A CHARACTERIZATION OF TWO WEIGHT NORM INEQUALITIES FOR MAXIMAL SINGULAR INTEGRALS

MICHAEL T. LACEY, ERIC T. SAWYER, AND IGNACIO URIARTE-TUERO

Abstract. Let σ and ω be positive Borel measures on $\mathbb{R}^n$ and let $1 < p < \infty$. We characterize boundedness of (dual pairs of) certain maximal singular integrals $T_\natural$ from $L^p(\sigma)$ to $L^p(\omega)$ in terms of two testing conditions. The first applies to a restricted class of functions and is a strong-type testing condition,

$$\int_Q T_\natural(\chi_E \sigma)(x)^p d\omega(x) \le C_1 \int_Q d\sigma(x), \qquad \text{for all } E \subset Q,$$

and the second is a weak-type or dual cube testing condition,

$$\int_Q T_\natural(\chi_Q f\sigma)(x) d\omega(x) \le C_2 \left(\int_Q |f(x)|^p d\sigma(x)\right)^{\frac{1}{p}} \left(\int_Q d\omega(x)\right)^{\frac{1}{p'}},$$

for all cubes Q in $\mathbb{R}^n$ and all functions $f \in L^p(\sigma)$. We also characterize the weak-type two weight inequality for $T_\natural$ in terms of the second condition and the A_p condition.

1. Introduction

Two weight inequalities for Maximal Functions and other positive operators have been characterized in [17], [16], [18], with these characterizations being given in terms of obviously necessary conditions, that the operators be uniformly bounded on a restricted class of functions, namely indicators of intervals and cubes. Thus, these characterizations have a form reminiscent of the $T1$ Theorem of David and Journé.

Corresponding results for even the Hilbert transform are not known, and evidently much harder to obtain. We comment in more detail on prior results below, including the innovative work of Nazarov, Treil and Volberg [7], [8], [9], [10].

Our focus is on providing characterizations of the boundedness of certain maximal truncations of a fixed operator of singular integral type. The singular integrals will be of the usual type, for example the Hilbert transform, or a generalized fractional integral, for instance the Cauchy transform in the plane. The characterizations are in terms of certain obviously necessary conditions, in which the class of functions being tested is simplified. For such examples, we prove unconditional

Research supported in part by the NSF, through grant DMS-0456538.
Research supported in part by NSERC.
Research supported in part by the NSF, through grant DMS-0901524.

characterizations of both strong-type and weak-type two weight inequalities for certain maximal truncations of these integrals.

As the precise statements of our characterizations are somewhat complicated, we illustrate them with two important cases here. Let T denote either the Hilbert or Beurling transform, let $T_\flat$ denote the usual maximal singular integral associated with T, and finally let $T_\natural$ denote the new *strongly* maximal singular integral associated with T that is defined below. Suppose σ and ω are two locally finite positive measures. Then we have the following weak and strong type characterizations which we emphasize hold for *all* $1 < p < \infty$.

- If σ and ω have no point masses in common, then the operator $T_\flat$ is *weak* type (p,p) with respect to (σ,ω), i.e.

$$\left\| T_\flat \left(f\sigma \right) \right\|_{L^{p,\infty}(\omega)} \leq C \left\| f \right\|_{L^p(\sigma)} , \tag{1.1}$$

  for all f bounded with compact support, *if and only if* the two weight A_p condition

$$\frac{1}{|Q|} \int_Q d\omega \left(\frac{1}{|Q|} \int_Q d\sigma \right)^{p-1} \leq C,$$

  holds for all cubes Q; and the dual $T_\flat$ cube testing condition

$$\int_Q T_\flat \left(\chi_Q f\sigma \right) d\omega \leq C \left(\int_Q |f|^p \, d\sigma \right)^{\frac{1}{p}} \left(\int_Q d\omega \right)^{\frac{1}{p'}} , \tag{1.2}$$

  holds for all cubes Q and $f \in L^p_Q(\sigma)$ (part 4 of Theorem 1.19). The same is true for $T_\natural$. It is easy to see that (1.2) is equivalent to the more familiar dual cube testing condition

$$\int_Q \left| L^* \left(\chi_Q \omega \right) \right|^{p'} d\sigma \leq C \int_Q d\omega, \tag{1.3}$$

  for all cubes Q and linearizations L of the maximal singular integral $T_\flat$ (see (2.14)).
- The operator $T_\natural$ is both *strong* type (p,p) with respect to (σ,ω) and *strong* type (p',p') with respect to (ω,σ), i.e.

$$\left\| T_\natural \left(f\sigma \right) \right\|_{L^p(\omega)} \leq C \left\| f \right\|_{L^p(\sigma)} ,$$
$$\left\| T_\natural \left(h\omega \right) \right\|_{L^{p'}(\sigma)} \leq C \left\| h \right\|_{L^{p'}(\omega)} ,$$

  for all f and h bounded with compact support, *if and only if* the dual $T_\natural$ cube testing conditions

$$\int_Q T_\natural \left(\chi_Q f\sigma \right) d\omega \leq C \left(\int_Q |f|^p \, d\sigma \right)^{\frac{1}{p}} \left(\int_Q d\omega \right)^{\frac{1}{p'}} ,$$
$$\int_Q T_\natural \left(\chi_Q h\omega \right) d\sigma \leq C \left(\int_Q |h|^{p'} \, d\omega \right)^{\frac{1}{p'}} \left(\int_Q d\sigma \right)^{\frac{1}{p}} ,$$

hold for all cubes Q and $f \in L^p_Q(\sigma), h \in L^{p'}_Q(\omega)$; and the forward $T_\natural$ testing conditions

$$\int_Q T_\natural (\chi_E \sigma)^p \, d\omega \leq C \int_Q d\sigma, \tag{1.4}$$
$$\int_Q T_\natural (\chi_E \omega)^{p'} \, d\sigma \leq C \int_Q d\omega,$$

hold for all cubes Q and all compact subsets E of Q (part 4 of Theorem 1.24). Note that in (1.4) we are required to test over all compact subsets E of Q on the left side, but retain the upper bound over the (larger) cube Q on the right side.

As these results indicate, the imposition of the weight σ on both sides of (1.1) is a standard part of weighted theory, in general necessary for the testing conditions to be sufficient. Compare to the characterization of the two weight maximal function inequalities in Theorem 1.6 below.

Problem 1.5. *In* (1.4)*, our testing condition is more complicated than one would like, in that one must test over all compact $E \subset Q$ in* (1.4)*. There is a corresponding feature of* (1.2)*, seen after one unwinds the definition of the linearization L^*. We do not know if these testing conditions can be further simplified. The form of these testing conditions is dictated by our use of what we call the 'maximum principle', see Lemma 2.9.*

We now recall the two weight inequalities for the Maximal Function as they are central to the new results of this paper. Define the Maximal Function

$$\mathcal{M}\nu(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |\nu| \,, \qquad x \in \mathbb{R},$$

where the supremum is taken over all cubes Q (by which we mean cubes with sides parallel to the coordinate axes) containing x.

Theorem on Maximal Function Inequalities 1.6. *Suppose that σ and ω are positive locally finite Borel measures on $\mathbb{R}^n$, and that $1 < p < \infty$. The maximal operator $\mathcal{M}$ satisfies the two weight norm inequality (*[16]*)*

$$\|\mathcal{M}(f\sigma)\|_{L^p(\omega)} \leq C \|f\|_{L^p(\sigma)} \,, \qquad f \in L^p(\sigma) \,, \tag{1.7}$$

if and only if *for all cubes $Q \subset \mathbb{R}^n$,*

$$\int_Q \mathcal{M}\left(\chi_Q \sigma\right)(x)^p d\omega(x) \leq C_1 \int_Q d\sigma(x). \tag{1.8}$$

The maximal operator $\mathcal{M}$ satisfies the weak-type *two weight norm inequality (*[5]*)*

$$\|\mathcal{M}(f\sigma)\|_{L^{p,\infty}(\omega)} \equiv \sup_{\lambda > 0} \lambda \left|\{\mathcal{M}(f\sigma) > \lambda\}\right|_\omega^{\frac{1}{p}} \leq C \|f\|_{L^p(\sigma)} \,, \qquad f \in L^p(\sigma) \,, \tag{1.9}$$

if and only if *the two weight A_p condition holds for all cubes $Q \subset \mathbb{R}^n$:*

$$\left[\frac{1}{|Q|}\int_Q d\omega\right]^{\frac{1}{p}}\left[\frac{1}{|Q|}\int_Q d\sigma\right]^{\frac{1}{p'}} \leq C_2. \tag{1.10}$$

The necessary and sufficient condition (1.8) for the strong type inequality (1.7) states that one need only test the strong type inequality for functions of the form $\chi_Q\sigma$. Not only that, but the full $L^p(\omega)$ norm of $\mathcal{M}(\chi_Q\sigma)$ need not be evaluated. There is a corresponding weak-type interpretation of the A_p condition (1.10). Finally, the proofs given in [16] and [5] for absolutely continuous weights carry over without difficulty for the locally finite measures considered here.

1.1. **Two Weight Inequalities for Singular Integrals.** Let us set notation for our Theorems. Consider a kernel function $K(x,y)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$ satisfying the following size and smoothness conditions,

$$\begin{aligned} &|K(x,y)| \leq C\,|x-y|^{-n}\,, \\ &|K(x,y)-K(x',y)| \leq C\delta\left(\frac{|x-x'|}{|x-y|}\right)|x-y|^{-n}\,, \qquad \frac{|x-x'|}{|x-y|}\leq\frac{1}{2}, \end{aligned} \tag{1.11}$$

where δ is a Dini modulus of continuity, i.e. a nondecreasing function on $[0,1]$ with $\delta(0)=0$ and $\int_0^1 \delta(s)\frac{ds}{s}<\infty$.

Next we describe the truncations we consider. Let ζ,η be fixed smooth functions on the real line satisfying

$$\begin{aligned} &\zeta(t)=0 \text{ for } t\leq\frac{1}{2} \text{ and } \zeta(t)=1 \text{ for } t\geq 1, \\ &\eta(t)=0 \text{ for } t\geq 2 \text{ and } \eta(t)=1 \text{ for } t\leq 1, \\ &\zeta \text{ is nondecreasing and } \eta \text{ is nonincreasing.} \end{aligned}$$

Given $0<\varepsilon<R<\infty$, set $\zeta_\varepsilon(t)=\zeta\left(\frac{t}{\varepsilon}\right)$ and $\eta_R(t)=\eta\left(\frac{t}{R}\right)$ and define the smoothly truncated operator $T_{\varepsilon,R}$ on $L^1_{loc}(\mathbb{R}^n)$ by the absolutely convergent integrals

$$T_{\varepsilon,R}f(x)=\int K(x,y)\zeta_\varepsilon(|x-y|)\eta_R(|x-y|)f(y)dy, \qquad f\in L^1_{loc}(\mathbb{R}^n)\,.$$

Define the *maximal* singular integral operator $T_\flat$ on $L^1_{loc}(\mathbb{R}^n)$ by

$$T_\flat f(x)=\sup_{0<\varepsilon<R<\infty}|T_{\varepsilon,R}f(x)|\,, \qquad x\in\mathbb{R}^n.$$

We also define a corresponding *new* notion of *strongly maximal* singular integral operator $T_\natural$ as follows. In dimension $n=1$ we set

$$T_\natural f(x)=\sup_{0<\varepsilon_i<R<\infty,\frac{1}{4}\leq\frac{\varepsilon_1}{\varepsilon_2}\leq 4}|T_{\boldsymbol{\varepsilon},R}f(x)|\,, \qquad x\in\mathbb{R},$$

where $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2)$ and

$$T_{\boldsymbol{\varepsilon},R}f(x) = \int K(x,y)\left\{\zeta_{\varepsilon_1}(x-y) + \zeta_{\varepsilon_2}(y-x)\right\}\eta_R(|x-y|)\, f(y)dy.$$

Thus the local singularity has been removed by a *noncentered* smooth cutoff - ε_1 to the left of x and ε_2 to the right of x, but with controlled eccentricity $\frac{\varepsilon_1}{\varepsilon_2}$. There is a similar definition of $T_\natural f$ in higher dimensions involving in place of $\zeta_\varepsilon(|x-y|)$, a product of smooth cutoffs,

$$\zeta_{\boldsymbol{\varepsilon}}(x-y) \equiv 1 - \prod_{k=1}^{n}\left[1 - \left\{\zeta_{\varepsilon_{2k-1}}(x_k - y_k) + \zeta_{\varepsilon_{2k}}(y_k - x_k)\right\}\right],$$

satisfying $\frac{1}{4} \leq \frac{\varepsilon_{2k-1}}{\varepsilon_{2k}} \leq 4$ for $1 \leq k \leq n$. The advantage of this larger operator $T_\natural$ is that in many cases boundedness of $T_\natural$ (or collections thereof) implies boundedness of the maximal operator $\mathcal{M}$. Our method of proving boundedness of $T_\flat$ and $T_\natural$ requires boundedness of the maximal operator $\mathcal{M}$ anyway, and as a result we can in some cases give necessary and sufficient conditions for strong boundedness of $T_\natural$. As for weak-type boundedness, we can in many more cases give necessary and sufficient conditions for weak boundedness of the usual truncations $T_\flat$.

Definition 1.12. We say that T is a *standard singular integral operator with kernel* K if T is a bounded linear operator on $L^q(\mathbb{R}^n)$ for some fixed $1 < q < \infty$, that is

$$\|Tf\|_{L^q(\mathbb{R}^n)} \leq C\,\|f\|_{L^q(\mathbb{R}^n)}, \qquad f \in L^q(\mathbb{R}^n), \tag{1.13}$$

if $K(x,y)$ is defined on $\mathbb{R}^n \times \mathbb{R}^n$ and satisfies both (1.11) and the Hörmander condition,

$$\int_{B(y,2\varepsilon)^c} |K(x,y) - K(x,y')|\, dx \leq C, \qquad y' \in B(y,\varepsilon), \varepsilon > 0, \tag{1.14}$$

and finally if T and K are related by

$$Tf(x) = \int K(x,y)f(y)dy, \qquad \text{a.e.-}x \notin \mathit{supp}\ f, \tag{1.15}$$

whenever $f \in L^q(\mathbb{R}^n)$ has compact support in $\mathbb{R}^n$. We call a kernel $K(x,y)$ *standard* if it satisfies (1.11) and (1.14).

Some of our results will apply to singular integral operators that are not standard. However, for standard singular operators, we have this classical result. (See the appendix on truncation of singular integrals on page 30 of [20] for the case $R = \infty$; the case $R < \infty$ is similar.)

Theorem 1.16. *Suppose that T is a standard singular integral operator. Then the map $f \to T_\flat f$ is of weak-type $(1,1)$, and bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$.*

There exist sequences $\varepsilon_j \to 0$ and $R_j \to \infty$ such that for $f \in L^p(\mathbb{R})$ with $1 \leq p < \infty$,

$$\lim_{j\to\infty} T_{\varepsilon_j,R_j} f(x) \equiv T_{0,\infty} f(x)$$

exists for a.e. $x \in \mathbb{R}$. Moreover, there is a bounded measurable function $a(x)$ (depending on the sequences) satisfying

$$Tf(x) = T_{0,\infty} f(x) + a(x) f(x), \qquad x \in \mathbb{R}^n.$$

We state a conjecture, so that the overarching goals of this subject are clear.

Conjecture 1.17. *Suppose that σ and ω are positive Borel measures on $\mathbb{R}^n$, $1 < p < \infty$, and T is a standard singular integral operator on $\mathbb{R}^n$. Then the following two statements are equivalent:*

$$\int |T(f\sigma)|^p \omega \leq C \int |f|^p \sigma\,, \qquad f \in C_0^\infty,$$

$$\begin{cases} \left[\frac{1}{|Q|}\int_Q d\omega\right]^{\frac{1}{p}} \left[\frac{1}{|Q|}\int_Q d\sigma\right]^{\frac{1}{p'}} \leq C\,, & \\ \int_Q |T\chi_Q\sigma|^p \leq C' \int_Q \sigma\,, & \textit{for all cubes } Q. \\ \int_Q |T^*\chi_Q\omega|^{p'}\sigma \leq C'' \int_Q \omega\,, & \end{cases}$$

Remark 1.18. The first of the three testing conditions above is the two-weight A_p condition. We would expect that this condition can be strengthened to a 'Posisson two-weight A_p condition'. See [9, 23].

The most important instances of this Conjecture occur when T is one of a few canonical singular integral operators, such as the Hilbert transform, the Beurling Transform, or the Riesz Transforms. This question occurs in different instances, such as the Sarason Conjecture concerning the composition of Hankel operators, or the semi-commutator of Toeplitz operators (see [3], [24]), Mathematical Physics [12], as well as perturbation theory of some self-adjoint operators. See references in [23].

To date, this has only been verified for positive operators, such as Poisson integrals, and fractional integral operators [17], [16] and [18]. The two weight Helson-Szego Theorem has been proved by Cotlar and Sadosky [1] and [2], thus the L^2 case of the Hilbert transform is completely solved, though not in a manner that can be described as real-variable.

Nazarov, Treil and Volberg [7], [9] have characterized those weights for which the class of Haar multipliers is bounded when $p = 2$. They also have a result for an important special class of singular integral operators, the 'well-localized' operators of [8]. Citing the specific result here would carry us too far afield, but this class includes the important Haar shift examples, such as the one found by S. Petermichl [13], and generalized in [14]. Consequently, characterizations are given in [23] and [9] for the Hilbert transform and Riesz transforms in weighted

L^2 spaces under various additional hypotheses. In particular they obtain an analogue of the case $p = 2$ of the strong type theorem below. Our results can be reformulated in the context there, which theme we do not pursue further here.

We now characterize the weak-type two weight norm inequality for both maximal singular integrals and strongly maximal singular integrals.

Theorem on Maximal Singular Integral Weak-Type Inequalities 1.19. *Suppose that σ and ω are positive locally finite Borel measures on $\mathbb{R}^n$, $1 < p < \infty$, and let $T_\flat$ and $T_\natural$ be the maximal singular integral operators as above with kernel $K(x,y)$ satisfying* (1.11).

(1) *Suppose that the maximal operator $\mathcal{M}$ satisfies (1.9). Then $T_\natural$ satisfies the weak-type two weight norm inequality*

$$\|T_\natural(f\sigma)\|_{L^{p,\infty}(\omega)} \leq C \|f\|_{L^p(\sigma)}, \qquad f \in L^p(\sigma), \tag{1.20}$$

if and only if

$$\int_Q T_\natural\left(\chi_Q f\sigma\right)(x) d\omega(x) \leq C_2 \left(\int_Q |f(x)|^p \, d\sigma(x)\right)^{\frac{1}{p}} \left(\int_Q d\omega(x)\right)^{\frac{1}{p'}}, \tag{1.21}$$

for all cubes $Q \subset \mathbb{R}^n$ and all functions $f \in L^p(\sigma)$.

(2) *The same characterization as above holds for $T_\flat$ in place of $T_\natural$ everywhere.*

(3) *Suppose that σ and ω are absolutely continuous with respect to Lebesgue measure, that the maximal operator $\mathcal{M}$ satisfies* (1.9)*, and that T is a standard singular integral operator with kernel K as above. If* (1.20) *holds for $T_\natural$ or $T_\flat$, then it also holds for T:*

$$\|T(f\sigma)\|_{L^{p,\infty}(\omega)} \leq C \|f\|_{L^p(\sigma)}, \qquad f \in L^p(\sigma), f\sigma \in L^\infty, \text{supp } f\sigma \text{ compact.} \tag{1.22}$$

(4) *Suppose $c > 0$ and that $\{K_j\}_{j=1}^J$ is a collection of* standard *kernels such that for* each *unit vector $\mathbf{u}$ there is j satisfying*

$$|K_j(x, x + t\mathbf{u})| \geq ct^{-n}, \qquad t \in \mathbb{R}. \tag{1.23}$$

Suppose also that σ and ω have no common point masses, i.e. $\sigma(\{x\}) = \omega(\{x\}) = 0$ for all $x \in \mathbb{R}^n$. Then

$$\left\|(T_j)_\flat(f\sigma)\right\|_{L^{p,\infty}(\omega)} \leq C \|f\|_{L^p(\sigma)}, \qquad f \in L^p(\sigma), \qquad 1 \leq j \leq J,$$

if and only if the two weight A_p condition (1.10) *holds and*

$$\int_Q (T_j)_\flat\left(\chi_Q f\sigma\right)(x) d\omega(x) \leq C_2 \left(\int_Q |f(x)|^p \, d\sigma(x)\right)^{\frac{1}{p}} \left(\int_Q d\omega(x)\right)^{\frac{1}{p'}},$$

$$f \in L^p(\sigma), \text{ cubes } Q \subset \mathbb{R}^n, \ 1 \leq j \leq J.$$

While in (1)—(3), we assume that the Maximal Function inequality holds, in point (4), we obtain an *unconditional* characterization of the weak-type inequality for a large class of families of (centered) maximal singular integral operators $T_\flat$. This class includes the individual maximal Hilbert transform in one dimension, the individual maximal Beurling transform in two dimensions, and the families of maximal Riesz transforms in higher dimensions, see Lemma 2.19.

Note that in (1) above, there is only size and smoothness assumptions placed on the kernel, so that it could for instance be a degenerate fractional integral operator, and therefore unbounded on $L^2(dx)$. But, the characterization still has content in this case, if ω and σ are not of full dimension.

In (3), we deduce a two weight inequality for standard singular integrals T without truncations when the measures are absolutely continuous. The proof of this is easy. From (1.20) and the pointwise inequality $T_{0,\infty}f\sigma(x) \leq T_\flat f\sigma(x) \leq T_\natural f\sigma(x)$, we obtain that for any limiting operator $T_{0,\infty}$ the map $f \to T_{0,\infty}f\sigma$ is bounded from $L^p(\sigma)$ to $L^{p,\infty}(\omega)$. By (1.9) $f \to \mathcal{M}f\sigma$ is bounded, hence $f \to f\sigma$ is bounded, and so Theorem 1.16 shows that $f \to Tf\sigma = T_{0,\infty}f\sigma + af\sigma$ is also bounded, provided we initially restrict attention to functions f for which $f\sigma$ is bounded with compact support.

The characterizing condition (1.21) is a weak-type condition, with the restriction that one only needs to test the weak-type condition for functions supported on a given cube, and test the weak-type norm over that given cube. It also has an interpretation as a dual inequality $\int_Q \left|L^*\left(\chi_Q\omega\right)\right|^{p'} d\sigma \leq C_2 \int_Q d\omega$, which we return to below, see (2.14) and (2.15).

We now characterize the two weight norm inequality for a strongly maximal singular integral $T_\natural$.

Theorem on Maximal Singular Integral Strong-Type Inequalities 1.24. *Suppose that σ and ω are positive locally finite Borel measures on $\mathbb{R}^n$, $1 < p < \infty$, and let $T_\flat$ and $T_\natural$ be the maximal singular integral operators as above with kernel $K(x,y)$ satisfying* (1.11).

(1) *Suppose that the maximal operator $\mathcal{M}$ satisfies (1.7) and also the 'dual' inequality*

$$\|\mathcal{M}(g\omega)\|_{L^{p'}(\sigma)} \leq C\,\|g\|_{L^{p'}(\omega)}, \qquad g \in L^{p'}(\omega). \tag{1.25}$$

Then $T_\natural$ satisfies the two weight norm inequality

$$\int_{\mathbb{R}^n} T_\natural(f\sigma)(x)^p d\omega(x) \leq C \int_{\mathbb{R}^n} |f(x)|^p \, d\sigma(x), \tag{1.26}$$

for all $f \in L^p(\sigma)$ that are bounded with compact support in $\mathbb{R}^n$, if and only if *both the dual cube testing condition* (1.21) *and the condition below hold:*

$$\int_Q T_\natural\left(\chi_Q g\sigma\right)(x)^p d\omega(x) \leq C_1 \int_Q d\sigma(x), \tag{1.27}$$

for all cubes $Q \subset \mathbb{R}^n$ and all functions $|g| \leq 1$.

(2) *The same characterization as above holds for* $T_\flat$ *in place of* $T_\natural$ *everywhere. In fact*

$$\left|T_\natural f\sigma(x) - T_\flat f\sigma(x)\right| \leq C\mathcal{M}\left(f\sigma\right)(x).$$

(3) *Suppose that* σ *and* ω *are absolutely continuous with respect to Lebesgue measure, that the maximal operator* $\mathcal{M}$ *satisfies* (1.7)*, and that* T *is a standard singular integral operator. If* (1.26) *holds for* $T_\natural$ *or* $T_\flat$*, then it also holds for* T*:*

$$\int_{\mathbb{R}^n} |T(f\sigma)(x)|^p \, d\omega(x) \leq C \int_{\mathbb{R}^n} |f(x)|^p \, d\sigma(x),$$
$$f \in L^p(\sigma), f\sigma \in L^\infty, \operatorname{supp}(f\sigma) \text{ compact.}$$

(4) *Suppose that* $\{K_j\}_{j=1}^n$ *is a collection of* standard *kernels satisfying for some* $c > 0$,

$$\pm \operatorname{Re} K_j(x,y) \geq \frac{c}{|x-y|^n}, \qquad \text{for } \pm(y_j - x_j) \geq \frac{1}{4}|x-y|, \tag{1.28}$$

where $x = (x_j)_{1\leq j\leq n}$*. Then* (1.26) *holds for* $(T_j)_\natural$ *and* $(T_j^*)_\natural$ *for all* $1 \leq j \leq n$*,* if and only if *both (1.27) and* (1.21) *hold for* $(T_j)_\natural$ *and* $(T_j^*)_\natural$ *for all* $1 \leq j \leq n$*.*

Note that the second condition (1.27) is a stronger condition than we would like: it is the L^p inequality, applied to functions *bounded by* 1 and supported on a cube Q, but with the $L^p(\sigma)$ norm of $\mathbf{1}_Q$ on the right side. It is easy to see that the bounded function g in (1.27) can be replaced by χ_E for every compact subset E of Q. Indeed if L ranges over all linearizations of $T_\natural$, then with $g_{h,Q} = \frac{L^*(\chi_Q h\omega)}{|L^*(\chi_Q h\omega)|}$ we have

$$\begin{aligned}
\sup_{|g|\leq 1} \int_Q T_\natural\left(\chi_Q g\sigma\right)^p \omega &= \sup_{|g|\leq 1} \sup_L \sup_{\|h\|_{L^{p'}(\omega)}\leq 1} \left|\int_Q L\left(\chi_Q g\sigma\right) h\omega\right| \\
&= \sup_L \sup_{\|h\|_{L^{p'}(\omega)}\leq 1} \sup_{|g|\leq 1} \left|\int_Q L^*\left(\chi_Q h\omega\right) g\sigma\right| \\
&= \sup_L \sup_{\|h\|_{L^{p'}(\omega)}\leq 1} \int_Q L^*\left(\chi_Q h\omega\right) g_{h,Q}\sigma \\
&= \sup_{\|h\|_{L^{p'}(\omega)}\leq 1} \sup_L \int_Q L\left(\chi_Q g_{h,Q}\right) h\omega\sigma \\
&\leq \sup_{\|h\|_{L^{p'}(\omega)}\leq 1} \int_Q T_\natural\left(\chi_Q g_{h,Q}\sigma\right)^p \omega.
\end{aligned}$$

Point (3) is again easy, just as in the previous weak-type theorem.

And in (4), we note that the truncations in the way that we formulate them, dominate the Maximal Function, so that our assumption on $\mathcal{M}$ in (1)—(3) is not unreasonable. The main result of [9] assumes $p = 2$ and that T is the Hilbert transform, and makes similar kinds of assumptions. In fact it is essentially the same as our result in the case $p = 2$, but only for T and not $T_\flat$ or $T_\natural$. Finally, we observe that by our definition of the truncation $T_\natural$, we obtain in point (4) an *unconditional* characterization of the strong-type inequality for appropriate families of standard singular integrals and their adjoints, including the Hilbert and Riesz transforms, see Lemma 2.22.

We do not know if the bounded function g in condition (1.27) can be replaced by the constant function 1.

1.2. **Two Weight Inequalities for Generalized Fractional Integrals.** Part 1 of Theorem 1.24 and part 1 of Theorem 1.19 extend to generalized fractional integrals, including the Cauchy integral in the plane. The setup is essentially the same as above but with a fractional variant of the size and smoothness conditions (1.11) on the kernel, and a fractional maximal function replacing the standard maximal function. Here are the details.

Let $0 \leq \alpha < n$. Consider a kernel function $K^\alpha(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the fractional size and smoothness conditions,

$$\begin{gathered} |K^\alpha(x,y)| \leq C\,|x-y|^{\alpha-n}\,, \\ |K^\alpha(x,y) - K^\alpha(x',y)| \leq C\delta\left(\frac{|x-x'|}{|x-y|}\right)|x-y|^{\alpha-n}\,, \qquad \frac{|x-x'|}{|x-y|} \leq \frac{1}{2}, \end{gathered} \tag{1.29}$$

where δ is a Dini modulus of continuity.

Example 1.30. *The Cauchy integral C^1 in the complex plane arises when $K(x,y) = \frac{1}{x-y}$, $x, y \in \mathbb{C}$. The fractional size and smoothness condition 1.29 holds with $n = 2$ and $\alpha = 1$ in this case.*

Define maximal fractional integrals $T^\alpha_\flat$ and $T^\alpha_\natural$ as above, but with K^α in place of K, and define the fractional maximal function by

$$\mathcal{M}^\alpha\nu(x) = \sup_{x\in Q} |Q|^{\frac{\alpha}{n}-1}\int_Q |\nu|\,, \qquad x \in \mathbb{R}. \tag{1.31}$$

We have the following solution from [16] to the two weight norm inequality for the fractional maximal operator $\mathcal{M}^\alpha$.

Theorem on Fractional Maximal Function Inequalities 1.32. *Let $0 \leq \alpha < n$. Suppose that σ and ω are positive locally finite Borel measures on $\mathbb{R}^n$, and $1 < p < \infty$. The fractional maximal operator $\mathcal{M}^\alpha$ satisfies the two weight norm inequality*

$$\|\mathcal{M}^\alpha(f\sigma)\|_{L^p(\omega)} \leq C\,\|f\|_{L^p(\sigma)}\,, \qquad f \in L^p(\sigma)\,, \tag{1.33}$$

if and only if *for all cubes* $Q \subset \mathbb{R}^n$,

$$\int_Q \mathcal{M}^\alpha \left(\chi_Q \sigma\right)(x)^p d\omega(x) \leq C_1 \int_Q d\sigma(x) < \infty.$$

The fractional maximal operator $\mathcal{M}^\alpha$ *satisfies the* weak-type *two weight norm inequality*

(1.34)

$$\left\|\mathcal{M}^\alpha(f\sigma)\right\|_{L^{p,\infty}(\omega)} \equiv \sup_{\lambda>0} \lambda \left|\left\{\mathcal{M}^\alpha(f\sigma) > \lambda\right\}\right|_\omega^{\frac{1}{p}} \leq C \left\|f\right\|_{L^p(\sigma)}, \qquad f \in L^p(\sigma),$$

if and only if *the two weight* A_p^α *condition holds for all cubes* $Q \subset \mathbb{R}^n$*:*

$$|Q|^{\frac{\alpha}{n}} \left(\frac{1}{|Q|}\int_Q d\omega\right)^{\frac{1}{p}} \left(\frac{1}{|Q|}\int_Q d\sigma\right)^{\frac{1}{p'}} \leq C_2.$$

The following theorem is proved in exactly the same way as part 1 of Theorems 1.24 and 1.19 above, but with $\mathcal{M}^\alpha$ in place of $\mathcal{M}$.

Theorem on Fractional Maximal Singular Integral Inequalities 1.35. *Let* $0 \leq \alpha < n$*. Suppose that* σ *and* ω *are positive locally finite Borel measures on* $\mathbb{R}^n$*,* $1 < p < \infty$*, and let* $T_\flat^\alpha$ *and* $T_\natural^\alpha$ *be maximal fractional integral operators as above with kernel* $K^\alpha(x,y)$ *satisfying* (1.29).

(1) *Suppose that the fractional maximal operator* $\mathcal{M}^\alpha$ *satisfies* (1.33) *and the corresponding 'dual' inequality*

$$\left\|\mathcal{M}^\alpha(g\omega)\right\|_{L^{p'}(\sigma)} \leq C \left\|g\right\|_{L^{p'}(\omega)}, \qquad g \in L^{p'}(\omega).$$

Then $T_\natural^\alpha$ *satisfies the two weight norm inequality*

$$\int_{\mathbb{R}^n} T_\natural^\alpha(f\sigma)(x)^p d\omega(x) \leq C \int_{\mathbb{R}^n} |f(x)|^p d\sigma(x), \tag{1.36}$$

for all $f \in L^p(\sigma)$ *that are bounded with compact support in* $\mathbb{R}^n$*,* if and only if *both*

$$\int_Q T_\natural^\alpha \left(\chi_Q g\sigma\right)(x)^p d\omega(x) \leq C_1 \int_Q d\sigma(x), \qquad \text{for all } |g| \leq 1, \tag{1.37}$$

and

$$\int_Q T_\natural^\alpha \left(\chi_Q f\sigma\right)(x) d\omega(x) \leq C_2 \left(\int_Q |f(x)|^p d\sigma(x)\right)^{\frac{1}{p}} \left(\int_Q d\omega(x)\right)^{\frac{1}{p'}}, \tag{1.38}$$

for all cubes $Q \subset \mathbb{R}^n$ *and all* $f \in L^p(\sigma)$*. The same holds with* $T_\flat^\alpha$ *in place of* $T_\natural^\alpha$ *in* (1.36)*,* (1.37) *and* (1.38).

(2) *Suppose that the fractional maximal operator* $\mathcal{M}^{\alpha}$ *satisfies* (1.34). *Then* $T_{\natural}^{\alpha}$ *satisfies the weak-type two weight norm inequality*

(1.39)
$$\left\|T_{\natural}^{\alpha}(f\sigma)\right\|_{L^{p,\infty}(\omega)} \leq C \left\|f\right\|_{L^{p}(\sigma)}, \quad f \in L^{p}(\sigma) \text{ bounded with compact support,}$$

if and only if (1.38) *holds. The same holds with* $T_{\flat}^{\alpha}$ *in place of* $T_{\natural}^{\alpha}$ *in (1.39) and* (1.38).

In particular, the Cauchy integral C^1 in the plane is bounded from one weighted space to another provided the fractional maximal function $\mathcal{M}_1$ is bounded and the two testing conditions (1.37) and (1.38) hold. Thus we see that the problem of deciding whether the testing conditions hold is the main issue here, and the cancellation conditions inherent in the Cauchy kernel play a decisive role. For more general fractional integrals, the appropriate notion of cancellation remains mysterious, and so we do not have a corresponding definition of a *standard* generalized fractional integral operator. Finally, we note that there are analogues of part 4 of both Theorem 1.24 and Theorem 1.19 which the interested reader can easily supply.

Acknowledgment. The authors completed this work during research stays at the Fields Institute, Toronto Canada, and the Centre de Recerca Matemàtica, Barcelona Spain. They thank these insitutions for their generous hospitality.

2. Overview of the Proofs, General Principles

If Q is a cube then $\ell(Q)$ is its side length, $|Q|$ is its Lebesgue measure and for a positive Borel measure ν, $|Q|_{\nu} = \int_{Q} d\nu$ is its ν-measure.

2.1. Calderón-Zygmund Decompositions.

Our starting place is the argument in [18] used to prove a two weight norm inequality for fractional integral operators on Euclidean space. Of course the fractional integral is a positive operator, with a monotone kernel, which properties we do not have in the current setting.

A central tool arises from the observation that for any positive Borel measure μ, one has the boundedness of a maximal function associated with μ. Define the dyadic μ-maximal operator $\mathcal{M}_{\mu}^{dy}$ by

$$\mathcal{M}_{\mu}^{dy} f(x) = \sup_{\substack{Q\in\mathcal{D}\\ x\in Q}} \frac{1}{|Q|_{\mu}} \int_{Q} |f| \, \mu, \tag{2.1}$$

with the supremum taken over all dyadic cubes $Q \in \mathcal{D}$ containing x. It is immediate to check that $\mathcal{M}_{\mu}^{dy}$ satisfies the weak-type $(1,1)$ inequality, and the $L^{\infty}(\mu)$ bound is obvious. Hence we have

$$\int \left(\mathcal{M}_{\mu}^{dy} f\right)^{p} \mu \leq C \int f^{p} \mu, \qquad f \geq 0 \text{ on } \mathbb{R}^{n}. \tag{2.2}$$

This observation places certain Calderón-Zygmund decompositions at our disposal. Exploitation of this brings in the testing condition (1.27) involving the bounded function g on a cube Q, and indeed, g turns out to be the "good" function in a Calderón-Zygmund decomposition of f on Q. The associated 'bad' function requires the dual testing condition (1.21) as well.

2.2. **Edge effects of dyadic grids.** Our operators are not dyadic operators, nor—in contrast to the fractional integral operators—can they be easily obtained from dyadic operators. This leads to the necessity of considering for instance triples of dyadic cubes, which are not dyadic.

Also, dyadic grids distinguish points by for instance making some points on the boundary of many cubes. As our measures are arbitrary, they could conspire to assign extra mass to some of these points. To address this point, Nazarov-Treil-Volberg [9, 10, 11] use a random shift of the grid.

A random approach would likely work for us as well, though the argument would be different from those in the cited papers above. Instead, we will use M. Christ's non-random technique of shifted dyadic grid from [6]. Define a *shifted dyadic grid* to be the collection of cubes

$$\mathcal{D}^\alpha = \left\{ 2^j(k + [0,1)^n + (-1)^j \alpha) \ : \ j \in \mathbb{Z}, k \in \mathbb{Z}^n \right\}, \qquad \alpha \in \{0, \tfrac{1}{3}, \tfrac{2}{3}\}^n . \tag{2.3}$$

The basic properties of these collections are these: In the first place, each $\mathcal{D}^\alpha$ is a grid, namely for $Q, Q' \in \mathcal{D}^\alpha$ we have $Q \cap Q' \in \{\emptyset\,,\, Q\,,\, Q'\}$ and Q is a union of 2^n elements of $\mathcal{D}^\alpha$ of equal volume. In the second place (and this is the novel property for us), for any cube $Q \subset \mathbb{R}^n$, there is a choice of some α and some $Q' \in \mathcal{D}_\alpha$ so that $Q \subset \frac{9}{10} Q'$ and $|Q'| \le C|Q|$.

We define the analogs of the dyadic maximal operator in (2.1), not to be confused with $\mathcal{M}^\alpha$ in (1.31), namely

$$\mathcal{M}^\alpha_\mu f(x) = \sup_{\substack{Q \in \mathcal{D}^\alpha \\ x \in Q}} \frac{1}{|Q|_\mu} \int_Q |f| \, \mu \, . \tag{2.4}$$

These operators clearly satisfy (2.2). Shifted dyadic grids will return in § 4.4.

2.3. **A Maximum Principle.** A second central tool is a 'maximum principle' (or good λ inequality) which will permit one to localize large values of a singular integral, provided the Maximal Function is bounded. It is convenient for us to describe this in conjunction with another fundamental tool of this paper, a family of Whitney decompositions.

We begin with the Whitney decompositions. Fix a finite measure ν with compact support on $\mathbb{R}^n$ and for $k \in \mathbb{Z}$ let

$$\Omega_k = \left\{ x \in \mathbb{R}^n : T_\natural \nu(x) > 2^k \right\} . \tag{2.5}$$

Note that $\Omega_k \ne \mathbb{R}^n$ has compact closure for such ν. Fix an integer $N \ge 3$. We can choose $R_W \ge 3$ sufficiently large, depending only on the dimension and N, such that there is a collection of cubes $\left\{Q_j^k\right\}_j$ which satisfy the following properties:

$$\text{(2.6)}\quad \begin{cases} \text{(disjoint cover)} & \Omega_k = \bigcup_j Q_j^k \text{ and } Q_j^k \cap Q_i^k = \emptyset \text{ if } i \neq j \\ \text{(Whitney condition)} & R_W Q_j^k \subset \Omega_k \text{ and } 3R_W Q_j^k \cap \Omega_k^c \neq \emptyset \text{ for all } k,j \\ \text{(finite overlap)} & \sum_j \chi_{NQ_j^k} \leq C\chi_{\Omega_k} \text{ for all } k \\ \text{(crowd control)} & \#\left\{Q_s^k : Q_s^k \cap NQ_j^k \neq \emptyset\right\} \leq C \text{ for all } k,j \\ \text{(nested property)} & Q_j^k \subsetneqq Q_i^\ell \text{ implies } k > \ell. \end{cases}$$

Indeed, one should choose the the $\left\{Q_j^k\right\}_j$ satisfying the Whitney condition, and then show that the other properties hold. The different combinatorial properties above are fundamental to the proof. And alternate Whitney decompositions are constructed in § 4.7.1 below.

Remark 2.7. Our use of the Whitney decomposition and the maximum principle are derived from the two weight fractional integral argument of Sawyer, see Sec 2 of [18]. In particular, the properties above are as in [18], aside from the the crowd control property above, which as $N = 3$ in *op. cit.*

Remark 2.8. In our notation for the Whitney cubes, the superscript indicates a 'height' and the subscript an arbitary enumeration of the cubes. We will use super- and sub-scripts below in this manner consistently throughout the paper. It is important to note that a fixed cube Q can arise in *many* Whitney decompositions: There are integers $K_-(Q) \leq K_+(Q)$ with $Q = Q_{j(k)}^k$ for some choice of $j(k)$ for all $K_-(Q) \leq k \leq K_+(Q)$. (The last point follows from the nested property.) There is no *a priori* upper bound on $K_+(Q) - K_-(Q)$.

Lemma 2.9. *[Maximum Principle] Let ν be a finite (signed) measure with compact support. For any cube Q_j^k as above we have the pointwise inequality*

$$\sup_{x\in Q_j^k} T_\natural \left(\chi_{(3Q_j^k)^c}\nu\right)(x) \leq 2^k + C\mathbf{P}\left(Q_j^k,\nu\right) \leq 2^k + CM\left(Q_j^k,\nu\right), \tag{2.10}$$

where $\mathbf{P}(Q,\nu)$ and $M(Q,\nu)$ are defined by

$$\begin{aligned} \mathbf{P}(Q,\nu) &\equiv \frac{1}{|Q|}\int_Q d|\nu| + \sum_{\ell=0}^{\infty} \frac{\delta\left(2^{-\ell}\right)}{|2^{\ell+1}Q|}\int_{2^{\ell+1}Q\setminus 2^\ell Q} d|\nu|, \\ M(Q,\nu) &\equiv \sup_{Q'\supset Q}\frac{1}{|Q'|}\int_{Q'} d|\nu|. \end{aligned} \tag{2.11}$$

The bound in terms of $\mathbf{P}(Q,\nu)$ should be regarded as one in terms of a modified Poisson integral. It is both slightly sharper than that of $M(Q,\nu)$, and a linear expression in $|\nu|$, which fact will be used in the proof of the strong type estimates.

Proof. To see this, take $x \in Q_j^k$ and note that for each $\eta > 0$ there is $\boldsymbol{\varepsilon}$ with $\ell(Q_j^k) < \max_{1 \le j \le n} \varepsilon_j < R < \infty$ and $\theta \in [0, 2\pi)$ such that

$$T_\natural \left(\chi_{(3Q_j^k)^c} \nu \right)(x) \le (1+\eta) \left| \int_{(3Q_j^k)^c} K(x,y) \zeta_{\boldsymbol{\varepsilon}}(x-y) \eta_R(x-y) d\nu(y) \right| = (1+\eta) e^{i\theta} T_{\boldsymbol{\varepsilon},R} \left(\chi_{(3Q_j^k)^c} \nu \right)(x).$$

For convenience we take $\eta = 0$ in the sequel. By the Whitney condition in (2.6), there is a point $z \in 3R_W Q_j^k \cap \Omega_k^c$ and it now follows that (remember that $\ell(Q_j^k) < \max_{1 \le j \le n} \varepsilon_j$),

$$\begin{aligned}
&\left| T_{\boldsymbol{\varepsilon},R} \left(\chi_{(3Q_j^k)^c} \nu \right)(x) - T_{\boldsymbol{\varepsilon},R} \nu(z) \right| \\
&\le C \frac{1}{|6R_W Q_j^k|} \int_{6R_W Q_j^k} d|\nu| + \left| T_{\boldsymbol{\varepsilon},R} \left(\chi_{(6R_W Q_j^k)^c} \nu \right)(x) - T_{\boldsymbol{\varepsilon},R} \left(\chi_{(6R_W Q_j^k)^c} \nu \right)(z) \right| \\
&= C \frac{1}{|6R_W Q_j^k|} \int_{6R_W Q_j^k} d|\nu| \\
&\quad + \int_{(6R_W Q_j^k)^c} |K(x,y)\zeta_{\boldsymbol{\varepsilon}}(x-y)\eta_R(x-y) - K(z,y)\zeta_{\boldsymbol{\varepsilon}}(z-y)\eta_R(z-y)| \, d|\nu|(y) \\
&\le C \frac{1}{|6R_W Q_j^k|} \int_{6R_W Q_j^k} d|\nu| + C \int_{(6R_W Q_j^k)^c} \delta\left(\frac{|x-z|}{|x-y|} \right) \frac{1}{|x-y|^n} d|\nu|(y) \\
&\le C \mathbf{P}\left(Q_j^k, \nu \right).
\end{aligned}$$

Thus

$$T_\natural \left(\chi_{(3Q_j^k)^c} \nu \right)(x) \le |T_\natural \nu(z)| + C\mathbf{P}\left(Q_j^k, \nu \right) \le 2^k + C\mathbf{P}\left(Q_j^k, \nu \right),$$

which yields (2.10) since $\mathbf{P}(Q,\nu) \le CM(Q,\nu)$. ∎

2.4. **Linearizations.** We now make comments on the linearizations of our maximal singular integral operators. We would like, at different points, to treat $T_\natural$ as a linear operator, which of course it is not. Nevertheless $T_\natural$ is a pointwise supremum of the linear truncation operators $T_{\boldsymbol{\varepsilon},R}$, and as such, the supremum can be linearized with measurable selection of the parameters $\boldsymbol{\varepsilon}$ and R, as was just done in the previous proof. We make this a definition.

Definition 2.12. We say that L is a linearization of $T_\natural$ if there are measurable functions $\boldsymbol{\varepsilon}(x) \in (0,\infty)^n$ and $R(x) \in (0,\infty)$ with $\frac{1}{4} \le \frac{\varepsilon_i}{\varepsilon_j} \le 4$, $\max_{1 \le i \le n} \varepsilon_i < R(x) < \infty$ and $\theta(x) \in [0, 2\pi)$ such that

$$Lf(x) = e^{i\theta(x)} T_{\boldsymbol{\varepsilon}(x),R(x)} f(x), \qquad x \in \mathbb{R}^n. \tag{2.13}$$

For fixed f and $\delta > 0$, we can always choose a linearization L so that $T_\natural f(x) \leq (1+\delta)\, Lf(x)$ for all x. In a typical application of this Lemma, one takes δ to be one.

Note that condition (1.27) is obtained from inequality (1.26) by testing over f of the form $f = \chi_Q g$ with $|g| \leq 1$, and then restricting integration on the left to Q. By passing to linearizations L, we can 'dualize' (1.21) to the testing conditions

$$\int_Q \left|L^*\left(\chi_Q \omega\right)(x)\right|^{p'} d\sigma(x) \leq C_2 \int_Q d\omega(x), \tag{2.14}$$

or equivalently (note that in (1.27) the presence of g makes a difference, but not here),

$$\int_Q \left|L^*\left(\chi_Q g\omega\right)(x)\right|^{p'} d\sigma(x) \leq C_2 \int_Q d\omega(x), \qquad |g| \leq 1, \tag{2.15}$$

with the requirement that these inequalities hold *uniformly* in all linearizations L of $T_\natural$.

While the smooth truncation operators $T_{\boldsymbol{\varepsilon},R}$ are essentially self-adjoint, the dual of a linearization L is generally complicated. Nevertheless, the dual L^* does satisfy one important property which plays a crucial role in the proof of Theorem 1.24, the L^p-norm inequalities.

Lemma 2.16. *$L^*\mu$ is δ-Hölder continuous (where δ is the Dini modulus of continuity of the kernel K) with constant $C\mathbf{P}(Q,\mu)$ on any cube Q satisfying $\int_{3Q} d|\mu| = 0$, i.e.*

$$\left|L^*\mu(y) - L^*\mu(y')\right| \leq C\mathbf{P}(Q,\mu)\,\delta\left(\frac{|y-y'|}{\ell(Q)}\right), \qquad y, y' \in Q. \tag{2.17}$$

Here, recall the definition (2.11) *and that* $\mathbf{P}(Q,\mu) \leq CM(Q,\mu)$.

Proof. Suppose L is as in (2.13). Then for any finite measure ν,

$$L\nu(x) = e^{i\theta(x)} \int \zeta_{\boldsymbol{\varepsilon}(x)}(x-y)\eta_{R(x)}(x-y)K(x,y)d\nu(y).$$

Fubini's theorem shows that the dual operator L^* is given on a finite measure μ by

$$L^*\mu(y) = \int \zeta_{\boldsymbol{\varepsilon}(x)}(x-y)\eta_{R(x)}(x-y)K(x,y)e^{i\theta(x)}d\mu(x). \tag{2.18}$$

For $y, y' \in Q$ and $|\mu|(3Q) = 0$, we thus have

$$\begin{aligned} &L^*\mu(y) - L^*\mu(y') \\ &= \int \left\{\left(\zeta_{\boldsymbol{\varepsilon}(x)}\eta_{R(x)}\right)(x-y) - \left(\zeta_{\boldsymbol{\varepsilon}(x)}\eta_{R(x)}\right)(x-y')\right\} K(x,y)e^{i\theta(x)}d\mu(x) \\ &\quad + \int \left(\zeta_{\boldsymbol{\varepsilon}(x)}\eta_{R(x)}\right)(x-y')\left\{K(x,y) - K(x,y')\right\} e^{i\theta(x)}d\mu(x), \end{aligned}$$

from which (2.17) follows easily if we split the two integrals in x over dyadic annuli centered at the center of Q. ∎

2.5. **Control of Maximal Functions.** Next we record the facts that T and $T_\natural$ control $\mathcal{M}$ for many (collections of) standard singular integrals T, including the Hilbert transform, the Beurling transform and the collection of Riesz transforms in higher dimensions.

Lemma 2.19. *Suppose that σ and ω have no point masses in common, and that $\{K_j\}_{j=1}^J$ is a collection of* standard *kernels satisfying* (1.11) *and* (1.23)*. If the corresponding operators T_j given by* (1.15) *satisfy*

$$\left\|\chi_E T_j(f\sigma)\right\|_{L^{p,\infty}(\omega)} \leq C\left\|f\right\|_{L^p(\sigma)}, \qquad E=\mathbb{R}^n\setminus supp\ f,$$

for $1\leq j\leq J$, then the two weight A_p condition (1.10) *holds, and hence also the weak-type two weight inequality* (1.9).

Proof. Part of the 'one weight' argument on page 211 of Stein [21] yields the *asymmetric* two weight A_p condition

$$|Q|_\omega\,|Q'|_\sigma^{p-1}\leq C\,|Q|^p\,, \tag{2.20}$$

where Q and Q' are cubes of equal side length r and distance approximately C_0r apart for some fixed large positive constant C_0 (for this argument we choose the unit vector $\mathbf{u}$ in (1.23) to point in the direction from the center of Q to the center of Q', and then with j as in (1.23), C_0 is chosen large enough by (1.11) that (1.23) holds for all unit vectors $\mathbf{u}$ pointing from a point in Q to a point in Q'). In the one weight case treated in [21] it is easy to obtain from this (even for a *single* direction $\mathbf{u}$) the usual (symmetric) A_p condition (1.10). Here we will instead use our assumption that σ and ω have no point masses in common for this purpose.

So fix an open cube Q in $\mathbb{R}^n$ and let $\{\mathsf{Q}_\alpha\}_\alpha$ be a Whitney decomposition (2.6) of the open set $(Q\times Q)\setminus\mathcal{D}$ relative to $\mathcal{D}$ where $\mathcal{D}$ is the diagonal in $\mathbb{R}^n\times\mathbb{R}^n$. Note that if $\mathsf{Q}_\alpha=Q_\alpha\times Q'_\alpha$, then (2.20) can be written

$$\mathcal{A}_p\left(\omega,\sigma;\mathsf{Q}_\alpha\right)\leq C\left|\mathsf{Q}_\alpha\right|^{\frac{p}{2}}. \tag{2.21}$$

where $\mathcal{A}_p\left(\omega,\sigma;\mathsf{Q}_\alpha\right)=|Q_\alpha|_\omega\,|Q'_\alpha|_\sigma^{p-1}$ ($\mathcal{A}_2\left(\omega,\sigma;\mathsf{Q}_\alpha\right)=|\mathsf{Q}_\alpha|_{\omega\times\sigma}$ where $\omega\times\sigma$ denotes product measure on $\mathbb{R}^n\times\mathbb{R}^n$). We choose R_W sufficiently large in (2.6), depending on C_0, such that (2.21) holds for all the Whitney cubes Q_α. For $1<p<\infty$ we easily compute that $\sum_\alpha|\mathsf{Q}_\alpha|^{\frac{p}{2}}\leq C\,|Q\times Q|^{\frac{p}{2}}=C\,|Q|^p$.

Suppose now that $1<p\leq 2$. We claim that if $\mathsf{R}=Q\times Q'$ is a rectangle in $\mathbb{R}^n\times\mathbb{R}^n$ (i.e. Q,Q' are cubes in $\mathbb{R}^n$), and if $\mathsf{R}=\dot{\cup}_\alpha\mathsf{R}_\alpha$ is a finite disjoint union of rectangles R_α, then with the obvious extension of $\mathcal{A}_p\left(\omega,\sigma;\mathsf{R}\right)$ to rectangles,

$$\mathcal{A}_p\left(\omega,\sigma;\mathsf{R}\right)\leq\sum_\alpha\mathcal{A}_p\left(\omega,\sigma;\mathsf{R}_\alpha\right).$$

This is easy to see using $0 < p-1 \leq 1$ if the disjoint union consists of just two rectangles, and the general case then follows by induction (the case $p=2$ is just countable additivity of product measure).

Since ω and σ have no point masses in common, a limiting argument using the above subadditivity of $\mathcal{A}_p$ shows that

$$\mathcal{A}_p(\omega,\sigma;Q\times Q) \leq \sum_\alpha \mathcal{A}_p(\omega,\sigma;\mathsf{Q}_\alpha) \leq C\sum_\alpha |\mathsf{Q}_\alpha|^{\frac{p}{2}} \leq C|Q|^p,$$

which is (1.10). The case $2 \leq p < \infty$ is proved in the same way using that (2.20) can be written

$$\mathcal{A}_{p'}(\sigma,\omega;\mathsf{Q}_\alpha) \leq C'|\mathsf{Q}_\alpha|^{\frac{p'}{2}}. \blacksquare$$

Lemma 2.22. *If $\{T_j\}_{j=1}^n$ satisfies (1.28), then*

$$\mathcal{M}\nu(x) \leq C\sum_{j=1}^n (T_j)_\natural\nu(x), \quad x\in\mathbb{R}^n,\ \nu\geq 0 \text{ a finite measure with compact support.}$$

Proof. We prove the case $n=1$, the general case being similar. Then with $T=T_1$ and $r>0$ we have

$$\begin{aligned}&\operatorname{Re}\left(T_{r,\frac{r}{4},100r}\nu(x) - T_{r,4r,100r}\nu(x)\right)\\&\quad= \int\left\{\zeta_{\frac{r}{4}}(y-x)-\zeta_{4r}(y-x)\right\}\operatorname{Re}K(x,y)d\nu(y) \geq \frac{c}{r}\int_{\left[x+\frac{r}{2},x+2r\right]} d\nu(y).\end{aligned}$$

Thus

$$T_\natural\nu(x) \geq \max\left\{\left|T_{r,\frac{r}{4},100r}\nu(x)\right|, \left|T_{r,4r,100r}\nu(x)\right|\right\} \geq \frac{c}{r}\int_{\left[x+\frac{r}{2},x+2r\right]} d\nu(y),$$

and similarly

$$T_\natural\nu(x) \geq \frac{c}{r}\int_{\left[x-2r,x-\frac{r}{2}\right]} d\nu(y).$$

It follows that

$$\begin{aligned}\mathcal{M}\nu(x) &\leq \sup_{r>0}\frac{1}{4r}\int_{[x-2r,x+2r]} d\nu(y)\\&= \sup_{r>0}\sum_{k=0}^\infty 2^{-k}\frac{1}{2^{2-k}r}\int_{\left[x-2^{1-k}r,x-2^{-1-k}r\right]\cup\left[x+2^{-1-k}r,x+2^{1-k}r\right]} d\nu(y)\\&\leq CT_\natural\nu(x). \blacksquare\end{aligned}$$

Finally, we will use the following covering lemma of Besicovitch type for multiples of dyadic cubes (the case of triples of dyadic cubes arises in (4.65) below).

Lemma 2.23. *Let M be an odd positive integer, and suppose that Φ is a collection of cubes P with bounded diameters and having the form $P = MQ$ where Q is dyadic (a product of clopen dyadic intervals). If Φ^* is the collection of* maximal *cubes in Φ, i.e. $P^* \in \Phi^*$ provided there is no strictly larger P in Φ that contains P^*, then the cubes in Φ^* have finite overlap at most M^n.*

Proof. Let $Q_0 = [0,1)^n$ and assign labels $1, 2, 3, \ldots, M^n$ to the dyadic subcubes of side length one of MQ_0. We say that the subcube labeled k is of type k, and we extend this definition by translation and dilation to the subcubes of MQ having side length that of Q. Now we simply observe that if $\{P_i^*\}_i$ is a set of cubes in Φ^* containing the point x, then for a given k, there is at most one P_i^* that contains x in its subcube of type k. The reason is that if P_j^* is another such cube and $\ell\left(P_j^*\right) \leq \ell\left(P_i^*\right)$, we must have $P_j^* \subset P_i^*$ (draw a picture in the plane for example). ∎

2.6. **Preliminary Precaution.** Given a positive locally finite Borel measure μ on $\mathbb{R}^n$, there exists a rotation such that all boundaries of rotated dyadic cubes have μ-measure zero (see [4] where they actually prove a stronger assertion when μ has no point masses, but our conclusion is obvious for a sum of point mass measures). We will assume that such a rotation has been made so that all boundaries of rotated dyadic cubes have $(\omega + \sigma)$-measure zero, where ω and σ are the positive Borel measures appearing in the theorems above. While this assumption is not essential for the proof, it relieves the reader of having to consider the possibility that boundaries of dyadic cubes have positive measure at each step of the argument below.

Recall also (see e.g. Theorem 2.18 in [15]) that any positive locally finite Borel measure on $\mathbb{R}^n$ is both inner and outer regular.

3. The proof of Theorem 1.19: Weak-type Inequalities

We begin with the necessity of condition (1.21):

$$\begin{aligned}
\int_Q T_\natural\left(\chi_Q f\sigma\right)\omega &= \int_0^\infty \min\left\{|Q|_\omega, \left|\left\{T_\natural\left(\chi_Q f\sigma\right) > \lambda\right\}\right|_\omega\right\} d\lambda \\
&\leq \left\{\int_0^A + \int_A^\infty\right\} \min\left\{|Q|_\omega, C\lambda^{-p}\int |f|^p\, d\sigma\right\} d\lambda \\
&\leq A\,|Q|_\omega + CA^{1-p}\int |f|^p\, d\sigma \\
&= (C+1)\,|Q|_\omega^{\frac{1}{p'}}\left(\int |f|^p\, d\sigma\right)^{\frac{1}{p}},
\end{aligned}$$

if we choose $A = \left(\frac{\int |f|^p d\sigma}{|Q|_\omega}\right)^{\frac{1}{p}}$.

Now we turn to proving (1.20), assuming both (1.21) and (1.9), and moreover that f is bounded with compact support. We will prove the quantitative estimate

$$\|T_\natural f\sigma\|_{L^{p,\infty}(\omega)} \le C\{\mathfrak{A}+\mathfrak{T}\}\,\|f\|_{L^p(\sigma)}\ , \tag{3.1}$$

$$\mathfrak{A} = \sup_Q \sup_{\|f\|_{L^p(\sigma)}=1} \sup_{\lambda>0} \lambda\,|\{\mathcal{M}\,(f\sigma)>\lambda\}|_\omega^{\frac{1}{p}}\ , \tag{3.2}$$

$$\mathfrak{T}_* = \sup_{\|f\|_{L^p(\sigma)}=1} \sup_Q |Q|_\omega^{-1/p'} \int_Q T_\natural\left(\chi_Q f\sigma\right)(x)d\omega(x)\,. \tag{3.3}$$

We should emphasize that the term (3.2) is comparable to the two weight A_p condition (1.10).

Standard considerations ([17]. Section 2) show that it suffices to prove the following good-λ inequality: There is a positive constant C so that for $\beta>0$ sufficiently small, and provided

$$\sup_{0<\lambda<\Lambda} \lambda^p\,|\{x\in\mathbb{R}^n : T_\natural f\sigma\,(x)>\lambda\}|_\omega < \infty\,, \qquad \Lambda<\infty\,, \tag{3.4}$$

we have this inequality:

$$\begin{aligned} &|\{x\in\mathbb{R}^n : T_\natural f\sigma\,(x)>2\lambda \text{ and } \mathcal{M}f\sigma\,(x)\le\beta\lambda\}|_\omega \\ &\qquad \le C\beta\mathfrak{T}_*^p\,|\{x\in\mathbb{R}^n : T_\natural f\sigma\,(x)>\lambda\}|_\omega + C\beta^{-p}\lambda^{-p}\int|f|^p\,d\sigma\,. \end{aligned} \tag{3.5}$$

Our presumption (3.4) holds due to the A_p condition (1.10) and the fact that

$$\{x\in\mathbb{R}^n : T_\natural f\sigma\,(x)>\lambda\}\subset B\left(0,c\lambda^{-\frac{1}{n}}\right), \qquad \lambda>0 \text{ small},$$

Hence it is enough to prove (3.5).

To prove (3.5) we choose $\lambda=2^k$, and apply the decomposition in (2.6). In this argument, we can take k to be fixed, so that we suppress its appearance as a superscript in this section. (When we come to L^p estimates, we will not have this luxury.)

Define

$$E_j = \{x\in Q_j : T_\natural f\sigma\,(x)>2\lambda \text{ and } \mathcal{M}f\sigma\,(x)\le\beta\lambda\}\,.$$

Then for $x\in E_j$, we can apply Lemma 2.9 to deduce

$$T_\natural\left(\chi_{(3Q_j)^c}f\sigma\right)(x)\le(1+C\beta)\,\lambda. \tag{3.6}$$

If we take $\beta>0$ so small that $1+C\beta\le\frac{3}{2}$, then (3.6) implies that for $x\in E_j$

$$\begin{aligned} 2\lambda < T_\natural f\sigma\,(x) &\le T_\flat\chi_{3Q_j}f\sigma\,(x)+T_\natural\chi_{(3Q_j)^c}f\sigma\,(x) \\ &\le T_\natural\chi_{3Q_j^k}f\sigma\,(x)+\tfrac{3}{2}\lambda. \end{aligned}$$

Integrating this inequality with respect to ω over E_j we obtain

$$\lambda\,|E_j|_\omega \le 2\int_{E_j}\left(T_\natural\chi_{3Q_j}f\sigma\right)\omega. \tag{3.7}$$

The disjoint cover condition in (2.6) shows that the sets E_j are disjoint, and this suggests we should sum their ω-measures. We split this sum into two parts, according to the size of $|E_j|_\omega / |3Q_j|_\omega$. The left-hand side of (3.5) satisfies

$$\begin{aligned}\sum_j |E_j|_\omega &\le \beta \sum_{j:|E_j|_\omega \le \beta |3Q_j|_\omega} |3Q_j|_\omega \\ &\quad + \beta^{-p} \sum_{j:|E_j|_\omega > \beta |3Q_j|_\omega} |E_j|_\omega \left(\frac{2}{\lambda} \frac{1}{|3Q_j|_\omega} \int_{E_j} \left(T_\natural \chi_{3Q_j} f\sigma \right) \omega \right)^p \\ &= I + II.\end{aligned}$$

Now

$$I \le \beta \sum_j \left|3Q_j^k\right|_\omega \le C\beta \left|\Omega\right|_\omega ,$$

by the finite overlap condition in (2.6). From (1.21) with $Q = 3Q_j$ we have

$$\begin{aligned} II &\le \left(\frac{2}{\beta\lambda}\right)^p \sum_j |E_j|_\omega \left(\frac{1}{|3Q_j|_\omega} \int_{E_j^k} \left(T_\natural \chi_{3Q_j} f\sigma \right) \omega \right)^p \\ &\le C \left(\frac{2}{\beta\lambda}\right)^p \mathfrak{T}_*^p \sum_j |E_j|_\omega \frac{1}{|3Q_j|_\omega^p} |3Q_j|_\omega^{p-1} \int_{3Q_j} |f|^p \, d\sigma \\ &\le C \left(\frac{2}{\beta\lambda}\right)^p \mathfrak{T}_*^p \int \left(\sum_j \chi_{3Q_j^k} \right) |f|^p \, d\sigma \\ &\le C \left(\frac{2}{\beta\lambda}\right)^p \mathfrak{T}_*^p \int |f|^p \, d\sigma, \end{aligned}$$

by the finite overlap condition in (2.6) again. This completes the proof of the good-λ inequality (3.5).

The proof of assertion 2 regarding $T_\flat$ is similar. Assertion 3 was discussed earlier and assertion 4 follows readily from assertion 2 and Lemma 2.19.

4. The proof of Theorem 1.24: Strong-type Inequalities

Since conditions (1.27) and (1.21) are obviously necessary for (1.26), we turn to proving the weighted inequality (1.26) for the strongly maximal singular integral $T_\natural$.

4.1. **The Quantitative Estimate.** In particular, we will prove

$$\left\| T_\natural f\sigma \right\|_{L^p(\omega)} \le C \left\{ \mathfrak{M} + \mathfrak{M}_* + \mathfrak{T} + \mathfrak{T}_* \right\} \|f\|_{L^p(\sigma)} , \tag{4.1}$$

$$\mathfrak{M} = \sup_{\|f\|_{L^p(\sigma)}=1} \left\| \mathcal{M}\left(f\sigma\right) \right\|_{L^p(\omega)} , \tag{4.2}$$

$$\mathfrak{M}_* = \sup_{\|g\|_{L^{p'}(\omega)}=1} \|\mathcal{M}(g\omega)\|_{L^p(\sigma)}, \tag{4.3}$$

$$\mathfrak{T} = \sup_Q \sup_{\|f\|_{L^\infty}\le 1} |Q|_\sigma^{-1/p} \|\chi_Q T_\natural (\chi_Q f\sigma)\|_{L^p(\omega)}, \tag{4.4}$$

$$\mathfrak{T}_* = \sup_{\|f\|_{L^p(\sigma)}=1} \sup_Q |Q|_\omega^{-1/p'} \int_Q T_\natural (\chi_Q f\sigma)(x) d\omega(x). \tag{4.5}$$

The norm estimates on the maximal function (4.2) and (4.3) are equivalent to the testing conditions in (1.8) and its dual formulation. The term $\mathfrak{T}_*$ also appeared in (3.3).

4.2. **The Initial Construction.** We suppose that both (1.27) and (1.21) hold, and that f is bounded with compact support on $\mathbb{R}^n$. Moreover, in the case (1.28) holds, we see that (1.27) implies (1.8) by Lemma 2.22, and so by Theorem 1.6 we may also assume that the maximal operator $\mathcal{M}$ satisfies the two weight norm inequality (1.7). It now follows that $\int (T_\natural f\sigma)^p \omega < \infty$ for f bounded with compact support. Indeed, $T_\natural f\sigma \le C\mathcal{M} f\sigma$ far away from the support of f, while $T_\natural f\sigma$ is controlled by the testing condition (1.27) near the support of f.

Let $\{Q_j^k\}$ be the cubes as in (2.5) and (2.6), with the measure ν that appears in there being $\nu = f\sigma$. We will use Lemma 2.9 with this choice of ν as well. Now define an 'exceptional set' associated to Q_j^k to be

$$E_j^k = Q_j^k \cap (\Omega_{k+1} \setminus \Omega_{k+2}).$$

See Figure 4.1. One might anticipate the definition of the exceptional set to be more simply $Q_j^k \cap \Omega_{k+1}$. We are guided to this choice by the work on fractional integrals [18]. And indeed, the choice of exceptional set above enters in a decisive way in the analysis of the bad function at the end of the proof.

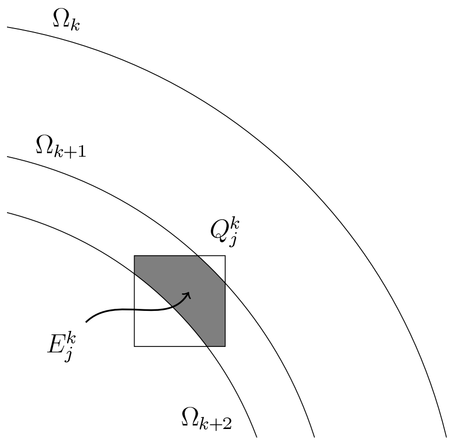

FIGURE 4.1. The set $E_j^k(Q)$.

We estimate the left side of (1.26) in terms of this family of dyadic cubes $\{Q_j^k\}_{k,j}$ by

$$\int (T_\natural f\sigma)^p \,\omega(dx) \leq \sum_{k\in\mathbb{Z}} (2^{k+2})^p \left|\Omega_{k+1} \setminus \Omega_{k+2}\right|_\omega \leq \sum_{k,j} (2^{k+2})^p \left|E_j^k\right|_\omega . \tag{4.6}$$

Choose a linearization L of $T_\natural$ as in (2.13) so that (recall $R(x)$ is the upper limit of truncation)

$$R(x) \leq \frac{1}{2}\ell(Q_j^k), \qquad x \in E_j^k, \tag{4.7}$$

$$\text{and} \quad T_\natural \left(\chi_{3Q_j^k} f\sigma\right)(x) \leq 2L\left(\chi_{3Q_j^k} f\sigma\right)(x) + C\frac{1}{\left|3Q_j^k\right|}\int_{3Q_j^k} |f|\,\sigma, \qquad x\in E_j^k.$$

For $x \in E_j^k$, the maximum principle (2.10) yields

$$\begin{aligned} T_\natural \chi_{3Q_j^k} f\sigma(x) &\geq T_\natural f\sigma(x) - T_\natural \chi_{(3Q_j^k)^c} f\sigma(x) \\ &> 2^{k+1} - 2^k - C\mathbf{P}\left(Q_j^k, f\sigma\right) \\ &= 2^k - C\mathbf{P}\left(Q_j^k, f\sigma\right). \end{aligned}$$

From (4.7) we conclude that

$$L\chi_{3Q_j^k} f\sigma(x) \geq 2^{k-1} - C\mathbf{P}\left(Q_j^k, f\sigma\right).$$

Thus either $2^k \leq 4\inf_{E_j^k} L\chi_{3Q_j^k} f\sigma$ or $2^k \leq 4C\mathbf{P}\left(Q_j^k, f\sigma\right) \leq 4CM\left(Q_j^k, f\sigma\right)$. So we obtain either

$$\left|E_j^k\right|_\omega \leq C2^{-k}\int_{E_j^k} (L\chi_{3Q_j^k} f\sigma)\omega(dx), \tag{4.8}$$

or

$$\left|E_j^k\right|_\omega \leq C2^{-pk}\left|E_j^k\right|_\omega M\left(Q_j^k, f\sigma\right)^p \leq C2^{-pk}\int_{E_j^k} (\mathcal{M}f\sigma)^p\,\omega(dx). \tag{4.9}$$

Now consider the following decomposition of the set of indices (k,j):

$$\begin{aligned} \mathbb{E} &= \left\{(k,j) : \left|E_j^k\right|_\omega \leq \beta\left|NQ_j^k\right|_\omega\right\}, \\ \mathbb{F} &= \{(k,j) : (4.9) \text{ holds}\}, \\ \mathbb{G} &= \left\{(k,j) : \left|E_j^k\right|_\omega > \beta\left|NQ_j^k\right|_\omega \text{ and (4.8) holds}\right\}, \end{aligned} \tag{4.10}$$

where $0 < \beta < 1$ will be chosen sufficiently small at the end of the argument. (It will be of the order of c^p for a small constant c.) By the 'crowd control' condition of (2.6), we have

$$\sum_j \chi_{NQ_j^k} \le C, \qquad k \in \mathbb{Z}. \tag{4.11}$$

We then have the corresponding decomposition:

$$\begin{aligned}
\int (T_\natural f\sigma)^p \,\omega &\le \left\{ \sum_{(k,j)\in\mathbb{E}} + \sum_{(k,j)\in\mathbb{F}} + \sum_{(k,j)\in\mathbb{G}} \right\} (2^{k+2})^p \left|E_j^k\right|_\omega \qquad (4.12)\\
&\le \beta \sum_{(k,j)\in\mathbb{E}} (2^{k+2})^p \left|NQ_j^k\right|_\omega + C \sum_{(k,j)\in\mathbb{F}} \int_{E_j^k} (\mathcal{M} f\sigma)^p \,\omega \\
&\quad + C \sum_{(k,j)\in\mathbb{G}} \left|E_j^k\right|_\omega \left(\frac{1}{\beta \left|NQ_j^k\right|_\omega} \int_{E_j^k} \left(L\chi_{3Q_j^k} f\sigma \right) \omega \right)^p \\
&= J(1) + J(2) + J(3) \\
&\le C_0 \left\{ \beta \int (T_\natural f\sigma)^p \,\omega + \beta^{-p} \int |f|^p \sigma \right\}, \qquad (4.13)
\end{aligned}$$

where $C_0 \le C\{\mathfrak{M} + \mathfrak{M}_* + \mathfrak{T} + \mathfrak{T}_*\}^p$. The last line is the claim that we take up in the remainder of the proof. Once it is proved, note that if we take $0 < C_0\beta < \frac{1}{2}$ and use the fact that $\int (T_\natural f\sigma)^p \,\omega < \infty$ for f bounded with compact support, we have proved assertion (1) of Theorem 1.24, and in particular (4.1).

The proof of the strong type inequality requires a complicated series of decompositions of the dominating sums, which are illustrated for the reader's convenience as a schematic tree in Figure 4.2.

4.3. **Two Easy Estimates.** Note that the first term $J(1)$ in (4.12) satisfies

$$J(1) = \beta \sum_{(k,j)\in\mathbb{E}} (2^{k+2})^p \left|NQ_j^k\right|_\omega \le C\beta \int (T_\natural f\sigma)^p \,\omega,$$

by the finite overlap condition (4.11). The second term $J(2)$ is dominated by

$$C \sum_{(k,j)\in\mathbb{F}} \int_{E_j^k} (\mathcal{M} f\sigma)^p \,\omega \le C\mathfrak{M}^p \left\|f\right\|_{L^p(\sigma)}^p ,$$

by our assumption (1.7). It is useful to note that this is the *only* time in the proof that we use the maximal function inequality (1.7) - from now on we use the *dual* maximal function inequality (1.25).

Remark 4.14. In the arguments below we can use Theorem 2 of [18] to replace the dual maximal function assumption $\mathfrak{M}_* < \infty$ with two assumptions, namely a 'Poisson two weight *Ap* condition' and the analogue of the dual pivotal condition of Nazarov, Treil and Volberg [9]. The Poisson two weight Ap condition is in fact

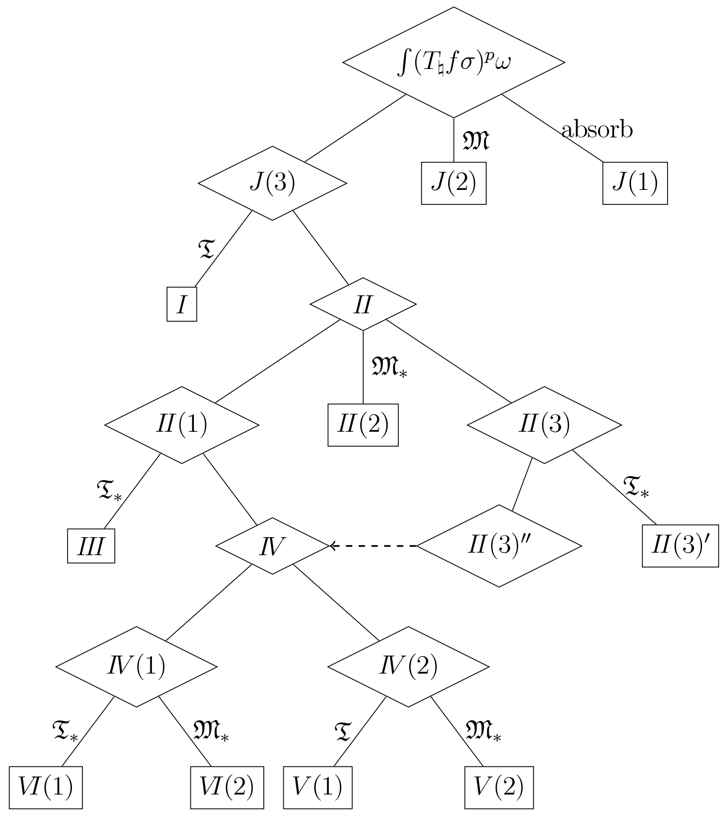

FIGURE 4.2. This a schematic tree of how the integral $\int (T_\natural f\sigma)^p\,\omega$ has been, and will continue to be, decomposed. We have suppressed superscripts, subscripts and sums in the tree. Terms in diamonds are further decomposed, while terms in rectangles are final estimates. The edges leading into rectangles are labelled by the $\mathfrak{M}$, $\mathfrak{M}_*$, $\mathfrak{T}$, or $\mathfrak{T}_*$ whose finiteness is used to control that term. The word 'absorb' leading into $J(1)$ indicates that this term is a small multiple of $\int (T_\natural f\sigma)^p\,\omega$ and can be absorbed into the left-hand side of the inequality. The sole horizontal dashed arrow in the schematic indicates that the term $\sum II_s^t(3)''$ is treated with the term $\sum IV_s^t$. As most of the terms involve the maximal theorem (2.2), we do not indicate its use in the schematic tree.

necessary for the two weight inequality, but the the necessity of the dual pivotal conditions for singular integral weighted inequalities is still an open question. On the other hand, the assumption $\mathfrak{M} < \infty$ cannot be weakened here, reflecting that our method requires the maximum principle in Lemma 2.9.

It is the third term $J(3)$ that is the most involved, see Figure 4.2. The remainder of the proof is taken up with the proof of

$$\sum_{(k,j)\in\mathbb{G}} R_j^k \left| \int_{E_j^k} \left(L\chi_{3Q_j^k} f\sigma \right) \omega \right|^p \leq C\{\mathfrak{M}_*^p + \mathfrak{T}^p + \mathfrak{T}_*^p\} \|f\|_{L^p\sigma(dx)}^p , \tag{4.15}$$

where

$$R_j^k = \frac{\left|E_j^k\right|_\omega}{\left|NQ_j^k\right|_\omega^p} . \tag{4.16}$$

Once this is done, the proof of (4.12) is complete, and the proof of assertion (1) is finished.

4.4. **The Calderón-Zygmund Decompositions.** To carry out this proof, we implement Calderón-Zygmund Decompositions relative to the measure σ. These Decompositions will be done at *all heights simultaneously*. We will use the shifted dyadic grids, see (2.3). For $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, let

$$\mathcal{M}_\sigma^\alpha f(x) = \sup_{x\in Q\in\mathcal{D}^\alpha} \frac{1}{|Q|_\sigma} \int_Q |f| \, d\sigma,$$

$$\Gamma_t^\alpha = \left\{ x \in \mathbb{R} : \mathcal{M}_\sigma^\alpha f(x) > 2^t \right\} = \bigcup_s G_s^{\alpha,t}, \tag{4.17}$$

where $\{G_s^{\alpha,t}\}_{t,s}$ are the maximal $\mathcal{D}^\alpha$ cubes in Γ_t^α. This implies that we have the nested property: If $G_s^{\alpha,t} \subsetneqq G_{s'}^{\alpha,t'}$ then $t > t'$. Moreover, if $t > t'$ there is some s' with $G_s^{\alpha,t} \subset G_{s'}^{\alpha,t'}$. These are the cubes used to make a Calderón-Zygmund Decomposition at height 2^t for the grid $\mathcal{D}^\alpha$ with respect to the measure σ.

Of course we have from the maximal inequality in (2.2)

$$\sum_{t,s} 2^{pt} |G_s^{\alpha,t}|_\sigma \leq C \|f\|_{L^p(\sigma)}^p . \tag{4.18}$$

The point of these next several definitions is associate to each dyadic cube Q, a good shifted dyadic grid, and an appropriate height, at which we will build our Calderón-Zygmund Decomposition.

Principal Labels: We identify a distinguished subset of the labelling set of the cubes $\{G_s^{\alpha,t}\}$, which we refere to as the 'principal labels', of the pairs (t,s) parameterizing the cubes $\{G_s^{\alpha,t}\}$. Define a set of indices (t,s) by

$$\mathbb{L}^\alpha = \left\{ (t,s) : \text{there is no cube } G_{s'}^{\alpha,t+1} \text{ equal to } G_s^{\alpha,t} \right\} . \tag{4.19}$$

In other words, if there is a maximal chain of equal cubes $G_{s_0}^{\alpha,t_0} = G_{s_1}^{\alpha,t_0+1} = \cdots = G_{s_N}^{\alpha,t_0+N}$ we discard all of these indices but (s_N, t_0+N), the one for which

$$2^{t_0+N} < \frac{1}{|G_{s_N}^{\alpha,t_0+N}|_\sigma} \int_{G_{s_N}^{\alpha,t_0+N}} |f| \, \sigma \leq 2^{t_0+N+1}. \tag{4.20}$$

We have this variant of (4.18).

$$\sum_{(t,s)\in\mathbb{L}^\alpha}\left(\frac{1}{|G_s^{\alpha,t}|_\sigma}\int_{G_s^{\alpha,t}}|f|\,\sigma\right)^p|G_s^{\alpha,t}|_\sigma\le C\,\|f\|^p_{L^p(\sigma)}\,, \tag{4.21}$$

For $(t,s)\in\mathbb{L}^\alpha$, and any s' we have $G_s^{\alpha,t}\cap G_{s'}^{\alpha,t+1}\in\{\emptyset,G_{s'}^{\alpha,t+1}\}$. We will refer to a cube $G_s^{\alpha,t}$ with principal label $(t,s)\in\mathbb{L}^\alpha$ as a *principal cube* (thus every cube $G_s^{\alpha,t}$ is a principal cube when properly labelled).

Select a shifted grid: Let $\vec{\alpha}\ :\ \mathcal{D}\longrightarrow\{0,\frac13,\frac23\}^n$ be a map so that for $Q\in\mathcal{D}$, there is a $\widehat{Q}\in\mathcal{D}^{\vec{\alpha}(Q)}$ so that $3Q\subset\frac{9}{10}\widehat{Q}$ and $|\widehat{Q}|\le C|Q|$. Here, C is an appropriate constant depending only on dimension. Thus, $\vec{\alpha}(Q)$ picks a 'good' shifted dyadic grid for Q. Note that

$$\widehat{Q}\subset MQ. \tag{4.22}$$

for some positive dimensional constant M. The cubes $\widehat{Q_j^k}$ will play a critical role below. See Figure 4.4

Select a principal cube: Define $\mathcal{A}(Q)$ to be the smallest cube from the collection $\{G_s^{\vec{\alpha}(Q),t}\ |\ (t,s)\in\mathbb{L}^\alpha\}$ that contains $3Q$; $\mathcal{A}(Q)$ is uniquely determined by Q and the choice of function $\vec{\alpha}$. Define

$$\mathbb{H}_s^{\alpha,t}=\left\{(k,j):\mathcal{A}(Q_j^k)=G_s^{\alpha,t}\right\}\,,\qquad (s,t)\in\mathbb{L}^\alpha\,. \tag{4.23}$$

This is an important definition for us. The combinatorial structure this places on the corresponding cubes is essential for this proof to work. Note that $3Q_j^k\subset\widehat{Q_j^k}\subset\mathcal{A}\left(Q_j^k\right)$.

Parents: For any of the shifted dyadic grids $\mathcal{D}^\alpha$, a $Q\in\mathcal{D}^\alpha$ has a unique parent denoted as $\mathcal{P}(Q)$, the smallest member of $\mathcal{D}^\alpha$ that strictly contains Q. We suppress the dependence upon α here.

Select maximal parents: Let

$$\mathbb{K}_s^{\alpha,t}=\left\{r\mid P(G_r^{\alpha,t+1})\text{ is maximal in }\{P(G_{r'}^{\alpha,t+1})\mid G_{r'}^{\alpha,t+1}\subset G_s^{\alpha,t}\}\right\}\,. \tag{4.24}$$

Note that the labels $(t+1,r)$ with $r\in\mathbb{K}_s^{\alpha,t}$ are not necessarily principal labels, although the actual cubes $G_r^{\alpha,t+1}$ are principal when properly labelled.

The good and bad functions: Let $A_{P(G_r^{\alpha,t+1})}=\frac{1}{\left|P(G_r^{\alpha,t+1})\right|_\sigma}\int_{P(G_r^{\alpha,t+1})}f\sigma$ be the σ-average of f on $P(G_r^{\alpha,t+1})$. Define functions $g_s^{\alpha,t}$ and $h_s^{\alpha,t}$ satisfying $f=g_s^{\alpha,t}+h_s^{\alpha,t}$ on $G_s^{\alpha,t}$ by

$$g_s^{\alpha,t}(x)=\begin{cases}A_{P(G_r^{\alpha,t+1})} & x\in P(G_r^{\alpha,t+1})\text{ with }r\in\mathbb{K}_s^{\alpha,t}\\ f(x) & x\in G_s^{\alpha,t}\setminus\bigcup\{P(G_r^{\alpha,t+1}):r\in\mathbb{K}_s^{\alpha,t}\}\,,\end{cases} \tag{4.25}$$

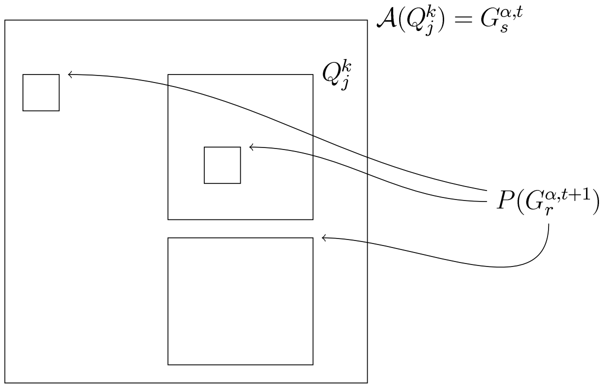

FIGURE 4.3. The relative positions for the rectangles Q_j^k, $\mathcal{A}(Q_j^k) = G_s^{\alpha,t}$, and cubes $P(G_r^{\alpha,t+1})$ for $r \in \mathbb{K}_s^{\alpha,t}$. Note that $g_s^{\alpha,t}$ is supported on $g_s^{\alpha,t}$, and has L^∞ norm at most 2^{t+1}, and that the function $h_s^{\alpha,t}$ is supported on the cubes $P(G_r^{\alpha,t+1})$, and has integral zero with respect to σ-measure on each such cube.

$$h_s^{\alpha,t}(x) = \begin{cases} f(x) - A_{P(G_r^{\alpha,t+1})} & x \in P(G_r^{\alpha,t+1}) \text{ with } r \in \mathbb{K}_s^{\alpha,t} \\ 0 & x \in G_s^{\alpha,t} \setminus \bigcup \{P(G_r^{\alpha,t+1}) : r \in \mathbb{K}_s^{\alpha,t}\}\,. \end{cases} \tag{4.26}$$

We extend both $g_s^{\alpha,t}$ and $h_s^{\alpha,t}$ to all of $\mathbb{R}^n$ by defining them to vanish outside $G_s^{\alpha,t}$. Now the average $A_{P(G_r^{\alpha,t+1})}$ is at most 2^{t+1} by maximality of the cubes in (4.17). Thus Lebesgue's differentiation theorem shows that (any of the standard proofs can be adapted to the dyadic setting for positive locally finite Borel measures on $\mathbb{R}^n$)

$$\left|g_s^{\alpha,t}(x)\right| \le 2^{t+1} < \frac{2}{\left|G_s^{\alpha,t}\right|_\sigma} \int_{G_s^{\alpha,t}} |f|\,\sigma, \qquad \sigma\text{-a.e. } x \in G_s^{\alpha,t},\ (t,s) \in \mathbb{L}^\alpha. \tag{4.27}$$

That is, $g_s^{\alpha,t}$ is the 'good' function and $h_s^{\alpha,t}$ is the 'bad' function. See Figure 4.3

We can now refine the final sum on the left side of (4.15) according to the decomposition of $\mathcal{M}_\sigma^\alpha f$. We carry out these steps. In the first step, we fix an $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$, and for the remainder of the proof, we only consider Q_j^k for which $\vec{\alpha}(Q_j^k) = \alpha$. Namely, we will modify the important definition of $\mathbb{G}$ in (4.10) to

$$\mathbb{G}^\alpha = \left\{(k,j) : \vec{\alpha}(Q_j^k) = \alpha\,,\ \left|E_j^k\right|_\omega > \beta \left|NQ_j^k\right|_\omega \text{ and (4.8) holds}\right\}, \tag{4.28}$$

In the second step, we partition the indices (k,j) into the sets $\mathbb{H}_s^{\alpha,t}$ in (4.23) for $(t,s) \in \mathbb{L}^\alpha$. In the third step, for $(k,j) \in \mathbb{H}_s^{\alpha,t}$, we split f into the corresponding good and bad parts. This yields the decomposition

$$\sum_{(k,j)\in\mathbb{G}^\alpha} R_j^k \left|\int_{E_j^k} \left(L\chi_{3Q_j^k} f\sigma\right) \omega(dx)\right|^p \leq C\,(I+II)\,, \tag{4.29}$$

$$I = \sum_{(t,s)\in\mathbb{L}^\alpha} I_s^t\,, \qquad II = \sum_{(t,s)\in\mathbb{L}^\alpha} II_s^t \tag{4.30}$$

$$I_s^t = \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left|\int_{E_j^k} \left(L\chi_{3Q_j^k} g_s^{\alpha,t}\sigma\right) \omega(dx)\right|^p \tag{4.31}$$

$$II_s^t = \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left|\int_{E_j^k} \left(L\chi_{3Q_j^k} h_s^{\alpha,t}\sigma\right) \omega(dx)\right|^p \tag{4.32}$$

$$\mathbb{I}_s^{\alpha,t} = \mathbb{G}^\alpha \cap \mathbb{H}_s^{\alpha,t} \tag{4.33}$$

Recall the definition of R_j^k in (4.16). In the definitions of I, I_s^t and II, II_s^t we will suppress the dependence on $\alpha \in \{0, \frac{1}{3}, \frac{2}{3}\}^n$. The same will be done for the subsequent decompositions of the (difficult) term II, although we usually retain the superscript α in the quantities arising in the estimates. In particular, we emphasize that the combinatorial properties of the cubes associated with $\mathbb{I}_s^{\alpha,t}$ are essential to completing this proof.

Term I requires only condition (1.27) and the maximal theorem (2.2), while term II will require both conditions (1.27) and (1.21), along with the dual maximal function inequality (1.25) and the maximal theorem (2.2). It will be convenient at this point to map out the various decompositions of the terms and the conditions used to control them.

4.5. **The Analysis of the good function.** We claim that

$$I \leq C\mathfrak{T}^p \|f\|_{L^p(\sigma)}^p\,. \tag{4.34}$$

Proof. We use boundedness of the 'good' function $g_s^{\alpha,t}$, as defined in (4.25), the testing condition (1.27) for $T_\natural$, see also (4.4), and finally the universal maximal function bound (2.2) with $\mu = \omega$. Here are the details. For $x \in E_j^k$, (4.7) implies that $L\chi_{3Q_j^k} g_s^{\alpha,t}\sigma\,(x) = L g_s^{\alpha,t}\sigma\,(x)$ and so

$$\begin{aligned} I &= \sum_{(t,s)\in\mathbb{L}^\alpha} I_s^t \\ &= C \sum_{(t,s)\in\mathbb{L}^\alpha} \sum_{(k,j)\in\mathbb{G}^\alpha\cap\mathbb{H}_s^{\alpha,t}} R_j^k \left|\int_{E_j^k} \left(L g_s^{\alpha,t}\sigma\right)\omega\right|^p \\ &\leq C \sum_{(t,s)\in\mathbb{L}^\alpha} \int \left|\mathcal{M}_\omega^{dy}\left(\chi_{G_s^{\alpha,t}} L g_s^{\alpha,t}\sigma\right)\right|^p \omega \end{aligned}$$

$$\leq C \sum_{(t,s)\in\mathbb{L}^\alpha} \int_{G_s^{\alpha,t}} \left|Lg_s^{\alpha,t}\sigma\right|^p \omega$$

$$\leq C \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \int_{G_s^{\alpha,t}} \left(T_\natural \frac{g_s^{\alpha,t}}{2^{t+1}}\sigma\right)^p \omega$$

$$\leq C\mathfrak{T}^p \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \left|G_s^{\alpha,t}\right|_\sigma,$$

where we have used (4.27) and (1.27) with $g = \frac{g_s^{\alpha,t}}{2^{t+1}}$ in the final inequality. This last sum is controlled by (4.18), and completes the proof of (4.34). ∎

4.6. **The Analysis of the Bad Function: Part 1.** It remains to estimate term II, as in (4.32), but this is in fact the harder term. Recall the definition of $\mathbb{K}_s^{\alpha,t}$ in (4.24). We now write

$$h_s^{\alpha,t} = \sum_{r\in\mathbb{K}_s^{\alpha,t}} \left[f - A_{P(G_r^{\alpha,t+1})}\right] \chi_{P(G_r^{\alpha,t+1})} \equiv \sum_{r\in\mathbb{K}_s^{\alpha,t}} b_r, \tag{4.35}$$

where the 'bad' functions b_r are supported in the cube $P(G_r^{\alpha,t+1})$ and have σ-mean zero, $\int_{P(G_r^{\alpha,t+1})} b_r\sigma = 0$. To take advantage of this, we will pass to the dual L^* below.

But first we must address the fact that the triples of the $\mathcal{D}^\alpha$ cubes $P(G_r^{\alpha,t+1})$ do not form a grid. For $(t,s) \in \mathbb{L}^\alpha$ and $(k,j) \in \mathbb{I}_s^{\alpha,t}$ let $\widetilde{Q_j^k}$ be the largest $\mathcal{D}^\alpha$ cube containing $\widehat{Q_j^k}$ and satisfying

$$R'_W\widetilde{Q_j^k} \subset \Omega_k. \tag{4.36}$$

where $R'_W = \frac{R_W}{M}$ and M is defined in (4.22). Note that such a cube $\widetilde{Q_j^k}$ exists since $\widehat{Q_j^k} \subset MQ_j^k$ by (4.22) and $R_W Q_j^k \subset \Omega_k$ by (2.6). Moreover, we can arrange to have

$$3\widetilde{Q_j^k} \subset NQ_j^k, \tag{4.37}$$

where N is as in Remark 2.7, by choosing R_W sufficiently large in (2.6). (Recall that the cubes $\widetilde{Q_j^k}$ are chosen at (4.22) above.) See Figure 4.4.

Now momentarily fix $(k,j) \in \mathbb{I}_s^{\alpha,t}$ and set

$$\mathbb{K}_s^{\alpha,t}(k,j) = \left\{r \in \mathbb{K}_s^{\alpha,t} : P(G_r^{\alpha,t+1}) \subset \widetilde{Q_j^k}\right\}. \tag{4.38}$$

Let

$$\mathcal{C}_s^{\alpha,t}(k,j) = \left\{3P\left(G_r^{\alpha,t+1}\right) : r \in \mathbb{K}_s^{\alpha,t}(k,j)\right\} \tag{4.39}$$

be the collection of triples of the $\mathcal{D}^\alpha$ cubes $P\left(G_r^{\alpha,t+1}\right)$ with $r \in \mathbb{K}_s^{\alpha,t}(k,j)$. We select the *maximal* triples $\left\{3P\left(G_{r_\ell}^{t+1}\right)\right\}_\ell \equiv \{T_\ell\}_\ell$ from the collection $\mathcal{C}_s^{\alpha,t}(k,j)$, and assign to each $r \in \mathbb{K}_s^{\alpha,t}(k,j)$ the maximal triple $T_\ell = T_{\ell(r)}$ containing

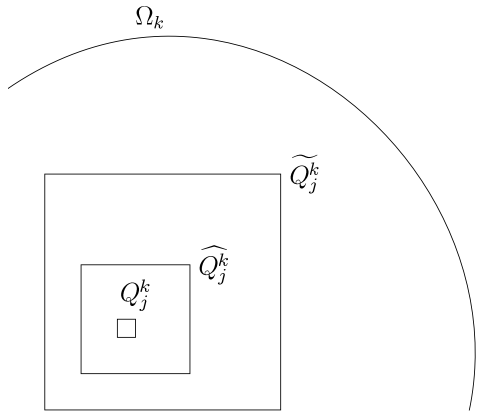

FIGURE 4.4. The relative positions of the cubes Q_j^k, $\widehat{Q_j^k}$, and $\widetilde{Q_j^k}$ inside a set Ω_k.

$3P\left(G_r^{\alpha,t+1}\right)$ with least ℓ. By Lemma 2.23 applied to $\mathcal{D}^\alpha$ the maximal triples $\{T_\ell\}_\ell$ have finite overlap 3^n, and this will prove crucial in (4.65) below. Note that $T_{\ell(r)}$ depends on (k,j) as well.

We will pass to the dual of the linearization.

$$\begin{aligned}(4.40)\qquad \int_{E_j^k}\left(Lh_s^{\alpha,t}\sigma\right)\omega(dx) &= \sum_{r\in\mathbb{K}_s^{\alpha,t}}\int_{E_j^k}(Lb_r\sigma)\,\omega \\ &= \sum_{r\in\mathbb{K}_s^{\alpha,t}}\int_{P(G_r^{\alpha,t+1})\cap 3Q_j^k}\left(L^*\chi_{E_j^k}\omega(dx)\right)b_r\sigma\end{aligned}$$

Note that (4.7) implies $L^*\nu$ is supported in $3Q_j^k$ if ν is supported in E_j^k, explaining the range of integration above. Continuing, we have

$$(4.41)\qquad (4.40)\le\left|\sum_{r\in\mathbb{K}_s^{\alpha,t}(k,j)}\int_{P(G_r^{\alpha,t+1})\cap 3Q_j^k}\left(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega\right)b_r\sigma\right|$$

$$(4.42)\qquad +C\sum_{r\in\mathbb{K}_s^{\alpha,t}}\mathbf{P}\left(P(G_r^{\alpha,t+1}),\chi_{E_j^k\setminus 3P(G_r^{\alpha,t+1})}\omega\right)\int_{P(G_r^{\alpha,t+1})}|f|\,\sigma$$

$$(4.43)\qquad +\left|\sum_{r\in\mathbb{K}_s^{\alpha,t}\setminus\mathbb{K}_s^{\alpha,t}(k,j)}\int_{P(G_r^{\alpha,t+1})\cap 3Q_j^k}\left(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega\right)b_r\sigma\right|.$$

To see the above inequality, note that for $r\in\mathbb{K}_s^{\alpha,t}$ we are splitting the set E_j^k into $E_j^k\cap T_{\ell(r)}$ and $E_j^k\setminus T_{\ell(r)}$. On the latter set, the hypotheses of Lemma 2.16 are in force, namely the set $E_j^k\setminus T_{\ell(r)}$ does not intersect $3P(G_r^{\alpha,t+1})$, whence we have an estimate on the δ-Hölder modulus of continuity of $L^*\chi_{E_j^k\setminus T_{\ell(r)}}\omega$. Combine this with the fact that b_r has σ-mean zero on $P(G_r^{\alpha,t+1})$ to derive the estimate below, in which y_r^{t+1} is the center of the cube $G_r^{\alpha,t+1}$.

$$\left|\int_{P(G_r^{\alpha,t+1})} \left(L^*\chi_{E_j^k \setminus T_{\ell(r)}}\omega\right) b_r\sigma\right|$$

$$= \left|\int_{P(G_r^{\alpha,t+1})} \left(L^*\chi_{E_j^k \setminus T_{\ell(r)}}\omega(y) - L^*\chi_{E_j^k \setminus T_{\ell(r)}}\omega(y_r^{t+1})\right)(b_r\sigma)\right|$$

(4.44)

$$\leq \int_{P(G_r^{\alpha,t+1})\cap 3Q_j^k} C\mathbf{P}\left(P(G_r^{\alpha,t+1}), \chi_{E_j^k \setminus T_{\ell(r)}}\omega\right) \delta\left(\frac{|y-y_r^{t+1}|}{\ell\left(P(G_r^{\alpha,t+1})\right)}\right) |b_r(y)|\, d\sigma(y)$$

$$\leq C\mathbf{P}\left(P(G_r^{\alpha,t+1}), \chi_{E_j^k \setminus 3P(G_r^{\alpha,t+1})}\omega\right) \int_{P(G_r^{\alpha,t+1})} |f|\, d\sigma.$$

We have after application of (4.41),

$$II_s^t = \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left[\int_{E_j^k} \left(Lh_s^{\alpha,t}\sigma\right)\omega\right]^p \tag{4.45}$$

$$\leq II_s^t(1) + II_s^t(2) + II_s^t(3)\,, \tag{4.46}$$

$$II_s^t(1) = \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left|\sum_{r\in\mathbb{K}_s^{\alpha,t}(k,j)} \int_{P(G_r^{\alpha,t+1})} \left(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega\right) b_r\sigma\right|^p, \tag{4.47}$$

$$II_s^t(2) = \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left[\sum_{r\in\mathbb{K}_s^{\alpha,t}} \mathbf{P}\left(P(G_r^{\alpha,t+1}), \chi_{E_j^k}\omega\right) \int_{P(G_r^{\alpha,t+1})} |f|\,\sigma\right]^p, \tag{4.48}$$

$$II_s^t(3) = \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left|\sum_{r\in\mathbb{K}_s^{\alpha,t}\setminus\mathbb{K}_s^{\alpha,t}(k,j)} \int_{P(G_r^{\alpha,t+1})} \left(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega\right) b_r\sigma\right|^p. \tag{4.49}$$

Note that we may further restrict the integrations in (4.47) and (4.49) to $P(G_r^{\alpha,t+1})\cap 3Q_j^k$ since $L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega$ is supported in $3Q_j^k$.

4.6.1. *Analysis of* $II(2)$. We claim that

$$\sum_{(t,s)\in\mathbb{L}^\alpha} II_s^t(2) \leq C\mathfrak{M}_*^p \int |f|^p\,\sigma\,. \tag{4.50}$$

Recall the definition of $\mathfrak{M}_*$ in (4.3).

Proof. We begin by defining a linear operator by

$$\mathbf{P}_j^k(\mu) \equiv \sum_{r\in\mathbb{K}_s^{\alpha,t}} \mathbf{P}\left(P(G_r^{\alpha,t+1}),\chi_{E_j^k}\mu\right)\chi_{P(G_r^{\alpha,t+1})}. \tag{4.51}$$

In this notation, we have for $(k,j)\in\mathbb{I}_s^{\alpha,t}$ (See (4.23) and (4.32)),

$$\begin{aligned}
&\sum_{r\in\mathbb{K}_s^{\alpha,t}} \mathbf{P}\Big(P(G_r^{\alpha,t+1}),\chi_{E_j^k}\omega(dx)\Big)\int_{P(G_r^{\alpha,t+1})}|f|\,\sigma \\
&\qquad = \sum_{r\in\mathbb{K}_s^{\alpha,t}} \mathbf{P}\Big(P(G_r^{\alpha,t+1}),\chi_{E_j^k}\omega(dx)\Big)\int_{P(G_r^{\alpha,t+1})}\sigma \\
&\qquad\quad \times\left\{\frac{1}{\left|P(G_r^{\alpha,t+1})\right|_\sigma}\int_{P(G_r^{\alpha,t+1})}|f|\,\sigma\right\} \\
&\qquad \le 2^{t+1}\int_{G_s^{\alpha,t}}\mathbf{P}_j^k(\omega)\sigma = 2^{t+1}\int_{E_j^k}(\mathbf{P}_j^k)^*(\chi_{G_s^{\alpha,t}}\sigma)\omega.
\end{aligned}$$

By assumption, the maximal function $\mathcal{M}(\omega\cdot)$ maps $L^{p'}(\omega)$ to $L^{p'}(\sigma)$, and we now note a particular consequence of this. In the definition (4.51) we were careful to insert $\chi_{E_j^k}$ on the right hand side. These sets are pairwise disjoint, whence we have the inequality below for measures μ.

$$\begin{aligned}
&\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} \mathbf{P}_j^k(\mu)(x) \\
&\quad \le \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}\sum_{r\in\mathbb{K}_s^{\alpha,t}}\sum_{\ell=0}^{\infty}\frac{\delta\left(2^{-\ell}\right)}{\left|2^\ell P(G_r^{\alpha,t+1})\right|}\left(\int_{2^\ell P(G_r^{\alpha,t+1})}\chi_{E_j^k}\mu\right)\chi_{P(G_r^{\alpha,t+1})}(x) \\
&\quad \le \sum_{\ell=0}^{\infty}\sum_{r\in\mathbb{K}_s^{\alpha,t}}\frac{\delta\left(2^{-\ell}\right)}{\left|2^\ell P(G_r^{\alpha,t+1})\right|}\left(\int_{2^\ell P(G_r^{\alpha,t+1})}\mu\right)\chi_{P(G_r^{\alpha,t+1})}(x) \\
&\quad \le C\chi_{G_s^{\alpha,t}}\mathcal{M}\left(\chi_{G_s^{\alpha,t}}\mu\right)(x).
\end{aligned} \tag{4.52}$$

Thus the inequality

$$\|\chi_{G_s^{\alpha,t}}\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}\mathbf{P}_j^k(|g|\omega)\|_{L^{p'}(\sigma)} \le C\mathfrak{M}_*\|\chi_{G_s^{\alpha,t}}g\|_{L^{p'}(\omega)}$$

follows immediately. By duality we then have

$$\left\|\chi_{G_s^{\alpha,t}}\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}(\mathbf{P}_j^k)^*(|f|\sigma)\right\|_{L^p(w)} \le C\mathfrak{M}_*\|\chi_{G_s^{\alpha,t}}f\|_{L^p(\sigma)}\ . \tag{4.53}$$

Note that it was the linearity that we wanted in (4.51), so that we could appeal to the dual maximal function assumption.

We thus obtain

$$II_s^t(2) \le 2^{p(t+1)} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left[\int_{Q_j^k} (\mathbf{P}_j^k)^* \left(\chi_{G_s^{\alpha,t}}\sigma\right) d\omega\right]^p .$$

Summing in (t,s) and using $(\mathbf{P}_j^k)^* \le \sum_{(\ell,i)\in\mathbb{I}_s^{\alpha,t}} (\mathbf{P}_i^\ell)^*$ for $(k,j)\in\mathbb{I}_s^{\alpha,t}$ we obtain

(4.54)

$$\sum_{(t,s)\in\mathbb{L}^\alpha} II_s^t(2) \le C \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left[\int_{Q_j^k} (\mathbf{P}_j^k)^* \left(\chi_{G_s^{\alpha,t}}\sigma\right) d\omega\right]^p$$

$$= C \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} \left|E_j^k\right|_\omega \left[\frac{1}{\left|NQ_j^k\right|_w} \int_{Q_j^k} (\mathbf{P}_j^k)^* \left(\chi_{G_s^{\alpha,t}}\sigma\right) \omega\right]^p$$

$$\le C \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \int \left[\mathcal{M}_\omega \left(\chi_{G_s^{\alpha,t}} \sum_{(\ell,i)\in\mathbb{I}_s^{\alpha,t}} (\mathbf{P}_i^\ell)^* \left(\chi_{G_s^{\alpha,t}}\sigma\right)\right)\right]^p \omega \tag{4.55}$$

$$\le C \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \int_{G_s^{\alpha,t}} \left[\sum_{(\ell,i)\in\mathbb{I}_s^{\alpha,t}} (\mathbf{P}_i^\ell)^* \left(\chi_{G_s^{\alpha,t}}\sigma\right)\right]^p \omega \tag{4.56}$$

$$\le C\mathfrak{M}_*^p \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \left|G_s^{\alpha,t}\right|_\sigma ,$$

which is bounded by $C\mathfrak{M}_*^p \int |f|^p \sigma$. In the last line we are applying (4.53). ∎

4.6.2. *Decomposition of* $II(3)$. We now dominate the term $II_s^t(3)$ in (4.49) by the sum of the two terms

(4.57)

$$II_s^t(3)' \equiv \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left| \sum_{r\in\mathbb{K}_s^{\alpha,t}\setminus\mathbb{K}_s^{\alpha,t}(k,j)} \int_{P(G_r^{\alpha,t+1})\setminus\Omega_{k+2}} \left(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega\right) b_r\sigma(dy)\right|^p$$

(4.58)

$$II_s^t(3)'' \equiv \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left| \sum_{r\in\mathbb{K}_s^{\alpha,t}\setminus\mathbb{K}_s^{\alpha,t}(k,j)} \int_{P(G_r^{\alpha,t+1})\cap\Omega_{k+2}} \left(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega\right) b_r\sigma(dy)\right|^p$$

The term $II_s^t(3)''$ is the same as term IV_s^t in (4.63) except that the sum in r is over $\mathbb{K}_s^{\alpha,t}\setminus\mathbb{K}_s^{\alpha,t}(k,j)$ in term $II_s^t(3)''$ and over $\mathbb{K}_s^{\alpha,t}(k,j)$ in term IV_s^t. These two terms will be handled together below.

Remark 4.59. The key difference between these two terms is the range of integration: $P(G_r^{\alpha,t+1})\setminus\Omega_{k+2}$ for $II_s^t(3)'$ and $P(G_r^{\alpha,t+1})\cap\Omega_{k+2}$ for $II_s^t(3)''$. Just as for the fractional integral case, it is the latter case that is harder, requiring combinatorial facts, which we come to at the end of the argument. An additional fact that we return to in different forms, is that the set $P(G_r^{\alpha,t+1})\cap\Omega_{k+2}$ can be further decomposed by the Whitney decompositions $\{Q_j^{k+2}\}_j$.

Recalling the definition in (4.5), we next claim that

$$\sum_{(t,s)\in\mathbb{L}^\alpha} II_s^t(3)'\leq C\mathfrak{T}_*^p\int|f|^p\,\sigma. \tag{4.60}$$

Proof. The term $II_s^t(3)'$ in (4.49) is handled by observing that if $r\in\mathbb{K}_s^{\alpha,t}\setminus\mathbb{K}_s^{\alpha,t}(k,j)$ then $P(G_r^{\alpha,t+1})\nsubseteq\widetilde{Q_j^k}$ yet $P(G_r^{\alpha,t+1})\cap 2Q_j^k\neq\phi$ as the support of $L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega$ is contained in $2Q_j^k$ by (4.7). The key point here is that there are only a *bounded* number (depending on n and N) of such cubes $P(G_r^{\alpha,t+1})$. We have using the form (2.15) of (1.21) with $g=\chi_{E_j^k}$ and $Q=3Q_j^k$ the following estimate for each such cube $P(G_r^{\alpha,t+1})$:

$$\begin{aligned}\left|\int_{P(G_r^{\alpha,t+1})\setminus\Omega_{k+2}}\left(L^*\chi_{E_j^k}\omega\right)b_r\sigma(dy)\right|^p &\\ \leq\left[\int_{3Q_j^k}\left|L^*\chi_{E_j^k}\omega\right|^{p'}\sigma\right]^{p-1}&\times\int_{\widetilde{E_j^k}}\left|h_s^{\alpha,t}\right|^p\sigma\\ \leq C\mathfrak{T}_*^p\left|3Q_j^k\right|_\omega^{p-1}&\int_{\widetilde{E_j^k}}\left|h_s^{\alpha,t}\right|^p\sigma,\end{aligned} \tag{4.61}$$

where $\widetilde{E_j^k}=3Q_j^k\setminus\Omega_{k+2}$ (note that $\widetilde{E_j^k}$ is much larger than E_j^k). Thus we have

$$\begin{aligned}II_s^t(3)'&\leq C\mathfrak{T}_*^p\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}\frac{\left|E_j^k\right|_\omega}{\left|NQ_j^k\right|_\omega^p}\left|3Q_j^k\right|_\omega^{p-1}\int_{\widetilde{E_j^k}}\left|h_s^{\alpha,t}\right|^p\sigma\\ &\leq C\mathfrak{T}_*^p\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}\int_{\widetilde{E_j^k}}\left|h_s^{\alpha,t}\right|^p\sigma\leq C\mathfrak{T}_*^p\sum_{(k,j)\in\mathbb{G}^\alpha\cap\mathbb{H}_s^{\alpha,t}}\int_{\widetilde{E_j^k}}\left(|f|^p+|\mathcal{M}_\sigma^\alpha f|^p\right)\sigma,\end{aligned}$$

since $|h_s^{\alpha,t}|\leq|f|+|\mathcal{M}_\sigma^\alpha f|$. Using

$$\sum_{(t,s)\in\mathbb{L}^\alpha}\sum_{(k,j)\in\mathbb{G}^\alpha\cap\mathbb{H}_s^{\alpha,t}}\chi_{\widetilde{E_j^k}}=\sum_{all\ k,j}\chi_{\widetilde{E_j^k}}\leq 2, \tag{4.62}$$

we thus obtain that

$$\sum_{(t,s)\in\mathbb{L}^\alpha} II_s^t(3)'\leq C\mathfrak{T}_*^p\int|f|^p\,\sigma.\quad\blacksquare$$

Next we note that the term $II_s^t(1)$ in (4.47) is dominated by $II_s^t(1) \leq III_s^t + IV_s^t$, where

$$III_s^t = \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left| \sum_{r\in\mathbb{K}_s^{\alpha,t}(k,j)} \int_{P(G_r^{\alpha,t+1})\setminus\Omega_{k+2}} \left(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega\right) b_r\sigma(dy)\right|^p$$

$$IV_s^t = \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left| \sum_{r\in\mathbb{K}_s^{\alpha,t}(k,j)} \int_{P(G_r^{\alpha,t+1})\cap\Omega_{k+2}} \left(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega\right) b_r\sigma(dy)\right|^p \tag{4.63}$$

The term III_s^t includes that part of b_r supported on $P(G_r^{\alpha,t+1}) \setminus \Omega_{k+2}$, and the term IV_s^t includes that part of b_r supported on $P(G_r^{\alpha,t+1}) \cap \Omega_{k+2}$, which is the more delicate case.

Recall the definition of $\mathfrak{T}_*$ in (4.5). We claim

$$\sum_{(t,s)\in\mathbb{L}^\alpha} III_s^t \leq C\mathfrak{T}_*^p \int |f|^p \,\sigma(dy). \tag{4.64}$$

Proof. Recall that $r \in \mathbb{K}_s^{\alpha,t}(k,j)$ in the sum defining III_s^t, the definition above (4.39). We will use the definition of R_j^k in (4.16), and the fact that

$$\sum_\ell \chi_{T_\ell} \leq 3^n \tag{4.65}$$

provided $N \geq 9$. We will apply the form (2.15) of (1.21) with $g = \chi_{E_j^k\cap T_\ell}$ and Q equal to $T_\ell \cap \widetilde{Q_j^k}$, also see (4.5), but we must first address the fact that $T_\ell \cap \widetilde{Q_j^k}$ may not be a cube. Recall that $\widehat{Q_j^k}$ is the cube in the shifted grid $\mathcal{D}^\alpha$ that is selected by Q_j^k as in the definition '**Select a shifted grid'** above, and that $\widetilde{Q_j^k}$ is defined by (4.36) and satisfies $3\widetilde{Q_j^k} \subset NQ_j^k$ - see (4.37). Now T_ℓ is a triple of a cube in the grid $\mathcal{D}^\alpha$ and $\widetilde{Q_j^k}$ is a cube in $\mathcal{D}^\alpha$. Thus if $T_\ell \cap \widetilde{Q_j^k}$ is *not* a cube, then we must have $T_\ell \subset 3\widetilde{Q_j^k}$, and in this case we apply (2.15) to the cube $3\widetilde{Q_j^k}$ with $g = \chi_{E_j^k\cap T_\ell}$.

$$\begin{aligned} III_s^t &\leq \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left[\sum_\ell \sum_{r\in\mathbb{K}_s^{\alpha,t}(k,j):\ell=\ell(r)} \int_{P(G_r^{\alpha,t+1})\cap\widetilde{E_j^k}} \left|L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega(dx)\right|^{p'}\sigma\right]^{p-1} \int_{\widetilde{E_j^k}} \left|h_s^{\alpha,t}\right|^p \sigma \\ &\leq \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left[\sum_\ell \int_{T_\ell\cap\widetilde{Q_j^k}} \left|L^*\chi_{E_j^k\cap T_\ell}\omega(dx)\right|^{p'}\sigma\right]^{p-1} \int_{\widetilde{E_j^k}} \left|h_s^{\alpha,t}\right|^p \sigma \\ &\leq \mathfrak{T}_*^p \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left[\sum_\ell \left|T_\ell \cap 3\widetilde{Q_j^k}\right|_\omega\right]^{p-1} \int_{\widetilde{E_j^k}} \left|h_s^{\alpha,t}\right|^p \sigma \end{aligned}$$

$$\leq \mathfrak{T}_*^p \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} \frac{\left|E_j^k\right|_\omega}{\left|NQ_j^k\right|_\omega} \left|NQ_j^k\right|_\omega^{p-1} \int_{\widetilde{E_j^k}} \left|h_s^{\alpha,t}\right|^p \sigma$$

$$\leq C\mathfrak{T}_*^p \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} \int_{\widetilde{E_j^k}} \left|h_s^{\alpha,t}\right|^p \sigma \leq C\mathfrak{T}_*^p \sum_{(k,j)\in\mathbb{G}^\alpha\cap\mathbb{H}_s^{\alpha,t}} \int_{\widetilde{E_j^k}} (|f|^p + |\mathcal{M}_\sigma^\alpha f|^p)\,\sigma,$$

and then (4.64) follows using (4.62) again. ∎

4.7. **The Analysis of the Bad Function: Part 2.** This is the most intricate and final case. We consider each of the terms IV_s^t and $II_s^t(3)''$, which differ only in that the sum in r is over $\mathbb{K}_s^{\alpha,t}(k,j)$ in term IV_s^t and over $\mathbb{K}_s^{\alpha,t}\setminus\mathbb{K}_s^{\alpha,t}(k,j)$ in term $II_s^t(3)''$. We recall these two terms here.

$$IV_s^t = \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left| \sum_{r\in\mathbb{K}_s^{\alpha,t}(k,j)} \int_{P(G_r^{\alpha,t+1})\cap\Omega_{k+2}} \left(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega\right) b_r\sigma(dy)\right|^p,$$

$$II_s^t(3)'' = \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left| \sum_{r\in\mathbb{K}_s^{\alpha,t}\setminus\mathbb{K}_s^{\alpha,t}(k,j)} \int_{P(G_r^{\alpha,t+1})\cap\Omega_{k+2}} \left(L^*\chi_{E_j^k\cap T_{\ell(r)}}\omega\right) b_r\sigma(dy)\right|^p.$$

For these terms we will prove

$$\sum_{(t,s)\in\mathbb{L}^\alpha} IV_s^t \leq C\{\mathfrak{T}^p + \mathfrak{T}_*^p + \mathfrak{M}_*^p\} \int |f|^p\,\sigma\,, \tag{4.66}$$

$$\sum_{(t,s)\in\mathbb{L}^\alpha} II_s^t(3)'' \leq C\{\mathfrak{T}^p + \mathfrak{T}_*^p + \mathfrak{M}_*^p\} \int |f|^p\,\sigma\,. \tag{4.67}$$

where $\mathfrak{T}$, $\mathfrak{T}_*$ and $\mathfrak{M}_*$ are defined in (4.4), (4.5) and (4.3) respectively. The estimates (4.34), (4.50), (4.60), (4.64), (4.66) prove (4.12), and so complete the proof of assertion 1 of the strong type characterization in Theorem 1.24. Assertions 2 and 3 of Theorem 1.24 follow as in the weak-type Theorem 1.19. Finally, to prove assertion 4 we note that Lemma 2.22 and condition (1.27) imply (1.8), which by Theorem 1.6 yields (1.7).

The argument below handles both terms $II_s^t(3)''$ and IV_s^t equally well. We will explicitly address the latter term, and discuss the variants needed for the former term in remarks below.

4.7.1. *Whitney decompositions with shifted grids.* We now use the shifted grid $\mathcal{D}^\alpha$ in place of the dyadic grid $\mathcal{D}$ to form a Whitney decomposition of Ω_{k+2} in the spirit of (2.6). More precisely, recalling the definition of $\widetilde{Q_i^{k+2}}$ in (4.36). Note that we could have for $i<j$

$$\widetilde{Q_i^{k+2}} = \widetilde{Q_j^{k+2}}\,. \tag{4.68}$$

Let us define

$$\mathbb{S}^{\alpha,k+2} = \{i \ : \ \text{for all } j, \text{ if (4.68) holds, then } i \leq j\} . \tag{4.69}$$

It is immediate that

$$\left\{ \widetilde{Q_i^{k+2}} \right\}_{i:\vec{\alpha}\left(Q_i^{k+2}\right)=\alpha}$$

is a pairwise disjoint collection of cubes in $\mathcal{D}^\alpha$. These cubes satisfy a modified Whitney-type condition, namely

$$3Q_i^{k+2} \subset \widehat{Q_i^{k+2}} \subset \widetilde{Q_i^{k+2}} \subset NQ_i^{k+2} \subset \Omega_{k+2},$$

and hence with $R'_W = \frac{R_W}{M}$, we have

$$R'_W \widetilde{Q_i^{k+2}} \subset \Omega_{k+2},$$

$$3R'_W \widetilde{Q_i^{k+2}} \cap \Omega_{k+2}^c \neq \emptyset,$$

$$\widetilde{Q_j^k} \subsetneqq \widetilde{Q_i^\ell}, \ j \in \mathbb{S}^{\alpha,k+2}, i \in \mathbb{S}^{\alpha,\ell+2} \quad \text{implies} \quad k > \ell.$$

We now *complete* this collection to a pairwise disjoint Whitney covering of Ω_{k+2} by cubes B_i^{k+2} in $\mathcal{D}^\alpha$ satisfying

$$R'_W B_i^{k+2} \subset \Omega_{k+2},$$

$$3R'_W B_i^{k+2} \cap \Omega_{k+2}^c \neq \emptyset,$$

and the following analogue of the nested property in (2.6):

$$B_j^k \subsetneqq B_i^\ell \text{ implies } k > \ell. \tag{4.70}$$

Indeed, we simply use the decomposition in (2.6) with $\mathcal{D}$ replaced by $\mathcal{D}^\alpha$ and R_W replaced by R'_W. In particular we have decomposed

$$\Omega_{k+2} = \dot{\bigcup_i} B_i^{k+2}$$

into a Whitney decomposition of pairwise disjoint cubes B_i^{k+2} in $\mathcal{D}^\alpha$ that include among them all of the cubes $\widetilde{Q_i^{k+2}}$ with $i \in \mathbb{S}^{\alpha,k+2}$, namely:

$$\widetilde{Q_i^{k+2}} = B_\ell^{k+2} \text{ for some } \ell \text{ if } \alpha = \vec{\alpha}\left(Q_i^{k+2}\right) \text{ and (4.36) holds.} \tag{4.71}$$

Note that the set of indices i arising in the decomposition of Ω_{k+2} into cubes B_i^{k+2} is *not* the same as the set of indices i arising in the decomposition of Ω_{k+2} into cubes Q_i^{k+2}, but this should not cause confusion.

Define

$$S_\ell = \bigcup \left\{ P(G_r^{\alpha,t+1}) : 3P(G_r^{\alpha,t+1}) \subset T_\ell \right\},$$

This is a union of pairwise disjoint cubes. Note that $S_\ell \subset T_\ell$ and now split the term IV_s^t into two pieces as follows:

$$IV_s^t = \sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} R_j^k \left| \sum_\ell \sum_{i\in \mathcal{I}_s^t : B_i^{k+2}\subset T_\ell} \int_{S_\ell \cap Q_i^{k+2}} \left(L^* \chi_{E_j^k \cap T_\ell}\omega \right) b_r \sigma \right|^p \tag{4.72}$$
$$+ \sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} R_j^k \left| \sum_\ell \sum_{i\in \mathcal{J}_s^t : B_i^{k+2}\subset T_\ell} \int_{S_\ell \cap Q_i^{k+2}} \left(L^* \chi_{E_j^k \cap T_\ell}\omega \right) b_r \sigma \right|^p$$
$$= IV_s^t(1) + IV_s^t(2),$$

where

$$\mathcal{I}_s^t = \left\{ i : \mathsf{A}_i^{k+2} > 2^{t+2} \right\} \text{ and } \mathcal{J}_s^t = \left\{ i : \mathsf{A}_i^{k+2} \le 2^{t+2} \right\}, \tag{4.73}$$

and where

$$\mathsf{A}_i^{k+2} = \frac{1}{\left| B_i^{k+2} \right|_\sigma} \int_{B_i^{k+2}} |f| \, d\sigma \tag{4.74}$$

denotes the σ-average of $|f|$ on the cube B_i^{k+2}. Thus $IV(1)$ corresponds to the case where the averages are 'big' and $IV(2)$ where the averages are 'small'. The analysis of $IV_s^t(1)$ in (4.72) is the hard case, taken up later.

4.7.2. *Replace bad functions by averages.* The first task in the analysis of these terms will be to replace part of the 'bad functions' b_r by their averages over B_i^{k+2}, or more exactly the averages A_i^{k+2}. We again appeal to the Hölder continuity of $L^* \chi_{E_j^k \cap T_\ell}\omega$. By construction, $3B_i^{k+2}$ does not meet E_j^k, so that Lemma 2.16 applies. We have for some constant c_i^{k+2} satisfying $\left| c_i^{k+2} \right| \le 2$

(4.75)
$$\left| \int_{S_\ell \cap B_i^{k+2}} \left(L^* \chi_{E_j^k \cap T_\ell}\omega \right) b_r \sigma - \left\{ c_i^{k+2} \int_{B_i^{k+2}} \left(L^* \chi_{E_j^k \cap T_\ell}\omega \right) \sigma \right\} \mathsf{A}_i^{k+2} \right|$$
$$\le C\mathbf{P}\left(B_i^{k+2}, \chi_{E_j^k \cap T_\ell}\omega \right) \int_{B_i^{k+2}} |f| \, \sigma .$$

Indeed, if z_i^{k+2} is the center of the cube B_i^{k+2}, we have

$$\int_{S_\ell \cap B_i^{k+2}} \left(L^* \chi_{E_j^k \cap T_\ell}\omega \right) b_r \sigma$$
$$= L^* \chi_{E_j^k \cap T_\ell}\omega \left(z_i^{k+2} \right) \int_{B_i^{k+2}} \chi_{S_\ell} b_r \sigma + O\left\{ \mathbf{P}\left(B_i^{k+2}, \chi_{E_j^k \cap T_\ell}\omega \right) \int_{B_i^{k+2}} |b_r| \, \sigma \right\}$$
$$= \left\{ \int_{B_i^{k+2}} \left(L^* \chi_{E_j^k \cap T_\ell}\omega \right) \sigma \right\} \frac{1}{\left| B_i^{k+2} \right|_\sigma} \int_{S_\ell \cap B_i^{k+2}} b_r \sigma$$

$$+O\left\{\mathbf{P}\left(B_i^{k+2},\chi_{E_j^k\cap T_\ell}\omega\right)\int_{B_i^{k+2}}|b_r|\,\sigma\right\}.$$

Now, the functions b_r are given in (4.35), and by construction, we note that

$$\frac{1}{\left|B_i^{k+2}\right|_\sigma}\left|\int_{S_\ell\cap B_i^{k+2}}b_r\sigma\right|\leq\frac{2}{\left|B_i^{k+2}\right|_\sigma}\int_{B_i^{k+2}}|f|\,\sigma=2\mathsf{A}_i^{k+2},$$

so that with

$$c_i^{k+2}=\frac{1}{\mathsf{A}_i^{k+2}}\frac{1}{\left|B_i^{k+2}\right|_\sigma}\int_{S_\ell\cap B_i^{k+2}}b_r\sigma,$$

we have $\left|c_i^{k+2}\right|\leq 2$ and

$$\int_{S_\ell\cap B_i^{k+2}}\left(L^*\chi_{E_j^k\cap T_\ell}\omega\right)b_r\sigma=\left\{c_i^{k+2}\int_{B_i^{k+2}}\left(L^*\chi_{E_j^k\cap T_\ell}\omega\right)\sigma\right\}\mathsf{A}_i^{k+2}$$
$$+O\left\{\mathbf{P}\left(B_i^{k+2},\chi_{E_j^k\cap T_\ell}\omega\right)\int_{B_i^{k+2}}|b_r|\,\sigma\right\}.$$

We apply (4.75) to be able to write

$$\begin{aligned}
IV_s^t(2)&=\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\left|\sum_\ell\sum_{i\in\mathcal{J}_s^t:B_i^{k+2}\subset T_\ell}\int_{S_\ell\cap B_i^{k+2}}\left(L^*\chi_{E_j^k\cap T_\ell}\omega\right)b_r\sigma\right|^p\\
&\leq\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\left|\sum_\ell\sum_{i\in\mathcal{J}_s^t:B_i^{k+2}\subset T_\ell}\left[\int_{B_i^{k+2}}\left(L^*\chi_{E_j^k\cap T_\ell}\omega\right)\sigma\right]c_i^{k+2}\mathsf{A}_i^{k+2}\right|^p\\
&\quad+\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\left|\sum_\ell\sum_{i\in\mathcal{J}_s^t:B_i^{k+2}\subset T_\ell\cap 3Q_j^k}\mathbf{P}\left(Q_i^{k+2},\chi_{E_j^k\cap T_\ell}\omega\right)\int_{Q_i^{k+2}}|f|\,\sigma\right|^p \qquad (4.76)\\
&=V_s^t(1)+V_s^t(2).
\end{aligned}$$

We claim that

$$\sum_{(t,s)\in\mathbb{L}^\alpha}V_s^t(1)\leq C\mathfrak{T}^p\|f\|_{L^p(\sigma)}^p. \qquad (4.77)$$

Proof. We estimate $V_s^t(1)$ by

$$V_s^t(1)=\sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}}R_j^k\left|\sum_\ell\int_{E_j^k\cap T_\ell}L\left(\sum_{i\in\mathcal{J}_s^t:B_i^{k+2}\subset T_\ell}c_i^{k+2}\mathsf{A}_i^{k+2}\chi_{B_i^{k+2}}\sigma\right)\omega\right|^p$$

$$\leq 2^{p(t+3)} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left(\sum_\ell \int_{E_j^k \cap T_\ell} T_\natural \left(\sum_{i\in\mathcal{J}_s^t : B_i^{k+2}\subset T_\ell} c_i^{k+2} \frac{\mathsf{A}_i^{k+2}}{2^{t+2}} \chi_{B_i^{k+2}}\sigma \right) \omega \right)^p$$

$$\leq 2^{p(t+3)} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left(\int_{3Q_j^k} T_\natural \left(\chi_{G_s^{\alpha,t}} h_{t,s,\ell}\sigma \right) \omega \right)^p$$

$$\leq 2^{p(t+3)} \int \mathcal{M}_\omega \left(T_\natural \left(\chi_{G_s^{\alpha,t}} h_{t,s,\ell}\sigma \right) \right)^p \omega$$

$$\leq 2^{p(t+3)} \int T_\natural \left(\chi_{G_s^{\alpha,t}} h_{t,s,\ell}\sigma \right)^p \omega \leq \mathfrak{T}^p 2^{p(t+3)} \left| G_s^{\alpha,t} \right|_\sigma$$

by the testing condition (1.27) on $T_\natural$, see (4.4). Here the function

$$h_{t,s,\ell} = \sum_{i\in\mathcal{J}_s^t : B_i^{k+2}\subset T_\ell} c_i^{k+2} \frac{\mathsf{A}_i^{k+2}}{2^{t+2}} \chi_{B_i^{k+2}}$$

has modulus bounded by 1.

We see that (4.18) then implies (4.77). ∎

4.7.3. *The bound for* $V(2)$. We next claim that

$$\sum_{(t,s)\in\mathbb{L}^\alpha} V_s^t(2) \leq C\mathfrak{M}_*^p \|f\|_{L^p(\sigma)}^p . \tag{4.78}$$

Here, $\mathfrak{M}_*$ is defined in (4.3).

Proof. The estimate for term $V_s^t(2)$ is similar to that of $II_s^t(2)$ above, see (4.50). We define

$$\mathbf{P}_j^k(\mu) \equiv \sum_\ell \sum_{i\in\mathcal{J}_s^t : B_i^{k+2}\subset T_\ell\cap 3Q_j^k} \mathbf{P}\left(B_i^{k+2}, \chi_{E_j^k\cap T_\ell}\mu \right) \chi_{B_i^{k+2}} .$$

We observe that this operator satisfies $\sum_{k,j} \mathbf{P}_j^k(\mu) \leq C\mathcal{M}(\mu)$ due to the fact that the sets E_j^k are pairwise disjoint and the cubes T_ℓ have bounded overlaps. See the discussion following (4.39).

With this notation, the summands in the definition of $V_s^t(2)$, as given in (4.76), are

$$\sum_\ell \sum_{i\in\mathcal{J}_s^t : B_i^{k+2}\subset T_\ell\cap 3Q_j^k} \mathbf{P}\left(B_i^{k+2}, \chi_{E_j^k\cap T_\ell}\omega \right) \left(\int_{B_i^{k+2}} \sigma \right) \left\{ \frac{1}{\left| B_i^{k+2} \right|_\sigma} \int_{B_i^{k+2}} |f|\,\sigma \right\}$$

$$\leq 2^{t+2} \int \mathbf{P}_j^k(\omega)\,\sigma \qquad (\text{since } i\in\mathcal{J}_s^t)$$

$$\leq 2^{t+2} \int_{Q_j^k} (\mathbf{P}_j^k)^*(\sigma)\,\omega . \tag{4.79}$$

We can restrict the integration to Q_j^k due to the definition of $\mathbf{P}_j^k$. Our assertation is that we have the inequality

$$\left\| \chi_{G_s^{\alpha,t}} \sum_{k,j} (\mathbf{P}_j^k)^* (\chi_{G_s^{\alpha,t}} |f|\sigma) \right\|_{L^p(\omega)} \leq C\mathfrak{M}_* \|\chi_{G_s^{\alpha,t}} f\|_{L^p(\sigma)} . \tag{4.80}$$

Indeed, this follows from the same reasoning as the proof of (4.53).

We then have from (4.76) and (4.79)

$$\begin{aligned}
\sum_{(t,s)\in\mathbb{L}^\alpha} V_s^t(2) &\leq C \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \Big| \int_{Q_j^k} (\mathbf{P}_j^k)^*(\sigma)\,\omega \Big|^p \\
&\leq C \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} |E_{j,k}|_\omega \left| |NQ_j^k|_\omega^{-1} \int_{Q_j^k} (\mathbf{P}_j^k)^*(\sigma)\,\omega \right|^p \\
&\leq C \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \int \Big[\mathcal{M}_\omega \Big(\chi_{G_s^{\alpha,t}} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} (\mathbf{P}_j^k)^*(\chi_{G_s^{\alpha,t}}\sigma) \Big) \Big]^p \omega \\
&\leq C \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \int \left[\chi_{G_s^{\alpha,t}} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} (\mathbf{P}_j^k)^*(\chi_{G_s^{\alpha,t}}\sigma) \right]^p \omega \\
&\leq C\mathfrak{M}_*^p \sum_{(t,s)\in\mathbb{L}^\alpha} 2^{pt} \left| G_s^{\alpha,t} \right|_\sigma \leq C\mathfrak{M}_*^p \int |f|^p \,\sigma \,.
\end{aligned}$$

In last line we are using (4.80) and (4.18). ∎

4.7.4. *The bound for IV and VI(2).* To estimate the first term $IV_s^t(1)$ in (4.72), we again apply (4.75) to be able to write

$$\begin{aligned}
IV_s^t(1) &\leq C \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left[\sum_\ell \sum_{i\in\mathcal{I}_s^t : B_i^{k+2}\subset T_\ell} \left[\int_{B_i^{k+2}} \left| L^* \chi_{E_j^k\cap T_\ell}\omega \right| \sigma \right] \mathsf{A}_i^{k+2} \right]^p && (4.81)\\
&\quad + C \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} R_j^k \left[\sum_\ell \sum_{i\in\mathcal{I}_s^t : B_i^{k+2}\subset T_\ell\cap\widetilde{Q_j^k}} \mathbf{P}\left(B_i^{k+2}, \chi_{E_j^k\cap T_\ell}\omega \right) \int_{B_i^{k+2}} |f|\,\sigma \right]^p && (4.82)\\
&= VI_s^t(1) + VI_s^t(2).
\end{aligned}$$

We comment that we are able to replace the averages of the bad function by $2\mathsf{A}_i^{k+2}$ since $i \in \mathcal{I}_s^t$ in this case, see (4.73), and note that membership in $\mathcal{I}_s^t$ implies that the average of $|b_r|$ is dominated by the average of $|f|$ over the cube B_i^{k+2}.

We claim that

$$\sum_{(t,s)\in\mathbb{L}^\alpha} VI_s^t(2) \le C\mathfrak{M}_*^p \|f\|_{L^p(\sigma)}^p \tag{4.83}$$

Proof. The term $VI_s^t(2)$ can be handled the same way as the term $II_s^t(2)$, see (4.50), and the term $V_s^t(2)$, see (4.78). We omit the details. ∎

4.7.5. *The bound for $VI(1)$.* Our final estimate in the proof of (4.66) is for the term $VI_s^t(1)$. We claim that

$$\sum_{(t,s)\in\mathbb{L}^\alpha} VI_s^t(1) \le C\mathfrak{T}_*^p \|f\|_{L^p(\sigma)}^p . \tag{4.84}$$

Proof. This proof will require combinatorial facts related to the principal cubes, and the definition of the collection $\mathbb{G}$ in (4.28). Also essential is the implementation of the shifted dyadic grids. We detail the arguments below.

Remark 4.85. A difference in the arguments for IV_s^t and $II_s^t(3)''$ in (4.66) and (4.67) arises. Recall that for the term IV_s^t every cube B_ℓ^{k+2} that arises is contained in the associated cube $\widetilde{Q_j^k}$ since $r \in \mathbb{K}_s^{\alpha,t}(k,j)$—see (4.38) and the definition of IV_s^t. We claim that the same is true for the term $II_s^t(3)''$, i.e. every cube B_ℓ^{k+2} that arises is contained in the associated cube $\widetilde{Q_j^k}$. To see this, note that Ω_{k+2} decomposes as a pairwise disjoint union of cubes B_i^{k+2} and thus we have

$$\int_{P(G_r^{\alpha,t+1})\cap\Omega_{k+2}} \left(L^*\chi_{E_j^k\cap 3P\left(G_r^{\alpha,t+1}\right)}\omega\right) b_r\sigma = \sum_{i:B_i^{k+2}\cap\widetilde{Q_j^k}\neq\emptyset} \int_{B_i^{k+2}} \left(L^*\chi_{E_j^k\cap 3P\left(G_r^{\alpha,t+1}\right)}\omega\right) b_r\sigma,$$

since the support of $L^*\chi_{E_j^k\cap 3P\left(G_r^{\alpha,t+1}\right)}\omega$ is contained in $2Q_j^k \subset \widehat{Q_j^k} \subset \widetilde{Q_j^k}$ by (4.7). Since both B_i^{k+2} and $\widetilde{Q_j^k}$ lie in the grid $\mathcal{D}^\alpha$, one of these cubes is contained in the other. Now B_i^{k+2} cannot *strictly* contain $\widetilde{Q_j^k}$ since $\widetilde{Q_j^k} = B_\ell^k$ for some ℓ and the cubes $\{B_j^k\}_{k,j}$ satisfy the nested property (4.70). It follows that we must have

$$B_i^{k+2} \subset \widetilde{Q_j^k}. \tag{4.86}$$

We now continue giving the proof the estimate for IV_s^t in (4.66), noting that the remaining arguments apply also to (4.67) because of (4.86).

We first estimate the sum in i inside term $VI_s^t(1)$. Recall that the sum in i is over those i such that $B_i^{k+2} \subset T_\ell$ for some ℓ where $\{T_\ell\}_\ell$ is the set of maximal cubes in the collection $\{3P(G_r^{\alpha,t+1}) : r \in \mathbb{K}_s^{\alpha,t}(k,j)\}$. See the discussion at (4.39), and (4.65). It is also important to note that since $L^*\chi_{E_j^k\cap T_\ell}\omega$ is supported in

$3Q_j^k \subset \widehat{Q_j^k} \subset \widetilde{Q_j^k}$, the sum in i deriving from term IV_s^t is also restricted to those i such that $B_i^{k+2} \subset \widetilde{Q_j^k}$, while the sum in i deriving from term $II_s^t(3)''$ is as well by (4.86). We have

$$\begin{aligned}
&\left|\sum_i \left[\int_{B_i^{k+2}} \left|L^* \chi_{E_j^k \cap T_\ell}\omega\right| \sigma\right] \mathsf{A}_i^{k+2}\right|^p \\
&\quad \le \sum_i \left|B_i^{k+2}\right|_\sigma (\mathsf{A}_i^{k+2})^p \left[\sum_i \left|B_i^{k+2}\right|_\sigma^{1-p'} \left[\int_{B_i^{k+2}} \left|L^* \chi_{E_j^k \cap T_\ell}\omega\right| \sigma\right]^{p'}\right]^{p-1} \\
&\quad \le \sum_i \left|B_i^{k+2}\right|_\sigma (\mathsf{A}_i^{k+2})^p \left[\sum_i \int_{B_i^{k+2}} \left|L^* \chi_{E_j^k \cap T_\ell}\omega\right|^{p'} \sigma\right]^{p-1} \\
&\quad \le C\mathfrak{T}_*^p \sum_i \left|B_i^{k+2}\right|_\sigma (\mathsf{A}_i^{k+2})^p \left[\sum_\ell |T_\ell|_\omega\right]^{p-1} \\
&\quad \le C\mathfrak{T}_*^p \sum_i \left|B_i^{k+2}\right|_\sigma (\mathsf{A}_i^{k+2})^p \left|NQ_j^k\right|_\omega^{p-1},
\end{aligned}$$

where the second to last inequality uses the form (2.15) of (1.21) with $g = \chi_{E_j^k \cap T_\ell}$ and $Q = T_\ell$. With this we obtain,

$$VI_s^t(1) \le C\beta^{-1}\mathfrak{T}_*^p \sum_{(k,j)\in \mathbb{I}_s^{\alpha,t}} R_j^k \sum_{i \in \mathcal{I}_s^t} \left|B_i^{k+2}\right|_\sigma (\mathsf{A}_i^{k+2})^p \left|NQ_j^k\right|_\omega^{p-1}, \tag{4.87}$$

Here we have also used the fact that a given cube Q occurs as B_i^{k+2} at most a bounded number of times in the sum for $VI_s^t(1)$. This property, listed below, is first of some properties that we should formalize at this point. ∎

4.7.6. *The combinatorial arguments.*

Bounded Occurrence of Cubes: A given cube Q can occur only a finite number of times as B_i^{k+2} in (4.87). Specifically, let $(k_1, i_1), \dots, (k_M, i_M) \in \mathbb{G}$, as defined in (4.10), be such that $Q = B_{i_\sigma}^{k_\sigma+2}$ for $1 \le \sigma \le M$. It follows that $M < C\beta^{-1}$, where β is the small constant chosen in the definition of $\mathbb{G}$. The constant C depends only on dimension.

Proof. The Whitney structure, see (2.6), is decisive here. First we show that a given Q can occur only a finite number of times as Q_i^{k+2}. The distinction between this claim and the property we are proving is that the cubes $\{B_i^{k+2}\}_i$ are the Whitney decomposition of Ω_{k+2} constructed in § 4.7.1.

Suppose that $(k_1, i_1), \dots, (k_M, I_M) \in \mathbb{G}$ are such that $Q = Q_{i_\sigma}^{k_\sigma+2}$ for $1 \le \sigma \le M$. Let $Q_{j_\sigma}^{k_\sigma}$ be such that $Q \subset 3Q_{j_\sigma}^{k_\sigma}$, with the indices (k_σ, j_σ) being distinct. Observe that the finite overlap property in (2.6) then gives us the observation that a single integer k can occur only a bounded number of times among the $k_1, \ldots, k_M$.

After a relabelling, we can assume that all the k_σ for $1 \le \sigma \le M'$ are distinct, listed in increasing order, and the number of k_σ satisfies $CM' > M$. The nested property of (2.6) assures us that Q is an element of the Whitney decomposition of Ω_k for *all* $k_1 \le k \le k_{M'}$.

Remark 4.88. Note that the k_σ are not necessarily consecutive since we require that $(k_\sigma, j_\sigma) \in \mathbb{G}^\alpha$. Nevertheless, the cube Q *does* occur among the Q_i^{k+2} for any k that lies between k_σ and $k_{\sigma+1}$. These latter occurrences of Q may be unbounded, but we are only concerned with bounding those for which $(k_\sigma, j_\sigma) \in \mathbb{G}^\alpha$, and it is these occurrences that our argument is treating.

Thus for $2 \le \sigma \le M'$, *both* Q and $Q_{i_\sigma}^{k_\sigma}$ are members of the Whitney decomposition of the open set Ω_{k_σ}. By the Whitney condition, we have $R_W Q_{j_\sigma}^{k_\sigma} \subset \Omega_k$ but $3R_W Q \not\subset \Omega_{k_\sigma}$, whence $3R_W Q \not\subset R_W Q_{j_\sigma}^{k_\sigma}$. Since we are free to take $R_W \ge 4$, this last conclusion shows that the number of possible locations for the cubes $Q_{j_\sigma}^{k_\sigma}$ is bounded by a constant depending only on dimension.

Apply the pigeonhole principle to the locations of the $Q_{j_\sigma}^{k_\sigma}$. After a relabelling, we can argue under the assumption that all $Q_{j_\sigma}^{k_\sigma}$ equal the same cube Q' for all choices of $1 \le \sigma \le M''$ where $CM'' > M$. There is another condition that the indices (k_σ, j_σ) must satisfy: They are members of $\mathbb{G}$, as given in (4.10). In particular we have $|E_{j_\sigma}^{k_\sigma}|_\omega \ge \beta |NQ'|_\omega$ where N is as in Remark 2.7. The k_σ are distinct, and the sets $E_{j_\sigma}^{k_\sigma} \subset Q'$ are pairwise disjoint, hence $M''\beta|NQ'|_\omega \le |Q'|_\omega$ implies $M'' \le \beta^{-1}$. Our proof of the bounded occurrence of Q as one of the Q_i^{k+2} is complete.

Since $Q_i^{k+2} \subset 3Q_i^{k+2} \subset \widehat{Q_i^{k+2}} \subset \widetilde{Q_i^{k+2}} = B_i^{k+2} \subset NQ_i^{k+2}$, the above argument shows that there are only a bounded number of times that a given cube Q can arise as B_i^{k+2} with $i \in \mathbb{S}^{\alpha,k+2}$, the latter collection defined in (4.69). Finally, to deal with those B_i^{k+2} that do not arise as some $\widetilde{Q_i^{k+2}}$, i.e. $i \notin \mathbb{S}^{\alpha,k+2}$, we simply apply the entire Whitney argument above using that the B_j^k are defined as in (2.6) but with $\mathcal{D}^\alpha$ in place of $\mathcal{D}$ and R'_W in place of R_W. ∎

Definition 4.89. We say that a cube B_i^{k+2} satisfying the defining condition in $VI_s^t(1)$, namely

$$\begin{aligned}
&\text{there is } (k,j) \in \mathbb{I}_s^{\alpha,t} = \mathbb{G}^\alpha \cap \mathbb{H}_s^{\alpha,t} \text{ such that} \\
&B_i^{k+2} \subset \widetilde{Q_j^k} \text{ and} \\
&B_i^{k+2} \subset \text{ some } P\left(G_r^{\alpha,t+1}\right) \subset G_s^{\alpha,t} \text{ satisfying } \mathsf{A}_i^{k+2} > 2^{t+2},
\end{aligned}$$

is a *final type* cube for the pair $(t,s) \in \mathbb{L}^\alpha$ generated from Q_j^k.

We can now complete the bound for $\sum_{(t,s)\in\mathbb{L}^\alpha} VI_s^t(1)$. The collection of cubes

$$\mathcal{F} \equiv \left\{ B_i^{k+2} : B_i^{k+2} \text{ is a } \textit{final type} \text{ cube generated from some } Q_j^k \right. \\ \left. \text{with } (k,j) \in \mathbb{I}_s^{\alpha,t} \text{ for some pair } (t,s) \in \mathbb{L}^\alpha \right\}.$$

satisfies the following three properties:

Property 1: $\mathcal{F}$ is a nested grid in the sense that given any two *distinct* cubes in $\mathcal{F}$, either one is strictly contained in the other, or they are disjoint (ignoring boundaries).

Property 2: If B_i^{k+2} and $B_{i'}^{k'+2}$ are two *distinct* cubes in $\mathcal{F}$ with $Q_{i'}^{k'+2} \subsetneqq Q_i^{k+2}$, and k, k' have the same parity, then

$$\mathsf{A}_{i'}^{k'+2} > 2\mathsf{A}_i^{k+2}.$$

Property 3: A given cube B_i^{k+2} can occur at most a bounded number of times in the grid $\mathcal{F}$.

Proof of Properties 1, 2 and 3. Property 1 is obvious from the properties of the dyadic shifted grid $\mathcal{D}^\alpha$. Property 3 follows from the 'Bounded Occurrence of Cubes' noted above. So we turn to Property 2. It is this Property that prompted the use of the shifted dyadic grids.

Indeed, since $B_{i'}^{k'+2} \subsetneqq B_i^{k+2}$, it follows from the nested property (4.70) that $k' > k$. By Definition 4.89 there are cubes $Q_{j'}^{k'}$ and Q_j^k satisfying

$$B_{i'}^{k'+2} \subset \widetilde{Q_{j'}^{k'}} \qquad \text{and} \qquad B_i^{k+2} \subset \widetilde{Q_j^k},$$

and also cubes $G_{s'}^{\alpha,t'} \subset G_s^{\alpha,t}$ such that $(k',j') \in \mathbb{I}_{s'}^{\alpha,t'}$ and $(k,j) \in \mathbb{I}_s^{\alpha,t}$ with $(t',s'), (t,s) \in \mathbb{L}^\alpha$, so that in particular,

$$\widetilde{Q_{j'}^{k'}} \subset G_{s'}^{\alpha,t'} \text{ and } \widetilde{Q_j^k} \subset G_s^{\alpha,t}.$$

Now $k' \geq k+2$ and in the extreme case where $k' = k+2$, it follows from (4.71) that the $\mathcal{D}^\alpha$-cube $\widetilde{Q_{j'}^{k'}}$ is one of the cubes B_ℓ^{k+2}, so in fact it must be B_i^{k+2} since $B_{i'}^{k'+2} \subset B_i^{k+2}$. Thus we have

$$B_{i'}^{k'+2} \subset \widetilde{Q_{j'}^{k'}} = B_i^{k+2}.$$

In the general case $k' \geq k+2$ we have instead

$$B_{i'}^{k'+2} \subset \widetilde{Q_{j'}^{k'}} \subset B_i^{k+2}.$$

Now $\mathsf{A}_i^{k+2} > 2^{t+2}$ by Definition 4.89, and so there is $t_0 \geq t+2$ determined by the condition

$$2^{t_0} < \mathsf{A}_i^{k+2} \leq 2^{t_0+1}, \tag{4.90}$$

and also s_0 such that

$$B_i^{k+2} \subset G_{s_0}^{\alpha,t_0} \subset G_s^{\alpha,t}.$$

Combining inclusions we have

$$\widetilde{Q_{j'}^{k'}} \subset B_i^{k+2} \subset G_{s_0}^{\alpha,t_0},$$

and since $(k',j') \in \mathbb{I}_{s'}^{\alpha,t'}$, we obtain $G_{s'}^{\alpha,t'} \subset G_{s_0}^{\alpha,t_0}$. Since $(t',s') \in \mathbb{L}^\alpha$ is a principal label, we have the key property that

$$t' \geq t_0. \tag{4.91}$$

Thus using (4.91) and (4.90) we obtain

$$\mathsf{A}_{i'}^{k'+2} > 2^{t'+2} \geq 2^{t_0+2} \geq 2\mathsf{A}_i^{k+2},$$

which is Property 2. ∎

Proof of (4.84). Now for $Q = B_i^{k+2} \in \mathcal{F}$ set

$$\mathsf{A}(Q) = \frac{1}{|Q|_\sigma} \int_Q |f|\,\sigma = \mathsf{A}_i^{k+2} = \frac{1}{\left|B_i^{k+2}\right|_\sigma} \int_{B_i^{k+2}} |f|\,\sigma.$$

With the above three properties we can continue from (4.87) to estimate as follows. Use the trivial inequality $R_j^k \left|NQ_j^k\right|_\omega^{p-1} \leq 1$. In the display below by $\sum_i^*$ we mean the sum over i such that B_i^{k+2} is contained in some $P\left(G_r^{\alpha,t+1}\right) \subset G_s^{\alpha,t}$, and also in some $\widetilde{Q_j^k}$ with $(k,j) \in \mathbb{I}_s^{\alpha,t}$, and satisfying $\mathsf{A}_i^{k+2} > 2^{t+2}$.

$$\begin{aligned}
\sum_{(t,s)\in\mathbb{L}^\alpha} VI_s^t(1) &\leq C\mathfrak{T}_*^p \sum_{(t,s)\in\mathbb{L}^\alpha} \sum_{(k,j)\in\mathbb{I}_s^{\alpha,t}} \sum_i^* \left|B_i^{k+2}\right|_\sigma (\mathsf{A}_i^{k+2})^p \\
&= C\mathfrak{T}_*^p \sum_{Q\in\mathcal{F}} |Q|_\sigma \mathsf{A}(Q)^p = \sum_{Q\in\mathcal{F}} |Q|_\sigma \left[\frac{1}{|Q|_\sigma}\int_Q |f|\,\sigma\right]^p \\
&= C\mathfrak{T}_*^p \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{F}} \chi_Q(x) \left[\frac{1}{|Q|_\sigma}\int_Q |f|\,\sigma\right]^p d\sigma(x) \\
&\leq C\mathfrak{T}_*^p \int_{\mathbb{R}^n} \sup_{x\in Q:Q\in\mathcal{F}} \left[\frac{1}{|Q|_\sigma}\int_Q |f|\,\sigma\right]^p d\sigma(x) \\
&\leq C\mathfrak{T}_*^p \int_{\mathbb{R}^n} \mathcal{M}_\sigma^\alpha f(x)^p\,\sigma(dx) \leq C\mathfrak{T}_*^p \int_{\mathbb{R}^n} |f(x)|^p\,d\sigma(x),
\end{aligned}$$

where the second to last line follows since for fixed $x \in \mathbb{R}^n$, the sum

$$\sum_{Q\in\mathcal{F}} \chi_Q(x) \left[\frac{1}{|Q|_\sigma}\int_Q |f|\,\sigma\right]^p$$

is supergeometric by properties 1, 2 and 3 above, i.e. for any two distinct cubes Q and Q' in $\mathcal{F}$ each containing x, the ratio of the corresponding values is bounded away from 1, more precisely,

$$\frac{\left[\frac{1}{|Q|_\sigma}\int_Q |f|\,\sigma\right]^p}{\left[\frac{1}{|Q'|_\sigma}\int_{Q'} |f|\,\sigma\right]^p} \notin [2^{-p}, 2^p).$$

This completes the proof of (4.84). ∎

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Michael T. Lacey
School of Mathematics
Georgia Institute of Technology
Atlanta GA 30332
E-mail address: lacey@math.gatech.edu

Eric T. Sawyer
Department of Mathematics & Statistics
McMaster University
1280 Main Street West, Hamilton
Ontario, Canada L8S 4K1
E-mail address: sawyer@mcmaster.ca

Ignacio Uriarte-Tuero
Department of Mathematics
Michigan State University
East Lansing MI
E-mail address: ignacio@math.msu.edu