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NORM CONVOLUTION INEQUALITIES IN LEBESGUE SPACES

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ABSTRACT. We obtain upper and lower estimates of the (p, q) norm of the convolution operator. The upper estimate sharpens the Young-type inequalities due to O'Neil and Stepanov.

1. INTRODUCTION

Let $1 \leqslant p \leqslant \infty$, $L_p \equiv L_p(\mathbb{R})$ and let the convolution operator be given by

$$(Af)(x) = (K * f)(x) = \int_{\mathbb{R}} K(x-y)f(y)dy. \tag{1.1}$$

The Young convolution inequality

$$\|A\|_{L_p \to L_q} \leqslant \|K\|_{L_r}, \qquad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}, \qquad 1 \leqslant p \leqslant q \leqslant \infty,$$

plays a very important role both in Harmonic Analysis and PDE theory. We note however that this estimate does not allow us to deal with power kernels such as $K(x) = |x|^{-\gamma}$, $\gamma > 0$.

Young's estimates were generalized by O'Neil [ON] who showed that for $1 < p < q < \infty$ and $1/r = 1 - 1/p + 1/q$

$$\|A\|_{L_p \to L_q} \leqslant C \|K\|_{L_{r,\infty}} := C \sup_{t>0} \ t^{1/r} K^*(t), \tag{1.2}$$

where $K^*(t) = \inf \left\{\sigma : \mu\{x \in \Omega : |f(x)| > \sigma\} \leqslant t\right\}$ is the decreasing rearrangement of K. In particular, this gives the Hardy-Littlewood fractional integration theorem, which corresponds to the model case of convolution by $K(x) = |x|^{-1/r}$.

Another extension of Young's inequality was proved by Stepanov [Stp] using the Wiener amalgam space $W\big(L_{r,\infty}[0,1], l_{r,\infty}(\mathbb{Z})\big)$ (see e.g. [Fe]): for $1 < p < q < +\infty$ and $1/r = 1 - 1/p + 1/q$ one has

$$\|A\|_{L_p \to L_q} \leqslant C \|K\|_{W(L_{r,\infty}[0,1],\, l_{r,\infty}(\mathbb{Z}))}, \tag{1.3}$$

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where

$$\|K\|_{W(L_{r,\infty}[0,1],\, l_{r,\infty}(\mathbb{Z}))} := \Big\| \|\tilde{K}\|_{L_{r,\infty}[0,1]} \Big\|_{l_{r,\infty}(\mathbb{Z})} := \sup_{n\in\mathbb{N}} \; n^{1/r} \left(\sup_{0\leqslant t\leqslant 1} \; t^{1/r}\tilde{K}^*(t,\cdot) \right)^*_n ,$$

and $\tilde{K}(x,m) := K(m+x), \quad m \in \mathbb{Z}, \; x \in [0,1]$. In [Stp] it was also shown that inequalities (1.2) and (1.3) are not comparable.

The aim of the present paper is to give upper and lower estimates of $\|A\|_{L_p\to L_q}$ so that the upper estimate improves both (1.2) and (1.3). To formulate our main results, we will need the following definitions.

Let I be an interval with $|I| = d$. Then $T_I = \{I + kd\}_{k\in\mathbb{Z}}$ is a partition of $\mathbb{R}$, i.e., $\mathbb{R} = \bigcup_{k\in\mathbb{Z}}(I + kd)$. We define two collections of sets $\mathfrak{L}(I) \subset \mathfrak{U}(I)$:

$$\mathfrak{L}(I) = \left\{ e \;:\; e = \bigcup_{k=1}^{m} \big([a,b] + kd\big), \quad [a,b] \subseteq I, \quad m \in \mathbb{N} \right\} \tag{1.4}$$

and

$$\mathfrak{U}(I) = \left\{ e \;:\; e = \bigcup_{k=1}^{m} \omega_k, \quad m \in \mathbb{N} \right\},$$

where $\{\omega_k\}_1^m$ is any collection of compact sets of equal measure $|\omega_k| \leqslant d$ and such that each ω_k belongs to a different elements of T_I.

Theorem. *Let $1 < p < q < \infty$ and $K \in L_{loc}$. Then for $Af = K * f$ we have*

$$C_1 \sup_I \sup_{e\in\mathfrak{L}(I)} \frac{1}{|e|^{1/p-1/q}} \left| \int_e K(x)dx \right| \leqslant \|A\|_{L_p\to L_q} \leqslant C_2 \inf_I \sup_{e\in\mathfrak{U}(I)} \frac{1}{|e|^{1/p-1/q}} \left| \int_e K(x)dx \right|, \tag{1.5}$$

where the constants C_1 and C_2 depend only on p and q.

For the certain regular kernels K, for instance, monotone or quasi-monotone, the upper and lower bounds in (1.5) coincide, that is, we get the equivalent relation for $\|A\|_{L_p\to L_q}$. More precisely, we call a locally integrable function $K(x)$ *weakly monotone* if there exists a constant $C > 0$ such that for any $x \in \mathbb{R} \setminus \{0\}$

$$|K(x)| \leqslant C \left| \frac{1}{x} \int_0^x K(t)dt \right|.$$

Corollary. *Let $1 < p < q < \infty$ and $K \in L_{loc}$ be a weakly monotone function. Then a necessary and sufficient condition for $Af = K * f$ to be bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ is*

$$\sup_{|x|>0} \frac{1}{|x|^{1/p-1/q}} \left| \int_0^x K(y)dy \right| < \infty.$$

Moreover,

$$\|A\|_{L_p\to L_q} \approx \sup_{|x|>0} \frac{1}{|x|^{1/p-1/q}} \left| \int_0^x K(y)dy \right|.$$

By C, C_i, c we will denote positive constants that may be different on different occasions. We write $F \approx G$ if $F \leqslant C_1 G$ and $G \leqslant C_2 F$ for some positive constants C_1 and C_2 independent of essential quantities involved in the expressions F and G.

The paper is organized as follows. In section 2 we obtain a required version of the Riesz Lemma for rearrangements (see, e.g., [St]). Section 3 and 4 are devoted to the estimates of $\|A\|_{L_p\to L_q}$ from above and below, correspondingly. We conclude with Section 5, where we show that the right-hand side estimate in (1.5) implies both (1.2) and (1.3) but the reverse does not hold in general.

2. Rearrangement inequalities

First, we denote the decreasing rearrangement of f on $\mathbb{Z}^n$ by f^*. We also denote $f^{**}(n) := \frac{1}{n}\sum_{k=1}^n f^*(k)$.

Lemma 2.1. *Let functions f, g, and K are defined on $\mathbb{Z}^n$; then*

$$\sum_{k\in\mathbb{Z}} g(k)(K*f)(k) \leqslant 2\sum_{r=1}^{\infty} r g^{**}(r) f^{**}(r) K^{**}(r). \tag{2.1}$$

Proof. From $f^{**}(n) = \sup_{\substack{|e|=n\\ e\subset\mathbb{Z}}} \frac{1}{|e|}\sum_{s\in e}|f(s)|$ (see [BS, Ch. 2, §3]) and the Hardy-Littlewood inequality [BS, p.44], we write

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} g(k)(K*f)(k) &\leqslant \sum_{r=1}^{\infty} g^*(r)(K*f)^{**}(r)\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}} \sum_{m\in\mathbb{Z}} |f(m)| \frac{1}{|e|}\sum_{s\in e}|K(s-m)|\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}} \sum_{m=1}^{\infty} f^*(m) \left(\frac{1}{|e|}\sum_{s\in e}|K(s-\cdot)|\right)^{**}(m)\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}} \sum_{m=1}^{\infty} f^*(m) \left(\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}} \frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sum_{s\in e}|K(s-t)|\right)\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sum_{m=1}^{\infty} f^*(m) \left(\sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}} \frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sum_{s\in e}|K(s-t)|\right).
\end{aligned}$$

We consider

$$\Phi(r,m)=\sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}}\frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sum_{s\in e}|K(s-t)|\,.$$

If $r\leqslant m$, then

$$\Phi(r,m)\leqslant\sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sum_{s\in e}\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}}\frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}|K(s-t)|=K^{**}(m)$$

and if $m\leqslant r$, then

$$\Phi(r,m)\leqslant\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}}\frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sum_{s\in e}|K(s-t)|=K^{**}(r).$$

Hence, we get

$$\Phi(r,m)\leqslant K^{**}(\max\{r,m\}).$$

Therefore,

$$\begin{aligned}\sum_{k\in\mathbb{Z}}^{\infty}g(k)(K*f)(k)&\leqslant\sum_{r=1}^{\infty}g^*(r)\sum_{m=1}^{\infty}f^*(m)K^{**}(\max\{r,m\})\\&=\sum_{r=1}^{\infty}g^*(r)K^{**}(r)\sum_{m=1}^{r}f(m)^*\\&\quad+\sum_{r=1}^{\infty}g^*(r)\sum_{m=r+1}^{\infty}f^*(m)K^{**}(m)\\&=\sum_{r=1}^{\infty}rg^*(r)K^{**}(r)f^{**}(r)+\sum_{m=1}^{\infty}f^*(m)K^{**}(m)\sum_{r=1}^{m}g^*(r)\\&\leqslant 2\sum_{r=1}^{\infty}rg^{**}(r)K^{**}(r)f^{**}(r).\end{aligned}$$

The proof is complete. □

The continuous analogue of the previous lemma is the following result.

Lemma 2.2. *Let f and g be measurable functions on $[0,d]$ and K be measurable on $[-d,d]$. Then*

$$\int_0^d g(y)\int_0^d f(x)K(y-x)\,dx\,dy \leqslant 2\int_0^d tg^{**}(t)f^{**}(t)\left(\sup_{\substack{e\subset[-d,d]\\ |e|=t}}\frac{1}{|e|}\int_e|K(x)|\,dx\right)dt. \tag{2.2}$$

Proof. Similarly to the proof of Lemma 2.1, we have

$$\int_0^d g(y)\left(K*f\right)(y)dy \leqslant \int_0^d g^*(s)\int_0^d f^*(t)\sup_{\substack{e\subset[0,d]\\|e|=s}}\sup_{\substack{\omega\subset[0,d]\\|\omega|=t}}\frac{1}{|e|}\frac{1}{|\omega|}\int_e\int_\omega |K(y-x)|\,dxdy$$
$$=\int_0^d g^*(s)\int_0^d f^*(t)\Phi(s,t)dtds.$$

Further, for $s\leqslant t$, we get

$$\Phi(s,t)\leqslant \sup_{\substack{e\subset[0,d)\\|e|=s}}\frac{1}{|e|}\int_e \sup_{\substack{\omega\subset[0,d]\\|\omega|=t}}\frac{1}{|\omega|}\int_\omega |K(y-x)|\,dxdy \leqslant \sup_{\substack{\omega\subset[-d,d]\\|\omega|=t}}\frac{1}{|\omega|}\int_\omega |K(x)|\,dx,$$

and for $s\geqslant t$,

$$\Phi(s,t)\leqslant \sup_{\substack{e\subset[-d,d]\\|e|=s}}\frac{1}{|e|}\int_e |K(y)|\,dy.$$

Finally, as in the proof of Lemma 2.1, we have

$$\int_0^d g(y)\left(K*f\right)(y)dy\leqslant 2\int_0^d tg^{**}(t)f^{**}(t)\sup_{\substack{e\subset[-d,d]\\|e|=t}}\frac{1}{|e|}\int_e |K(x)|\,dx. \qquad \square$$

3. Proof of upper bound for $\|A\|_{L_p\to L_q}$

Let $d>0$, $I=[0,d)$, and $T_I=\{(md,(m+1)d]\}_{m\in\mathbb{Z}}$ be the corresponding partition of $\mathbb{R}$. For a locally integrable function $K(x)$ we put $K(x)=K_1(x,d)+K_2(x,d)$, where

$$K_1(x,d)=\begin{cases} K(x), & \text{if } \ x\in(2md,(2m+1)d]\,, \quad m\in\mathbb{Z}\\ 0, & \text{if } \ x\in((2m-1)d,2md]\,, \quad m\in\mathbb{Z}\end{cases}$$

and

$$K_2(x,d)=\begin{cases} 0, & \text{if } \ x\in(2md,(2m+1)d]\,, \quad m\in\mathbb{Z}\\ K(x), & \text{if } \ x\in((2m-1)d,2md]\,, \quad m\in\mathbb{Z}.\end{cases}$$

Then we write the convolution operator $Af=f*K$ as $A=A_1+A_2$, where $A_if=f*K_i$, $i=1,2$, we have

$$\|A\|_{L_p\to L_q}\leqslant 2\max_{i=1,2}\|A_i\|_{L_p\to L_q}. \tag{3.1}$$

Let $d>0$ for $k\in\mathbb{Z}$ and $x\in[0,d]$, we denote

$$\widetilde{f}(x,k):=f(x+kd),$$
$$\widetilde{g}(x,k):=g(x+kd),$$
$$\widetilde{K}_i(x,k):=K_i(x+kd).$$

We are going to estimate the following quantity

$$J_i:=\int_{\mathbb{R}} g(y)\int_{\mathbb{R}} f(x)K_i(y-x)\,dxdy,\quad i=1,2.$$

Let us write it as follows

$$J_i=\sum_{k\in\mathbb{Z}}\int_0^d g(y+kd)\sum_{m\in\mathbb{Z}}\int_0^d f(x+md)K_i\Big((y-x)+(k-m)d\Big)\,dxdy$$

$$\equiv\sum_{k\in\mathbb{Z}}\int_0^d \tilde{g}(y,k)\sum_{m\in\mathbb{Z}}\int_0^d \tilde{f}(x,m)\tilde{K}_i(y-x,k-m)\,dxdy. \tag{3.2}$$

To estimate this functional, we first use Lemma 2.2:

$$J_i\leqslant 2\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_0^d t\tilde{f}^{(**)_1}(t,m)\tilde{g}^{(**)_1}(t,k)\sup_{\substack{e\subset[-d,d]\\|e|=t}}\frac{1}{|e|}\left|\int_e \tilde{K}_i(x,k-m)dx\right|dt$$

$$=2\int_0^d t\left(\sum_{k\in\mathbb{Z}}\tilde{g}^{(**)_1}(t,k)\sum_{m\in\mathbb{Z}}\tilde{f}^{(**)_1}(t,m)\sup_{\substack{e\subset[-d,d]\\|e|=t}}\frac{1}{|e|}\int_e\left|\tilde{K}_i(x,k-m)\right|dx\right)dt,$$

where

$$\tilde{f}^{(**)_1}(t,m)=\frac{1}{t}\int_0^t \tilde{f}^{*_1}(t,m)dt,\quad m\in\mathbb{Z},$$

$$\tilde{g}^{(**)_1}(t,k)=\frac{1}{t}\int_0^t \tilde{g}^{*_1}(t,k)dt,\quad k\in\mathbb{Z},$$

and $\tilde{f}^{*_1}(t,m),\tilde{g}^{*_1}(t,k)$ are decreasing rearrangements of $\tilde{f}(x,m),\tilde{g}(x,k)$ with respect to x and with fixed m and k, correspondingly.

Applying now Lemma 2.1, we get

$$J_i\leqslant 4\int_0^d t\sum_{s=1}^{\infty}s\tilde{f}^{**}(t,s)\tilde{g}^{**}(t,s)\left(\sup_{\substack{|\omega|=s\\\omega\subset\mathbb{Z}}}\frac{1}{|\omega|}\sum_{m\in e}\sup_{\substack{e\subset[-d,d]\\|e|=t}}\frac{1}{|e|}\int_e\left|\tilde{K}_i(x,m)\right|dx\right)dt$$

$$\equiv 4\int_0^d t\sum_{s=1}^{+\infty}s\tilde{f}^{**}(t,s)\tilde{g}^{**}(t,s)\,F_d(t,s;K_i)\,dt,$$

where

$$\tilde{f}^{**}(t,s) = \frac{1}{s}\sum_{l=1}^{s}\left(\tilde{f}^{(**)_1}(t,\cdot)\right)_l^{*_2},$$

$$\tilde{g}^{**}(t,s) = \frac{1}{s}\sum_{l=1}^{s}\left(\tilde{g}^{(**)_1}(t,\cdot)\right)_l^{*_2}.$$

Then writing

$$(ts)\tilde{g}^{**}(t,s)\tilde{f}^{**}(t,s)F_d(t,s;K_i) \leqslant \left((ts)^{\frac{1}{p}-\frac{1}{q}}\tilde{f}^{**}(t,s)\right)\left(\tilde{g}^{**}(t,s)\right)\left(\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{\frac{1}{r}}F_d(t,s;K_i)\right)$$

and using Hölder's inequality with parameters q and q' and the fact that $L_{pq} \hookrightarrow L_{pq_1}$ for $q \leqslant q_1$, we get

$$\int_0^d \sum_{s=1}^{\infty} ts\tilde{f}^{**}(t,s)\tilde{g}^{**}(t,s)F_d(t,s;K_i)dt \leqslant 4\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_i)\left(\sum_{s\in\mathbb{N}}\int_0^d\left(\tilde{g}^{**}(t,s)\right)^{q'}dt\right)^{1/q'} \left(\sum_{s\in\mathbb{N}}\int_0^d\left(\tilde{f}^{**}(t,s)\right)^{p}dt\right)^{1/p}.$$

Then by Hardy's inequality,

$$\left|\int_{\mathbb{R}} g(y)\int_{\mathbb{R}} f(x)K_i(y-x)dxdy\right| \leqslant C\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_i)\left\|f\right\|_{L_p}\left\|g\right\|_{L_{q'}}.$$

Thus,

$$\|A_i\|_{L_p\to L_q} \leqslant C\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_i), \quad i=1,2. \tag{3.3}$$

Note that by definition, K_1 and K_2 satisfy

$$\operatorname{supp}\tilde{K}_1(x,s)\subset[-d,0]\times\mathbb{Z}, \qquad \operatorname{supp}\tilde{K}_2(x,s)\subset[0,d]\times\mathbb{Z}.$$

Therefore,

$$\sup_{\substack{e\subset[-d,d]\\ |e|=t}}\frac{1}{|e|}\int_e|K_1(x,k)|\,dx = \sup_{\substack{e\subset[-d,0]\\ |e|=t}}\frac{1}{|e|}\int_e\left|\tilde{K}_1(x,k)\right|dx, \quad k\in\mathbb{Z}$$

and

$$\sup_{\substack{e\subset[-d,d]\\ |e|=t}}\frac{1}{|e|}\int_e|K_2(x,k)|\,dx = \sup_{\substack{e\subset[0,d]\\ |e|=t}}\frac{1}{|e|}\int_e\left|\tilde{K}_2(x,k)\right|dx, \quad k\in\mathbb{Z}.$$

Then

$$\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_1)=\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}\sup_{|\omega|=s}\frac{1}{(ts)^{\frac{1}{p}-\frac{1}{q}}}\sum_{m\in\omega}\sup_{\substack{|e|=t\\ e\subset[-d,0]}}\int_e\left|\tilde{K}_1(x,m)\right|dx.$$

For any $m\in\mathbb{Z}$ and $t\in(0,d]$ we find $e_{m,t}\subset[-d,0]$ such that $|e_{m,t}|=t$ and

$$\sup_{|e|=t}\int_e\left|\tilde{K}_1(x,m)\right|dx\leqslant 2\int_{e_{m,t}}\left|\tilde{K}_1(x,m)\right|dx$$

$$=2\int_{e_{m,t}}|K_1(x+md)|\,dx=2\int_{e_{m,t}+md}|K(x)|\,dx.$$

The set $\eta_m=e_{m,t}+md$ of measure t for different m belongs to different elements of $T_d=\{nd,(n+1)d\}_{n\in\mathbb{Z}}$. So, for $0<t\leqslant d$ and $r\in\mathbb{N}$ we have

$$\sup_{|\omega|=s}\frac{1}{(ts)^{\frac{1}{p}-\frac{1}{q}}}\sum_{m\in\omega}\sup_{|e|=t}\int_e\left|\tilde{K}_1(x,m)\right|dx\leqslant 2\sup_{e\in\mathfrak{U}([0,d])}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx.$$

Therefore, we obtain

$$\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_1)\leqslant 2\sup_{e\in\mathfrak{U}([0,d])}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx$$

and, similarly,

$$\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{N}}}(ts)^{1-\left(\frac{1}{p}-\frac{1}{q}\right)}F_d(t,s;K_2)\leqslant 2\sup_{e\in\mathfrak{U}([0,d])}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx.$$

Combining this with (3.1) and (3.3), we get

$$\|A\|_{L_p\to L_q}\leqslant C\sup_{e\in\mathfrak{U}([0,d])}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx$$

and using an arbitrary choice of $d>0$,

$$\|A\|_{L_p\to L_q}\leqslant C\sup_{d>0}\sup_{e\in\mathfrak{U}([0,d])}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx$$

with a constant C depending on p and q. Since the norms of operators $Af=K*f$ and $A_tf=K_t*f$, where $K_t(x)=K(x+t)$, $t\geqslant 0$ coincide, the last estimate implies

$$\|A\|_{L_p\to L_q}\leqslant C\sup_I\sup_{e\in\mathfrak{U}(I)}\frac{1}{|e|^{\frac{1}{p}-\frac{1}{q}}}\int_e|K(x)|\,dx.$$

To finish this proof, it is sufficient to show the following

Lemma 3.1. *Let $0<\gamma\leqslant 1$ and K be locally integrable. Then for any $e\in\mathfrak{U}(I)$ there exists $e'\in\mathfrak{U}(I)$ such that*

$$\frac{1}{|e|^{\gamma}}\int_e|K(x)|dx\leqslant 2^{3-\gamma}\frac{1}{|e'|^{\gamma}}\left|\int_{e'}K(x)dx\right|.$$

Proof. Since $e \in \mathfrak{U}(I)$ we have $e = \bigcup_{k=1}^m \omega_k$, where $|\omega_k| = \omega < d, k = \overline{1,m}$ and ω_k belong to different elements of $T_I = \{I + kd\}_{r\in\mathbb{Z}}$.

For any w_k let us define

$$\omega_k^1 := \Big\{x \in w_k : K(x) \geqslant 0\Big\} \qquad \text{and} \qquad \omega_k^2 = \Big\{x \in \omega_k : K(x) < 0\Big\}.$$

Then

$$\int\limits_{\omega_k} |K(x)|dx = \int\limits_{\omega_k^1} K(x)dx - \int\limits_{\omega_k^2} K(x)dx \leqslant 2\max\left\{\Big|\int\limits_{\omega_k^1} K(x)dx\Big|, \Big|\int\limits_{\omega_k^2} K(x)dx\Big|\right\}.$$

We can assume that

$$\Big|\int\limits_{\omega_k^1} K(x)dx\Big| \geqslant \Big|\int\limits_{\omega_k^2} K(x)dx\Big|.$$

Let us consider two cases: $|\omega_k^1| \geqslant \frac{w}{2}$ and $|\omega_k^1| < \frac{w}{2}$. In the first case, there exists $\tilde{\omega}_k \subset \omega_k^1$ such that $|\tilde{\omega}_k| = \frac{w}{2}$ and

$$2\Big|\int\limits_{\tilde{\omega}_k} K(x)dx\Big| \geqslant \Big|\int\limits_{\omega_k^1} K(x)dx\Big|.$$

In the second case, $|\omega_k^1| < \frac{w}{2}$ and there exist η_k^1 and η_k^2 such that $|\eta_k^1 \bigcap \eta_k^2| = 0$, $\eta_k^1 \bigcup \eta_k^2 = \omega_k^2$, and $|\eta_k^i| = \frac{|\omega_k^2|}{2}$. Since $K(x)$ keeps its sign on ω_k^2, we have

$$\begin{aligned}\Big|\int\limits_{\omega_k^1} K(x)dx\Big| &\geqslant \Big|\int\limits_{\omega_k^2} K(x)dx\Big| = \Big|\int\limits_{\eta_k^1} K(x)dx\Big| + \Big|\int\limits_{\eta_k^2} K(x)dx\Big| \\ &\geqslant 2\min\left(\Big|\int\limits_{\eta_k^1} K(x)dx\Big|, \Big|\int\limits_{\omega_2^2} K(x)dx\Big|\right) = 2\Big|\int\limits_{\eta_k^{i_0}} K(x)dx\Big|.\end{aligned}$$

Here $\eta_k^{i_0}$ are sets where the infimum is attained. Then we consider $\eta_k \subset \eta_k^{i_0}$ such that $|\eta_k| = \frac{w}{2} - |\omega_k^1|$.

Let now $\tilde{\omega}_k = \eta_k \cup \omega_k^1$, then $|\tilde{\omega}_k| = \frac{w}{2}$ and

$$\begin{aligned}\Big|\int\limits_{\tilde{\omega}_k} K(x)dx\Big| &= \Big|\int\limits_{\omega_k^1} K(x)dx + \int\limits_{\eta_k} K(x)dx\Big| \\ &\geqslant \Big|\int\limits_{\omega_k^1} K(x)dx\Big| - \Big|\int\limits_{\eta_k} K(x)dx\Big| \\ &\geqslant \Big|\int\limits_{\omega_k^1} K(x)dx\Big| - \Big|\int\limits_{\eta_k^{i_0}} K(x)dx\Big| \geqslant \frac{1}{2}\Big|\int\limits_{\omega_1} K(x)dx\Big|.\end{aligned}$$

Therefore, in both cases, we have

$$\int\limits_{\omega_k} |K(x)|dx \leqslant 2\Big|\int\limits_{\omega_1} K(x)dx\Big| \leqslant 4\Big|\int\limits_{\tilde{\omega}_k} K(x)dx\Big|,$$

Suppose

$$J_+ = \{k : \int_{\tilde{\omega}_k} K(x)dx \geqslant 0\}, \qquad J_- = \{k : \int_{\tilde{\omega}_k} K(x)dx \leqslant 0\};$$

then

$$2\max\left\{\Big|\int_{\bigcup_{k\in J_+}\tilde{\omega}_k} K(x)dx\Big|, \Big|\int_{\bigcup_{k\in J_-}\tilde{\omega}_k} K(x)dx\Big|\right\}$$
$$\geqslant \sum_{k=1}^{m}\Big|\int_{\omega_k} K(x)dx\Big| \geqslant \frac{1}{4}\int_e |K(x)|dx.$$

Taking as e' the set $\bigcup_{k\in J_+}\tilde{\omega}_k$ or $\bigcup_{k\in J_-}\tilde{\omega}_k$, where the maximum is attained, we get

$$\frac{1}{|e|^\gamma}\int_e |K(x)|dx \leqslant 2^3\frac{1}{|e|^\gamma}\left|\int_{e'} K(x)dx\right| \leqslant 2^{3-\gamma}\frac{1}{|e'|^\gamma}\left|\int_{e'} K(x)dx\right|.$$

By construction, $\tilde{\omega}_k \subset \omega_k$ and $|\tilde{\omega}_k| = \frac{\omega}{2}$, and therefore $e' \in \mathfrak{U}(I)$. □

4. Proof of lower bound for $\|A\|_{L_p\to L_q}$

Let $1 < p < q < \infty$, $\frac{1}{r} = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)$, and $Af = K * f$ is bounded from $L_p(\mathbb{R})$ in $L_q(\mathbb{R})$. We are going to prove that for any number d and an interval I, $|I| = d$, there holds

$$\sup_{e\in\mathfrak{L}(I)}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right| \leqslant c(p,q)\|A\|_{L_p\to L_q}, \tag{4.1}$$

where the collection $\mathfrak{L}(I)$ is given by (1.4). We define $\mathfrak{L}'(I) \subset \mathfrak{L}(I)$ as follows

$$\mathfrak{L}'(I) = \left\{e = \bigcup_{k=0}^{m}([a,b]+kd) : m\in\mathbb{N}, [a,b]\subset I, b-a \leqslant d/2\right\}.$$

Note that for any locally summable function $K(x)$ we have

$$\sup_{e\in\mathfrak{L}'(I)}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right| \leqslant \sup_{e\in\mathfrak{L}(I)}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|$$
$$\leqslant 2^{1/r}\sup_{e\in\mathfrak{L}'(I)}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|.$$

Indeed, the left-hand side inequality is clear since $\mathfrak{L}'(I)\subset\mathfrak{L}(I)$. To prove the right-hand side, we consider $e\in\mathfrak{L}(I)$, that is, $e=\bigcup_{i=0}^{m}([a,b]+id)=\bigcup_{i=0}^{m}([a,\frac{a+b}{2}]+id)$ $\cup\bigcup_{i=0}^{m}([\frac{a+b}{2},b]+id)=e_1\cup e_2$. Clearly, $|e_i|=|e|/2$, $e_i\in\mathfrak{L}'(I)$, $i=1,2$. Therefore,

$$\begin{aligned}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right| &\leqslant \frac{2}{|e|^{1/r'}}\max_{i=1,2}\left|\int_{e_i} K(x)dx\right|\\ &= 2^{1-1/r'}\max_{i=1,2}\frac{1}{|e_i|^{1/r'}}\left|\int_{e_i} K(x)dx\right|\\ &\leqslant 2^{1/r}\sup_{e\in\mathfrak{L}'(I)}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|.\end{aligned}$$

Hence, it is sufficient to verify

$$\sup_{e\in\mathfrak{L}'(I)}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|\leqslant c\|A\|_{L_p\to L_q}.$$

Let us first assume that K is bounded, that is, $|K(x)|\leqslant D$, $x\in\mathbb{R}$. For $s>0$ we define

$$\alpha_s=\sup_{\substack{e\in\mathfrak{L}'(I)\\ |e|\leqslant s}}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|.$$

This is well-defined since for any $e\in\mathfrak{L}'(I)$ and $|e|\leqslant s$ we get

$$\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|\leqslant D\,|e|^{1-1/r'}\leqslant Ds^{1-1/r'}.$$

Then we consider $e_0\in\mathfrak{L}'(I)$, $|e_0|\leqslant s$ such that

$$\frac{1}{|e_0|^{1/r'}}\left|\int_{e_0} K(x)dx\right|\geqslant\frac{\alpha_s}{2}.$$

Since the convolution is translation invariant, then we assume that e_0 is of form

$$e_0=\bigcup_{i=0}^{m}([0,b]+id),$$

where $b\leqslant d/2$, $m\in\mathbb{N}\cup\{0\}$.

Let us take $0<\delta<\frac{1}{2}$ to be specified later. We define the following sets $e_{1+\delta}$ and e_δ:

$$e_{1+\delta}=\bigcup_{i=0}^{[(1+\delta)m]}([0,(1+\delta)b]+id),$$

$$e_\delta=\bigcup_{i=0}^{[\delta m]}([0,\delta b]+id).$$

Since $e_0 \in \mathfrak{L}'(I)$, we have $e_{1+\delta} \in \mathfrak{L}(I)$. Then taking $f_0 = \chi_{e_{1+\delta}}$, boundedness of the operator A implies

$$\begin{aligned} \|K * f_0\|_{L_q} &\leqslant \|A\|_{L_p \to L_q} \|f_0\|_{L_p} \\ &= \|A\|_{L_p \to L_q} |e_{1+\delta}|^{1/p} \leqslant 2 \|A\|_{L_p \to L_q} (1+\delta)^{2/p} |e_0|^{1/p}. \end{aligned} \tag{4.2}$$

On the other hand,

$$\begin{aligned} \|K * f_0\|_{L_q} &= \left(\int_{-\infty}^{\infty} \left| \int_{e_{1+\delta}} K(x-y)dx \right|^q dy \right)^{1/q} \\ &= \left(\sum_{j \in \mathbb{Z}} \int_0^d \left| \sum_{i=0}^{(1+\delta)m} \int_0^{b(1+\delta)} K((i-j)d + (x-y))dx \right|^q dy \right)^{1/q} \\ &\geqslant \left(\sum_{j=0}^{[\delta m]} \int_0^{\delta b} \left| \sum_{i=0}^{(1+\delta)m} \int_0^{(1+\delta)b} K((i-j)d + (x-y))dx \right|^q dy \right)^{1/q} \\ &= \left(\sum_{j=0}^{[\delta m]} \int_0^{\delta b} \left| \sum_{i=-j}^{(1+\delta)m-j} \int_{-y}^{(1+\delta)b-y} K(id+x)dx \right|^q dy \right)^{1/q} \\ &\geqslant \left(\sum_{j=0}^{[\delta m]} \int_0^{\delta b} \left[\left| \sum_{i=0}^{m} \int_0^{b} K(id+x)dx \right| - \left| \sum_{i=-j}^{-1} \int_{-y}^{(1+\delta)b-y} K(id+x)dx \right| \right. \right. \\ &\quad - \left| \sum_{i=m+1}^{[(1+\delta)m]-j} \int_{-y}^{(1+\delta)b-y} K(id+x)dx \right| - \left| \sum_{i=0}^{m} \int_{-y}^{0} K(id+x)dx \right| \\ &\quad \left. \left. - \left| \sum_{i=0}^{m} \int_{b}^{(1+\delta)b-y} K(id+x)dx \right| \right]^q dy \right)^{1/q} \\ &=: \left(\int_{e_\delta} \left[\left| \int_{e_0} K(x)dx \right| - \sum_{i=1}^{4} \left| \int_{e_i} K(x)dx \right| \right]^q dy \right)^{1/q}, \end{aligned}$$

where $e_i \in \mathfrak{L}'(I)$ such that $|e_i| \leqslant 2\delta |e_0|$, $i = 1, 2, 3, 4$.

We put $\delta = \left(2(16^{r'})\right)^{-1} < \frac{1}{2}$. Then $|e_i| < |e_0| \leqslant s$ and

$$\frac{1}{|e_0|^{1/r'}} \left| \int_{e_0} K(x)dx \right| \geqslant \frac{\alpha_s}{2} \geqslant \frac{1}{2|e_i|^{1/r'}} \left| \int_{|e_i|} K(x)dx \right|$$

and therefore

$$\left|\int_{e_i} K(x)dx\right| \leqslant \frac{2\,|e_i|^{1/r'}}{|e_0|^{1/r'}}\left|\int_{e_0} K(x)dx\right|.$$

Taking into account $|e_i| \leqslant 2\delta\,|e_0|$, we get

$$\begin{aligned}\|K * f_0\|_{L_q} &\geqslant \left(\int_{e_\delta}\left[\left|\int_{e_0} K(x)dx\right|\left(1-2\sum_{i=1}^{4}\left(\frac{|e_i|}{|e_0|}\right)^{1/r'}\right)\right]^q dy\right)^{1/q} \\ &\geqslant |e_\delta|^{1/q}\left|\int_{e_0} K(x)dx\right|\left(1-8(2\delta)^{1/r'}\right) \\ &\geqslant \frac{1}{2}\delta^{2/q}\,|e_0|^{1/q}\left|\int_{e_0} K(x)dx\right|.\end{aligned}$$

Using (4.2), we have

$$\|A\|_{L_p\to L_q} \geqslant C_{p,q}\frac{1}{|e_0|^{1/r'}}\left|\int_{e_0} K(x)dx\right| \geqslant \frac{C_{p,q}}{2}\sup_{\substack{e\in\mathfrak{L}'(I)\\ |e|\leqslant s}}\frac{1}{|e|^{1/r'}}\left|\int_{e} K(x)dx\right|$$

Thus, for the bounded K and for any $s > 0$ we obtain

$$\sup_{e\in\mathfrak{L}(I)}\frac{1}{|e|^{1/r'}}\left|\int_{e} K(x)dx\right| \leqslant C\,\|A\|_{L_p\to L_q}, \tag{4.3}$$

where C depends on p and q.

To prove this in a general case of locally integrable K not necessary bounded, we consider

$$K_N(x) = \begin{cases} N, & K(x) > N, \\ K(x), & K(x) \leqslant N \end{cases}, \qquad N \in \mathbb{N}$$

and

$$K_{N,M}(x) = \begin{cases} K_N(x) & K(x) \geqslant -M, \\ M & K(x) < -M \end{cases}, \qquad N, M \in \mathbb{N}.$$

As we have proved before,

$$\sup_{e\in\mathfrak{L}(I)}\frac{1}{|e|^{1/r'}}\left|\int_{e} K_{N,M}(x)dx\right| \leqslant C\,\|A_{N,M}\|_{L_p\to L_q}, \qquad A_{N,M}f = K_{N,M} * f,$$

where a constant C does not depend on N and M.

Noting that Banach-Steinhaus' theorem implies $\|A_{N,M}\|_{L_p\to L_q} \leqslant D (D > 0)$ for some $D > 0$ and using the monotonicity properties of $K_{N,M}$, namely,

$$K_{N,1}(x) \geqslant K_{N,2} \geqslant \ldots \geqslant K_{N,M} \geqslant \ldots$$

and

$$K_1(x) \leqslant K_2(x) \leqslant \ldots K_N \leqslant \ldots,$$

we apply Levi's theorem:

$$\sup_{e\in\mathfrak{L}(I)} \frac{1}{|e|^{1/r'}} \left|\int_e K(x)dx\right| \leqslant cD < \infty.$$

Finally, repeating the proof of (4.3), we arrive at required inequality (4.1). □

We would like to mention that attempts have already been made at proving the lower estimate for the convolution operator in [NS], although they require stronger hypotheses than those used here.

5. Comparison with O'Neil and Stepanov's inequalities

Let us first show that the right-hand side estimate in (1.5) implies both (1.2) and (1.3). Indeed, it is known that ([BS, Ch. 2, §3])

$$\sup_{t>0} t^{1/r}K^*(t) \approx \sup_{t>0} t^{1/r}K^{**}(t) \approx \sup_{0<|e|<\infty} \frac{1}{|e|^{1/r'}} \int_e |K(x)|\,dx, \tag{5.1}$$

and therefore

$$\sup_{e\in\mathfrak{U}(I)} \frac{1}{|e|^{1/r'}} \left|\int_e K(x)dx\right| \leqslant C \sup_{t>0} t^{1/r}K^*(t).$$

Let $1/r = 1-(1/p-1/q) < 1$, $r' = r/(r-1)$, and let I be an interval with $|I|=1$. Assume that $e\in\mathfrak{U}(I)$, that is, $e=\bigcup_{s=1}^m w_s$ such that $w_s\in I+i_s, |w_s| = |w_j| = g\leqslant 1, i_s\neq i_j$. Then

$$\begin{aligned}
\frac{1}{|e|^{1/r'}} \left|\int_e K(x)dx\right| &= \left|\frac{1}{(mg)^{1/r'}} \sum_{s=1}^m \int_{w_s} K(x)dx\right| \\
&= \frac{1}{(mg)^{1/r'}} \left|\sum_{s=1}^m \int_0^1 K(x+i_s)\chi_{w_s}(x+i_s)dx\right| \\
&\leqslant \frac{1}{(mg)^{1/r'}} \sum_{s=1}^m \int_0^1 |\tilde{K}(x,i_s)\tilde{\chi}_{w_s}(x,i_s)|dx \\
&= \frac{1}{(mg)^{1/r'}} \sum_{s=1}^m \int_0^g \tilde{K}^*(t,i_s)dt \\
&\leqslant \frac{1}{(mg)^{1/r'}} \sum_{s=1}^m \sup_{0<t\leqslant 1} t^{1/r}\tilde{K}^*(t,i_s) \int_0^g t^{-1/r}dt \\
&\leqslant (r')^2 \sup_{n\in\mathbb{N}} s^{1/r} \left(\sup_{0<t\leqslant 1} t^{1/r}\tilde{K}^*(t,\cdot)\right)_s^* .
\end{aligned}$$

Thus, (1.5) is stronger than either (1.2) or (1.3). We now give an example capturing the difference between these estimates.

Example. *Let* $1 < p < q < \infty$ *and* $0 < 1/r = 1 - (1/p - 1/q)$*. Define the function* $K(x)$ *on* $\mathbb{R}$ *as follows:*

$$K(x) = \begin{cases} 2^{k/r}, & \text{for } x \in [-k, -k + 2^{-k}], \quad k \in \mathbb{N}; \\ 1, & \text{for } x \in [k, k + 1/k), \quad k \in \mathbb{N}; \\ 0, & \text{otherwise.} \end{cases}$$

This function satisfies

$$\sup_{e \in \mathfrak{U}([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_e K(x) dx \right| < \infty, \tag{5.2}$$

but

$$\|K\|_{L_{r,\infty}} = \infty \tag{5.3}$$

and

$$\|K\|_{W(L_{r,\infty}[0,1],\, l_{r,\infty}(\mathbb{Z}))} = \sup_{n \in \mathbb{N}} n^{1/r} \left(\sup_{0 \leqslant t \leqslant 1} \quad t^{1/r} \tilde{K}^*(t, \cdot) \right)_n^* = \infty. \tag{5.4}$$

Indeed, let us show (5.2). Let $K_+(x) = K(x)\chi_{[0,\infty)}(x)$, $K_-(x) = K(x)\chi_{(-\infty,0)}(x)$, then $K_+(x) + K_-(x) = K(x)$ and therefore,

$$\sup_{e \in M([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_e K(x) dx \right| \leqslant \sup_{e \in M([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_e K_+(x) dx \right| + \sup_{e \in M([0,1])} \frac{1}{|e|^{1/r'}} \left| \int_e K_-(x) dx \right|.$$

Let $e \in \mathfrak{U}([0,1])$. Then $e = \cup_{k \in u} w_k$, where $|w_k| = w < 1$ and $u \subset \mathbb{Z}$, $|u| = m$. We have

$$\begin{aligned}
\frac{1}{|e|^{1/r'}} \int_e K_+(x)\, dx &= \frac{1}{|m|^{1/r'}} \frac{1}{w^{1/r'}} \sum_{k \in u} \left| \int_{w_k} K_+(x) dx \right| \\
&= \frac{1}{m^{1/r'}} \frac{1}{w^{1/r}} \sum_{n=1}^{m} \int_0^w K_+(x+n) dx \\
&= \frac{1}{(wm)^{1/r'}} \left(\sum_{n=1}^{1/w} \int_0^w K_+(x+n) dx + \sum_{n=1/w}^{m} \int_0^{1/n} K_+(x+n) dx \right) \\
&= \frac{1}{(wm)^{1/r'}} \left(\sum_{n=1}^{1/w} w + \sum_{n=1/w}^{n} \frac{1}{n} \right) \\
&\leqslant \frac{2}{(wm)^{1/r'}} \big(1 + \ln(mw)\big) \leqslant 2r'.
\end{aligned}$$

Further,

$$\begin{aligned}\frac{1}{|e|^{1/r'}}\int_e K_-(x)\,dx &= \frac{1}{|m|^{1/r'}}\frac{1}{w^{1/r'}}\sum_{k\in u}\left|\int_{w_k} K_-(x)dx\right| \\ &= \frac{1}{(mw)^{1/r'}}\frac{1}{w^{1/r}}\sum_{n\in u}\int_0^w K_-(x+n)dx \\ &\leqslant \frac{1}{(wm)^{1/r'}}\left(\sum_{\substack{n\in u\\ |n|<\log_2\frac{1}{w}}}\int_0^w K_-(x-|n|)dx+\right. \\ &\qquad\left.\sum_{\substack{n\in u\\ |n|\geqslant\log_2\frac{1}{w}}}\int_0^{2^{-|n|}} K_-(x-|n|)dx\right) \\ &= \frac{1}{(wm)^{1/r'}}\left(\sum_{\substack{n\in u\\ |n|<\log_2\frac{1}{w}}} 2^{|n|/r}w+\sum_{\substack{n\in u\\ |n|\geqslant\log_2\frac{1}{w}}} 2^{-|n|/r'}\right) \\ &\leqslant 2\frac{1}{(wm)^{1/r'}}\left(w^{1/r'}+w^{1/r'}\right)\leqslant 4.\end{aligned}$$

Combining these estimates, we get

$$\sup_{e\in\mathfrak{U}([0,1])}\frac{1}{|e|^{1/r'}}\left|\int_e K(x)dx\right|\leqslant 4+2r'.$$

To show (5.3), we note that $K_+^*(t)\equiv 1$. Hence,

$$\sup_{t>0}\ t^{1/r}K^*(t)\geqslant\sup_{t>0}\ t^{1/r}K_+^*(t)=\infty.$$

To show (5.4), we note

$$\begin{aligned}\|K\|_{W(L_{r,\infty}[0,1],\,l_{r,\infty}(\mathbb{Z}))} &\geqslant \|K_-\|_{W(L_{r,\infty}[0,1],\,l_{r,\infty}(\mathbb{Z}))} \\ &= \sup_{n\in\mathbb{N}} n^{1/r}\left(\sup_{0<t\leqslant 1} t^{1/r}(\widetilde{K_N^-})^*(t,n)\right)_n^* \\ &= \sup_{n\in\mathbb{N}} n^{1/r}\left(\sup_{0<t\leqslant 2^{-n}} t^{1/r}2^{n/r}\right) \\ &= \sup_{n\in\mathbb{N}} n^{1/r}=\infty.\end{aligned}$$

□

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