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SINGULARITY THEORY AND FORCED SYMMETRY BREAKING IN EQUATIONS

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Abstract

A theory of bifurcation equivalence for forced symmetry breaking bifurcation problems is developed. We classify $(\mathrm{O}(2), \mathbf{1})$ problems of corank 2 of low codimension and discuss examples of bifurcation problems leading to such symmetry breaking.

1. Introduction

The importance of symmetries in understanding and influencing bifurcation problems has been recognised basically since the beginning of bifurcation theory. Symmetries influence bifurcation problems via two broad classes of mechanisms:

- spontaneous symmetry breaking,
- forced (or induced) symmetry breaking.

Spontaneous symmetry breaking occurs when the symmetry of the equations is *constant* whereas solutions bifurcate and lose (or gain) internal symmetry as the parameters vary. From the many problems leading to bifurcation equations with spontaneous symmetry breaking, we mention for later reference, the analysis of the behaviour of an homogeneous, isotropic elastic cube under uniform traction (its symmetry group is the permutation group S_3, see [**20**]) or the bifurcation of periodic orbits in autonomous systems (its symmetry group for the non-resonant degenerate Hopf bifurcation is the rotation group S^1, see [**19**]). A singularity theory approach to the local study of the bifurcation diagrams, based on

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equivariant contact equivalence, has been developed by Golubitsky and Schaeffer ([**19**], [**20**]). The original set-up is as follows.

1.1. Singularity Theory for Spontaneous Symmetry Breaking.

First, because such analysis is local, it is sensible to consider germs at the origin to be able to state results that will persist on any neighbourhood of the origin. A **germ of a function** around a point x_0 is an equivalence class when two functions are identified if they coincide in a neighbourhood of that point. We use the notation $f\colon (\mathbb{R}^n, x_0) \to \mathbb{R}^m$ to denote the germ of f around x_0. To make sense of equations like $f = 0$, when $f\colon (\mathbb{R}^n, x_0) \to \mathbb{R}^m$, we define **germs of sets**, or **germs of varieties**, at x_0 using the same identification process as for germ of functions at x_0. Let Γ be a compact group acting on $\mathbb{R}^n$ and $\mathbb{R}^m$, we say that $f\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^m, 0)$ is **Γ-equivariant**, resp. **Γ-invariant**, if $f(\gamma x) = \gamma f(x)$, resp. $f(\gamma x) = f(x)$, $\forall\, x \in (\mathbb{R}^n, 0)$, $\forall\, \gamma \in \Gamma$. A **bifurcation germ** with l parameters is a germ $f\colon (\mathbb{R}^{n+l}, 0) \to (\mathbb{R}^m, 0)$. Its **bifurcation diagram** is its zero-set $f^{-1}(0)$. Let Γ be a compact group acting on $\mathbb{R}^n$ and $\mathbb{R}^m$, trivially on parameters, two Γ-equivariant bifurcation germs (with one distinguished bifurcation parameter) $f, g\colon (\mathbb{R}^{n+1}, 0) \to (\mathbb{R}^m, 0)$ are **bifurcation**, or $\mathcal{K}^\Gamma_\lambda$, **equivalent** if there exist Γ-equivariant changes of coordinates (T, X, L) such that

$$f(x, \lambda) = T(x, \lambda)\, g(X(x, \lambda), L(\lambda)), \tag{1.1}$$

where $T(x, \lambda)$ is an invertible matrix and (X, L) a local diffeomorphism around the origin. Note the special role of λ which means that λ-slices of the bifurcation diagrams of f and g are diffeomorphic via (X, L). The use of singularity theory leads to a systematic understanding and systematic calculation of the relative roles of the different terms of a bifurcation germ, and, via the notion of miniversal unfolding, of its perturbations. The theory based on (1.1) established itself for the local study of bifurcation germs, giving efficient algebraic calculations to establish classifications of **normal forms**, special polynomial members of the orbits under bifurcation equivalence, and their **miniversal deformations**, perturbations of the normal forms with the minimal number of parameters necessary to represent all possible perturbations modulo changes of co-ordinates. Given a bifurcation germ f, the key algebraic ingredient of the theory is the **extended tangent space** of f, denoted by $\mathcal{T}_e\mathcal{K}^\Gamma_\lambda(f)$. It has a so-called mixed module structure because it is the sum of a module generated over the ring of Γ-invariant germs in (x, λ) and of a module generated over germs in λ only. An important number associated with f is the codimension of $\mathcal{T}_e\mathcal{K}_\lambda^\Gamma(f)$ as a vector subspace of the space of Γ-equivariant maps. It is called the **$\mathcal{K}_\lambda^\Gamma$-codimension** of f, denoted by $\mathcal{K}_\lambda^\Gamma$-cod$(f)$. When $\mathcal{K}_\lambda^\Gamma$-cod$(f)$ is finite, f is $\mathcal{K}_\lambda^\Gamma$-equivalent to a polynomial normal form and has a miniversal unfolding with $\mathcal{K}_\lambda^\Gamma$-cod$(f)$ parameters, formed using a basis a complement of $\mathcal{T}_e\mathcal{K}_\lambda^\Gamma(f)$ in the space of Γ-equivariant bifurcation maps, the **normal space** of f, denoted by $\mathcal{N}_e\mathcal{K}_\lambda^\Gamma(f)$.

1.2. Forced Symmetry Breaking.

In the second mechanism, that of forced symmetry breaking, the symmetries of the equations *change* when some parameters are switched on. For example, in the elastic cube problem, the traction load distribution could change first to remain equal only on two opposite faces before losing any symmetry with a second additional parameter. The forced symmetry breaking is then $\mathrm{S}_3 \to \mathrm{Z}_2 \to \mathbf{1}$ (with one, then two additional parameters coming into play, see [**13**]).

There are many other buckling problems in elasticity exhibiting forced symmetry breaking because there is often an interaction between the internal symmetries of the material, the geometry of the object and of the externally applied forces. An example is the work of Pierce [**31**] on the bifurcation of straight circular rods subject to an axially symmetric compressive load and perturbations by additional loads breaking the axial symmetry. In previous works on prismatic rods, like in [**3**], the typical diagrams are to be found in the miniversal unfolding of the non degenerate double cusp because the symmetry group is finite. When the rod is circular the double cusp is degenerate because the acting symmetry group is now O(2) and so the problem has no miniversal unfolding within the classical theory.

Similarly, the forcing of an autonomous equation $\ddot{u} + q(u, \lambda) = 0$ by either a T-periodic function p, like in the following model problem

$$\ddot{u} + q(u, \lambda) + \mu\, p(t) = 0, \tag{1.2}$$

or non linear boundary conditions of the type (g_1, g_2 are (non local) non linear functions of u):

$$u(0) - u(1) = \mu g_1(u, \lambda, \mu), \quad \dot{u}(0) - \dot{u}(1) = \mu g_2(u, \lambda, \mu),$$

gives rise to a bifurcation problem with the forced symmetry breaking $\mathrm{O}(2) \to \mathbf{1}$ near the values of λ where the linear mode of the autonomous part is of period-T also. We can extend the symmetry breaking bifurcation for any pair of compact subgroups $\Delta \subset \Gamma \subset \mathrm{O}(2)$, by considering a generalisation of the differential equation (1.2) with time periodic terms g (nonlinear) and p with rationally dependent periods (see [**17**] for a classical analysis).

1.3. Main Results.

A special version of bifurcation equivalence, stronger than (1.1), is called for to classify systematically the bifurcation germs and their unfoldings arising in forced symmetry breaking problems.

1.3.1. Section 2: abstract theory.

In Section 2, we present a general theory of unfoldings, finite determinacy and the recognition problem for forced symmetry breaking bifurcation germs of the type

$$f\colon (\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}^n, 0), \quad (x, \lambda, \mu) \mapsto f(x, \lambda, \mu),$$

where f is Γ-equivariant when $\mu = 0$ and Δ-equivariant when $\mu \neq 0$, for Δ a closed subgroup of Γ. The essential part of the theory was advanced in [**18**] but not much action was taken on it, the main theory drifting instead towards the very rich field of spontaneous symmetry breaking. But, fundamentally, Damon in [**8**] adapted his general framework to clear the way for the abstract theory to work successfully in that case. So, we may write

$$f(x, \lambda, \mu) = f_1(x, \lambda) + \mu f_2(x, \lambda, \mu) \tag{1.3}$$

with f_1 Γ-equivariant and f_2 Δ-equivariant. A first approach to study such problems, is to view f as a perturbation of f_1 in the $\mathcal{K}^{\Delta}_{(\lambda,\mu)}$-theory, the bifurcation equivalence of Δ-equivariant maps with two bifurcation parameters (λ, μ). In general that approach fails. For instance, when Γ and Δ have different dimensions, the $\mathcal{K}^{\Delta}_{(\lambda,\mu)}$-codimension of f_1 is not finite (even when Γ is finite, it may fail). So, the group of change of coordinates $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ we define in Section 2.3 will have the property that (T, X, L) is Γ-equivariant when $\mu = 0$, but only Δ-equivariant when $\mu \neq 0$. This (Γ, Δ)-equivariant structure of the group of contact equivalences will be transported to the tangent spaces (see Section 2.3.4). They are modules over Γ-invariant functions when $\mu = 0$ and over Δ-invariant functions when $\mu \neq 0$. This is unusual but, in his general framework, Damon did define in [**8**] the necessary extended concepts to deal with the new situation.

The first point is to make sure that a version of the Preparation Theorem applies. As a consequence, we can work with the algebraic structure of the tangent spaces. For the usual theory of $\mathcal{K}^{\Gamma}_{\lambda}$-equivalence, the ring of invariant functions has a structure of differentiable DA-algebra (see

Section 2.2), and so the Preparation Theorem holds true. When Γ is a continuous Lie group and Δ is finite, our rings of invariant functions will not be DA-algebras, but Damon showed that the main properties of DA-algebras can be extended to this situation of so-called *extended DA-algebras*. In Corollary 2.2 we give an explicit criterion we use to test if the rings of invariants under consideration are actually DA-algebras.

In a second step, we look into the structure of our group of contact equivalences $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ and the tangent spaces. In Section 2.3 we show that $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ decomposes into three subgroups: $\hat{\mathrm{K}}(\Gamma,\Delta)$, $\hat{\mathrm{M}}(\Gamma,\Delta)$ and $\hat{\mathrm{S}}(\Gamma,\Delta)$. When Γ is continuous and Δ is finite, for instance, each subgroup add a non trivial contribution to the extended tangent space of $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$. In Section 2.3.5 we show that $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ is a geometric subgroup of $\mathcal{K}$ (contact equivalences), hence it satisfies the abstract theorems of Damon [**8**] about miniversal unfoldings and finite determinacy. We show in Theorem 2.7 that the explicit description of $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ in [**18**] corresponds to the best possible situation, because its extended tangent space contains the extended tangent space of any other geometric subgroup of $\mathcal{K}$ fixing globally the Γ-equivariant maps when $\mu = 0$.

Finally, we discuss topological $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalence where the changes of co-ordinates (T, X, L) are only continuous, not smooth, germs. In Section 2.5, we summarise the results of Damon [**10**], [**11**] we need. Topological equivalence is more efficient in our context, because it preserves the topological properties of the bifurcation germs and their (smooth) miniversal unfoldings. Our smooth normal forms and their smooth miniversal unfoldings have many moduli (parameters without topological significance). They appear because the geometry of the bifurcation diagrams is intricate (see Figure 4). As most two parameter situations, the $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalence preserves the respective position of the regions of the parameter plane with a different zero structure. In our case the only one dimensional slice to be preserved is $f_1^{-1}(0)$, when $\mu = 0$.

1.3.2. Section 3: $(\mathrm{O}(2), \mathbf{1})$-symmetry breaking classification.

As an example, we classify $(\mathrm{O}(2), \mathbf{1})$-symmetry breaking problems. Although we stop at topological codimension 1, we still have germs of high smooth codimension with an intricate region structure for the zero-set (see Figure 4 in Section 4, for instance) because the normal forms can have many moduli parameters (parameters invariant under smooth $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalence). In Theorem 3.4 (about topological $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalence) we show that there are 3 normal forms up to topological codimension 1. They all satisfy $f_2(0,0,0) \neq 0$, and their smooth codimension vary from 0 to 4 as shown in Theorem 3.5 (about smooth $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalence).

We believe that we have a complete list up to topological codimension 2, adding 4 more normal forms, but we cannot fully prove it at present, so we simply mention that list as a remark following Theorem 3.4. Interestingly, one normal form has $f_2(0,0,0) = 0$. Note that the highest smooth codimension of those germs can reach is 12. As a consequence of our computations of the tangent spaces, the list of the $(\mathrm{Z}_2, \mathbf{1})$-symmetry breaking germs f in one dimension, that is, $f(y,\lambda) = f_1(y^2,\lambda)\,y + \mu f_2(y,\lambda,\mu)$, is basically the same as the $(\mathrm{O}(2), \mathbf{1})$-classification list when $f_2(0,0,0) \neq 0$. Nevertheless, the stability properties of the solutions are different (see [**12**]). In the corank two case, the symmetry breaking term selects only one stable solution from the O(2)-orbit. In the one dimensional case, the obvious obstructions from the unstable solutions mean that the stability properties of the pairs of solutions remain unchanged.

1.3.3. Section 4: $(\mathbf{O(2)}, \mathbf{1})$-symmetry breaking bifurcation diagrams.

In this section, we look at the bifurcation diagrams for the $(\mathrm{O}(2), \mathbf{1})$-symmetry breaking bifurcation germs f we classified in Theorem 3.4, cases I_0, II and III. In Figure 2 of Section 4.2.1, we give a simple illustration on a potential dynamics of what happen when orbits of steady states are destroyed by the perturbation of the miniversal unfolding of case I_0. More generally, we describe the regions of parameter space (λ, μ) where the zero set structure of f is invariant, including the stability of the solution with respect to the sign of the eigenvalues of the linearisation of f. They are portrayed in Figures 1, 3 and 4, respectively. For bifurcation equivalent germs, those regions are diffeomorphic but the only one dimensional slice preserved is $\mu = 0$.

1.3.4. Section 5: examples.

In the final section we describe examples leading to bifurcation equations satisfying our framework. Our results are readily applicable to the bifurcation of period-2π solutions of

$$\ddot{u} + u + u\,q(u,\lambda) + \mu\,p(t,u,\dot{u},\lambda,\mu) = 0$$

where p is 2π-periodic in t and $q(0,0) = 0$. This example is technically easy to manipulate. As it is well-known, when $|\int_{-\pi}^{\pi} p(t,\mathbf{0})\,e^{it}\,dt| \neq 0$, the symmetry breaking term selects a pair of solutions (that is, selects a phase shift) from the O(2)-orbit, keeping only one solution possibly stable. In that case the problem reduces to the corank one symmetry breaking $(\mathrm{Z}_2, \mathbf{1})$.

Another problem is concerned with the problem of contact resistance in homogeneous metal rings obtained from welding the end points of a piece of metal. Usually the joining is not perfect and give rise to some 'contact resistance' to heat conduction. This resistance is characterised by a continuous gradient of the temperature across the join (the material is the same across the join on the left and the right) and a loss of temperature at first approximation proportional to its gradient. More explicitly, if the piece of metal is $[0, 1]$, the simplest boundary conditions are

$$u(0) - u(1) = \mu\dot{u}(1) \quad \text{and} \quad \dot{u}(1) = \dot{u}(0). \tag{1.4}$$

The boundary conditions are periodic if $\mu = 0$ and have no symmetries when $\mu \neq 0$. Combined with an autonomous non linear heat equation they give rise to a forced $(\mathrm{O}(2), \mathbf{1})$-symmetry breaking (see [**34**] and [**29**]).

Another type of problems are reaction-diffusion equations. Let u be the density of a population living in the unit disk. Suppose u satisfies a parametrised semilinear parabolic equation

$$u_t = \Delta u + g(u, \bar{u}, \lambda) + \mu\, p(x, y, u, \bar{u}, \lambda, \mu) \tag{1.5}$$

subject to non-flux boundary conditions $\frac{\partial u}{\partial n} = 0$ on the unit circle. The term $\bar{u}$ represents a non local contribution of the averaged density $\bar{u}(t) = \frac{1}{|\Omega|}\int_\Omega u(s, t)\, ds$. Under the hypotheses of Section 4, (1.5) has a constant steady state which looses stability with a 2-D kernel for the linearisation. For general p, the resulting bifurcation equation fits our framework. As an example of $(\mathrm{Z}_2, \mathbf{1})$-symmetry breaking, take a generic thin rectangular plate under a uniform compression along its boundary. Apply a generic normal load distribution and use the von Kármán approximation for the buckling equations (see [**36**]). The unperturbed equations have a Z_2-symmetry, destroyed by the perturbing load distribution.

1.4. Comments and Some Related Work.

1.4.1. Variational problems.

Bifurcation equations in elasticity are often the gradients of some parametrised functional (see [**31**]). Bifurcation equivalences do not preserve in general the gradient structure, but they still induce an *equivalence relation* on the set of gradient bifurcation germs and their unfoldings. A theory using a path formulation has been developed in [**2**], but it is not clear how to extend those ideas to the $(\mathrm{O}(2),\mathbf{1})$-symmetry breaking, but no more powerful theory is needed here. Our normal forms (with case I_2 for some values of its moduli) and their miniversal unfoldings are already *gradients.* If any other gradient bifurcation germ satisfies the recognition conditions, then it is bifurcation equivalent to the representative normal form, which *is* a gradient. Moreover, the same applies to the miniversal unfoldings that are also gradients. This means that the classification of gradient problems mirrors the general classification.

1.4.2. Symmetries on parameters.

Forced symmetry breaking arises also when some symmetries are retained by the parameters (see [**15**]). A typical example is when we have a non trivial 'diagonal' action of Γ on the space of (x,λ), $\gamma\mapsto(\gamma x,\gamma\lambda)$. In that case the theory in this paper is not necessary. The general structure of the problem remains 'classical' because the invariants form a DA-algebra (see [**15**]). But, if we combine both ideas: some symmetry is lost, some is retained by the parameters when they are switched on, we get a theory which is still contained into the abstract framework we describe here because the invariants still form a DA-algebra. When combined with the symmetry breaking terms, we have a structure of extended DA-algebras for the invariants. More explicitly, the problems represented by the functions $f(x,\lambda,\mu)=f_1(x,\lambda)+f_2(x,\lambda,\mu)$, with $f_1(\gamma x,\gamma\lambda)=\gamma f_1(x,\lambda)$ and $f_2(\delta x,\delta\lambda,\delta\mu)=\delta f_2(x,\lambda,\mu)$, fall into a similar framework.

1.4.3. Bifurcations from orbits of solutions under perturbations.

Some previous work have been concerned with the symmetry of bifurcating solutions of (1.3). In the Γ-equivariant situation one can sometimes show that *all* bifurcating solutions have at least isotropy Δ for some subgroup $\Delta\subset\Gamma$ (see [**36**] for an abstract analysis). An example of such analysis is the generalised Duffing's equation (see [**22**], [**23**])

$$\ddot{u}+u+uq(u,\lambda)+\mu\, p(t)=0 \tag{1.6}$$

where $q(0,0)=0$ and p is an even 2π-periodic function. One can then show that all solutions of (1.6) are also even. Similar results hold when q is an odd function and p is odd in time.

Systematic approaches can be traced back to Vanderbauwhede [**35**] and Lauterbach and Roberts [**26**], giving rise to Hou and Golubitsky [**25**] (see Chapter 10 of [**6**] for a more recent discussion). The bifurcation from Γ-orbits of solutions is studied when a Δ-equivariant perturbation is applied to the original Γ-equivariant equations. For instance, a simple example in tune with previous example is the equation $\ddot{u} + q(u, \lambda) + \mu h(t, u, \dot{u}) = 0$ when h is 2π-periodic in t and $\ddot{u} + q(u) = 0$ has nontrivial periodic orbits (see [**36**] and [**4**]). More generally, there has been more recently interest about the effect of forced symmetry breaking on the dynamics around some special solutions, invariant manifolds (Galante and Rodrigues [**17**], Chillingworth and Lauterbach [**5**], Comanici [**7**], Parker et al. [**30**] to mention but a few). In these papers, somewhat different techniques are used, linked with equivariant differential geometry. A complete unfolding of the underlying singularity near bifurcation is difficult. However, in this paper, we are able to provide just such analysis.

2. General Theory

In this section we present a general theory of unfoldings, finite determinacy and the recognition problem for the bifurcation problems with forced symmetry breaking. The main ideas and definitions were sketched in [**18**] and [**8**]. Here we give a full, careful and organised account of the theory with some extended results. This needs many ingredients with many definitions and results. The abstract theory of Damon [**8**] works with modules over systems of rings that are DA-algebras. In general, the ring of forced symmetry breaking invariant functions has only the structure of an *extended* DA-algebra. In Corollary 2.2 we give a criterion we use to test if an extended DA-algebra is actually a DA-algebra. In that case we do not need any extension of the abstract theory. On the contrary, like for the $(\mathrm{O}(2), \mathbf{1})$-symmetry breaking, an extended theory is required. In Section 2.3, we discuss the structure of the group of bifurcation equivalence $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ we are going to use. It requires the composition of any finite string of elements of three subgroups. One subgroup, $\hat{\mathrm{K}}(\Gamma, \Delta)$, is natural, the two others, $\hat{\mathrm{M}}(\Gamma, \Delta)$ and $\hat{\mathrm{S}}(\Gamma, \Delta)$, are more surprising. In Lemma 2.4 we show that the order of the elements is irrelevant: we can always recombine any string as 3 elements, one in each subgroup. The group $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ is a geometric subgroup in the sense of [**8**] over an adequately ordered extended system of DA-algebras. To characterise the tangent spaces we calculate the unusual contributions of $\hat{\mathrm{S}}(\Gamma, \Delta)$ and $\hat{\mathrm{M}}(\Gamma, \Delta)$ in Lemmas 2.5 and 2.6, respectively. In Theorem 2.7 we show that the tangent space of any other group of equivalence respecting the forced symmetry breaking structure cannot be larger than the tangent space of $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$, indicating that the choices of $\hat{\mathrm{S}}(\Gamma, \Delta)$ and $\hat{\mathrm{M}}(\Gamma, \Delta)$ are optimal. In Section 2.4 we state the main results we can deduce from the abstract theory of [**8**] about the unfolding (Theorem 2.8) and determinacy (Theorem 2.9) theories for $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$. Finally, most normal forms have moduli for the smooth equivalence and so we use the topological equivalence theory of [**10**], [**11**] to regroup orbits of the smooth classification into classes with equivalent topological behaviour. The main results are stated in Theorems 2.13 and 2.14.

2.1. Notation and Preliminary Definitions.

The state variable is $x \in (\mathbb{R}^n, 0)$ and the distinguished bifurcation parameters are $(\lambda, \mu) \in (\mathbb{R}^2, 0)$. The derivatives are denoted by subscripts, f_x for $\frac{\partial f}{\partial x}, \ldots$ and the superscript ^{o} denotes the value of any function at the origin, $f^o = f(0)$, $f^o_x = f_x(0), \ldots$. Let $\mathcal{E}_x$ denote the ring of smooth germs $f\colon (\mathbb{R}^n, 0) \to \mathbb{R}$ and $\mathcal{M}_x$ its maximal ideal. For $y \in \mathbb{R}^m$, let $\mathcal{E}_{x,y}$ denote the $\mathcal{E}_x$-module of smooth germs $g\colon (\mathbb{R}^n, 0) \to \mathbb{R}^m$, and $\mathcal{M}_{x,y}$ the submodule of germs vanishing at the origin. When y is clear from the context, we denote $\mathcal{E}_{x,y}$ by $\vec{\mathcal{E}}_x$ and $\mathcal{M}_{x,y}$ by $\vec{\mathcal{M}}_x$. When we would like to emphasise only the dimension of the source we denote $\mathcal{E}_x$, $\mathcal{E}_y$ by $\mathcal{E}_n$, $\mathcal{E}_m$, etc. To represent invariant and equivariant germs in terms of invariant polynomials, it is convenient to use the following concept. Let X, Y, Z be sets, $f\colon X \to Y$, $g\colon Z \to Y$, $h\colon X \to Z$ be maps, f is the **pullback** of g by h, denoted by $f = h^*g$, if $f(x) = g(h(x))$, $\forall\, x \in X$.

Let $\mathrm{GL}(n)$ be the group of all invertible $n \times n$-real matrices and $\mathrm{O}(n)$ the n-dimensional orthogonal group. Let Γ be a compact Lie group acting on $\mathbb{R}^n$ via an orthogonal representation $\rho\colon \Gamma \to \mathrm{O}(n)$. We denote by Γ_0 the connected component of the identity in Γ, identify γ with $\rho(\gamma)$, $\forall\, \gamma \in \Gamma$, and denote by γ the action on $\mathbb{R}^n$ induced by ρ. We denote by $\mathrm{GL}_\Gamma(n)$ the group $\{M \in \mathrm{GL}(n) : M\gamma = \gamma M, \forall\, \gamma \in \Gamma\}$ of Γ-equivariant matrices in $\mathrm{GL}(n)$ and by $\mathcal{L}^o_\Gamma(n)$ its connected component of the identity.

2.1.1. Invariant functions.

Let $\mathcal{E}^\Gamma_{(x,\lambda)}$ be the ring of smooth Γ-invariant germs $h\colon (\mathbb{R}^{n+1}, 0) \to \mathbb{R}$, $h(\gamma x, \lambda) = h(x, \lambda)$, $\forall\, \gamma \in \Gamma$, and $\mathcal{M}^\Gamma_{(x,\lambda)}$ its maximal ideal. Because Γ does not act on λ, there exists a finite set of Γ-invariant polynomials $\{\bar{u}_i(x)\}_{i=1}^r$ such that any element $h \in \mathcal{E}^\Gamma_{(x,\lambda)}$ can be written as the pullback by $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_r, \lambda)$ of a function of $u = (u_1, \ldots, u_r)$ and λ, that is, $\mathcal{E}^\Gamma_{(x,\lambda)} = \bar{u}^*\mathcal{E}_{(u,\lambda)}$ [**33**]. Similarly, for a closed subgroup $\Delta \subset \Gamma$, we define $\mathcal{E}^\Delta_{(x,\lambda,\mu)}$ as the ring of Δ-invariant germs $h\colon (\mathbb{R}^{n+2}, 0) \to \mathbb{R}$ which is the pullback by the Δ-invariant generators $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_s, \lambda, \mu)$ of a germ of $v = (v_1, \ldots, v_s)$, λ and μ, that is, $\mathcal{E}^\Delta_{(x,\lambda,\mu)} = \bar{v}^*\mathcal{E}_{(v,\lambda,\mu)}$. Finally, we define the local ring $\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu)} = \mathcal{E}^{\Gamma}_{(x,\lambda)} + \mu\,\mathcal{E}^{\Delta}_{(x,\lambda,\mu)}$ of maximal ideal $\mathcal{M}^{\Gamma,\Delta}_{(x,\lambda,\mu)} = \mathcal{M}^{\Gamma}_{(x,\lambda)} + \mu\,\mathcal{E}^{\Delta}_{(x,\lambda,\mu)}$.

2.1.2. Equivariant maps.

Let $\vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$ be the $\mathcal{E}^{\Gamma}_{(x,\lambda)}$-module of smooth Γ-equivariant germs $f_1\colon (\mathbb{R}^{n+1},0) \to \mathbb{R}^n$, $f_1(\gamma x,\lambda) = \gamma f_1(x,\lambda)$, $\forall\ \gamma \in \Gamma$. From [**32**], it is generated over $\mathcal{E}^{\Gamma}_{(x,\lambda)}$ by a finite set of Γ-equivariant polynomial maps $\{Y_i(x)\}_{i=1}^{J}$. Hence, for any $f \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$ there exist some $\{h_j\}_{j=1}^{J} \subset \mathcal{E}_{(u,\lambda)}$ with $f = \bar{u}^*(h_1Y_1 + \cdots + h_JY_J)$. Thus, we identify $\vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$ with $\bar{u}^*\mathcal{E}^{J}_{(u,\lambda)}$ (in general that module is not free on $\mathcal{E}^{\Gamma}_{(x,\lambda)}$). Similarly, we define $\vec{\mathcal{E}}^{\Delta}_{(x,\lambda,\mu)}$ and $\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)} = \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)} + \mu\,\vec{\mathcal{E}}^{\Delta}_{(x,\lambda,\mu)}$, the space of bifurcation germs we are going to consider. It is a module over the ring $\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu)} = \mathcal{E}^{\Gamma}_{(x,\lambda)} + \mu\,\mathcal{E}^{\Delta}_{(x,\lambda,\mu)}$. We also define $\vec{\mathcal{M}}^{\Gamma}_{(x,\lambda)} = \{X_1 \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)} : X_1^o = 0\}$ and $\vec{\mathcal{M}}^{\Gamma,\Delta}_{(x,\lambda,\mu)} = \vec{\mathcal{M}}^{\Gamma}_{(x,\lambda)} + \mu\,\vec{\mathcal{E}}^{\Delta}_{(x,\lambda,\mu)}$.

2.2. More Algebraic Structures.

Before we discuss the contact equivalence we need, we turn our attention to some important algebraic properties of $\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$. We need them to be able to use some basic definitions and results due to Damon [**8**] to establish the Unfolding and Finite Determinacy Theorems. A **DA-algebra** A (**differentiable algebra**) consists of an $\mathbb{R}$-algebra A and a surjective algebra homomorphism $\psi\colon \mathcal{E}_n \to A$ (for some n). These algebras are local rings with maximal ideals $\mathcal{M}_A$. If $\phi\colon \mathcal{E}_m \to B$ defines another DA-algebra, then a **homomorphism of DA-algebras** $\alpha\colon A \to B$ is an algebra homomorphism which lifts to $\tilde{\alpha}\colon \mathcal{E}_n \to \mathcal{E}_m$ with $\tilde{\alpha} = g^*$ for some $g \in \mathcal{E}_{n,m}$ and $\alpha \circ \psi = \phi \circ \tilde{\alpha}$.

Example. From Schwarz's Theorem [**33**], $\mathcal{E}^{\Gamma}_{(x,\lambda)}$ and $\mathcal{E}^{\Delta}_{(x,\lambda,\mu)}$ are DA-algebras. As an example, in the first case, let $\phi\colon (\mathbb{R}^{n+1},0) \to (\mathbb{R}^{r+1},0)$ be defined as $\phi(x,\lambda) = (\bar{u}_1(x),\ldots,\bar{u}_r(x),\lambda)$. Since ϕ induces the surjective algebra homomorphism $\phi^*\colon \mathcal{E}_{(u,\lambda)} \to \mathcal{E}^{\Gamma}_{(x,\lambda)}$, $h \mapsto h \circ \phi$, we can conclude.

The ring $\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is not necessarily a DA-algebra. Nevertheless, it behaves quite well. Damon has introduced the notion of extended DA-algebra to fit his general framework.

2.2.1. Extended DA-algebra.

Let $A \subset B$ be two DA-algebras and $I \subset B$ a B-finitely generated ideal. Then $R = A + I$ is still a local $\mathbb{R}$-algebra with maximal ideal

$\mathcal{M}_R = \mathcal{M}_A + I$ where $\mathcal{M}_A$ is the maximal ideal of A. R is called an **extended DA-algebra**. Clearly $\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu)} = \mathcal{E}^{\Gamma}_{(x,\lambda)} + \mu\,\mathcal{E}^{\Delta}_{(x,\lambda,\mu)}$ is an extended DA-algebra with $A = \mathcal{E}^{\Gamma}_{(x,\lambda)}$ and $I = \mu\,\mathcal{E}^{\Delta}_{(x,\lambda,\mu)} \subset \mathcal{E}^{\Delta}_{(x,\lambda,\mu)} = B$. A **homomorphism ϕ: $R \to S$ of extended DA-algebras**, $R = A + I \subset B$ and $S = C + J \subset D$ is a restriction of a homomorphism of DA-algebras $\psi\colon B \to D$ such that $\psi(A) \subset C$ and that $\psi(I) \subset J$. A **module over an extended DA-algebra** is an R-module which satisfies the additional property that the B-module I induces a B-module structure over $I \cdot M$. This is always the case if M is contained in a B-module. The next result is fundamental to ascertain if an extended DA-algebra is actually still a DA-algebra.

Lemma 2.1. *$\mathcal{M}_R$ is a finitely generated R-ideal if and only if I/I^2 is a finitely generated A-module.*

Proof: Let $\{h_i\}_{i=1}^L$ be a set of generators for $\mathcal{M}_R$. We may write $h_i = \bar{a}_i + \sigma_i$ with $\bar{a}_i \in \mathcal{M}_A$ and $\sigma_i \in I$, $1 \le i \le L$. Because $\mathcal{M}_A$ is finitely generated and $A \subset R$ we can assume that $\bar{a}_i$ or $\sigma_i = 0$, $1 \le i \le L$. For any $g \in I \subset R$ there exists G such that $g = G(\sigma_1, \dots, \sigma_L) = \sum_{i=1}^L a_i\sigma_i + I^2$, and so $\{\sigma_i\}_{i=1}^L$ is a set of generators for I/I^2 as an A-module.

Conversely, we need only to show that I is a finitely generated R-ideal. Let $\{\sigma_i\}_{i=1}^l$ be the generators of I/I^2 as an A-module. Let $\{a_i\}_{i=1}^k$ be the set of generators of $\mathcal{M}_A$. Then we see that the union of those two sets of generators generates any element of $\mathcal{M}_R$ up to flat germs and the result follows from the theory of graded modules. □

When $\mathcal{M}_R$ is not finitely generated, $\mathcal{M}_R^k$ is not of finite codimension for $k > 1$. For later use in the discussion of finite determinacy, Damon defines in [**8**] the ideal

$$\mathcal{M}_R^{(k)} = \mathcal{M}_A^k + \mathcal{M}_B^k \cdot I$$

which has now finite codimension, because $R/\mathcal{M}_R^{(k)} \subset A/\mathcal{M}_A^k \oplus B/\mathcal{M}_B^k$.

Corollary 2.2. (a) *R is a DA-algebra if and only if I/I^2 is a finitely generated A-module.*

(b) *The extended DA-algebra $\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is a DA-algebra if and only if $\mathcal{E}^{\Delta}_x$ is a finite $\mathcal{E}^{\Gamma}_x$-module.*

Proof: (a) Let R be a DA-algebra with surjective homomorphism $\phi\colon \mathcal{E}_y \to \mathcal{R}$. As $\mathcal{M}_y$ is finitely generated, $\mathcal{M}_R$ is itself finitely R-generated and so I/I^2 is a finitely generated A-module.

Conversely, if I/I^2 is a finitely generated A-module, we know from Lemma 2.1 that $\mathcal{M}_R$ is finitely generated, say by $\{h_i\}_{i=1}^L$. Define $\sigma\colon \mathbb{R}^n \to \mathbb{R}^L$ by $x \mapsto (h_1(x), \ldots, h_L(x))$, then $\sigma^*\colon \mathcal{E}_L \to R$ is a surjective homomorphism and so R is a DA-algebra.

(b) For $\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$, $I = \mu\mathcal{E}^{\Delta}_{(x,\lambda,\mu)}$ and $A = \mathcal{E}^{\Gamma}_{(x,\lambda)}$ and so $I/I^2 \approx \mathcal{E}^{\Delta}_{(x,\lambda)}$. As our groups do not act on the parameters, I/I^2 is a finite A-module if and only if $\mathcal{E}^{\Delta}_x$ is a finite $\mathcal{E}^{\Gamma}_x$-module. □

Examples. (1) Let $\Gamma = \mathrm{O}(2)$ acting on $\mathbb{C}$ via the usual orthogonal representation on $\mathbb{C}$ generated by $\theta \cdot z = e^{i\theta} z$, $\theta \in \mathrm{S}^1$, and $\kappa \cdot z = \bar{z}$. Take $\Delta = \mathbf{1}$, the trivial subgroup of Γ. The ring of invariants $\mathcal{E}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)} = \langle z\bar{z}, \lambda\rangle + \mu\mathcal{E}_{(z,\lambda,\mu)}$. It follows from Corollary 2.2 that $\mathcal{E}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)}$ is not a DA-algebra because $\mathcal{E}_z$ is not a finite $\mathcal{E}^{\mathrm{O}(2)}_z$-module because the local algebra $\mathcal{E}_z/\langle z\bar{z}\rangle$ cannot be finite.

(2) When Γ is a finite group acting on $\mathbb{R}^n$, the origin is an isolated singularity for the quotient space in the complexification, and so $\mathcal{E}^{\Delta}_x$ is a finite $\mathcal{E}^{\Gamma}_x$-module. When Γ is a faithful representation of a compact continuous Lie group and Δ is finite the result is false.

(3) When $\Gamma = \mathrm{Z}_2$ acts on $\mathbb{R}$ via $(-1)x \mapsto (-x)$, and $\Delta = \mathbf{1}$, $\mathcal{E}^{\mathrm{Z}_2,\mathbf{1}}_{(x,\lambda,\mu)}$ is a DA-algebra with generators x^2, λ, μ and μx. In case $\Gamma = \mathrm{D}_3$, acting on $\mathbb{R}^2$ via the rotation of $2\pi/3$ and the reflection $(x, y) \to (x, -y)$, and $\Delta = \mathbf{1}$, $\mathcal{E}^{\mathrm{D}_3,\mathbf{1}}_{(x,y,\lambda,\mu)}$ is a DA-algebra generated by x^2+y^2, $x^3 - 3xy^2$, λ, μ, μx, μy, μxy, μy^2 and μy^3.

2.2.2. Systems of algebras.

Let $(\mathcal{D}, \leq)$ be a finite partially ordered set of indices. A **system of (extended) DA-algebras** consists of a set of (extended) DA-algebras $\{R_\alpha\}_{\alpha\in\mathcal{D}}$ together with homomorphisms of (extended) DA-algebras $\phi_{\alpha\beta}\colon R_\alpha \to R_\beta$ defined for $\alpha \leq \beta$ so that $\phi_{\gamma\beta}\circ\phi_{\alpha\gamma} = \phi_{\alpha\beta}$, for $\alpha \leq \beta \leq \gamma$, and $\phi_{\alpha\alpha} = \mathrm{id}$. We only allow extended DA-algebras for indices *maximal* in $\mathcal{D}$. In our context,

$$\{R_{(x,\lambda,\mu)}\} = \{\mathcal{E}_{(\lambda,\mu)}, \mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu)}\} \tag{2.1}$$

is a system of (extended) DA-algebras. A **system of ideals** $\{\mathcal{I}_\alpha\}_{\alpha\in\mathcal{D}}$ of $\{R_\alpha\}_{\alpha\in\mathcal{D}}$ consists of ideals $\mathcal{I}_\alpha$ of R_α so that $\phi_{\alpha\beta}(\mathcal{I}_\alpha) \subseteq \mathcal{I}_\beta$ for $\alpha \leq \beta$. Then, $\{(R_\alpha, \mathcal{I}_\alpha)\}_{\alpha\in\mathcal{D}}$ denotes a system of (extended) DA-algebras and ideals. We use for the unfolding theory the system of (extended) DA-algebras $\{\mathcal{E}_\beta, \mathcal{E}_{(\lambda,\mu,\beta)}, \mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}\}$. To specify in this last system the

maximal ideals we define
(2.2)
$$\{R_{(x,\lambda,\mu,\beta)}\} = \{(\mathcal{E}_\beta, \mathcal{M}_\beta), (\mathcal{E}_{(\lambda,\mu,\beta)}, \mathcal{M}_{(\lambda,\mu,\beta)}), (\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}, \mathcal{M}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)})\}.$$

An $\{R_{(x,\lambda,\mu,\beta)}\}$**-module** M consists of a direct sum $M_1 \oplus M_2 \oplus M_3$ of an $\mathcal{E}_\beta$-module, an $\mathcal{E}_{(\lambda,\mu,\beta)}$-module and an $\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}$-module, respectively. The module M is said to be **finitely generated** if each M_i is a finitely generated module over the corresponding ring R_i, $1 \le i \le 3$, but M is **almost finitely generated** if only M_1, M_2 are finitely generated. An $\{R_{(x,\lambda,\mu,\beta)}\}$**-module homomorphism** $\psi\colon M \to N$ consists of a sum of homomorphisms $\psi_{ij}\colon cM_i \to N_j$, for $1 \le i \le j \le 3$, which are homomorphisms over the appropriate connecting homomorphisms ϕ^*_{ij} of the system $\{R_{(x,\lambda,\mu,\beta)}\}$. We say that N is an $\{R_{(x,\lambda,\mu,\beta)}\}$**-submodule** of M if $N = N_1 \oplus N_2 \oplus N_3$, where N_i is a submodule of M_i for all $1 \le i \le 3$. If $\{\mathcal{I}_{(x,\lambda,\mu,\beta)}\} = \{\mathcal{I}_\beta, \mathcal{I}_{(\lambda,\mu,\beta)}, \mathcal{I}_{(x,\lambda,\mu,\beta)}\}$ is a system of ideals of $\{R_{(x,\lambda,\mu,\beta)}\}$, we define the $\{R_{(x,\lambda,\mu,\beta)}\}$-submodule of M, $\{\mathcal{I}_{(x,\lambda,\mu,\beta)}\} \cdot M = \mathcal{I}_\beta \cdot M_1 \oplus \mathcal{I}_{(\lambda,\mu,\beta)} \cdot M_2 \oplus \mathcal{I}_{(x,\lambda,\mu,\beta)} \cdot M_3$. Similar definitions can be made for the system $\{R_{(x,\lambda,\mu)}\}$ by setting $\beta = 0$. Note that each ring in $\{R_{(x,\lambda,\mu,\beta)}\}$ is a $\mathcal{E}_\beta$-algebra. We say that $\{R_{(x,\lambda,\mu,\beta)}\}$ is an **adequately ordered system of (extended) DA-algebras over** $\mathcal{E}_\beta$ if each connecting homomorphism is an $\mathcal{E}_\beta$-algebra homomorphism and each ring has one predecessor. For such systems a version of the Preparation Theorem holds true.

Theorem 2.3 (Damon [**8**]). *If $\{R_{(x,\lambda,\mu,\beta)}\}$ is an adequately ordered system of (extended) DA-algebras then, if $\Psi\colon N \to M$ is a homomorphism of $\{R_{(x,\lambda,\mu,\beta)}\}$-modules, with M finitely generated and N (finitely) almost finitely generated, such that $\Psi(N) + \{I_{(x,\lambda,\mu,\beta)}\} \cdot M = M$, then $\Psi(N) = M$ and $\Psi(\{I_{(x,\lambda,\mu,\beta)}\} \cdot N) = \{I_{(x,\lambda,\mu,\beta)}\} \cdot M$.*

2.3. Bifurcation Equivalence.

Next we introduce some equivalence relations we need to organise the theory.

2.3.1. $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalence.

Let $M_n(\mathbb{R})$ be the set of real $n \times n$-matrices. Let $\mathbf{M}^\Gamma_{(x,\lambda)}$ be the $\mathcal{E}^\Gamma_{(x,\lambda)}$-module of smooth Γ-equivariant matrix-valued maps $T_1\colon (\mathbb{R}^{n+1}, 0) \to M_n(\mathbb{R})$, $T_1(\gamma x, \lambda)\,\gamma = \gamma\, T_1(x,\lambda)$, $\forall\ \gamma \in \Gamma$, and let $\mathbf{M}^\Delta_{(x,\lambda,\mu)}$ be the $\mathcal{E}^\Delta_{(x,\lambda,\mu)}$-module of smooth Δ-equivariant matrix-valued maps. Let $\mathbf{M}^{\Gamma,\Delta}_{(x,\lambda,\mu)} = \mathbf{M}^\Gamma_{(x,\lambda)} + \mu \mathbf{M}^\Delta_{(x,\lambda,\mu)}$. We denote by $\mathrm{GL}^\Gamma_{(x,\lambda)}$, $\mathrm{GL}^\Delta_{(x,\lambda,\mu)}$ and by $\mathrm{GL}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ the corresponding subsets of matrix-valued functions with values in $\mathrm{GL}(n)$. We also define $\vec{\mathcal{M}}_\Lambda = \vec{\mathcal{M}}_\lambda + \mu\vec{\mathcal{E}}_{(\lambda,\mu)}$. The group of equivalences we are going to work with must preserve the Γ-equivariance of the $(\mu = 0)$-slice. A straightforward group of equivalences is obtained by combining $\mathcal{K}^{\Gamma}_{(x,\lambda)}$ and $\mathcal{K}^{\Delta}_{(x,\lambda,\mu)}$ in the group $\hat{\mathrm{K}}(\Gamma,\Delta)$, defined as the connected component of the identity in

$$\{(T, X, L) : T \in \mathbf{M}^{\Gamma,\Delta}_{(x,\lambda,\mu)},\ X \in \vec{\mathcal{M}}^{\Gamma,\Delta}_{(x,\lambda,\mu)},\ L \in \vec{\mathcal{M}}_\Lambda\}.$$

But, equivalences which leave germs in $\vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$ point-wise invariant may provide a fundamental non trivial contribution in the μ-dependent part of $\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$. This key fact when Γ is a continuous group was already recognised in [**18**]. We identify two groups of such equivalences: $\hat{\mathrm{M}}(\Gamma,\Delta) = \{M\colon (\mathbb{R}^{n+2}, 0) \to \Gamma_0 : M \in \mathrm{GL}^{\Delta}_{(x,\lambda,\mu)}\}$ and $\hat{\mathrm{S}}(\Gamma,\Delta)$ as the identity component in $\{S \in \mathrm{GL}^{\Delta}_{(x,\lambda,\mu)} : S(x,\lambda,\mu)\, g(x,\lambda) = g(x,\lambda),\ \forall\ g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}\}$. In Lemmas 2.5 and 2.6 we give alternative descriptions.

Remark. We can always assume that the changes of coordinates in $\hat{\mathrm{M}}(\Gamma,\Delta)$ and $\hat{\mathrm{S}}(\Gamma,\Delta)$ are Δ-equivariant because nothing is gained by relaxing this condition (because we can always average non equivariant change of coordinates over Δ).

Finally, considering words of finite length on the elements of the three subgroups $\hat{\mathrm{M}}(\Gamma,\Delta)$, $\hat{\mathrm{S}}(\Gamma,\Delta)$ and $\hat{\mathrm{K}}(\Gamma,\Delta)$, our changes of coordinates form a group, denoted by $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$, defined as the free product ([**24**])

$$\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)} = \hat{\mathrm{M}}(\Gamma,\Delta) * \hat{\mathrm{S}}(\Gamma,\Delta) * \hat{\mathrm{K}}(\Gamma,\Delta).$$

The action of $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ on $f = f_1 + \mu f_2 \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is defined in the following way:

- for $(T, X, L) \in \hat{\mathrm{K}}(\Gamma,\Delta)$, we have $(T, X, L)\cdot f = Tf(X, L)$, in co-ordinates
$$\begin{aligned}((T,X,L)\cdot f)\,(x,\lambda,\mu) = {} & T_1(x,\lambda)\, f_1(X(x,\lambda,\mu), L_1(\lambda,\mu)) \\ & + \mu\,[T_1(x,\lambda)\, f_2(X(x,\lambda,\mu), L(\lambda,\mu)) \\ & \quad + T_2(x,\lambda,\mu)\, f(X(x,\lambda,\mu), L(\lambda,\mu))],\end{aligned}$$
- for $T \in \hat{\mathrm{M}}(\Gamma,\Delta)$, we have $T \cdot f = T^{-1} f(T)$, in co-ordinates
$$(T\cdot f)(x,\lambda,\mu) = f_1(x,\lambda) + \mu\, T^{-1}(x,\lambda,\mu)\, f_2(T(x,\lambda,\mu)x, \lambda, \mu),$$
- for $S \in \hat{\mathrm{S}}(\Gamma,\Delta)$, we have $S \cdot f = S$, in co-ordinates
$$(S\cdot f)(x,\lambda,\mu) = f_1(x,\lambda) + \mu\, S(x,\lambda,\mu)\, f_2(x,\lambda,\mu).$$

The groups of contact equivalences $\mathcal{K}$, or $\mathcal{K}_\lambda^\Gamma$, are semi-direct products. What about $\mathcal{K}_{(\lambda,\mu)}^{\Gamma,\Delta}$? Although $\hat{\mathrm{M}}(\Gamma,\Delta)$ and $\hat{\mathrm{S}}(\Gamma,\Delta)$ are not normal subgroups in $\mathcal{K}_{(\lambda,\mu)}^{\Gamma,\Delta}$, we show next that we can use the changes of coordinates in $\mathcal{K}_{(\lambda,\mu)}^{\Gamma,\Delta}$ in any order we may wish, and so we can reduce the words of any length in $\mathcal{K}_{(\lambda,\mu)}^{\Gamma,\Delta}$ into words of length at most 3 with one element in each subgroup.

Lemma 2.4. *Any element in $\mathcal{K}_{(\lambda,\mu)}^{\Gamma,\Delta}$ can be written as a product of only three elements, one in each of $\hat{\mathrm{M}}(\Gamma,\Delta)$, $\hat{\mathrm{S}}(\Gamma,\Delta)$, $\hat{\mathrm{K}}(\Gamma,\Delta)$.*

Proof: First, as for $\mathcal{K}_\lambda^\Gamma$, $\hat{\mathrm{K}}(\Gamma,\Delta)$ is the direct product of $\hat{\Lambda}=\{(\mathbf{I}_n,\mathbf{I}_n,L(\lambda,\mu))\in\hat{\mathrm{K}}(\Gamma,\Delta)\}$ with the semi-direct product of $\hat{\mathrm{T}}(\Gamma,\Delta)=\{(T,\mathbf{I}_n,\mathbf{I}_2)\in\hat{\mathrm{K}}(\Gamma,\Delta)\}$ and $\hat{\mathrm{X}}(\Gamma,\Delta)=\{(\mathbf{I}_n,X,\mathbf{I}_2)\in\hat{\mathrm{K}}(\Gamma,\Delta)\}$. The elements of $\hat{\Lambda}$ commute with the elements of the other groups, so we do not need to consider their influence. We deal with the relations between $\hat{\mathrm{M}}(\Gamma,\Delta)$, $\hat{\mathrm{S}}(\Gamma,\Delta)$, $\hat{\mathrm{X}}(\Gamma,\Delta)$ and $\hat{\mathrm{T}}(\Gamma,\Delta)$, showing how we can make their elements to interchange, to combine finally together the different terms of each of them. In the following we consider the different cases, in turn. The proofs are straightforward verifications from the definitions.

(1) First, $\hat{\mathrm{M}}(\Gamma,\Delta)$ is in the normaliser of $\hat{\mathrm{T}}(\Gamma,\Delta)$. For $M\in\hat{\mathrm{M}}(\Gamma,\Delta)$ and $T\in\hat{\mathrm{T}}(\Gamma,\Delta)$, define $U(x)=M^{-1}(x)T(M(x)x)M(x)$. Then, $U\in\hat{\mathrm{T}}(\Gamma,\Delta)$ and $MT=UM$.

(2) Let $M\in\hat{\mathrm{M}}(\Gamma,\Delta)$ and $X\in\hat{\mathrm{X}}(\Gamma,\Delta)$. Define $Y(x)=M^{-1}(x)X(M(x)x)$ and $N(x)=M(Y^{-1}(x))$. Then, $Y\in\hat{\mathrm{X}}(\Gamma,\Delta)$, $N\in\hat{\mathrm{M}}(\Gamma,\Delta)$ and $MX=YN$.

(3) The group $\hat{\mathrm{M}}(\Gamma,\Delta)$ is in the normaliser of $\hat{\mathrm{S}}(\Gamma,\Delta)$. Let $S\in\hat{\mathrm{S}}(\Gamma,\Delta)$ and $M\in\hat{\mathrm{M}}(\Gamma,\Delta)$, define $R(x)=M^{-1}(x)S(M(x)x)M(x)$. Then, $R\in\hat{\mathrm{S}}(\Gamma,\Delta)$ and $MS=RM$.

(4) The group $\hat{\mathrm{S}}(\Gamma,\Delta)$ is in the normaliser of $\hat{\mathrm{X}}(\Gamma,\Delta)$. Let $S\in\hat{\mathrm{S}}(\Gamma,\Delta)$ and $X\in\hat{\mathrm{X}}(\Gamma,\Delta)$, define $R(x)=S(X(x))$. Then, $R\in\hat{\mathrm{S}}(\Gamma,\Delta)$ and $XS=RX$.

(5) Finally, let $S\in\hat{\mathrm{S}}(\Gamma,\Delta)$ and $T\in\hat{\mathrm{T}}(\Gamma,\Delta)$. Define $R(x)=T_1(x)S(x)T_1^{-1}(x)$ and $U(x)=R^{-1}(x)T(x)S(x)$. Then, $R\in\hat{\mathrm{S}}(\Gamma,\Delta)$, $U\in\hat{\mathrm{T}}(\Gamma,\Delta)$ and $TS=RU$. □

Example. To show why composing elements of $\hat{\mathrm{S}}(\Gamma,\Delta)$ and $\hat{\mathrm{T}}(\Gamma,\Delta)$ does not form a direct product, consider the case we are going to look at in Section 3, $\Gamma=\mathrm{O}(2)$ and $\Delta=\mathbf{1}$. Let $S=\begin{pmatrix}1-y & x\\ 0 & 1\end{pmatrix}\in\hat{\mathrm{S}}(\mathrm{O}(2),\mathbf{1})$

and $T = \begin{pmatrix} x^2-y^2 & 2xy \\ 2xy & y^2-x^2 \end{pmatrix} \in \hat{\mathrm{T}}(\mathrm{O}(2), \mathbf{1})$. Then, although it acts like T on $\vec{\mathcal{E}}^{\mathrm{O}(2)}_{(z,\lambda)}$, $S^{-1}TS$ is not O(2)-equivariant because it has cubic terms in x and y. In the other direction, take $\bar{T} = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix} \in \hat{\mathrm{T}}(\mathrm{O}(2), \mathbf{1})$, then $\bar{T}^{-1} S \bar{T}$ is not in $\hat{\mathrm{S}}(\mathrm{O}(2), \mathbf{1})$.

2.3.2. $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu,\beta)}$-equivalence.

Let $\beta \in (\mathbb{R}^q, 0)$. The definitions of Section 2.3.1 clearly extend to their β-parametrised versions for unfoldings: $\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}$, $\mathbf{M}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}$, $\vec{\mathcal{M}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}$ and $\vec{\mathcal{M}}_{(\lambda,\mu,\beta)}$. Perturbations of any $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ are described by ***q*-parameter unfoldings** of f. They are map-germs $F \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}$ such that $F(x,\lambda,\mu,0) = f(x,\lambda,\mu)$ and $F(x,\lambda,0,0) = f(x,\lambda,0)$, that is, $F(x,\lambda,0,\beta)$ is a q-parameter unfolding of $f(x,\lambda,0)$. We denote by $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu,\beta)}$ the group of (Γ,Δ)-equivariant equivalences for q-parameter unfoldings. It is a natural extension of $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ in the following sense:

$$\begin{aligned}\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu,\beta)} = \{(T,X,L,\Phi) \in \mathbf{M}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)} \times \vec{\mathcal{M}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)} \times \vec{\mathcal{M}}_{(\lambda,\mu,\beta)} \times \mathcal{M}_{\beta,\beta} :\\ (T,X,L) \text{ is a } q\text{-parameter unfolding of an element of } \mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}\\ \text{and } \Phi \text{ is a diffeomorphism germ}\}.\end{aligned}$$

The action of $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu,\beta)}$ on $F \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}$ is defined by

$$(T,X,\Lambda,\Phi)\cdot F(x,\lambda,\mu,\beta) = T(x,\lambda,\mu,\beta)\, F(X(x,\lambda,\mu,\beta), L(\lambda,\mu,\beta), \Phi(\beta)).$$

We say that $F, G \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}$ are $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu,\beta)}$**-equivalent** if they belong to the same $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu,\beta)}$-orbit.

2.3.3. Characterisations of $\hat{\mathrm{M}}(\Gamma,\Delta)$, $\hat{\mathrm{S}}(\Gamma,\Delta)$ and their tangent spaces.

We begin by showing that the elements of $\hat{\mathrm{S}}(\Gamma,\Delta)$ are matrices of $\mathrm{GL}^{\Delta}_{(x,\lambda,\mu)}$ keeping each generator $\{Y_i\}_{i=1}^J$ of $\vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$ invariant. Then its tangent space $\hat{s}(\Gamma,\Delta)$ is formed of the matrices $M_n^{\Delta}(\mathbb{R})$ annihilating all the generators of $\vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$.

Lemma 2.5. $\hat{\mathrm{S}}(\Gamma,\Delta) = \{S \in \mathrm{GL}^{\Delta}_{(x,\lambda,\mu)} : S(x,\lambda,\mu)\, Y_j(x) = Y_j(x),\ 1 \le j \le J\}$ *and the tangent space of* $\hat{\mathrm{S}}(\Gamma,\Delta)$, $\hat{s}(\Gamma,\Delta) = \{S\colon (\mathbb{R}^{n+2},0) \to M_n^{\Delta}(\mathbb{R}) : S \cdot Y_j = 0,\ 1 \le j \le J\}$.

Proof: Let $S \in \hat{\mathrm{S}}(\Gamma,\Delta)$, that is, $S \cdot g = g$, $\forall\ g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$. In particular, $S \cdot Y_j = Y_j$, $1 \le j \le J$. Conversely, let $S \in \{T \in \mathrm{GL}^{\Delta}_{(x,\lambda,\mu)} : T(x,\lambda,\mu)\, Y_j(x) = Y_j(x),\ 1 \le j \le J\}$. For any $g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$, we may write $g(x,\lambda) = \sum_{j=1}^{J} p_j(\bar{u}_1(x),\ldots,\bar{u}_r(x),\lambda)\, Y_j(x)$. And so, $S(x,\lambda,\mu)\, g(x,\lambda) = g(x,\lambda)$.

Let $\psi\colon (-\epsilon,\epsilon) \to \hat{\mathrm{S}}(\Gamma,\Delta)$ be a path given by $\psi(t) = \bar{S}(x,\lambda,\mu,t)$ with $\psi^o = \mathbf{I}_n$ ($\bar{S}\, Y_j = Y_j$, $1 \le j \le J$). Note that $\dot{\psi}^o = \frac{\partial \bar{S}}{\partial t}\big|_{t=0} = S \in \hat{s}(\Gamma,\Delta)$ and so $S \cdot Y_j = 0$, $1 \le j \le J$. □

Next, consider $\hat{\mathrm{M}}(\Gamma,\Delta)$. Recall that $\hat{\mathrm{M}}(\Gamma,\Delta) = \{M\colon (\mathbb{R}^{n+2},0) \to \Gamma_0 : M \in \mathrm{GL}^{\Delta}_{(x,\lambda,\mu)}\}$.

Lemma 2.6. (1) *The tangent space of* $\hat{\mathrm{M}}(\Gamma,\Delta)$ *is*

$$\hat{m}(\Gamma,\Delta) = \{T\colon (\mathbb{R}^{n+2},0) \to \ell_\Gamma(n) : T \text{ is } \Delta\text{-equivariant}\} \tag{2.3}$$

where $\ell_\Gamma(n)$ *is the Lie algebra of* Γ. *The action of* $T \in \hat{m}(\Gamma,\Delta)$ *on* $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ *is given by*

$$T \cdot f = -Tf + f_x T x. \tag{2.4}$$

(2) $\hat{\mathrm{M}}(\Gamma,\Delta)$ *is the connected component of the identity of*

$$\tilde{\mathrm{M}} = \{M \in \mathrm{GL}^{\Delta}_{(x,\lambda,\mu)} : M^{-1}(x,\lambda,\mu)\, g(M(x,\lambda,\mu)x,\lambda) = g(x,\lambda),\quad \forall\ g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}\}.$$

Proof: (1) Let $\gamma\colon (-\epsilon,\epsilon) \to \hat{\mathrm{M}}(\Gamma,\Delta)$ be a path given by $\gamma(t) = \bar{T}(x,\lambda,\mu,t)$ with $\gamma^o = \mathbf{I}_n$. It acts as $t \mapsto \bar{T}^{-1}(t) g(\bar{T}(t)x)$. Then, from the Chain Rule, $\dot{\gamma}^o = \frac{\partial \bar{T}}{\partial t}^o = T \in \hat{m}(\Gamma,\Delta)$ is a Δ-equivariant map with values in $\ell_\Gamma(n)$, and satisfies $-Tg + g_x T x = 0$ for all $g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$. Hence its action on $\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is given by $T \cdot f = -Tf + f_x T x$ for all $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ as claimed in (2.4).

(2) Clearly $\hat{\mathrm{M}}(\Gamma,\Delta) \subset \tilde{\mathrm{M}}$ and is connected and contains the identity. For the converse, we establish first that we can test $M \in \tilde{\mathrm{M}}$ on its behaviour on the Γ-invariant generators and Γ-equivariant generators:

$$\tilde{\mathrm{M}} = \{M \in \mathrm{GL}^{\Delta}_{(x,\lambda,\mu)} : \bar{u}_i(M(x,\lambda,\mu)x) = \bar{u}_i(x),\ 1 \le i \le r, \quad \text{and} \quad M^{-1}(x,\lambda,\mu)\, Y_j(M(x,\lambda,\mu)x) = Y_j(x),\ 2 \le j \le J\}.$$

Let $M \in \mathrm{GL}^{\Delta}_{(x,\lambda,\mu)}$ such that $\bar{u}_i(M(x,\lambda,\mu)x) = \bar{u}_i(x)$, $1 \le i \le r$, and $M^{-1} Y_j(Mx) = Y_j(x)$, $2 \le j \le J$. Let $g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$. We may write

$g(x,\lambda) = \sum_{j=1}^{J} p_j(\bar{u}_1(x),\ldots,\bar{u}_r(x),\lambda)\, Y_j(x)$, for some $\{p_j\}_{j=1}^{J}$. So,

$$\begin{aligned} g(M(x,\lambda,\mu)x,\lambda) = \sum_{j=1}^{J} & p_j(\bar{u}_1(M(x,\lambda,\mu)x),\ldots \\ & \ldots,\bar{u}_r(M(x,\lambda,\mu)x),\lambda)\, Y_j(M(x,\lambda,\mu)x) \\ = {} & M(x,\lambda,\mu)g(x,\lambda), \end{aligned}$$

hence $M \in \tilde{\mathrm{M}}$. The converse is true by considering separately $\bar{u}_i Y_1$, $1 \le i \le r$, and Y_i, $2 \le i \le J$.

Let $t \in (\mathbb{R},0)$ and $M(x,\lambda,\mu,t)$ be a family of elements in $\tilde{\mathrm{M}}$, unfolding the identity. Differentiating the conditions $\bar{u}_i(M(x,\lambda,\mu,t)x) = \bar{u}_i(x)$, $1 \le i \le r$, with respect to t (at the origin), the tangent space of $\tilde{\mathrm{M}}$ consists of Δ-equivariant maps m such that $\langle \nabla_x \bar{u}_i(x), m(x)x\rangle = 0$, $1 \le i \le r$. If we introduce the orbit-map for Γ, $\Phi\colon \mathbb{R}^n/\Gamma \to \mathbb{R}^r$, defined by $\Phi(\Gamma \cdot x) = (\bar{u}_1(x),\ldots,\bar{u}_r(x))$, then $\langle \nabla_x \Phi(x), m(x)x\rangle = 0$ which means that $m(x) \in \ell_\Gamma(n)$ so that $\tilde{\mathrm{M}}$ and $\hat{\mathrm{M}}(\Gamma,\Delta)$ have the same tangent spaces. The local changes of coordinates form a local Lie group (see [**28**]). By uniqueness between the Lie algebras and the connected component of the identity of local Lie groups we can conclude that every element in the connected component of the identity in $\tilde{\mathrm{M}}$ takes its value in Γ_0 and so the conclusion. □

Remark. When Γ is finite, $\hat{m}(\Gamma,\Delta) = 0$ (because $\hat{\mathrm{M}}(\Gamma,\Delta)$ is trivial as $\Gamma_0 = \{\mathbf{I}_n\}$) and $\hat{s}(\Gamma,\Delta) = 0$ (because the range of any $S \in \hat{\mathrm{S}}(\Gamma,\Delta)$ is finite, hence locally constant). If the rank of the $n\times J$-matrix $(Y_1,\ldots,Y_J)$ is smaller than n in a (punctured) open sectorial neighbourhood of the origin, it is then possible to find a nontrivial $\hat{s}(\Gamma,\Delta)$.

2.3.4. Tangent spaces.

Associated with $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalence we define the **extended tangent space** of $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$:

$$\begin{aligned} \mathcal{T}^{\Gamma,\Delta}_e(f) = \{ & T_1 f_1 + f_{1x} X_1 + f_{1\lambda} L_{11} \\ & + \mu(T_2 f_1 + f_{1x} X_2 + f_{1\lambda} L_{12} + T_1 f_2 + f_{2x} X_1 + f_{2\lambda} L_{11} + f_2 L_{22}) \\ & + \mu^2 (T_2 f_2 + f_{2x} X_2 + f_{2\lambda} L_{12} + f_{2\mu} L_{22}) + \mu\,\hat{m}(f_2) + \mu\,\hat{s}(f_2)\} \end{aligned}$$

where $T = T_1 + \mu T_2 \in \mathbf{M}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$, $X = X_1 + \mu X_2 \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$, $L_{11} \in \mathcal{E}_\lambda$, $L_{12}, L_{22} \in \mathcal{E}_{(\lambda,\mu)}$, and $\hat{m}(\Gamma,\Delta)$, $\hat{s}(\Gamma,\Delta)$, are, respectively, the tangent spaces of $\hat{\mathrm{M}}(\Gamma,\Delta)$ and $\hat{\mathrm{S}}(\Gamma,\Delta)$ as described in Lemmas 2.5 and 2.6. The modules $\mathbf{M}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$, $\vec{\mathcal{M}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$, $\hat{m}(\Gamma,\Delta)$ and $\hat{s}(\Gamma,\Delta)$ are finitely generated over the ring $\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ and $\vec{\mathcal{E}}_{(\lambda,\mu)}$ is finitely generated over $\mathcal{E}_{(\lambda,\mu)}$.

The extended tangent space $\mathcal{T}^{\Gamma,\Delta}_e(f)$ is a direct sum of finitely generated modules over the system of rings $\{R_{(x,\lambda,\mu)}\}$ (see (2.1)). The **extended normal space** to f is defined by $\mathcal{N}^{\Gamma,\Delta}_e(f) = \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}/\mathcal{T}^{\Gamma,\Delta}_e(f)$ and the **(Γ,Δ)-codimension** of $f\in\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is $\mathrm{cod}^{\Gamma}_{\Delta}(f)=\dim_{\mathbb{R}}\mathcal{N}^{\Gamma,\Delta}_e(f)$. Another tangent space to consider is the **unipotent tangent space** $\mathcal{TU}^{\Gamma,\Delta}_{(\lambda,\mu)}(f)$ which is fundamental in the calculations of higher order terms, defined in (2.6).

2.3.5. Geometric subgroups: the 'best' situation.

We have now all the ingredient needed to check that our theory satisfies the abstract framework of Damon [**8**], so that the major theorems about unfolding and determinacy theories hold true. Efficient summaries of what is needed can be found in [**9**] or [**1**, pp. 170–178]. We refer to Damon's notation in [**9**], followed in [**1**]. The results and discussions from the previous subsections show that $\mathcal{G} = \mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ is a geometric subgroup of $\mathcal{K}$, that is, it satisfies the four conditions: naturality (with respect to pullback), algebraic structure of tangent spaces, exponential map and filtration property. Our group $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ acts on the linear subspace $\mathcal{F} = \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$. For any $q\in\mathbb{N}$, let $\beta\in(\mathbb{R}^q,0)$, the linear subspace of q-parameter unfoldings are $\mathcal{F}_{un}(q) = \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}$. The subgroups acting on $\mathcal{F}_{un}(q)$ are $\mathcal{G}_{un}(q) = \{(T,X,L,\Phi)\in\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu,\beta)} : \Phi = \mathrm{I}_\beta\}$ and, finally,

$$\mathcal{G}_{eq}(q) = \{(T,X,L,\Phi)\in\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu,\beta)} : (T,X,L) \text{ is an unfolding of the identity in } \mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}\}.$$

The algebraic structure of the tangent spaces is achieved via the (extended) system of DA-algebras defined in (2.1), (2.2), that are **nice** (adequately ordered). Explicitly, $\{R_{(x,\lambda,\mu)}\} = \{\mathcal{E}_{(\lambda,\mu)},\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu)}\}$ and $\{R_{(x,\lambda,\mu,\beta)}\} = \{\mathcal{E}_\beta,\mathcal{E}_{(\lambda,\mu,\beta)},\mathcal{E}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}\}$.

In an ideal world, we would be able to show that if $f,g\in\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ are bifurcation equivalent with (T,X,L) only in $\mathcal{K}^{\Delta}_{(\lambda,\mu)}$, then there exists a $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalence between f and g. We cannot prove it, but we can show that the definition of $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ corresponds to the largest extended tangent space in the class of geometric subgroups respecting the structure of $\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$.

Theorem 2.7. *Let $\tilde{\mathcal{K}}$ be any geometric subgroup of contact equivalences acting on $\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ and preserving globally its slice $\mathcal{E}^{\Gamma}_{(x,\lambda)}$ where $\mu = 0$. Then its extended tangent space $\tilde{k}$ is contained in the extended tangent space of $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$.*

Proof: Without additional restrictions, we may assume that $\tilde{\mathcal{K}} \subset \mathcal{K}^{\Delta}_{(\lambda,\mu)}$. Hence, its tangent space $\tilde{k}$ is contained in the tangent space $k^{\Delta}_{(\lambda,\mu)}$ of $\mathcal{K}^{\Delta}_{(\lambda,\mu)}$. The point is to identify all possible $(T, X) \in k^{\Delta}_{(\lambda,\mu)}$ such that

$$Tg + g_x X \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}, \quad \forall\, g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}. \tag{2.5}$$

We show that they are a sum of elements of the tangent spaces $\hat{k}(\Gamma,\Delta)$, $\hat{m}(\Gamma,\Delta)$ and $\hat{s}(\Gamma,\Delta)$ of $\hat{\mathrm{K}}(\Gamma,\Delta)$, $\hat{\mathrm{M}}(\Gamma,\Delta)$ and $\hat{\mathrm{S}}(\Gamma,\Delta)$, respectively.

First note that T is define modulo $S \in \hat{\mathrm{S}}(\Gamma,\Delta)$ because $Sg = 0$, $\forall\, g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$. Then, take $g(x) = x$ in (2.5). We get $Tx + X \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$, and so there exists $p \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$ such that $X = p - Tx$. Hence, (2.5) becomes $Tg + g_x p - g_x Tx \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$, $\forall\, g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$. Because $g_x p \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$, we have

$$Tg - g_x Tx \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}, \quad \forall\, g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)},$$

that is, $Tg - g_x Tx = q$ for some $q \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}$, depending on g. To eliminate q, consider $Tg - g_x Tx = q$ at γx and use the Γ-equivariance. We get

$$T(\gamma x)\gamma g(x) - \gamma g_x(x)\gamma^{-1}T(\gamma x)\gamma\, x = \gamma q(x).$$

Multiplying through by γ^{-1}, averaging over Γ and introducing $T^{\Gamma}(x,\lambda) = \int_{\Gamma} \gamma^{-1}T(\gamma x,\lambda,\mu)\gamma\, d\gamma$, we find $T^{\Gamma}g - g_x\, T^{\Gamma}x = q$. Subtracting from $Tg - g_x Tx = q$ to eliminate q, we find

$$(T - T^{\Gamma})\, g - g_x(T - T^{\Gamma})\, x = 0, \quad \forall\, g \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda)}.$$

Therefore, $T - T^{\Gamma} = M$, $M \in \hat{m}(\Gamma,\Delta)$ (see Lemma 2.6), hence $T = T^{\Gamma} + M + S$. We find that $(T, X) = (T^{\Gamma}, p - T^{\Gamma}x) + (M, 0) + (S, 0)$, hence $\tilde{k} \subset \hat{k}(\Gamma,\Delta) + \hat{m}(\Gamma,\Delta) + \hat{s}(\Gamma,\Delta)$. ☐

2.4. Unfolding and Determinacy Theory.

The Unfolding and Finite Determinacy Theorems are a rewording of the corresponding results in the general theory of [**8**]. Let $F \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)}$, resp. $G \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\alpha)}$, be k-, resp. r-parameter unfoldings of $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$.

We say that $\boldsymbol{G}$ **maps into** $\boldsymbol{F}$, or $\boldsymbol{G}$ **factors through** $\boldsymbol{F}$, if there exist $T \in \mathbf{M}^{\Gamma,\Delta}_{(x,\lambda,\alpha)}$, $X \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\alpha)}$, $L \in \vec{\mathcal{E}}_{(\lambda,\mu,\alpha)}$ and $A\colon (\mathbb{R}^r,0) \to (\mathbb{R}^k,0)$ satisfying $T(x,\lambda,\mu,0) = \mathbf{I}_n$, $X(x,\lambda,\mu,0) = x$ and $L(\lambda,\mu,0) = (\lambda,\mu)$, such that

$$G(x,\lambda,\mu,\alpha) = T(x,\lambda,\mu,\alpha)\, F(X(x,\lambda,\mu,\alpha), L(\lambda,\mu,\alpha), A(\alpha)).$$

The unfolding F is called **versal** if any unfolding G of f maps into F. A versal unfolding with minimal number of parameter is called **miniversal**. As a direct consequence of the abstract results of Damon [**8**, Theorems 9.3 and 11.4, and Corollary 9.9], we get the usual results from unfolding theory.

Theorem 2.8 (Unfolding Theorem)**.** *Let $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ and $F \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\alpha)}$ be a k-parameter unfolding of f, $\alpha = (\alpha_1,\ldots,\alpha_k)$. Then*

(1) *F is versal if and only if*

$$\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)} = \mathcal{T}^{\Gamma,\Delta}_e(f) + \langle F_{\alpha_1}(.,.,0),\ldots,F_{\alpha_k}(.,.,0)\rangle_{\mathbb{R}}.$$

(2) *Two versal unfoldings of a germ in $\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ are equivalent as unfoldings if and only if they have the same number of unfolding parameters.*

(3) *Let $W \subset \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ be a finite dimensional complement of $\mathcal{N}^{\Gamma,\Delta}_e(f)$ as a vector space. Let $\{p_i\}_{i=1}^{\operatorname{cod}^{\Gamma}_{\Delta}(f)}$ be a basis for W. Then a miniversal unfolding of g is*

$$F(x,\lambda,\mu,\alpha) = f(x,\lambda,\mu) + \sum_{j=1}^{\operatorname{cod}^{\Gamma}_{\Delta}(f)} \alpha_j\, p_j(x,\lambda,\mu).$$

(4) *If f and $g \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ are two $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalent germs of finite codimension and F and $G \in \vec{\mathcal{E}}^{\Gamma}_{(x,\lambda,\alpha)}$, with $\alpha = (\alpha_1,\ldots,\alpha_k)$, are two miniversal unfoldings of f and g, respectively, then F and G are $\mathcal{K}^{\Gamma}_{un}(k)$-equivalent.*

For any germ of map f at the origin, we denote by $j^k(f)$ the Taylor polynomial of order k (or *k-jet*) of f at the origin. A germ $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is k-$\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$**-determined** if every germ $g \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ with $j^k(g) = j^k(f)$ is $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalent to f. A germ is **finitely** $\boldsymbol{\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}}$**-determined** if it is k-$\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-determined for some integer k. As usual, there is a close relationship between being finitely determined and being of finite codimension. The first theorem follows from the general theory (see [**8**, Theorems 10.2 and 11.4]).

Theorem 2.9 (Finite Determinacy Theorem). *A germ $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is finitely $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-determined if and only if $\operatorname{cod}^{\Gamma}_{\Delta}(f)$ is finite.*

2.4.1. The recognition problem, higher order terms.

The **recognition problem** seeks conditions under which a germ $g \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalent to a given normal form. To solve a particular recognition problem means to characterise the $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalence class in terms of a finite number of polynomial equalities and inequalities to be satisfied by the Taylor coefficients of the elements of that class. Let $\Phi = (T, X, L) \in \mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ and consider the mapping $f \mapsto \Phi(f) = T \cdot f(X, L)$. A subspace $M \subset \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is **intrinsic** if $\Phi(f) \in M$, $\forall\, f \in M$, $\forall\, \Phi \in \mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$. The **intrinsic part** of $V \subset \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$, denoted by $\operatorname{Itr} V$, is the largest intrinsic subspace of $\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ contained in V. Note that any power of the maximal ideal is intrinsic because the contact equivalence preserves the gradation of the jet space by degrees.

Let $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$. The 'perturbation term' $p \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is of **higher order** with respect to f if $g + p$ is $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalent to f for every g that is $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalent to f. By definition, such a perturbation cannot enter into a solution of the recognition problem for f. We denote by $\mathcal{P}(f)$ the set of all higher order terms of f, that is, $\mathcal{P}(f) = \left\{p \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)} : g + p \sim f,\ \forall\, g \sim f\right\}$ where $\sim$ denotes $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalence. With the usual proof (see [**16**, Proposition 1.7]), one gets the following characterisation.

Proposition 2.10. *For each $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$, the set $\mathcal{P}(f)$ is an intrinsic submodule of $\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$.*

To evaluate $\mathcal{P}(f)$ we need the subgroup $\mathcal{U}^{\Gamma,\Delta}_{(\lambda,\mu)}$ of $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ of unipotent equivalences. To construct that group, let $\mathcal{N}_1$ be a maximal unipotent subgroup of $\mathcal{L}^o_\Gamma(n)$, the connected component of the identity in the subset of $\mathrm{GL}(n)$ of Γ-equivariant matrices, $\mathcal{N}_2$ a similar maximal subgroup of $\mathcal{L}^o_\Delta(n)$ and $\mathcal{N}_3 = \{\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right) \mid a \in \mathbb{R}\}$. Note that if Γ acts absolutely irreducibly, $\mathcal{N}_1$ is trivial. Consider now the projection maps π_1 sending $(T, X, L) \in \hat{\mathrm{K}}(\Gamma, \Delta)$ onto $(T^o, X^o_x, L^o_{(\lambda,\mu)})$, π_2 sending $T \in \hat{\mathrm{M}}(\Gamma, \Delta)$ onto T^o and π_3 sending $S \in \hat{\mathrm{S}}(\Gamma, \Delta)$ onto S^o. We define $\mathcal{U}^{\Gamma,\Delta}_{(\lambda,\mu)}$ as being generated by the combinations of the inverse images $\pi_1^{-1}(\mathcal{N}_1, \mathcal{N}_1, \mathcal{N}_3)$, $\pi_2^{-1}(\mathcal{N}_2)$ and $\pi_3^{-1}(\mathcal{N}_2)$. It is a normal subgroup of $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ consisting of unipotent diffeomorphisms, and is called the subgroup of **unipotent $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalences**. Its associated tangent space at $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ is

(2.6)

$$\begin{aligned} \mathcal{TU}^{\Gamma,\Delta}_{(\lambda,\mu)}(f) = \{ & T_1 f_1 + f_{1x} X_1 + f_{1\lambda} L_{11} \\ & + \mu(T_2 f_1 + f_{1x} X_2 + f_{1\lambda} L_{12} + T_1 f_2 + f_{2x} X_1 + f_{2\lambda} L_{11} + f_2 L_{22}) \\ & + \mu^2(T_2 f_2 + f_{2x} X_2 + f_{2\lambda} L_{12} + f_{2\mu} L_{22}) + \mu\, \widehat{um}(f_2) + \mu\, \widehat{us}(f_2)\} \end{aligned}$$

where $T \in \mathbf{M}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ such that $T^o(= T^o_1) = 0$, $X \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ such that $X^o(= X^o_1) = 0$ and $X^o_x = 0$, $L_{11} \in \mathcal{M}^2_\lambda$, $L_{12} \in \mathcal{E}_{(\lambda,\mu)}$, $L_{22} \in \mathcal{M}_{(\lambda,\mu)}$ and $\widehat{um}$ and $\widehat{us}$ are, respectively, the tangent spaces of the kernels of π_2 and π_3. Following the proof of Theorem 1.17 ([**16**]) we get the following.

Proposition 2.11. *Let $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ be of finite $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-codimension. Then $\mathcal{P}(f) \supset \mathrm{Itr}\, \mathcal{TU}^{\Gamma,\Delta}_{(\lambda,\mu)}(f)$.*

Corollary 2.12. *Let $p \in \mathrm{Itr}\, \mathcal{TU}^{\Gamma,\Delta}_{(\lambda,\mu)}(f)$. Then $f + p$ is $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalent to f.*

2.5. Topological Equivalence.

We have described a theory of smooth equivalences to classify the bifurcation diagrams in $\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ and their deformations. It is a well-known problem of smooth equivalences that they often lead to too fine a classification, containing parametrised families of normal forms. The continuous parameters of such families are called moduli. Although we are mainly interested in topological properties of the diagrams, like number and position of solutions (and some stability questions), because we are also interested in the deformations of those diagrams, we could not simply use homeomorphisms for change of coordinates. For instance, the homeomorphic diagrams of the type $x^2 - \lambda = 0$ and $x^4 - \lambda = 0$ are considered different because their perturbations are different. On the other hand, there are also higher order terms in the smooth normal forms (or their unfoldings) with no topological influence or values of the moduli which can be lumped together because the germs (and their miniversal unfolding) are topologically equivalent. The theory of Damon [**10**], [**11**] deals with two preoccupations:

- when are two smooth normal forms topologically equivalent?
- when is an unfolding topologically versal?

In this paper we use the theory in an utilitarian way; to deal with our germs which can have a large smooth codimension (up to 12 when we cap the topological codimension at 2). We present here the main results we are going to need to carry out the classification and check the topological versality of the unfoldings (see Theorem 3.4). Our cases are very simple, basically the homogeneous terms of lower order with respect to a weight filtration of the variables and parameters determine the topological type of the normal form (and of its miniversal unfolding).

For the rest of the section, we have a set of weights (a, b, c) for (x, λ, μ), with $a, b, c \in \mathbb{N}$. Let $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$, the **weight** of f, $wt(f)$, is the minimum weight of the nonzero monomials in the Taylor series expansion of f. The **initial part** of f, denoted by f_{in}, is the sum of those monomials of degree exactly $wt(f)$. We say that two germs $f, g \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ are **topologically $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalent** if the changes of coordinates involved in $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$ are only homeomorphisms instead of being diffeomorphisms. Similarly, we define the notion of **topological mapping** between unfoldings which induces the notion of **topological $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-versality**.

To discuss the topological versality of a $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-miniversal unfolding F of f, we construct the following sub-unfolding of F. Suppose $F(x, \lambda, \mu, \alpha) = f(x, \lambda, \mu) + \sum_{i=1}^{\mathrm{cod}^{\Gamma}_{\Delta}} \alpha_i\, p_i(x, \lambda, \mu)$ with $\{p_i\}_{i=1}^{\mathrm{cod}^{\Gamma}_{\Delta}}$ projecting onto a basis of $\mathcal{N}^{\Gamma,\Delta}_e(f)$. We reorganise that basis such that we can assume that the p_i's are weighted homogeneous. We define then $F_{\mathrm{in}}(x, \lambda, \mu, \beta) = f(x, \lambda, \mu) + \sum_{i=1}^{j} \beta_i\, p_i(x, \lambda, \mu)$ where $wt(p_i) < wt(f)$. We say that F is **versal in positive weight** if

$$\vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu,\beta)} / \mathcal{T}^{\Gamma,\Delta}_{e,un}(F_{\mathrm{in}}) + \langle p_1, \ldots, p_j \rangle_{\mathcal{E}_\beta}$$

is of finite dimension as a $\mathbb{R}$-vector space. Suppose that F is versal in positive weight then the number j is called the **topological codimension** of f.

Theorem 2.13 (Topological Determinacy, [**10**], [**11**]). *Let $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ with $\mathrm{cod}^{\Gamma}_{\Delta}(f_{\mathrm{in}}) < \infty$. Then, any $g \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ such that $g_{\mathrm{in}} = f_{\mathrm{in}}$ is topologically $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-equivalent to f.*

Theorem 2.14 (Topological Versality, [**10**], [**11**]). *Let $f \in \vec{\mathcal{E}}^{\Gamma,\Delta}_{(x,\lambda,\mu)}$ with $\mathrm{cod}^{\Gamma}_{\Delta}(f_{\mathrm{in}}) < \infty$. Let F be an unfolding of f, versal in positive weight, then F is topologically $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-versal. Moreover, F is topologically $\mathcal{K}^{\Gamma,\Delta}_{(\lambda,\mu)}$-trivial along the directions of weight larger or equal to $wt(f)$.*

3. $(\mathrm{O}(2),\mathbf{1})$-Symmetry Breaking Theory

Via the notation $z = x + iy$ we identify $\mathbb{C}$ and $\mathbb{R}^2$. We would like to classify the $(\mathrm{O}(2),\mathbf{1})$-symmetry breaking problems on $\mathbb{R}^2$ with the usual orthogonal representation of $\mathrm{O}(2)$ on $\mathbb{C}$, generated by $\theta\colon z \mapsto e^{i\theta} z$, $\theta \in \mathrm{S}^1$, and $\kappa\colon z \mapsto \bar{z}$. We first look at the $\mathrm{O}(2)$-part of the problem.

3.1. O(2)-Theory in $\vec{\mathcal{E}}^{\mathrm{O}(2)}_{(z,\lambda)}$.

We recall the theory in [**20**]. The only invariant of the action is $\bar{u} = z\bar{z}$, that is, $\mathcal{E}^{\mathrm{O}(2)}_{(z,\lambda)} = \bar{u}^*\mathcal{E}_{(u,\lambda)}$. Moreover, the module of equivariant maps $\vec{\mathcal{E}}^{\mathrm{O}(2)}_{(z,\lambda)}$ is generated over $\mathcal{E}^{\mathrm{O}(2)}_{(z,\lambda)}$ by $Y_1(z) = z$ and $\mathrm{M}^{\mathrm{O}(2)}_{(z,\lambda)}$ is generated over $\mathcal{E}^{\mathrm{O}(2)}_{(z,\lambda)}$ by $S_1(z,\lambda)\,w = w$ and $S_2(z,\lambda)\,w = z^2\bar{w}$. And so, identifying $\vec{\mathcal{E}}^{\mathrm{O}(2)}_{(z,\lambda)}$ and $\mathcal{E}_{(u,\lambda)}$, for $f(z,\lambda) = h(u,\lambda)\,z$, $\mathcal{T}_e^{\mathrm{O}(2)}(f)$ is identified with $\langle h, uh_u\rangle_{\mathcal{E}_{(u,\lambda)}} + \langle h_\lambda\rangle_{\mathcal{E}_\lambda}$ and $\mathcal{P}(f)$ with $\mathrm{Itr}\big\{\langle uh, \lambda h, u^2h_u, u\lambda h_u\rangle_{\mathcal{E}_{(u,\lambda)}} + \langle \lambda^2 h_\lambda\rangle_{\mathcal{E}_\lambda}\big\}$.

Finally, we have the following list of normal forms $h_i \in \vec{\mathcal{E}}^{\mathrm{O}(2)}_{(z,\lambda)}$, $i \leq i \leq 6$, of (topological) $\mathrm{O}(2)$-codimension less or equal to 2 with their miniversal unfoldings H_i, $1 \leq i \leq 6$:

I: $h_1(u,\lambda)\,z = (\epsilon_1 u + \delta_1\lambda)\,z$ of codimension 0,

II: $h_2(u,\lambda)\,z = (\epsilon_1 u + \delta_2\lambda^2)\,z$ of codimension 1,
$H_2(u,\lambda,\alpha)\,z = (\epsilon_1 u + \delta_2\lambda^2 + \alpha)\,z$,

III: $h_3(u,\lambda)\,z = (\epsilon_2 u^2 + \delta_1\lambda)\,z$ of codimension 1,
$H_3(u,\lambda,\alpha)\,z = (\epsilon_2 u^2 + \delta_1\lambda + \alpha u)\,z$,

IV: $h_4(u,\lambda)\,z = (\epsilon_3 u^3 + \delta_1\lambda)\,z$ of codimension 2,
$H_4(u,\lambda,\alpha,\beta)\,z = (\epsilon_3 u^3 + \delta_1\lambda + \alpha u + \beta u^2)\,z$,

V: $h_5(u,\lambda)\,z = (\epsilon_2 u^2 + m\lambda u + \delta_2\lambda^2)\,z$ of topological codimension 2 (m is a modal parameter),
$H_5(u,\lambda,\alpha,\beta)\,z = (\epsilon_2 u^2 + m\lambda u + \delta_2\lambda^2 + \alpha + \beta u)\,z$,

VI: $h_6(u,\lambda)\,z = (\epsilon_1 u + \delta_3\lambda^3)\,z$ of codimension 2,
$H_6(u,\lambda,\alpha,\beta)\,z = (\epsilon_1 u + \delta_3\lambda^3 + \alpha + \beta\lambda)\,z$,

where $\epsilon_i = \operatorname{sign} h^o_{u^i}$, $\delta_i = \operatorname{sign} h^o_{\lambda^i}$ and $m = \frac{2h^o_{u\lambda}}{\sqrt{|h^o_{uu}h^o_{\lambda\lambda}|}}$. We require that $\epsilon_i^2 = \delta_i^2 = 1$, $1 \leq i \leq 3$, and $m^2 - 4\epsilon_2\delta_2 \neq 0$ for h_5 (that is, $(h^o_{u\lambda})^2 \neq h^o_{uu}h^o_{\lambda\lambda}$). The unfolding parameters are α and β.

3.2. The Groups $\hat{\mathrm{M}}(\mathbf{O(2),1})$, $\hat{\mathrm{S}}(\mathbf{O(2),1})$ and their Tangent Spaces.

From the previous results, $\mathcal{E}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)} = \mathcal{E}^{\mathrm{O}(2)}_{(z,\lambda)} + \mu\,\mathcal{E}_{(z,\lambda,\mu)}$ and $f \in \vec{\mathcal{E}}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)} = \vec{\mathcal{E}}^{\mathrm{O}(2)}_{(z,\lambda)} + \mu\,\vec{\mathcal{E}}_{(z,\lambda,\mu)}$ can be written as $f(z,\lambda,\mu) = h(u,\lambda)\,z + \mu \begin{pmatrix} r(x,y,\lambda,\mu) \\ s(x,y,\lambda,\mu) \end{pmatrix}$ for some $h\colon (\mathbb{R}^2,0) \to (\mathbb{R},0)$ and $r,s\colon (\mathbb{R}^4,0) \to (\mathbb{R},0)$. The additional groups are $\hat{\mathrm{M}}(\mathrm{O}(2),\mathbf{1}) = \{T\colon (\mathbb{R}^{2+2},0) \to \mathrm{S}^1\}$ and $\hat{\mathrm{S}}(\mathrm{O}(2),\mathbf{1})$, given as follows.

Proposition 3.1. $\hat{\mathrm{S}}(\mathrm{O}(2),\mathbf{1}) = \left\{ \begin{pmatrix} 1+yh & -xh \\ yg & -xg+1 \end{pmatrix} : g,h \in \mathcal{E}_{(z,\lambda,\mu)} \right\}$.

Proof: We use the equivalent definition of $\hat{\mathrm{S}}(\mathrm{O}(2),\mathbf{1})$ given in Lemma 2.5. Let $S \in \hat{\mathrm{S}}(\mathrm{O}(2),\mathbf{1})$, then S can be written as $Sw = aw + b\bar{w}$ for a, b two complex valued functions of (z,λ,μ). As $Sz = z$, we see that $(a-1)\,z + b\bar{z} \equiv 0$. Writing $a = a_1 + ia_2$ and $b = b_1 + ib_2$, we see that $(a_1+b_1-1)\,x + (-a_2+b_2)\,y \equiv 0$ and that $(a_2+b_2)\,x + (a_1-b_1-1)\,y \equiv 0$. We can then conclude of the existence of $g,h \in \mathcal{E}_{(z,\lambda,\mu)}$ such that $a_1+b_1-1 = h\,y$, $-a_2+b_2 = -h\,x$, $a_1-b_1-1 = -g\,x$ and that $a_2+b_2 = g\,y$. Thus, in complex notation, we find that

$$Sw = w + \tfrac{1}{2}(hy-gx)\,w + \tfrac{1}{2}(gy+hx)\,iw + \tfrac{1}{2}(gx+hy)\,\bar{w} + \tfrac{1}{2}(gy-hx)\,i\bar{w},$$

and the result follows passing to the real coordinates. □

Proposition 3.2. (1) *The tangent space* $\hat{m}$ *of* $\hat{\mathrm{M}}(\mathrm{O}(2),\mathbf{1})$ *is generated by* $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ *on* $\mathcal{E}_{(z,\lambda,\mu)}$ *which acts on* $\vec{\mathcal{E}}_{(z,\lambda,\mu)}$ *by* $g \equiv (g_1,g_2) \mapsto \begin{pmatrix} g_2 - yg_{1x} + xg_{1y} \\ -g_1 - yg_{2x} + xg_{2y} \end{pmatrix}$.

(2) *The tangent space* $\hat{s}$ *of* $\hat{\mathrm{S}}(\mathrm{O}(2),\mathbf{1})$ *is generated on* $\mathcal{E}_{(z,\lambda,\mu)}$ *by* $\begin{pmatrix} -y & x \\ 0 & 0 \end{pmatrix}$ *and* $\begin{pmatrix} 0 & 0 \\ -y & x \end{pmatrix}$.

Proof: (1) We use Lemma 2.6. Here $\Delta = \mathbf{1}$ so, from (2.3), $\hat{m}$ is generated by the generator of $\ell_{\mathrm{O}(2)}$: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The action of $\hat{m}$ is given by (2.4).

(2) From Lemma 2.5, any element S of the tangent space $\hat{s}$ should satisfy $Sz \equiv 0$. And so it is generated on $\mathcal{E}_{(z,\lambda,\mu)}$ by $\begin{pmatrix} -y & x \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ -y & x \end{pmatrix}$. □

3.3. Tangent Spaces.

From the results of Section 2.3.4, the extended tangent space of $f = f_1 + \mu f_2 \in \vec{\mathcal{E}}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)}$ is

$$\mathcal{T}_e^{\mathrm{O}(2),\mathbf{1}}(f) = \{T_1\, f_1 + f_{1z}\, X_1 + f_{1\lambda} L_{11} + \mu(T_2\, f_1 + f_{1z}\, X_2 + f_{1\lambda} L_{12} + T_1\, f_2 + f_{2z}\, X_1 + f_{2\lambda} L_{11} + f_2 L_{22}) + \mu^2(T_2\, f_2 + f_{2z}\, X_2 + f_{2\lambda} L_{12} + f_{2\mu} L_{22}) + \mu\, \hat{m}(f_2) + \mu\, \hat{s}(f_2)\}$$

where $T = T_1 + \mu T_2 \in \mathbf{M}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)}$, $X = X_1 + \mu X_2 \in \vec{\mathcal{E}}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)}$, $L_{11} \in \mathcal{M}_\lambda$, $L_{12}, L_{22} \in \mathcal{E}_{(\lambda,\mu)}$ and $\hat{m}$, $\hat{s}$, are, respectively, the tangent spaces of $\hat{\mathrm{M}}(\mathrm{O}(2),\mathbf{1})$ and $\hat{\mathrm{S}}(\mathrm{O}(2),\mathbf{1})$.

The first part of the tangent space represents the usual extended tangent space in $\vec{\mathcal{E}}^{\mathrm{O}(2)}_{(z,\lambda)}$, image of the tangent map $\phi_1^{f_1} : k_\lambda^{\mathrm{O}(2)} \to \vec{\mathcal{E}}^{\mathrm{O}(2)}_{(z,\lambda)}$, $(T_1, X_1, L_{11}) \mapsto T_1 f_1 + f_{1z} X_1 + f_{1\lambda} L_{11}$. By definition the kernel of $\phi_1^{f_1}$, denoted by Φ_1, has no action on f_1, so its elements can make effective, independent contributions to the μ-dependent part of the tangent space when applied to f_2; let $\zeta \in \Phi_1$, then $\zeta(f) = \zeta(f_1 + \mu f_2) = \mu\zeta(f_2)$. In general, Φ_1 is the sum of $\mathcal{E}_\lambda$ and $\mathcal{E}_{(u,\lambda)}$-modules. In Section 3.5.3 we shall determine Φ_1 for our normal forms. The calculations for the extended tangent space $\mathcal{T}_e^{\mathrm{O}(2),\mathbf{1}}(f)$ of $f = h\, z + \mu(r,s)$ can be split into two parts. First

$$\mathcal{T}_e(h) \approx \langle h, uh_u\rangle_{\mathcal{E}_{(u,\lambda)}} + \langle h_\lambda\rangle_{\mathcal{E}_\lambda} \tag{3.1}$$

in $\mathcal{E}^{\mathrm{O}(2)}_{(z,\lambda)}$ ($\approx \mathcal{E}_{(u,\lambda)}$) and the second in $\vec{\mathcal{E}}_{(z,\lambda,\mu)}$

(3.2)

$$\Big\langle \begin{pmatrix} xh+\mu r \\ 0\end{pmatrix}, \begin{pmatrix} yh+\mu s \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ xh+\mu r\end{pmatrix}, \begin{pmatrix} 0 \\ yh+\mu s\end{pmatrix}, \ (T_2 \text{ contribution})$$

$$\begin{pmatrix} h+2x^2h_u+\mu r_x \\ 2xyh_u+\mu s_x\end{pmatrix}, \begin{pmatrix} 2xyh_u+\mu r_y \\ h+2y^2h_u+\mu s_y\end{pmatrix}, \ (X_2 \text{ contribution})$$

$$\begin{pmatrix} -yr+xs \\ 0\end{pmatrix}, \begin{pmatrix} 0 \\ -yr+xs\end{pmatrix}, \begin{pmatrix} s-yr_x+xr_y \\ -r-ys_x+xs_y\end{pmatrix}\Big\rangle_{\mathcal{E}_{(z,\lambda,\mu)}} \ (\hat{s}, \hat{m} \text{ contribution})$$

$$+ \Big\langle \begin{pmatrix} xh_\lambda+\mu r_\lambda \\ yh_\lambda+\mu s_\lambda\end{pmatrix}, \begin{pmatrix} r+\mu r_\mu \\ s+\mu s_\mu\end{pmatrix}\Big\rangle_{\mathcal{E}_{(\lambda,\mu)}} + \Phi_1(f)$$

where $\Phi_1(f) = \mu\Phi_1(f_2)$ represents the result of the action of the elements of Φ_1 on f.

Remark. The generators of (3.1) also give non trivial contributions in $\mu\mathcal{E}_{(z,\lambda,\mu)}$ but they are not relevant for the extended normal space because we consider the quotient $\vec{\mathcal{E}}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)}/\mathcal{T}_e^{\mathrm{O}(2),\mathbf{1}}(f)$. The kernel of $\phi_1^{f_1}$ provides those contributions in $\mu\mathcal{E}_{(z,\lambda,\mu)}$. We can say that $\mathcal{N}_e^{\mathrm{O}(2),\mathbf{1}}(f) \equiv \mathcal{N}_e^{\mathrm{O}(2)}(f_1) \oplus \vec{\mathcal{E}}_{(z,\lambda,\mu)}/\vec{M}$ where $\vec{M}$ represents all the elements of the T_2, X_2, $\hat{m}$, $\hat{s}$ and (3.2) contributions.

The unipotent tangent space $\mathcal{TU}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}(f)$ is formed from the action of the unipotent subgroup $\mathcal{U}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}$ of $\mathcal{K}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}$. Although its structure is similar to that of the extended tangent space, we are interested in its totality to be able to estimate its maximal intrinsic subspace (hence $\mathcal{P}(f)$) where Φ_2 represents the kernel of the tangent map $\phi_2^{f_1} : u_\lambda^{\mathrm{O}(2)} \to \vec{\mathcal{E}}^{\mathrm{O}(2)}_{(z,\lambda)}$. In our cases, Φ_2 can be larger than $\mathcal{M}_{(u,\lambda,\mu)} \cdot \Phi_1$ (see Proposition 3.8). The contributions by T_2, X_2 and $\hat{s}$ are the same as previously, but, for the last ones, we get $\mathcal{M}_{(z,\lambda,\mu)} \cdot \hat{m}(f)$ and for (3.2) we have

$$\left\langle \begin{pmatrix} xh_\lambda + \mu r_\lambda \\ yh_\lambda + \mu s_\lambda \end{pmatrix} \right\rangle_{\mathcal{E}_{(\lambda,\mu)}} + \left\langle \begin{pmatrix} r + \mu r_\mu \\ s + \mu s_\mu \end{pmatrix} \right\rangle_{\mathcal{M}_{(\lambda,\mu)}} + \Phi_2(f).$$

3.4. Classification.

The germs $f \in \vec{\mathcal{E}}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)}$ of topological codimension less or equal to 1 satisfy $f_2^o \neq 0$. Under that condition we can put f into a simpler form which will prove useful for the calculations.

3.4.1. Pre-normal forms.

Proposition 3.3. *When $f_2^o \neq 0$ there exists a change of coordinates in $\mathcal{K}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}$ casting f into the following pre-normal form for some $s \in \mathcal{E}_{(y,\lambda)}$ with $s(0,0) = 1$,*

$$f(z,\lambda,\mu) = h(u,\lambda)\, z + \mu(0, s(y,\lambda)). \tag{3.3}$$

Proof: The proof is in Section 3.5.2. □

Remark. The pre-normal form (3.3) is variational, that is, it is a gradient with respect to (x, y), as are the O(2)-miniversal unfoldings for cases II–VI: $H_2, \ldots, H_6$.

3.4.2. Classification theorems.

The smooth classification involves many moduli and is rapidly complicated. Moreover, the main information of practical importance for the study of the bifurcation diagrams is the topological type of germs *and* unfoldings. So, our first result is about the topological classification of germs.

Theorem 3.4 (Topological Classification)**.** *The normal forms of the germs in* $\vec{\mathcal{E}}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)}$ *of topological codimension* 0 *or* 1 *are given in the following table of the miniversal unfoldings with (topologically relevant) unfolding parameter* α*. The normal forms are obtained by setting* $\alpha = 0$*.*

CASE	MINIVERSAL UNFOLDING/NORMAL FORM	top-cod	cod
I_0	$(\epsilon_1 u + \delta_1 \lambda)\, z + \mu \binom{0}{1}$	0	0
II	$(\epsilon_1 u + \delta_2 \lambda^2 + \alpha)\, z + \mu \binom{0}{1+ay}$	1	$1-2$
III	$(\epsilon_2 u^2 + \delta_1 \lambda + \alpha u)\, z + \mu \binom{0}{1+ay^2+(b+c\lambda)y^3}$	1	$2-4$

The Greek letters δ_i *and* ϵ_j *represent normalised non-zero coefficients* (± 1)*. The coefficients* a*,* b*,* c *are topologically irrelevant. They can be put to* 0*, normalised or be genuine moduli depending on the situation (see Theorem 3.5 for more details in the cases* I_0*,* II *and* III*).*

Remarks. 1. We believe that we have also a complete list up to topological codimension 2, but we cannot prove that the following unfoldings are of topological codimension 2.

CASE	MINIVERSAL UNFOLDING/NORMAL FORM	top-cod	cod
IV	$(\epsilon_3 u^3 + \delta_1 \lambda + \alpha u + \beta u^2)\, z + \mu \binom{0}{1+ey^2+e_1 y^3+e_2 y^4+e_3 y^5}$	2	$5-12$
V	$(\epsilon_2 u^2 + m\lambda u + \delta_2 \lambda^2 + \alpha + \beta u)\, z + \mu \binom{0}{1+ey+e_1 y^2+e_2 y^3}$	2	$4-9$
VI	$(\epsilon_1 u + \delta_3 \lambda^3 + \alpha + \beta\lambda)\, z + \mu \binom{0}{1+(a+b\lambda)y}$	2	$3-4$
I_2	$(\epsilon_1 u + \delta_1 \lambda)\, z + \mu \binom{\alpha+\nu_{11} y}{\beta+n_{12}x+n_{13}y+\nu_{14}\mu}$	2	4

As previously we list the miniversal unfoldings with (topologically relevant) unfolding parameters α and β. The normal forms are obtained by setting $\alpha = \beta = 0$. The symbols δ_i, ϵ_j, ν_{11} and ν_{14} represent normalised non-zero coefficients (± 1) as m is a modulus with topological implication. The coefficients a, b, c, d,e, n_{12}, n_{13} and the functions $e_i(\lambda)$ are topologically irrelevant. They can be put to 0, normalised or be genuine moduli depending on the situation. We discuss in more details those cases in [**12**].

2. Cases I_0 to VI correspond to the classification of the $(\mathrm{Z}_2, \mathbf{1})$-symmetry breaking problems such that $f_2^o \neq 0$. The normal forms are obtained by setting $x = 0$ in the second component of the $(\mathrm{O}(2), \mathbf{1})$-normal forms and they have the same topological and smooth codimensions. When $f_2^o = 0$, for case I_2, we have a richer structure than the $(\mathrm{Z}_2, \mathbf{1})$-symmetry breaking problems in one dimension.

In the next theorem we give the smooth classification of the families of germs I, II and III of topological codimension 0 and 1.

Theorem 3.5 (Smooth Classification)**.** *Let $g \in \vec{\mathcal{E}}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)}$ be of $\mathcal{K}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}$-topological codimension at most one. Then g is $\mathcal{K}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}$-equivalent to*

$$f(z, \lambda, \mu) = h(u, \lambda)\, z + \mu \begin{pmatrix} 0 \\ s(y, \lambda) \end{pmatrix},$$

where the different possibilities for h and s are given by the following.

The Greek letters, δ_i, ϵ_j, are normalised coefficients (± 1), the n_k's are modal parameters.

I_0*:* $h(u, \lambda) = \epsilon_1 u + \delta_1 \lambda$, $s \equiv 1$ *of codimension* 0*,*

II_1*:* $h(u, \lambda) = \epsilon_1 u + \delta_2 \lambda^2$, $s(y) = 1 + \nu_0 y$ *of codimension* 1 *with* $\nu_0^2 = 1$*, with miniversal unfolding*

$$F_2(z, \lambda, \mu, \alpha) = (\epsilon_1 u + \delta_2 \lambda^2 + \alpha)\, z + \mu \begin{pmatrix} 0 \\ 1 + \nu_0 y \end{pmatrix},$$

II_2*:* $h(u, \lambda) = \epsilon_1 u + \delta_2 \lambda^2$, $s \equiv 1$ *of codimension* 2 *with miniversal unfolding*

$$F_3(z, \lambda, \mu, \alpha, \beta) = (\epsilon_1 u + \delta_2 \lambda^2 + \alpha)\, z + \mu \begin{pmatrix} 0 \\ 1 + \beta\, y \end{pmatrix},$$

III_1: *$h(u,\lambda)=\epsilon_2u^2+\delta_1\lambda$, $s\equiv 1+\nu_2y^2+n_1y^3$ of codimension 2 with $\nu_2^2=1$, with miniversal unfolding*

$$F_4(z,\lambda,\mu,\alpha)=(\epsilon_2u^2+\delta_1\lambda+\alpha u)\,z+\mu\begin{pmatrix}0\\1+\nu_2y^2+n_1y^3\end{pmatrix},$$

III_2: *$h(u,\lambda)=\epsilon_2u^2+\delta_1\lambda$, $s\equiv 1+\nu_3y^3$ of codimension 2 with $\nu_3^2=1$, with miniversal unfolding*

$$F_5(z,\lambda,\mu,\alpha,\beta)=(\epsilon_2u^2+\delta_1\lambda+\alpha u)\,z+\mu\begin{pmatrix}0\\1+\beta y^2+\nu_3y^3\end{pmatrix},$$

III_3: *$h(u,\lambda)=\epsilon_2u^2+\delta_1\lambda$, $s\equiv 1+\nu_4\lambda y^3$ of codimension 3 with $\nu_4^2=1$, with miniversal unfolding*

$$F_6(z,\lambda,\mu,\alpha,\beta)=(\epsilon_2u^2+\delta_1\lambda+\alpha u)\,z+\mu\begin{pmatrix}0\\1+\beta y^2+(\gamma+\nu_4\lambda)\,y^3\end{pmatrix},$$

III_4: *$h(u,\lambda)=\epsilon_2u^2+\delta_1\lambda$, $s\equiv 1$ of codimension 4 with miniversal unfolding*

$$F_7(z,\lambda,\mu,\alpha,\beta)=(\epsilon_2u^2+\delta_1\lambda+\alpha u)\,z+\mu\begin{pmatrix}0\\1+\beta y^2+(\gamma_1+\gamma_2\lambda)y^3\end{pmatrix}.$$

3.5. Proofs.

We first prove our main theorems. Then, in subsequent subsections, we deal with the pre-normal forms and the kernels Φ_1 and Φ_2.

3.5.1. Proofs of the classification results.

Proposition 3.6. (1) *The germs of type $f(z,\lambda,\mu)=h(u,\lambda)z+\mu f_2(z,\lambda,\mu)$ with $h(u,\lambda)=\epsilon_1u+\delta_m\lambda^m$ (for some integer $m\geq 1$) and $f_2^o\neq 0$ are $\mathcal{K}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}$-equivalent to*

$$g(z,\lambda,\mu)=(\epsilon_1u+\delta_m\lambda^m)\,z+\mu\begin{pmatrix}0\\s(y,\lambda)\end{pmatrix}, \tag{3.4}$$

with $s(0,0)=1$. The function s has no influence on the topological type of g.

(2) *The germs of type $f(z,\lambda,\mu)=h(u,\lambda)\,z+\mu f_2(z,\lambda,\mu)$ with $h(u,\lambda)=\epsilon_nu^n+\delta_1\lambda$ (for some integer $n\geq 1$) and $f_2^o\neq 0$ are $\mathcal{K}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}$-equivalent to*

$$g(z,\lambda,\mu)=(\epsilon_nu^n+\delta_1\lambda)\,z+\mu\begin{pmatrix}0\\s(y,\lambda)\end{pmatrix}, \tag{3.5}$$

with $s(0,0)=1$. *The function* s *has no influence on the topological type of* g.

(3) *Both normal forms* (3.4), (3.5) *are of finite smooth codimension. We express the higher order terms of* g *as higher order terms of* s*; terms which can be removed from* s *for any representative of the* $\mathcal{K}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}$*-orbit of* g. *We denote them by* $\mathcal{P}(s)$ *and they satisfy*

$$\mathcal{P}(s)\supset\mathcal{E}_{(y,\lambda)}\cdot\langle y\rangle^2+\mathcal{M}_\lambda^{m-1}\cdot\langle y\rangle+\mathcal{M}_\lambda,$$

for (3.4), *and for* $k=2n-2$, *for* (3.5),

$$\mathcal{P}(s)\supset\mathcal{E}_{(y,\lambda)}\cdot\langle y\rangle^{2n}+\sum_{i=1}^{2n-1}\mathcal{M}_\lambda^k\cdot\langle y\rangle^i+\mathcal{M}_\lambda.$$

Proof: (1) The pre-normal form (3.4) is a direct consequence of Proposition 3.3. That s has no topological influence follows from Damon's theory of topological equivalence (see Section 2.5). For (3.4), we introduce the weights $wt(z,\lambda,\mu)=(m,2,3m)$. With respect to that grading, $g_{\mathrm{in}}(z,\lambda,\mu)=(\epsilon_1u+\delta_m\lambda^m)\,z+\mu\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. To prove the result, we show that g_{in} is of finite $\mathcal{K}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}$-codimension. Recall that the calculations for the various tangent spaces of f can be split into two parts. In $\mathcal{E}^{\mathrm{O}(2)}_{(z,\lambda)}$, the extended tangent space for h is given in (3.1). In $\vec{\mathcal{E}}_{(z,\lambda,\mu)}$, using the pre-normal form $g(z,\lambda,\mu)=h(u,\lambda)\,z+\mu\left(\begin{smallmatrix}0\\s(y,\lambda)\end{smallmatrix}\right)$, a calculation shows that the generators for the extended tangent space in $\vec{\mathcal{E}}_{(z,\lambda,\mu)}$ (given in Section 3.3) simplify to the following:

(3.6)

$$\mathcal{E}_{(z,\lambda,\mu)}\cdot\left\langle\begin{pmatrix}1\\0\end{pmatrix}(\hat{m}),\begin{pmatrix}0\\x\end{pmatrix}(\hat{s}),\begin{pmatrix}0\\yh+\mu s\end{pmatrix}(T_2),\begin{pmatrix}0\\h+2y^2h_u+\mu s_y\end{pmatrix}(X_2)\right\rangle$$

$$+\mathcal{E}_{(\lambda,\mu)}\cdot\left\langle\begin{pmatrix}0\\yh_\lambda+\mu s_\lambda\end{pmatrix},\begin{pmatrix}0\\s\end{pmatrix}\right\rangle+\mathcal{E}_{(y^2,\lambda)}\cdot\langle\Phi_{11}(0,s)\rangle+\mathcal{E}_\lambda\cdot\langle\Phi_{12}(0,s)\rangle.$$

Using the results of Proposition 3.7 in Section 3.5.3, the contributions of the vectors $\Phi_{11}(0,s)$ and $\Phi_{12}(0,s)$ do not bring any new information for our cases. Because h is of finite O(2)-codimension, we only need to check the part of the tangent space given by (3.6):

$$\mathcal{E}_{(z,\lambda,\mu)}\cdot\left\langle\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\x\end{pmatrix},\begin{pmatrix}0\\yh+\mu\end{pmatrix},\begin{pmatrix}0\\h+2\epsilon_1y^2\end{pmatrix}\right\rangle+\mathcal{E}_{(\lambda,\mu)}\cdot\left\langle\begin{pmatrix}0\\\lambda^{m-1}y\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\rangle,$$

for our g_{in}. Therefore, the part of $\mathcal{N}_e^{\mathrm{O}(2),\mathbf{1}}(g_{\mathrm{in}})$ coming from (3.6) is

$$\mathcal{E}_{(y,\lambda,\mu)}/(\mathcal{E}_{(y,\lambda,\mu)}\cdot\langle\epsilon_1y^3+\delta_m\lambda^my+\mu,3\epsilon_1y^2+\delta_m\lambda^m\rangle+\mathcal{E}_{(\lambda,\mu)}\cdot\langle\lambda^{m-1}y,1\rangle).$$

Using the module structure on y and eliminating the higher powers using the quotient, we find

$$\mathcal{E}_{(y,\lambda,\mu)}/(\mathcal{E}_{(y,\lambda,\mu)} \cdot \langle 3\epsilon_1 y^2 + \delta_m \lambda^m, 2\delta_m \lambda^m y + 3\mu\rangle + \mathcal{E}_{(\lambda,\mu)} \cdot \langle \lambda^{m-1} y, 1\rangle),$$

simplifying into $(\mathcal{E}_{(\lambda,\mu)} \cdot \langle 1, y\rangle)/(\mathcal{E}_{(\lambda,\mu)} \cdot \langle \mu y, \lambda^{m-1} y\rangle + \mathcal{E}_{(\lambda,\mu)})$, that is, $\mathcal{N}_e^{\mathrm{O}(2),\mathbf{1}}(g_{\mathrm{in}}) = \{0\}$ when $m = 1$, and $\mathcal{N}_e^{\mathrm{O}(2),\mathbf{1}}(g_{\mathrm{in}}) = \mathcal{N}_e^{\mathrm{O}(2)}(h) + \mathbb{R} \cdot \langle y, \dots, \lambda^{m-2} y\rangle$, $m \geq 2$. They are of finite codimension. Using Theorem 2.13, we can conclude that s has no influence on the topological type of g.

(2) For the second family of germs (3.5), we proceed analogously. We introduce the weights $wt(z, \lambda, \mu) = (1, 2n, 2n+1)$. We need to show that the initial part of g with respect to that grading $g_{\mathrm{in}}(z, \lambda, \mu) = (\epsilon_n u^n + \delta_1 \lambda)\, z + \mu \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$ is of finite $\mathcal{K}_{(\lambda,\mu)}^{\mathrm{O}(2),\mathbf{1}}$-codimension. Therefore, the part of the normal space coming from (3.6) simplifies, when $n \geq 2$, into the quotient of $\mathcal{E}_{(\lambda,\mu)} \cdot \langle y^2, \dots, y^{2n-1}\rangle$ by the submodule

$$\mathcal{E}_{(\lambda,\mu)} \cdot \langle \lambda y^2, 2n\delta_1 \lambda y^3 + (2n+1)\mu y^2, \dots, 2n\delta_1 \lambda y^{2n-1} + (2n+1)\mu y^{2n-2}, \mu y^{2n-1}\rangle,$$

that is, $\mathcal{N}_e^{\mathrm{O}(2),\mathbf{1}}(g_{\mathrm{in}}) = \{0\}$, $n = 1$, and

$$\mathcal{N}_e^{\mathrm{O}(2),\mathbf{1}}(g_{\mathrm{in}}) \subset \mathcal{N}_e^{\mathrm{O}(2)}(h)$$

$$+ \left\{ \sum_{i,j,k} a_{ijk} \lambda^i \mu^j y^k : a_{ijk} \in \mathbb{R},\ i+j \leq 2n-1,\ 1 \leq k \leq 2n-1 \right\}, \quad n \geq 2,$$

which are of finite codimension, hence s has no influence on the topological type of g.

(3) That both (3.4), (3.5) have finite smooth codimension follows from our previous calculations because their initial parts have finite smooth codimension. The generators of the part of the unipotent tangent space of g inside $\vec{\mathcal{E}}_{(z,\lambda,\mu)}$ simplify into the following

$$\mathcal{E}_{(z,\lambda,\mu)} \cdot \left\langle \begin{pmatrix}0\\x\end{pmatrix}, \begin{pmatrix}0\\yh+\mu s\end{pmatrix}, \begin{pmatrix}0\\h+2y^2 h_u + \mu s_y\end{pmatrix} \right\rangle + \mathcal{M}_{(z,\lambda,\mu)} \cdot \left\langle \begin{pmatrix}1\\0\end{pmatrix} \right\rangle$$

$$+ \mathcal{E}_{(\lambda,\mu)} \cdot \left\langle \begin{pmatrix}0\\yh_\lambda + \mu s_\lambda\end{pmatrix} \right\rangle + \mathcal{M}_{(\lambda,\mu)} \cdot \left\langle \begin{pmatrix}0\\s\end{pmatrix} \right\rangle \tag{3.7}$$

$$+ \mathcal{E}_{(y^2,\lambda)} \cdot \langle \Phi_{11}(0, s)\rangle + \mathcal{M}_\lambda \cdot \langle \Phi_{12}(0, s)\rangle.$$

From (3.7) we can evaluate the intrinsic part of the unipotent tangent space of g which gives us an estimate of the terms we can ignore in s. □

Proof of Theorem 3.4: The first point to remark is that the O(2)-equivariant part of f, h, give a lower bound for the codimension. That is,

$$\text{(top-)cod}^{\mathrm{O}(2)}(h) \leq \text{(top-)cod}_{\mathbf{1}}^{\mathrm{O}(2)}(f). \tag{3.8}$$

As a consequence of Damon's theory of topological equivalence we are going to see that the two codimensions in (3.8) are equal for our cases. We can cast f_1 into one of the normal forms mentioned in Section 3.1. Using Proposition 3.3 we can cast f into the pre-normal form (3.3) for some s and use Proposition 3.6 to conclude in all cases about the germs. Finally, to verify that the topological codimension of our three cases is at most one, we use Theorem 2.14.

Case I_0 is obvious as it is already of codimension 0. For case II we need to show that

$$G_{\text{in}}(z, \lambda, \mu, \alpha) = (\epsilon_1 u + \delta_2 \lambda^2 + \alpha)\, z + \mu \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

is a topologically versal unfolding in positive weight of g_{in} in (3.4). We compute its tangent space in $\vec{\mathcal{E}}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu,\alpha)}$ with respect to the natural extension $\mathcal{K}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu,\alpha)}$ of the group of equivalence $\mathcal{K}^{\mathrm{O}(2),\mathbf{1}}_{(\lambda,\mu)}$. Let $H_2(u, \lambda, \alpha) = \epsilon_1 u + \delta_2 \lambda^2 + \alpha$. Because $(\epsilon_1 u + \delta_2 \lambda^2) z$ is of finite O(2)-codimension, we only need to check the part of the tangent space given by the equivalent of (3.6) in our extended situation:

$$\mathcal{E}_{(z,\lambda,\mu,\alpha)} \cdot \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix}, \begin{pmatrix} 0 \\ yH_2 + \mu \end{pmatrix}, \begin{pmatrix} 0 \\ H_2 + 2\epsilon_1 y^2 \end{pmatrix} \right\rangle$$
$$+ \mathcal{E}_{(\lambda,\mu,\alpha)} \cdot \left\langle \begin{pmatrix} 0 \\ \lambda y \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle.$$

The first two generators of that part of the extended tangent space of G_{in} mean that we can ignore the first component and the x-dependence of the second in the quotient giving the extended normal space. Therefore, the part of $\mathcal{N}_e^{\mathrm{O}(2),\mathbf{1}}(G_{\text{in}})$ coming from (3.6) is

$$\mathcal{E}_{(y,\lambda,\mu,\alpha)} / (\mathcal{E}_{(y,\lambda,\mu,\alpha)} \cdot \langle 3\epsilon_1 y^2 + (\delta_2 \lambda^2 + \alpha), 2(\delta_2 \lambda^2 + \alpha) y + 3\mu \rangle$$
$$+ \mathcal{E}_{(\lambda,\mu,\alpha)} \cdot \langle \lambda y, 1 \rangle)$$

which simplifies into $(\mathcal{E}_{(\lambda,\mu,\alpha)} \cdot \langle y \rangle) / (\mathcal{E}_{(\lambda,\mu,\alpha)} \cdot \langle \mu y, \lambda y, \alpha y \rangle)$, that is, the normal space $\mathcal{N}_e^{\mathrm{O}(2),\mathbf{1}}(G_{\text{in}}) = \mathcal{N}_e^{\mathrm{O}(2)}(H_2 z) + \mathbb{R} \cdot \langle y \rangle$, is of finite codimension.

Finally, for cases III, we need to show that $G_{\text{in}}(z, \lambda, \mu, \alpha) = (\epsilon_2 u^2 + \delta_1 \lambda + \alpha u)\, z + \mu \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is a topologically versal unfolding in positive weight of $G_{\rm in}$ in (3.5). As previously, we only need to check the part given by (3.6). With $H_3(u,\lambda,\alpha) = \epsilon_2 u^2 + \delta_1\lambda + \alpha u$, it becomes

$$\mathcal{E}_{(z,\lambda,\mu,\alpha)} \cdot \left\langle \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\x\end{pmatrix}, \begin{pmatrix}0\\ yH_3+\mu\end{pmatrix}, \begin{pmatrix}0\\ H_3 + 4\epsilon_2 y^4\end{pmatrix} \right\rangle$$

$$+ \mathcal{E}_{(\lambda,\mu,\alpha)} \cdot \left\langle \begin{pmatrix}0\\y\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix} \right\rangle.$$

Therefore, the part of the normal space coming from (3.6) is

$$\mathcal{E}_{(y,\lambda,\mu,\alpha)}/(\mathcal{E}_{(y,\lambda,\mu,\alpha)} \cdot \langle 2\alpha y^3 + 4\delta_1\lambda y + 5\mu, 5\epsilon_2 y^4 + 3\alpha y^2 + \delta_1\lambda\rangle + \mathcal{E}_{(\lambda,\mu,\alpha)} \cdot \langle y, 1\rangle).$$

It simplifies into the quotient of $\mathcal{E}_{(\lambda,\mu,\alpha)} \cdot \langle y^2, y^3\rangle$ by the submodule

$$\mathcal{E}_{(\lambda,\mu,\alpha)} \cdot \langle \alpha y^3, (20\delta_1\lambda - 6\epsilon_2\alpha^2)\, y^2, (20\delta_1\lambda - 6\epsilon_2\alpha^2)\, y^3 + 25\mu y^2, 125\mu y^3 - (70\epsilon_2\delta_1\lambda\alpha - 18\alpha^3)\, y^2\rangle.$$

Because $\lambda^2 y^2$, $\mu^2 y^2$, $\alpha^4 y^2$, $\lambda^3 y^3$, $\mu^2 y^3$ are in $\mathcal{N}_e^{\mathrm{O}(2),\mathbf{1}}(G_{\rm in})$, it is of finite codimension. □

Proof of Theorem 3.5: The calculations for the various tangent spaces can be carried out like in the proof of Proposition 3.6. Here the terms coming from $\Phi_{11}(0,s)$ and $\Phi_{12}(0,s)$ cannot be ignored. First we compute an estimate on the set of higher order terms $\mathcal{P}(f)$ using the intrinsic part of $\mathcal{TU}_{(\lambda,\mu)}^{\mathrm{O}(2),\mathbf{1}}(f)$. In our situation, because of the component mixing role of $\hat{\mathrm{M}}(\mathrm{O}(2),\mathbf{1})$, we take the two second components of intrinsic submodules to be equal. So we find that the following intrinsic submodules $\mathcal{I} + \mu\left(\begin{smallmatrix}\mathcal{J}\\ \mathcal{J}\end{smallmatrix}\right)$ are in $\mathcal{P}(f)$:

for I_0: $\mathcal{I} = \mathcal{M}_{(u,\lambda)}^2$ and $\mathcal{J} = \mathcal{M}_{(y,\lambda,\mu)}$,
for $\mathrm{II}_{1,2}$: $\mathcal{I} = \mathcal{M}_{(u,\lambda)} \cdot \langle u\rangle + \langle\lambda\rangle^3$ and $\mathcal{J} = \mathcal{M}_{(y,\lambda,\mu)}^2 + \langle\lambda,\mu\rangle$,
for $\mathrm{III}_{1,2,3,4}$: $\mathcal{I} = \mathcal{M}_{(u,\lambda)}^3 + \mathcal{M}_{(u,\lambda)} \cdot \langle\lambda\rangle$ and $\mathcal{J} = \mathcal{M}_{(y,\lambda,\mu)}^4 + \langle\lambda,\mu\rangle$.

We use those estimates to a priori cut off all irrelevant terms from the Taylor series expansion of f. For the cases III, Mather's Lemma [**27**] shows that the terms in y are irrelevant in s.

Let $s = 1 + ay + by^2 + cy^3 + d\lambda y^3$ and define $f_{a,b,c,d} = h + \mu s$. A calculation shows that the unipotent tangent space of $f_{a,b,c,d}$ is equal to the unipotent tangent space of $f_{0,b,c,d}$. Because $\frac{\partial f_{a,b,c,d}}{\partial a} = (0,y)$ is in the unipotent tangent space of $f_{a,b,c,d}$, it follows from Mather's Lemma that $f_{a,b,c,d}$ and $f_{0,b,c,d}$ are unipotent equivalent, a fortiori $\mathcal{K}_{(\lambda,\mu)}^{\mathrm{O}(2),\mathbf{1}}$-equivalent.

Then scaling of the coordinates gives the normal form. The last calculation is for the miniversal unfolding when we need to compute the normal space of our normal forms. □

3.5.2. Pre-normal forms.

Proof of Proposition 3.3: We are going to proceed in 3 steps.

Step 1. We are going first to show that we can eliminate the explicit dependence of f_2 on μ by only using a premultiplication by some perturbation of the identity $T \in \mathbf{M}^{\mathrm{O}(2),\mathbf{1}}_{(z,\lambda,\mu)}$. Let $T(z,\lambda,\mu) = \mathbf{I}_2 + \mu T_2(z,\lambda,\mu)$ and compute $T \cdot f$. Rearranging by powers of μ we find

$$T \cdot f = f_1 + \mu \left[T_2(f_1 + \mu f_2) + f_2 \right].$$

We need to determine a T_2 to eliminate the μ-dependence of $T_2(f_1 + \mu f_2) + f_2$.

First, assume that both r^o and s^o are non zero. It is enough to take T_2 as a diagonal matrix, of diagonal coefficients a and d, say. The system we want to solve in function of a and d is

$$a(z,\lambda,\mu)[xh(u,\lambda) + \mu r(z,\lambda,\mu)] + r(z,\lambda,\mu) = \hat{r}(z,\lambda), \tag{3.9}$$

$$d(z,\lambda,\mu)[yh(u,\lambda) + \mu s(z,\lambda,\mu)] + s(z,\lambda,\mu) = \hat{s}(z,\lambda), \tag{3.10}$$

with $\hat{r}$ and $\hat{s}$ to be determined with no conditions on a^o and d^o. Both equations can be solved in the same way. We show how to handle (3.9). Because $r^o \neq 0$, from the Implicit Function Theorem there is a unique solution $\mu = \bar{\mu}(z,\lambda)$ to the equation $xh(u,\lambda) + \mu r(z,\lambda,\mu) = 0$. Consider the following module homomorphism $\phi\colon \mathcal{E}_{(z,\lambda,\mu)} \to \mathcal{E}_{(z,\lambda)}$ defined by $z \mapsto z$, $\lambda \mapsto \lambda$ and $\mu \mapsto \bar{\mu}(z,\lambda)$. The kernel of ϕ is the ideal generated by $xh + \mu r$ and $\mathcal{E}_{(z,\lambda,\mu)}/\ker\phi$ is isomorphic to $\mathcal{E}_{(z,\lambda)}$ because ϕ is surjective. So, for any $r \in \mathcal{E}_{(z,\lambda,\mu)}$, there exist $a \in \mathcal{E}_{(z,\lambda,\mu)}$ and $\hat{r} \in \mathcal{E}_{(z,\lambda)}$ such that $a\,(xh + \mu r) - \hat{r} = -r$, which is what we need to solve (3.9). Moreover, $\hat{r}^0 = r^0 \neq 0$.

When one of r^o or s^o is zero, say $r^o = 0$ and $s^o \neq 0$, we can proceed similarly but this time we need to choose $T_2 = \left(\begin{smallmatrix} 1 & b \\ 0 & d \end{smallmatrix}\right)$. It means that the system (3.9), (3.10) is not diagonal anymore but upper triangular. We can solve both equations like previously (note that $\hat{s}^o = s^o \neq 0$)

$$b(z,\lambda,\mu)[yh(u,\lambda) + \mu s(z,\lambda,\mu)] + [xh(u,\lambda) + (\mu+1)r(z,\lambda,\mu)] = \hat{r}(z,\lambda),$$
$$d(z,\lambda,\mu)[yh(u,\lambda) + \mu s(z,\lambda,\mu)] + s(z,\lambda,\mu) = \hat{s}(z,\lambda).$$

After this first step the pre-normal form for f is (we still have $(\hat{r}^o, \hat{s}^o) \neq (0,0)$)

$$f_{pn_1}(z,\lambda,\mu) = h(u,\lambda)\, z + \mu \begin{pmatrix} \hat{r}(z,\lambda) \\ \hat{s}(z,\lambda) \end{pmatrix}.$$

Step 2. Recall that if $M \in \hat{\mathrm{M}}(\mathrm{O}(2),\mathbf{1})$ then $M(z,\lambda,\mu) \in \mathrm{O}(2)$, and so we are going to bring $\hat{r}$ to 0 and $\hat{s}$ to some $\bar{s}(z,\lambda)$ using the rotation aspect of M. Using the Implicit Function Theorem we can find $\theta \in \mathcal{E}_{(z,\lambda,\mu)}$ such that the first component of $M \cdot f_{pn_1}$ is 0. Thus we find

$$f_{pn_2}(z,\lambda,\mu) = h(u,\lambda)\, z + \mu \begin{pmatrix} 0 \\ \bar{s}(z,\lambda) \end{pmatrix}. \tag{3.11}$$

Step 3. Finally, let $S \in \hat{\mathrm{S}}(\mathrm{O}(2),\mathbf{1})$ be of the form $\left(\begin{smallmatrix} 1 & 0 \\ -yg & xg+1 \end{smallmatrix}\right)$ for some $g \in \mathcal{E}_{(z,\lambda)}$. To eliminate the variable x in (3.11), we consider

$$S \cdot f_{np_2}(z,\lambda,\mu) = h(u,\lambda)\, z + \mu \begin{pmatrix} 0 \\ (xg(z,\lambda)+1)\, \bar{s}(z,\lambda) \end{pmatrix}.$$

Decompose $\bar{s}(z,\lambda) = \bar{s}_1(y,\lambda) + x\, \bar{s}_2(z,\lambda)$. The new second member of $S \cdot f_{np_2}$ can be written as $(xg+1)\,\bar{s} = \bar{s}_1 + x\,(gs_1 + s_2 + xgs_2)$. Now, define $H\colon (\mathbb{R}^4,0) \to \mathbb{R}$ by $H(g,z,\lambda) = g\bar{s}_1(y) + \bar{s}_2(z,\lambda)(1+xg)$. It is clear that if $g^o = -\frac{\bar{s}_2^o}{\bar{s}_1^o}$ then $H(g^o,0,0) = 0$ and $H_g(g^o,0,0) = \bar{s}_1^o = \bar{s}^o \neq 0$. From the Implicit Function Theorem there exists $\bar{g} \in \mathcal{E}_{(z,\lambda)}$ such that $H(\bar{g}(z,\lambda),z,\lambda) = 0$. And so we have the desired normal form (3.3) where we have rescaled μ so that $\bar{s}_1^o = 1$. □

3.5.3. Determination of Φ_1, Φ_2.

We determine first Φ_1. Recall that $\Phi_1 \subset k_\lambda^{\mathrm{O}(2)}$ and so any element $(T_1, X_1, L_1) \in \Phi_1$ can be represented by germs (a,b,c,d) where $a,b,c \in \mathcal{E}_{(u,\lambda)}$ and $d \in \mathcal{E}_\lambda$ because $T_1(z,\lambda)w = a\,w + bz^2\bar{w}$, $X_1(z,\lambda) = c\,z$ and $L_1(\lambda) = d\,\lambda$. Next we give the generators of Φ_1 explicitly as a sum of an $\mathcal{E}_{(u,\lambda)}$-module and an $\mathcal{E}_\lambda$-module for different normal forms in $\vec{\mathcal{E}}_{(z,\lambda)}^{\mathrm{O}(2)}$ including our cases.

Proposition 3.7. (1) *For a normal form of the type* $h(u,\lambda)\,z = (\epsilon_n u^n + \delta_1 \lambda)\, z$, Φ_1 *is generated over* $\mathcal{E}_{(u,\lambda)}$ *by* $(-u,1,0,0)$ *(already in* $\hat{s}$*), by* $\Phi_{11} = (-(2n+1)\,\delta_1 u^n - \epsilon_n \lambda, 0, \epsilon_n \lambda + \delta_1 u^n, 0)$ *and over* $\mathcal{E}_\lambda$ *by* $\Phi_{12} = (-(2n+1), 0, 1, 2n)$.

(2) *For a normal form of the type* $h(u,\lambda)\,z = (\epsilon_1 u + \delta_m \lambda^m)\, z$, Φ_1 *is generated over* $\mathcal{E}_{(u,\lambda)}$ *by* $(-u,1,0,0)$ *(already in* $\hat{s}$*), by* $\Phi_{11} = (3\,\delta_m u + \epsilon_1 \lambda^m, 0, -\epsilon_1 \lambda^m - \delta_m u, 0)$ *and over* $\mathcal{E}_\lambda$ *by* $\Phi_{12} = (-3m, 0, m, 2)$.

Proof: The two cases can be treated with one set of calculations. Let

$$h(u,\lambda) = u^n p(u,\lambda) + \lambda^m q(\lambda)$$

for some integers n, m and $p^o, q^o \neq 0$. We would like to find $(T,X,L) \in k_\lambda^{O(2)}$ such that $(T,X,L)\cdot h \equiv 0$ where $T(z,\lambda)w = a\,w + bz^2\bar{w}$, $X(z,\lambda) = c\,z$ and $L(\lambda) = d\,\lambda$ with $a,b,c \in \mathcal{E}_{(u,\lambda)}$, $d \in \mathcal{E}_\lambda$. We represent such (T,X,L) by (a,b,c,d). By explicit calculations we find that

$$\begin{aligned}(T,X,L)\cdot h = u^n(ap + ubp + cp + 2ncp + 2cup_u + d\lambda p_\lambda)&\\ + \lambda^m(aq + cq + ubq + mdq + d\lambda q_\lambda) = 0,&\end{aligned}$$

hence, for any $r \in \mathcal{E}_{(u,\lambda)}$, we have that

$$\begin{aligned}ap + [(2n+1)p + 2up_u]\,c &= \lambda^m r - ubp - d\lambda p_\lambda\\ aq + cq &= -u^n r - ubq - mdq - d\lambda q_\lambda.\end{aligned} \tag{3.12}$$

Because $p^o, q^o \neq 0$, there exist an unique pair of solutions of (3.12) $\bar{a}(r,b,d)$ and $\bar{c}(r,b,d)$. Note that the problem (3.12) is linear in the dependent variables and so $\bar{a}$ and $\bar{c}$ are linear in (r,b,d). Therefore, three generators are then found by taking in turn the $\mathcal{E}_{(u,\lambda)}$-modules generated by $(1,0,0)$, $(0,1,0)$ (for r,b) and the $\mathcal{E}_\lambda$-module generated by $(0,0,1)$ (for d). □

Remark. The generators of Φ_1 (when $b=0$) act on $f_2 = (r,s)$ as

$$\mu\begin{pmatrix} ar + c(xr_x + yr_y) + d\lambda r_\lambda \\ as + c(xs_x + ys_y) + d\lambda s_\lambda\end{pmatrix}.$$

Proposition 3.8. *For our normal forms h_i, $i \neq 5$, the part of Φ_2, not in $\hat{s}$, is generated over $\mathcal{E}_{(u,\lambda)}$ by Φ_{11} and over $\mathcal{E}_\lambda$ by $\lambda\Phi_{12}$.*

Proof: We proceed as in the previous proof, but now we take $T(z,\lambda)w = (au + A\lambda)\,w + bz^2\bar{w}$, $X(z,\lambda) = (cu + C\lambda)\,z$ and $L_{11}(\lambda) = D\lambda^2$ where $a,b,c \in \mathcal{E}_{(u,\lambda)}$ and $A,C,D \in \mathcal{E}_\lambda$. Then

$$\begin{aligned}(T,X,L)\cdot h = (a+b+c)\,uh + 2cu^2h_u + 2C\lambda u\,h_u&\\ + (A+C)\,\lambda h + D\lambda^2 h_\lambda.&\end{aligned} \tag{3.13}$$

From the inspection of (3.13) for our h_i's, $i \neq 5$, we can conclude. □

4. Bifurcation Diagrams

In this paper we describe the bifurcation diagrams of the miniversal unfoldings $F(z,\lambda,\mu,\alpha) = F_1(z,\lambda,\alpha) + \mu F_2(z,\lambda,\mu,\alpha)$ for the cases I_0, II and III.

4.1. General Properties of Bifurcation Diagrams.

The miniversal unfolding F of the normal forms of Theorem 3.4 are given by

$$F(z,\lambda,\mu,\alpha) = \begin{pmatrix} x\,H(u,\lambda,\alpha) \\ y\,H(u,\lambda,\alpha) + \mu s(y,\lambda) \end{pmatrix}, \tag{4.1}$$

where H is an O(2)-invariant unfolding and s satisfies $s^o = 1$. Note that F in (4.1) is the gradient of the potential

$$V_{\lambda,\mu,\alpha}(z) = \tfrac{1}{2}\hat{H}(u,\lambda,\alpha) + \mu S(y,\lambda) \tag{4.2}$$

where $\hat{H}_u = H$ and $S_y = s$. Hence, the solutions of $F = 0$ are the critical points of $V_{\lambda,\mu,\alpha}$.

The solutions of $F = 0$ belong to two families. When $\mu = 0$, we get the origin $x = y = 0$ and the S^1-orbits $x^2 + y^2 = u$ determined by the solutions u of $H(u,\lambda,\alpha) = 0$ of the original O(2)-equivariant diagrams. When $\mu \neq 0$, the symmetry breaking solutions appear. For the normal form (4.1), they are in the hyperplane $x = 0$. In that hyperplane, we have the perturbed branches $y\,H(y^2,\lambda,\alpha) + \mu s(y,\lambda) = 0$. In general, this means that the symmetry breaking selects one hypersurface in the (x,y,λ,μ,α)-space where the symmetry breaking solutions live.

The stability of the solutions of $F = 0$ are determined from the Jacobian F_z of (4.1)

$$F_z = \begin{pmatrix} H + 2x^2H_u & 2xyH_u \\ 2xyH_u & H + 2y^2H_u + \mu s_y \end{pmatrix}.$$

When $\mu = 0$, the eigenvalues of the O(2)-equivariant problem reduce to

$$\sigma(F_z) = \{0, 2uH_u\}. \tag{4.3}$$

When $x = y = 0$ is a solution of $F = 0$, the eigenvalues are $\sigma(F_z) = \{H(0,\lambda,\alpha), H(0,\lambda,\alpha) + \mu s_y(0,\lambda)\}$. When $x = 0$, $y \neq 0$, the eigenvalues are a perturbation of (4.3), given by

$$\sigma(F_z) = \left\{ -\mu\frac{s}{y}, 2y^2H_u + \mu\frac{(ys_y - s)}{y} \right\}. \tag{4.4}$$

Note that the sign of the first eigenvalue of (4.4) *changes* with the sign of y, which means that only one of the pair of solutions $\pm y$ of the O(2)-problem remains stable after the symmetry breaking. The second eigenvalue is only a small perturbation of $2uH_u$. We shall come back to that issue when analysing case I_0.

Remark. Using the approach of Proposition 3.3, we can show that (4.1) is a general form for all miniversal unfoldings of normal forms with $f_2^o \neq 0$. Therefore, the results of this section will also apply to those cases.

4.2. Bifurcation Diagrams for Germs up to Topological Codimension 1.

In the following we describe explicitly the bifurcation diagrams associated with the cases of topological codimension at most 1, cases I_0, II and III. Bifurcation equivalence preserves the $\mu = 0$-slice and the distribution of regions in the (λ, μ)-plane with the same pattern of solutions. The boundaries of those regions form the **bifurcation variety** (at fixed values of α). It is constituted of the values of (λ, μ) where there is at least one fold in the bifurcation diagram, that is, the values of (λ, μ) such that there exist solutions of $F = 0$ with $\det(F_z) = 0$. Bifurcation equivalence does not preserve one dimensional slices of the bifurcation diagrams apart from along the λ-axis (when $\mu = 0$). We sketch the bifurcation varieties in the (λ, μ)-plane and the set of critical points of $V_{\lambda,\mu,\alpha}$ in the connected component of the complement of the bifurcation variety.

4.2.1. Case $\mathbf{I_0}$.

For case I_0, $H(u, \lambda) = \epsilon_1 u + \delta_1 \lambda$ and $s = 1$, and so the potential (4.2) is

$$V^{\mathrm{I}_0}_{\lambda,\mu}(x, y) = \tfrac{1}{4}\epsilon_1(x^2 + y^2)^2 + \tfrac{1}{2}\delta_1\lambda(x^2 + y^2) + \mu y.$$

The part of the zero-set of case I_0 when $\mu \neq 0$ is $\epsilon_1 y^3 + \delta_1 \lambda y + \mu = 0$ and so the bifurcation variety for case I_0 is $(27\mu^2 + 4\epsilon_1\delta_1\lambda^3)\,\mu = 0$. When $\epsilon_1 = -\delta_1 = 1$, the bifurcation variety is sketched in Figure 1. We have also indicated the number of critical points of $V^{\mathrm{I}_0}_{\lambda,\mu}$ with their stability, stable if it is a minimum, unstable otherwise.

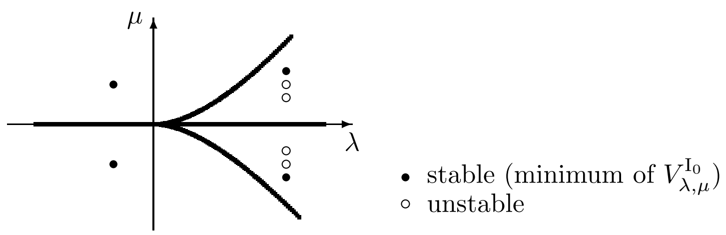
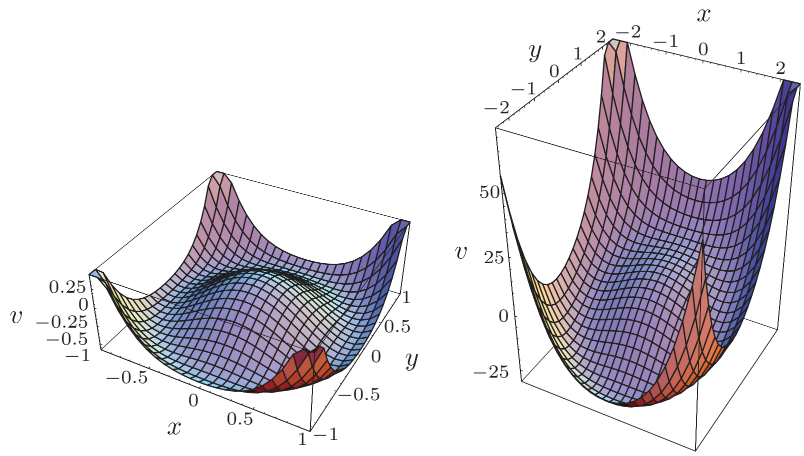


FIGURE 1. Bifurcation variety for case I_0 and critical points of $V^{\mathrm{I}_0}_{\lambda,\mu}$, $\epsilon_1 = -\delta_1 = 1$.

The corank one $(\mathrm{Z}_2, \mathbf{1})$-symmetry breaking problems when $f_2^o \neq 0$ have the same normal forms ([**12**]), but stability is preserved under perturbation when $\mu \neq 0$. This is not the case here, only one solution of the O(2)-orbit remains stable. To understand why, we represent in Figure 2 the potential $V_{\mu,\lambda}^{\mathrm{I}_0}$. When $\mu = 0$, $V_{0,\lambda}^{\mathrm{I}_0}$ has a unique minimum for $\lambda \leq 0$, but has a 'cardinal hat'-like shape for $\lambda > 0$ (see Figure 2 on the right). The influence of the term μy is to add a planar contribution to $V_{\mu,\lambda}^{\mathrm{I}_0}$, so $V_{\mu,\lambda}^{\mathrm{I}_0}$ will look tilted into the y-direction. Obviously this has only a marginal effect when $\lambda \leq 0$, so we still have a unique minimum there. But, when λ is positive, one can see that the minimum rim of the hat is now tilted, only one minimum is left from the original circle orbit (see Figure 2 on the right). This is a typical planar phenomena. In corank one, the potential of the normal form corresponding to case I_0 is $W_{\mu,\lambda}(y) = V_{\lambda,\mu}^{\mathrm{I}_0}(0, y) = \frac{1}{4}\epsilon_1 y^4 + \frac{1}{2}\delta_1 \lambda y^2 + \mu y$, but, when slightly tilted when $\mu \neq 0$, the local minima retain their nature.



$\lambda > 0,\ \mu = 0$ $\qquad\qquad$ $\lambda, \mu > 0$

Figure 2. Potential $V_{\lambda,0}^{\mathrm{I}_0}$ with $\lambda > 0$, and $V_{\lambda,\mu}^{\mathrm{I}_0}$ for $\lambda, \mu > 0$, $\epsilon_1 = -\delta_1 = 1$.

4.2.2. Case II.

For case II, $H(u, \lambda, \alpha) = \epsilon_1 u + \delta_2 \lambda^2 + \alpha$ and $s = 1 + ay$, and so the potential (4.2) is

$$V_{\lambda,\mu,\alpha}^{\mathrm{II}}(x, y) = \tfrac{1}{4}\epsilon_1 (x^2 + y^2)^2 + \tfrac{1}{2}(\delta_2 \lambda^2 + \alpha)(x^2 + y^2) + \mu(y + \tfrac{1}{2} a y^2).$$

The part of the zero-set of case II when $\mu \neq 0$ is $\epsilon_1 y^3 + (\delta_2\lambda^2 + \alpha + a\mu)y + \mu = 0$ and so the bifurcation variety for case II is

$$\left(27\mu^2 + 4\epsilon_1(\delta_2\lambda^2 + \alpha + a\mu)^3\right)\mu = 0.$$

The parameter a is topologically irrelevant. It breaks the Z_2-symmetry in μ of the bifurcation variety. The Taylor series expansion in term of a of the extremum value of μ on the bifurcation variety when $\lambda = 0$ is $\mu_\pm(a) = \pm\sqrt{-4\epsilon_1\alpha^3/27} - (2\epsilon_1\alpha^2/9)\,a + \cdots$, when $\mathrm{sign}(\epsilon_1\alpha) = -1$. For $\epsilon_1 = \delta_2 = 1$, the bifurcation variety is sketched in Figure 3 for $\alpha < 0$. When $\alpha > 0$, the bifurcation variety is $\mu = 0$. We get a unique minimum for $V^{\mathrm{II}}_{\lambda,\mu}$, perturbation of the trivial branch when $\mu = 0$.

The main difference between cases I_0 and II is in the behaviour of the parameters. Indeed, the potential $V^{\mathrm{II}}_{\lambda,\mu}$ behaves like in Figure 2, but for a more complicated dependence in the parameters, as can be seen in Figure 3. Note that the bifurcation variety for case II is formed from the gluing of two bifurcation varieties for case I_0 in opposite directions. When $\epsilon_1\delta_2 = 1$, the bifurcation variety is bounded when $\mu \neq 0$ ('elliptic' case). On the contrary, when $\epsilon_1\delta_2 = -1$, the bifurcation variety is unbounded for all values of μ ('hyperbolic' case).

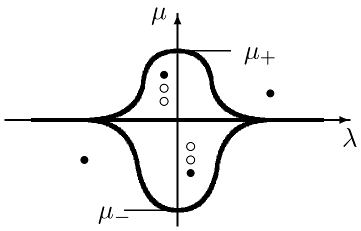


FIGURE 3. Bifurcation variety for case II when $\alpha < 0$ and critical points of $V^{\mathrm{II}}_{\lambda,\mu,\alpha}$, $\epsilon_1 = \delta_2 = 1$.

4.2.3. Case III.

Finally, for case III, $H(u,\lambda,\alpha) = \epsilon_2 u^2 + \delta_1\lambda + \alpha u$ and $s = 1 + ay^2 + (b + c\lambda)y^3$, and so the potential (4.2) is

$$V^{\mathrm{III}}_{\lambda,\mu,\alpha}(x,y) = \tfrac{1}{6}\epsilon_2(x^2+y^2)^3 + \tfrac{1}{4}\alpha(x^2+y^2)^2 + \tfrac{1}{2}\delta_1\lambda(x^2+y^2) + \mu\left(y + \tfrac{1}{3}ay^3 + \tfrac{1}{4}(b+c\lambda)\,y^4\right).$$

The part of the zero-set of case III when $\mu \neq 0$ is

$$\epsilon_2 y^5 + (\alpha + (b + c\lambda)\mu)y^3 + a\mu y^2 + \delta_1\lambda y + \mu = 0. \tag{4.5}$$

The parameters a, b, c are topologically irrelevant. The bifurcation variety of (4.5) is the union of the line $\mu = 0$ with a homeomorphic deformation of the (C, D)-slice of the bifurcation variety of the butterfly $\epsilon_2 y^5 + Ay^3 + By^2 + Cy + D = 0$ when $B = 0$. The total bifurcation variety is

$$\begin{aligned}256\epsilon_2 C^5 &- 128A^2C^4 + 16\epsilon_2 A^4C^3 + 3125D^4\\ &+ 108\epsilon_2 A^5D^2 - 900A^3CD^2 + 2000\epsilon_2 AC^2D^2\\ &- \epsilon_2 B(3750AD^3 + 1600C^3D - 560\epsilon_2 A^2C^2D + 72A^4CD)\\ &+ B^2(825A^2D^2 - 4\epsilon_2 A^3C^2 + 2250CD^2 + 144AC^3)\\ &+ B^3(16\epsilon_2 A^3D - 630ACD) - 27B^4C^2 + 108B^5D = 0,\end{aligned}$$

which simplifies when $B = 0$ to

$$\begin{aligned}-256C^5 - 128A^2C^4 - 16A^4C^3 + 3125D^4&\\ - 108A^5D^2 - 900A^3CD^2 - 2000AC^2D^2 = 0.&\end{aligned} \tag{4.6}$$

Via the rescalings $C = A^2x$, $D = A^{\frac{5}{2}}y$, the variety (4.6) is a topological cone of axis A on the curve $3125y^4 - 4y^2(27 + 225x + 500x^2) - 16x^3(1 + 4x)^2 = 0$. When $\alpha > 0$ and $\delta_1 = -1$, the bifurcation variety is as in Figure 1. There is a transition from one stable critical point to three critical points when crossing the bifurcation variety, only one of them stable. When $\alpha < 0$, the geometry is more intricate. The bifurcation variety is sketched in Figure 4.

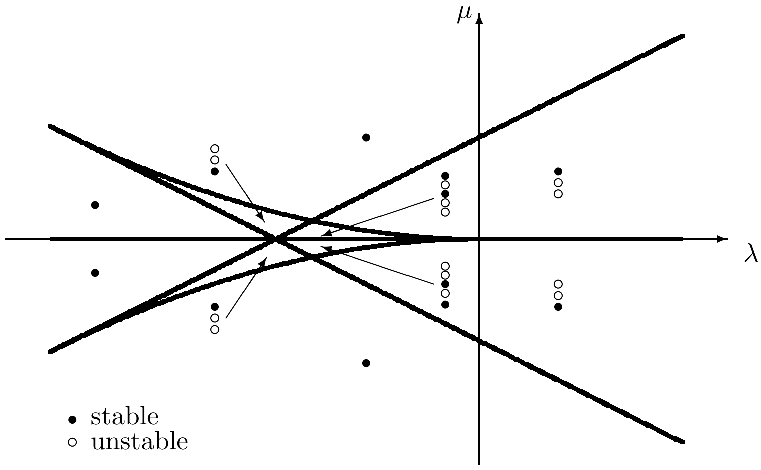


Figure 4. Bifurcation variety for case III when $\alpha < 0$ and critical points of $V^{\mathrm{III}}_{\lambda,\mu,\alpha}$, $\epsilon_2 = -\delta_1 = 1$.

There can be up to 5 critical points for $V^{\rm III}_{\lambda,\mu,\alpha}$, a maximum of two of them minima (stable) (see Figure 5).

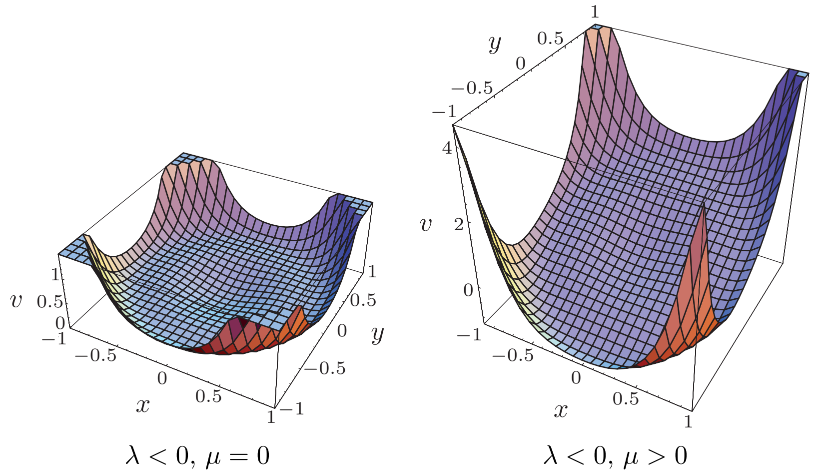


FIGURE 5. Potential $V^{\rm III}_{\lambda,\mu,0}$ with $\lambda < 0$, and $V^{\rm III}_{\lambda,\mu,\alpha}$ for $\lambda < 0$, $\mu > 0$, $\epsilon_2 = -\delta_1 = 1$.

5. Examples

In this final section we consider a few examples leading to bifurcation equations of the type we analysed in Section 4.

5.1. Periodic Solutions of Autonomous Equations.

We want to study the existence of period-2π solutions of

$$\ddot{u} + u + u\,g(u,\lambda,\alpha) + \mu\,p(t,u,\dot{u},\lambda,\mu,\alpha) = 0 \tag{5.1}$$

where p is 2π-periodic in t and $g(0,0) = 0$. As usual we assume that we have made a change in coordinates in u so that the origin is the trivial solution for all λ.

The autonomous part of equation (5.1) is O(2)-equivariant with the shift action $t \mapsto t + \theta$, $\theta \in S^1$, and time-reversal action $t \mapsto -t$ for the symmetry generator. It has been normalised so that the bifurcation point is at $\lambda = 0$, with the kernel of the linearisation $\ddot{u} + u = 0$ generated by $\cos t$ and $\sin t$. If p has period 2π in t, the equation (5.1) has generically no additional symmetries (see [**14**] for a broader discussion of subharmonic symmetry breaking). When

$$\left|\int_{-\pi}^{\pi} p(t,\mathbf{0})\, e^{it}\, dt\right| \neq 0,$$

we are in one of our cases I_0, II or III provided that the bifurcation equation of the autonomous equation is of topological codimension not more than 1. The coefficients we need to monitor for the cases classified in Theorem 3.4 are more or less well-known and are given by:

$$\delta_1 = \operatorname{sign} g^o_\lambda, \quad \delta_2 = \operatorname{sign} g^o_{\lambda\lambda}, \quad \epsilon_1 = \operatorname{sign}(9g^o_{vvv} - 20{g^o_{vv}}^2),$$
$$\epsilon_2 = \operatorname{sign}(8g^o_{vvvvv} - 56g^o_{vv}g^o_{vvvv} + 102g^o_{vvv}{g^o_{vv}}^2 + {g^o_{vvv}}^2 - 110{g^o_{vv}}^4).$$

Depending on those values, we find the diagrams we have illustrated in Section 4. When $\epsilon_1 = -\delta_1 = 1$, we have case I_0 illustrated in Figure 1. When $\epsilon_1 \cdot \delta_1 = 0$, we get problems of topological codimension at least 1. We assume that ϵ_1 or $\delta_1 = 0$ due to the variation of g on another parameter we denote by α. Then we get Figure 3 when $\delta_1 = 0$ but $\epsilon_1 = \delta_2 = 1$ and $g^o_{\lambda\alpha} < 0$, and we get Figure 4 when $\epsilon_1 = 0$, but $\epsilon_2 = -\delta_1 = 1$ and $\left(9g^o_{vvv} - 20{g^o_{vv}}^2\right)_\alpha > 0$. In each case the symmetry breaking term selects a pair of solutions from the O(2)-orbits, keeping only one solution possibly stable.

5.2. Non Linear Boundary Value Problem.

We consider the following nonlinear second order boundary value problem

$$\begin{aligned}\ddot{u} + \lambda_k u + q(u,\lambda,\mu) &= 0,\\ u(0) - u(1) &= \mu g_1(u,\lambda,\mu),\\ \dot{u}(0) - \dot{u}(1) &= \mu g_2(u,\lambda,\mu),\end{aligned}$$

where q, g_1, g_2 are non linear maps and $\lambda_k = 4(k\pi)^2$. To be precise, let $X = C^2[0,1]$ and $Y = C^o[0,1]$, then $q\colon X\times\mathbb{R}^2 \to Y$ and $g_1, g_2\colon X\times\mathbb{R}^2 \to \mathbb{R}$ are smooth enough, possibly involving non local terms. The system is defined as the zero-set of a smooth map $M\colon X\times\mathbb{R}^2 \to Y\times\mathbb{R}^2$ defined by

$$\begin{pmatrix} \ddot{u} + \lambda_k u + q(u,\lambda,\mu) \\ u(0) - u(1) - \mu g_1(u,\lambda,\mu) \\ \dot{u}(0) - \dot{u}(1) - \mu g_2(u,\lambda,\mu) \end{pmatrix}.$$

Note that when $\mu = 0$ the system is O(2)-equivariant and has no symmetry in general when $\mu \neq 0$. If $q^o = q^o_u = 0$, then the Implicit Function Theorem implies that M has a trivial branch of solutions near $(0, \lambda_k, 0)$. Using a change of co-ordinates, the trivial branch can be represented by $(0, \lambda + \lambda_k, 0)$, $\lambda \in (\mathbb{R}, 0)$. The derivative M_u at the trivial solution $(0, \lambda_k, 0)$ is given by $DM \colon X \to Y \times \mathbb{R}^2$ where $DMu = (\mathcal{L}_k u, u(0) - u(1), \dot{u}(0) - \dot{u}(1))$, with $\mathcal{L}_k u = \ddot{u} + \lambda_k u$. The kernel of DM is two dimensional, spanned by $\phi_1(t) = \frac{1}{\sqrt{2}} \sin 2k\pi t$ and $\phi_2(t) = \frac{1}{\sqrt{2}} \cos 2k\pi t$, where $\|\phi_{1,2}\| = 1$ using the usual L^2-norm on $[0, 1]$. Because the general solution of $\mathcal{L}_k u = f$ is

$$u = \mathcal{L}_k^{-1} f = a\phi_1 + b\phi_2 - \frac{1}{k\pi} \left(\int_0^t f(s)\phi_1(s)\, ds \right) \phi_2 + \frac{1}{k\pi} \left(\int_0^t f(s)\phi_2(s)\, ds \right) \phi_1, \quad a, b \in \mathbb{R},$$

the image of DM is given by $\{(v, c, d) \in Y \times \mathbb{R}^2 : l_1(v, c, d) = l_2(v, c, d) = 0\}$ where $l_1(v, c, d) = \sqrt{2}k\pi c - \langle v, \phi_1 \rangle$ and $l_2(v, c, d) = d + \sqrt{2}\langle v, \phi_2 \rangle$, with the scalar product $\langle , \rangle$ on $L^2(0, 1)$.

Explicitly,

$$M(u, \lambda, \mu) = DMu + \lambda Au + F(u, \lambda, \mu) + \mu H(u, \lambda, \mu)$$

where $Au = (u, 0, 0)$, $F(u, \lambda, \mu) = (q(u, \lambda, \mu), 0, 0)$ and $H(u, \lambda, \mu) = (0, -g_1(u, \lambda, \mu), -g_2(u, \lambda, \mu))$. We apply the Lyapounov-Schmidt reduction to M. Let

$$Q(v, c, d) = (v + l_1(v, c, d)\, \phi_1 - l_2(v, c, d)\, \phi_2, c, d)$$

define the projector $Q \colon Y \times \mathbb{R}^2 \to \mathrm{Im} DM$. And so, $I - Q$ is given by

$$(I - Q)(v, c, d) = (-l_1(v, c, d), l_2(v, c, d)).$$

The reduced bifurcation equations $m \colon (\mathbb{R}^{2+2}, 0) \to (\mathbb{R}^2, 0)$ are

$$m(v, \lambda, \mu) = (I - Q)\, M(v + \bar{w}(v, \lambda, \mu), \lambda, \mu) \tag{5.2}$$

where $\bar{w}(v, \lambda, \mu)$ is defined implicitly from

$$Q\, M(v + \bar{w}, \lambda, \mu) = 0. \tag{5.3}$$

To study the normal form of (5.2) we need to get the Taylor series expansion of m at the origin. In particular, the term m^o_μ is important. If it is non zero, we expect the cases I_0, II or III up topological codimension 1. If it is zero then the first possible normal form is I_2, of topological codimension 2. To evaluate the derivatives of m at the origin, we use the Chain Rule on (5.2) and on (5.3) to get the derivatives of $\bar{w}$. A routine calculation gives that $m^o_\mu = (I-Q)M^o_\mu$. Because $M^o_\mu = (q^o_\mu, -g^o_1, -g^o_2)$, we get

$$m^o_\mu = (I-Q)(q^o_\mu, -g^o_1, -g^o_2) = (-l_1(q^o_\mu, -g^o_1, -g^o_2), l_2(q^o_\mu, -g^o_1, -g^o_2)) = (\sqrt{2}k\pi g^o_1, -g^o_2).$$

And so, if $g^o_1 = g^o_2 = 0$, we cannot find the normal forms up to topological codimension 1 for (5.2). In particular, this is the case if the perturbed boundary conditions are not linear in μ at the origin, for example for (1.4) where $g_1(u,\lambda,\mu) = \dot{u}(1)$ and $g_2 = 0$.

5.3. Reaction-Diffusion Equations.

Consider a population of density u living in a bounded regular domain $\Omega \subset \mathbb{R}^n$. Suppose that u satisfies the following parametrised semilinear parabolic evolution equation

$$u_t = \Delta u + g(u, \bar{u}, \lambda) \tag{5.4}$$

subject to non-flux (Neumann) boundary conditions $\frac{\partial u}{\partial n} = 0$. The term $\bar{u}$ represents a nonlocal contribution of the averaged density $\bar{u} = \frac{1}{|\Omega|}\int_\Omega u(s)\,ds$, for instance if there is some sharing of resources on a time scale faster than the reproductive cycle of u. The diffusion coefficient has been normalised to 1 by rescaling time.

We assume that there is a constant positive steady state $u_0(\lambda)$. To study the bifurcation from u_0 of nontrivial steady states, consider the linearisation of the steady state equation of (5.4), subject to Neumann boundary conditions:

$$\mathcal{L}v \stackrel{\text{def}}{=} \Delta v + a(\lambda)\,v + b(\lambda)\,\bar{v},$$

where $a(\lambda) = g_u(u_0(\lambda), \bar{u}_0(\lambda), \lambda)$ and $b(\lambda) = g_{\bar{u}}(u_0(\lambda), \bar{u}_0(\lambda), \lambda)$. The values of λ for which $\mathcal{L}$ has a non-trivial kernel are candidates for bifurcation points. Moreover, the stability of the steady state solutions of (5.4) obeys the Principle of Linearised Stability. We are looking for values of λ when u_0 looses stability to nontrivial steady-states.

Take Ω to be the unit disk in the plane. Then (5.4) is O(2)-equivariant under the action $(\gamma u)(x) = u(\gamma x)$, where γ is the usual orthogonal action of O(2) in the plane, and so the bifurcation equations are O(2)-equivariant. When the kernel of $\mathcal{L}$ is two dimensional we can apply our results to see how the perturbation of the symmetry (both in the domain or in the equation) affects the solution set. Because we have Neumann boundary conditions, the study of the spectrum of the nonlocal operator $\mathcal{L}$ can be explicitly carried out.

Let $\hat{\mathcal{L}}v = \Delta v + a^2\, v$ for $a \in \mathbb{R}$. Then, using polar coordinates and separation of variables, we can find the eigenvalues and eigenfunctions of $\hat{\mathcal{L}}$ in terms of Fourier series in the angle variable θ and Bessel functions J_n of the radius r. Let $v(r,\theta) = R(r)\Theta(\theta)$ with Θ 2π-periodic and R bounded on $[0,1]$ such that $R'(1) = 0$. The eigenvalue problem $\hat{\mathcal{L}}v = \mu v$ becomes

$$r^2\frac{R''}{R} + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = (\mu - a^2)\, r^2.$$

And so, for $k = 0, \dots,$ Θ satisfies $\Theta'' + k^2\Theta = 0$ and R the following Bessel equation

$$r^2R'' + rR' + ((a^2-\mu)\, r^2 - k^2)\, R = 0. \tag{5.5}$$

Therefore, we have that $\Theta_k = 1$ $(k=0)$ or $\Theta_k^1(\theta) = \cos k\theta$ and $\Theta_k^2(\theta) = \sin k\theta$ $(k \geq 1)$. The bounded solutions on $[0,1]$ of (5.5) are multiples of $J_k(\nu r)$ for $\nu^2 = a^2 - \mu$. To satisfy the boundary condition on R, we impose that ν is a root of J_k'. Let $\{\nu_{ij}\}_{j=1}^{\infty}$ be the roots of $J_i'(x) = 0$. So, the eigenvalues of $\hat{\mathcal{L}}v = \mu v$ are given by the double sequence $\{\hat{\mu}_{ij}\}_{i,j=0}^{\infty}$, where $\hat{\mu}_{00} = 0$, of eigenfunction $\hat{v}_{00} \equiv 1$, for $j \geq 1$, $\hat{\mu}_{0j} = \nu_{0j}^2$, of eigenfunction $J_0(\nu_{0j}r)$, and for $i,j \geq 1$, $\hat{\mu}_{ij} = \nu_{ij}^2$, of eigenfunctions $\{J_i(\nu_{ij}r)\cos i\theta, J_i(\nu_{ij}r)\sin i\theta\}$. Moreover, note that, for any function h of r and $k \geq 1$, $\int_\Omega h(r)\sin k\theta\, rdr\, d\theta = \int_\Omega h(r)\cos k\theta\, r\, dr\, d\theta = 0$, and, for all $i \geq 1$,

$$\int_\Omega J_0(\nu_{0,i}r)\, r\, dr\, d\theta = 2\pi\int_0^1 J_0(\nu_{0,i}r)\, r\, dr = \frac{2\pi}{\nu_{0,i}^2}\int_0^{\nu_{0,i}} J_0(x)\, x\, dx = 0$$

because of the relation $xJ_1'(x) = -J_1(x) + xJ_0(x)$ and that $J_1(\nu_{0,i}) = 0$ for $i \geq 1$. Therefore, the eigenvalues of $\mathcal{L}$ are given by $\mu_{00} = a(\lambda) + b(\lambda)$ and, for $i \geq 0$, $j \geq 1$, $\mu_{ij} = a(\lambda) - \nu_{ij}^2$ with the same eigenfunctions as $\hat{\mathcal{L}}$. Remark that the first zero of the derivative of J_1 occurs before all the other zeroes of the derivatives of J_0 and of the other Bessel functions J_i, $i \geq 1$. It is now clear that there can be problems where $\mu_{11}(\lambda_0) = 0$ with $\mu_{00}(\lambda_0) < 0$, $\mu_{0i}(\lambda_0) < 0$, $i \geq 1$, with all eigenvalues being in the negative half-plane for $\lambda < \lambda_0$. In that case, u_0 loses stability first at $\lambda = \lambda_0$ with a two dimensional kernel generated by $J_1(\nu_{11}r)\cos\theta$ and $J_1(\nu_{11}r)\sin\theta$.

Suppose now that (5.4) is subjected to a non-symmetric perturbation μp

$$u_t = \Delta u + g(u, \bar{u}, \lambda) + \mu\, p(r, \theta, u, \bar{u}, \lambda, \mu)$$

with non-flux (Neumann) boundary conditions $\frac{\partial u}{\partial n} = 0$. The bifurcation equations will have the structure of (1.3) near $\lambda = \lambda_0$, $\mu = 0$. All normal forms we classified can occur depending on integrals involving p and the eigenfunctions.

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