal, then $g \circ f$ is $\mu Y[\mathcal{F}, \mathcal{G}]$-universal.*
(b) *If $f$ is $ARI[\mathcal{F})$ in $(M, N)$ and $g$ is both $\mathcal{G}$-universal and extendable in $(N, P)$, then $g \circ f$ is $\mu_{(M,P)} Y(\mathcal{F}, \mathcal{G}]$-universal.*
(c) *If $f$ is $ARI[\mathcal{F})$ in $(M, N)$ and $g$ is both $\mathcal{G}$-e-universal and extendable in $(N, P)$, then $g \circ f$ is $\mu_{(M,P)} Y(\mathcal{F}, \mathcal{G})$-e-universal.*

**Proof:** (*b*). Let a $\sigma \in \tilde{Z}$ and a $Y(\mathcal{F}, \mathcal{G}]$-e-movable in $(M, P)$ map $k : X \to A$ with $A$ a subset of $Z$ be given. Select an $\eta \in \tilde{P}$ such that $\eta|_Z$ refines $\sigma$. Let $\xi \in \eta^*$. Let $\nu \in \xi^{**}$.

Since $k$ is $Y(\mathcal{F}, \mathcal{G}]$-e-movable in $(M, P)$, there is a $U \in n(X, M)$ and a map $K : U \to P$ such that $k \overset{\nu}{=} K|_X$ and for every $a \in \mathcal{F}_Y^U$ there a $b \in \mathcal{G}_Y^A$ with $b \overset{\nu}{=} K \circ a$.

Choose a $V \in n(Y, N)$ and a map $G : V \to P$ with $g \overset{\xi}{=} G|_Y$. Let $\theta = G^{-1}(\nu)$. Since $f$ is $ARI[\mathcal{F})$ in $(M, N)$, there is a $W \in n(X, U)$ and a map

$F : W \to N$ such that $f \overset{\theta}{=} F|_X$ and for every $L \in n(X, W)$ there is an $s \in \mathcal{F}_Y^L$ with $F \circ s \overset{\theta}{=} i_{Y,N}$.

Let $R \in n(X, W)$. By assumption, we know there is an $s \in \mathcal{F}_Y^R$ with $F \circ s \overset{\theta}{=} i_{Y,N}$. Choose a $t \in \mathcal{G}_Y^A$ with $t \overset{\nu}{=} K \circ s$. Since $g$ is $\mathcal{G}$-universal, $t \overset{\nu}{\approx} g$. Now, $K \circ s \overset{\nu}{=} t \overset{\nu}{\approx} g$ and $G|_Y \overset{\nu}{=} G \circ F \circ s$. It follows that $K|_R \overset{\xi}{\approx} G \circ F|_R$. But, since $R$ is arbitrary, we get $K|_X \overset{\xi}{\approx} G \circ F|_X$. Hence, $k \overset{\sigma}{\approx} g \circ f$.

(c). Let $\sigma \in \tilde{P}_Z$. Let $\eta \in \sigma^*$ and $\xi \in \eta^{**}$. By asumptions on $g$, there is a $J \in n(Z, P)$, a $V \in n(Y, N)$, and a map $G : V \to P$ such that $g \overset{\xi}{\approx} h$ for every $h \in \mathcal{G}_Y^J$ and $g \overset{\xi}{=} G|_Y$. Let $\theta = G^{-1}(\xi)$.

Consider a $Y(\mathcal{F}, \mathcal{G})$-e-movable in $(M, P)$ map $k : X \to A$ and assume that $A$ is a subset of $J$. Select a $U \in n(X, M)$ and a map $K : U \to P$ such that $k \overset{\xi}{=} K|_X$ and for every $a \in \mathcal{F}_Y^U$ and every $L \in n(Z, P)$ there is a $b \in \mathcal{G}_Y^L$ with $b \overset{\xi}{=} K|_A$. Since $f$ is $ARI[\mathcal{F})$ in $(M, N)$, there is a $W \in n(X, U)$ and a map $F : W \to P$ such that $f \overset{\theta}{=} F|_X$ and for every $R \in n(X, W)$ there is an $s \in \mathcal{F}_Y^R$ and a $t \in \mathcal{G}_Y^J$ with $t \overset{\xi}{=} K \circ s$ and $F \circ s \overset{\theta}{=} i_{Y,N}$. The way in which $J$ was selected implies $t \overset{\xi}{\approx} g$. Now, we have the following chain of relations $K \circ s \overset{\xi}{=} t \overset{\xi}{\approx} g \overset{\xi}{=} G|_Y \overset{\xi}{=} G \circ F \circ s$. It follows that $K|_R \overset{\eta}{\approx} G \circ F|_R$. Since $R$ was arbitrary, we get $K|_X \overset{\eta}{\approx} G \circ F|_X$. Hence, $k \overset{\sigma}{\approx} g \circ f$. ■

We shall now establish partial converses of the above theorems. This time we assume that the composition $g \circ f$ is universal and try to get that either $g$ or $f$ is universal.

**Theorem 10.**

(a) *If $g \circ f$ is $\mathcal{F}$-universal and $g$ is both an embedding and $X[\mathcal{F}, \mathcal{G}]$-e-movable, then $f$ is $\mathcal{G}$-universal.*

(b) *If $g \circ f$ is $\mathcal{F}$-universal and $g$ is both an embedding and $X(\mathcal{F}, \mathcal{G}]$-e-movable in $(N, P)$, then $f$ is $\mathcal{G}$-e-universal in $N$.*

(c) *If $g \circ f$ is $\mathcal{F}$-e-universal in $P$ and $g$ is both an embedding and $X(\mathcal{F}, \mathcal{G})$-e-movable in $(N, P)$, then $f$ is $\mathcal{G}$-e-universal in $N$.*

*Proof:* (c). Let $\sigma \in \tilde{N}_Y$. Since $g$ is an embedding, there is a $\xi \in \tilde{P}_Z$ such that for every $U \in n(Y, N)$ and every map $G : U \to P$ with $g \overset{\xi}{=} G|_Y$ there is a $V \in n(Y, U)$ such that $G(x) \overset{\xi}{=} G(y)$ for $x, y \in V$ implies $x \overset{\sigma}{=} y$. Let $\eta \in \xi^*$. We use now the second assumption on $g$ to choose a $U \in n(Y, N)$ and a map $G : U \to P$ such that $g \overset{\eta}{=} G|_Y$ and for every $h \in \mathcal{G}_X^U$ and every $W \in n(Z, P)$ there is a $k \in \mathcal{F}_X^W$ with $k \overset{\eta}{=} G \circ h$. Since $g \circ f$ is $\mathcal{F}$-e-universal in $P$, there is a $W \in n(Z, P)$ such that $k \overset{\eta}{\approx} g \circ f$ for every $k \in \mathcal{F}_X^W$. Finally, pick a $V \in n(Y, U)$ using the way in which $\xi$ was selected.

Let $h \in \mathcal{G}_X^V$. Choose a $k \in \mathcal{F}_X^W$ with $k \overset{\eta}{=} G \circ h$. By assumption, $k \overset{\eta}{\approx} g \circ f$ and $g \circ f \overset{\eta}{=} G \circ f$. Hence, $G \circ f \overset{\xi}{\approx} G \circ h$ and $f \overset{\sigma}{\approx} h$. ■

**Theorem 11.** *Let $\mathcal{F}$ be a solid class of maps.*

(a) *If $g \circ f$ is $\mathcal{F}$-universal and $g$ is both an embedding and an $[\mathcal{F}]$-map, then $f$ is $\mathcal{A}$-universal.*
(b) *If $g \circ f$ is $\mathcal{F}$-e-universal in $P$ and $g$ is both an embedding and an $[\mathcal{F}, P)$–map, then $f$ is $\mathcal{A}$-universal.*
(c) *If $g \circ f$ is $\mathcal{F}$-e-universal in $P$ and $g$ is both an embedding and an $(\mathcal{F}, N, P)$–map, then $f$ is $\mathcal{A}$-e-universal in $N$.*

*Proof:* (c). Let $\sigma \in \tilde{N}_Y$. Select $\xi$ and $\eta$ as in the proof of the previous theorem. Since $g \circ f$ is $\mathcal{F}$-e-universal in $P$, there is a $V \in n(Z, P)$ such that $k \overset{\eta}{\approx} g \circ f$ for every $k \in \mathcal{F}_X^V$. By the second assumption on $g$, there is a $U \in kn(Y, N)$ and a $G \in \mathcal{F}_U^V$ with $g \overset{\eta}{=} G|_Y$. Let a $W \in n(Y, U)$ be selected using the way in which $\xi$ was chosen.

Let $h : X \to W$ be a map. Since $\mathcal{F}$ is a solid class of maps, $G \circ h \in \mathcal{F}_X^V$. It follows that $G \circ h \overset{\eta}{\approx} g \circ f \overset{\eta}{=} G \circ f$. Hence, $G \circ h \overset{\xi}{\approx} G \circ f$ and $h \overset{\sigma}{\approx} f$. ■

**Theorem 12.** *Let $\mathcal{F}$ be a solid class of maps.*

(a) *If $g \circ f$ is $\mathcal{G}$-universal and $g$ is both an embedding and an $[\mathcal{F}]$–map, then $f$ is $\lambda_Z[\mathcal{F}, \mathcal{G}]$-universal.*
(b) *If $g \circ f$ is $\mathcal{G}$-universal and $g$ is both an embedding and an $[\mathcal{F}, P)$–map, then $f$ is $\lambda_{(P,Z)}(\mathcal{F}, \mathcal{G}]$-universal.*
(c) *If $g \circ f$ is $\mathcal{G}$-e-universal in $P$ and $g$ is both an embedding and an $[\mathcal{F}, P)$–map, then $f$ is $\lambda_{(P,Z)}(\mathcal{F}, \mathcal{G})$-universal.*

*Proof:* (c). Let $h : X \to A$ be an $(\mathcal{F}, \mathcal{G})$-e-liftable in $(P, Z)$ map and assume that $A$ is a subset of $Y$. In order to show that $f \approx h$ it suffices to see that $f \overset{\sigma}{\approx} h$ for every $\sigma \in \tilde{Y}$.

Let $\sigma \in \tilde{Y}$. Since $g$ is an embedding, there is a $\xi \in \tilde{Z}$ such that $g(x) \overset{\xi}{=} g(y)$ for $x, y \in Y$ implies $x \overset{\sigma}{=} y$. Let $\eta \in \tilde{P}$ has the property that $\eta|_Z$ refines $\xi$. Let $\mu \in \eta^*$. By assumption on $g \circ f$, there is a $U \in n(Z, P)$ with $k \overset{\mu}{\approx} g \circ f$ for every $k \in \mathcal{G}_X^U$. Since $h$ is $(\mathcal{F}, \mathcal{G})$-e-liftable in $(P, Z)$, there is a $V \in n(Z, U)$ such that for every $a \in \mathcal{F}_A^V$ there is a $b \in \mathcal{G}_X^U$ with $b \overset{\mu}{=} a \circ h$. By the second assumption on $g$, there is an $m \in \mathcal{F}_Y^V$ such that $g \overset{\mu}{=} m|_Y$.

The restriction $m|_A$ is in $\mathcal{F}_A^V$ because the class $\mathcal{F}$ is solid. It follows that there is a $k \in \mathcal{G}_X^U$ with $k \overset{\mu}{=} m \circ h$. But, $k \overset{\mu}{\approx} g \circ f$. Hence, $g \circ h \overset{\xi}{\approx} g \circ f$ and $h \overset{\sigma}{\approx} f$. ■

**Theorem 13.**

(a) *If $g \circ f$ is $\mathcal{F}$-universal and $f$ is $[\mathcal{F}, \mathcal{G}]$-e-liftable in $Z$, then $g$ is $\mathcal{G}$-universal.*
(b) *If $g \circ f$ is $\mathcal{F}$-e-universal in $P$ and $f$ is $[\mathcal{F}, \mathcal{G})$-e-liftable in $(P, Z)$, then $g$ is $\mathcal{G}$-universal.*
(c) *If $g \circ f$ is $\mathcal{F}$-universal in $P$ and $f$ is $(\mathcal{F}, \mathcal{G}]$-e-liftable in $(P, Z)$, then $g$ is $\mathcal{G}$-e-universal in $P$.*
(d) *If $g \circ f$ is $\mathcal{F}$-e-universal in $P$ and $f$ is $(\mathcal{F}, \mathcal{G})$-e-liftable in $(P, Z)$, then $g$ is $\mathcal{G}$-e-universal in $P$.*

*Proof:* (*d*). Let $\sigma \in \tilde{P}_Z$. Let $\xi \in \sigma^*$. Since $g \circ f$ is $\mathcal{F}$-e-universal in $P$, there is a $U \in n(Z, P)$ such that $k \overset{\xi}{\approx} g \circ f$ for every $k \in \mathcal{F}_X^U$. By assumption on $f$, there is a $V \in n(Z, U)$ such that for every $a \in \mathcal{G}_Y^V$ there is a $b \in \mathcal{F}_X^U$ with $b \overset{\xi}{=} a \circ f$.

Let $h \in \mathcal{G}_Y^V$. Choose a $k \in \mathcal{F}_X^U$ with $k \overset{\xi}{=} h \circ f$. Observe that $k \overset{\xi}{\approx} g \circ f$. Hence, $h \circ f \overset{\xi}{=} k \overset{\xi}{\approx} g \circ f$ and $h \overset{\sigma}{\approx} g$. ■

**Theorem 14.** *Let $\mathcal{F}$ be a legal class of maps.*

(a) *If $g \circ f$ is $\mathcal{F}$-universal and $f$ is an $[\mathcal{F}]$–map, then the mapping $g$ is $\mathcal{A}$-universal.*
(b) *If $g \circ f$ is $\mathcal{F}$-e-universal in $P$, $f$ is an $[\mathcal{F}, N)$–map, and $P$ is an approximate polyhedron, then $g$ is $\mathcal{A}$-e-universal in $P$.*

*Proof:* (*b*). Let $\sigma \in \tilde{P}_Z$. Let $\xi \in \sigma^*$. Since $g \circ f$ is $\mathcal{F}$-e-universal in $P$, there is a $U \in n(Z, P)$ such that $k \overset{\xi}{\approx} g \circ f$ for every $k \in \mathcal{F}_X^U$.

Let $h \in \mathcal{A}_Y^U$. Since $P$ is an approximate polyhedron, there is a $V \in n(Y, N)$ and a continuous function $H : V \to U$ with $h \overset{\xi}{=} H|_Y$. Let $\theta = H^{-1}(\xi)$. Choose an $a \in \mathcal{F}_X^V$ with $a \overset{\theta}{=} f$. The assumptions about $\mathcal{F}$ and $U$ imply $H \circ a \in \mathcal{F}_X^U$ and $g \circ f \overset{\xi}{\approx} H \circ a$. But, $H \circ a \overset{\xi}{=} H \circ f \overset{\xi}{=} h \circ f$. Hence, $g \overset{\sigma}{\approx} h$. ■

**Theorem 15.**

(a) *If $f$ is an $[\mathcal{F}]$–map and $g \circ f$ is $\mathcal{G}$-universal, then $g$ is $\mu X[\mathcal{F}, \mathcal{G}]$-universal.*
(b) *If $f$ is an $[\mathcal{F}, N)$–map and $g \circ f$ is $\mathcal{G}$-universal, then the mapping $g$ is $\mu_{(N,P)} X(\mathcal{F}, \mathcal{G}]$-universal.*
(c) *If $f$ is an $[\mathcal{F}, N)$–map and $g \circ f$ is $\mathcal{G}$-e-universal in $P$, then $g$ is $\mu_{(N,P)} X(\mathcal{F}, \mathcal{G})$-e-universal.*

*Proof:* (*c*). Let $\sigma \in \tilde{P}_Z$. Let $\xi \in \sigma^{**}$. Since $g \circ f$ is $\mathcal{G}$-e-universal in $P$, there is a $U \in n(Z, P)$ such that $k \overset{\xi}{\approx} g \circ f$ for every $k \in \mathcal{G}_X^U$.

Consider an $X(\mathcal{F},\mathcal{G})$-e-movable in $(N,P)$ map $h : Y \to A$ and assume that $A$ is contained in $U$. Choose a $V \in n(Y,N)$ and a map $H : V \to P$ such that $h \overset{\xi}{=} H|_Y$ and for every $a \in \mathcal{F}_X^V$ and every $W \in n(Z,P)$ there is a $b \in \mathcal{G}_X^W$ with $b \overset{\xi}{=} H \circ a$. Let $\theta = H^{-1}(\xi)$. Since $f$ is an $[\mathcal{F},N)$–map, there is an $a \in \mathcal{F}_X^V$ with $a \overset{\theta}{=} f$. By assumption, there is a $b \in \mathcal{G}_X^W$ with $b \overset{\xi}{=} H \circ a$ and $b \overset{\xi}{\approx} g \circ f$. Hence, $g \circ f \overset{\xi}{\approx} b$ and $b \overset{\xi}{=} H \circ a \overset{\xi}{=} H \circ f \overset{\xi}{=} h \circ f$ and $g \overset{\sigma}{\approx} h$. ∎

## References


1. F. Al-Musallam and S.B. Nadler, Jr., Almost universal maps and the almost fixed point property, in "*Differential geometry, calculus of variations, and their applications,*" Lecture Notes in Pure and Appl. Math. **100**, Dekker, New York, 1985, pp. 19–35.
2. K. Borsuk, On nearly extendable maps, *Bull. Acad. Polon. Sci., Sér. sci. math., astronom. et phys.* **23** (1975), 753–760.
3. Z. Čerin, On global properties of maps I, *Suppl. Rendiconti Circ. Mat. Palermo* **12** (1986), 19–45.
4. G. Gauthier, La théorie des retracts approximatifs et le théorème des points fixes de Lefschetz, *Dissertationes Math.* **217** (1983), 1–78.
5. W. Holsztyński, Une généralisation du théorème de Brouwer sur les points invariants, *Bull. Acad. Polon. Sci., Sér. sci. math., astronom. et phys.* **12** (1964), 603–606.
6. W. Holsztyński, Universal mappings and fixed point theorems, *Bull. Acad. Pol. Sci., Sér. sci. math., astronom. et phys.* **15** (1967), 433–438.
7. W. Holsztyński, On the composition and products of universal mappings, *Fund. Math.* **64** (1969), 181–188.
8. G. Kozlowski, Images of ANR's, *Trans. Amer. Math. Soc.* (to appear).
9. S. Mardešić, Approximate polyhedra, resolutions of maps, and shape fibrations, *Fund. Math.* **114** (1981), 53–78.
10. J. M. R. Sanjurjo, Stability of the fixed point property and universal maps, *Proc. Amer. Math. Soc.* **105** (1989), 221–230.


Department of Mathematics
University of Zagreb
41001 Zagreb, p.p. 187
CROATIA

Rebut el 24 d'Abril de 1990