www.ontologia.net/studies

Ontology Studies 9, 2009 **209-220**

Actual and Virtual Events in the Quantum Domain

Fred Kronz[1]

University of Texas at Austin

Philosophy Department

Reception date / Fecha de recepción: 28-02-2009
Acceptation date / Fecha de aceptación: 06-05-2009

Abstract

The actual/virtual distinction is used to give an alternative account of quantum interference by way of a new theory of probability. The new theory is obtained by changing one of the axioms of the canonical theory of probability while keeping the other axioms fixed. It is used to give an alternative account of constructive quantum interference in the two-slit experiment. The account crucially involves a distinction between actual and virtual probabilities. Although actual probabilities are operational and virtual probabilities are not, there is a substantial connection between them. Virtual probabilities may be obtained indirectly from actual probabilities. In showing this, interpretive considerations are brought to bear including some having to do with quantum non-locality. Directions for future research are discussed in closing.

Key words: quantum, interference, theory of probability, two-slit experiment, locality.

Resumen. *Eventos reales y virtuales en el dominio cuántico*

La distinción actual/virtual se utiliza para dar una explicación alternativa de la interferencia cuántica a través de una nueva teoría de la probabilidad. La nueva teoría se ha obtenido cambiando uno de los axiomas de la teoría canónica de la probabilidad y manteniendo los otros axiomas fijos. Se usa para dar una explicación alternativa de la interferencia cuántica constructiva en el experimento de la doble rendija. El informe fundamentalmente implica una distinción entre las probabilidades actuales y virtuales. Aunque las probabilidades actuales son operativas y las probabilidades virtuales no, hay una relación sustancial entre ellas. Probabilidades virtuales pueden obtenerse indirectamente a partir de las probabilidades reales. Para demostrar esta interpretación hay que tener en cuenta las consideraciones de interpretación que se ejercen entre ellos y que tienen relación con la no-localidad cuántica. Directrices para futuras investigaciones se abordan al final del artículo.

Palabras clave: cuántica, interferencia, probabilidad, doble rendija, localidad.

1. I would very much like to thank the organizers of the VII International Ontology Congress in San Sebastián from 2-6 October 2006 for their kind invitation to participate in the congress, which was very stimulating in many ways,. This paper is based on my presentation at the congress. Please direct correspondence about it to kronz@mail.utexas.edu.

1. Introduction

One effective strategy for generating new frameworks that lead to important insights is to take an existing framework in axiomatic form and change slightly one or more of the axioms while keeping the others fixed. There are well-known examples in geometry and logic. In Euclidean geometry the parallel postulate was modified to form Lobachevskian geometry, and in Boolean logic the distributive law was modified to form quantum logic. In what follows the positivity axiom of Kolmogorov probability is modified to form non-monotonic probability—the positivity axiom places the lower bound for probabilities at 0 and the new theory places that bound at –1. It was inspired by interference phenomena in the quantum domain. It is shown in (Kronz, 2006) and (Kronz, 2007) that the new framework captures important aspects of quantum interference in experiments on the simplest of quantum systems, two-state systems (so-named because they involve measurable physical quantities that have just two possible values such as 1 and 0, or +1 and –1).[2]

Negative probabilities have been used before to capture physical phenomena. Muckenheim's (1986) review of extended probabilities discusses several such contexts such as phase-space formulations of quantum mechanics and indefinite metric space formulations of quantum electrodynamics. Negative probabilities have also been used to formulate local hidden-variables theories for quantum mechanics (Rothman and Sudarshan, 2000) and to provide a non-standard account of quantum interference (Feynman, 1987). A formal framework for Feynman's approach was first presented in (Kronz, 2006), which provides a rigorous elaboration and simplification of Feynman's work. In (Kronz, forthcoming) the non-monotonic theory is generalized for *n*-state systems, for any counting number *n*. The discussion below highlights the role of the virtual/actual distinction in applying the theory via a simplified version of the two-slit experiment.

2. Probability theory

Suppose that E is a given experiment having two possible outcomes.[3] A simple example to bear in mind is a coin flip that is repeated many times, the two possible outcomes being heads and tails. One way to think about probability is in terms of what happens when E is repeated many times (under the assumption that the outcome of one experiment does not affect another). The probability of an outcome (say heads) is often connected with

2. In the works cited above, the two-state systems involve the observables spin-1/2 and photon polarization, respectively. Below, the two-slit experiment is treated as a two-state system. The observable involves a region d on a detecting plate; the value 1 corresponds to detection in d and 0 to detection outside d.

3. What follows is easily generalized to one in which there is discrete number of possible outcomes but the special case of two possible ocutocmes will suffice for the purposes here.

the long run relative frequency of occurrence of the outcome. If E is repeated n times and the outcome of interest occurs k times, then the relative frequency of occurrence of the outcome is k/n. The term *long run* is meant to convey that the number of trials n must be suitably large before the relative frequency of occurrence may be regarded as the probability for the outcome.

A coin-flip experiment can be used to give a brief explanation of why the meaning of *long run* is a subtle matter. Take an unbiased coin, one that it is symmetrical and well balanced so that the probability of heads is ½, and consider what can happen in the *short run*, where the number of experiments is relatively small. In that case, the probability value for a given outcome does not impose restrictions on what happens. In ten flips of a fair coin, the outcome need not be heads in five of them. There are many sequences of ten flips in which the number of heads is other than five: one in which all outcomes are heads, ten having nine heads and one tails, and so on. The term "maverick" may be used to describe such sequences, meaning those in which the relative frequency of occurrence is not equal to the probability. As the number of experiments (the sequence length) increases, so does the number of mavericks. One may be inclined to think that mavericks eventually overwhelm non-mavericks, but the reverse is the case. The key result is Bernoulli's theorem, a deep theorem in the theory of probability, which says that the frequency of non-mavericks eventually overwhelms that of the mavericks.[4] For what follows, it is okay to suppose that n is large enough and set aside questions as to whether the long run is to be associated with some appropriately large number (that may depend on the probability value) or with the limit as n goes to infinity.

The positivity axiom together with some others axioms entails that probability values range between 0 and 1.[5] This range is reasonable when probabilities are uniformly associated with long-range relative frequencies of occurrence, an association that is weakened below. By definition, if the outcome O is obtained k times in n experiments, the *relative frequency* of O is k/n, which must be between 0 and 1 since k and n are counting numbers with $k \leq n$. It is 1 if the outcome is always O, 0 if it is never O, and some proper fraction if the outcome is sometimes O and sometimes not. If one modifies the positivity axiom by reducing

4. A particularly clear discussion of Bernoulli's theorem is Chapter 6 of (Gnedenko & Khinchin 1961), pp. 53-61.

5. The other axioms alluded to here are Total Probability, Conditional Probability, and Additivity. Total Probability says that the probability of a tautological event is one. Conditional Probability defines the probability of one event given a second as equal to the probability of their conjunction divided by the probability of the second (if the latter is not equal to zero). Additivity says that the probability of the disjunction of two events is equal to the sum of the probability of the disjuncts whenever the disjuncts are mutually exclusive. One other axiom of the standard theory is not invoked here, Countable Additivity, which has to do with the probability of a disjunction having a countably infinite number of disjuncts.

the lower bound to –1, then the meaning of probability and related notions change.[6] For example, a broader range of possible probability values follows from the new axiom together with the rest; the range becomes –1 to 2 instead of 0 to 1. Relative frequencies and probabilities can still be associated together in the new theory, but that association must be suitably tempered. That is accomplished here by way of the actual/virtual distinction. Actual probabilities are operational, meaning that they can be directly measured as relative frequencies of occurrence in an actual experiment. Virtual probabilities are not operational in this sense. However, they can be derived from actual probabilities together with some interpretive identities. The character of these identities and the role that they are to play are explained below. It follows immediately that actual probabilities have the standard range for probabilities. It turns out that virtual probabilities have the extended range implied by the axioms of the nonstandard theory.

3. Suggestive aspects of the two-slit experiment

The canonical example of quantum interference is the two-slit experiment. It involves a source, a screen with two slits, and a detecting plate. It will be informative to discuss that experiment as a two-state system by focusing on one region of the detecting plate where constructive interference occurs.

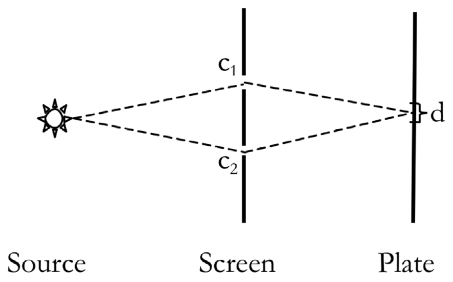

Diagram 1. The Two-Slit Experiment

Three configurations of the experiment are relevant—see Table 1.

State	Configuration
C_1	c_1 is open & c_2 is open
C_2	c_1 is open & c_2 is closed
C_3	c_1 is closed & c_2 is open

Table 1: Three configurations of the two-slit experiment.

6. Similar changes occur in geometry and logic, respectively, when the parallel postulate and the distributive law are modified. For example, the meaning of "straight line" changes when the axioms permit there to be at least two distinct lines through a given point that are each parallel to a given line.

To say that constructive interference occurs at d means that more photons are detected at d given C_1 than the sum of what is detected at d given C_2 and what is detected at d given C_3. That is, there are more photons that hit d when both slits are open for a given length of time T than do so when photons are collected over 2T, over T with only c_1 open and then again over T with only c_2 open.

For the purposes here, the situation above is now characterized in terms of probabilities. What follows is a preliminary characterization; it will later undergo modification due to subtleties that arise. Let A denote that the photon passes through c_1, B that it passes through c_2, and D that it is detected at d. The case of constructive interference above may now be characterized as a situation in which the probability that the photon passes through either one slit or the other and lands at d, P((AvB)&D), is greater than the sum of the probabilities for its passing through each of them, P(A&D)+P(B&D). The idea is that the three probabilities formally indicated (two of them in the sum) are actual; they are operationalized via the three configurations of the two-slit experiment indicated in Table 1. This formal characterization is incorrect; nevertheless, it is instructive to proceed for the moment along this line since doing so reveals the possible utility of allowing for non-standard probability values.

The relevance of negative probability values becomes apparent via general additivity, a theorem that relates the three probability expressions above with a fourth. The theorem follows.

$$P((A \vee B)\&D) = P(A\&D) + P(B\&D) - P((A\&B)\&D)$$

The theorem is counter-intuitive since it is not obvious why the third term on the right is required. The reason is that if A and B are not mutually exclusive, then the probability associated with A&B is counted twice, once in determining P(A&D) and again in determining P(B&D).

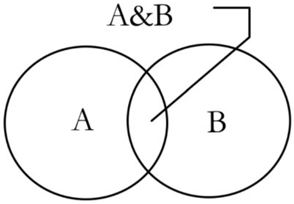

Diagram 2. The region A&B is counted twice when A and B are added together.

Constructive interference is a situation in which $P((A \vee B)\&D) > P(A\&D) + P(B\&D)$, as indicated above, which suggests that might be accounted for, if one allows P((A&B)&R) to be less than zero.

4. One subtle aspect of the two-slit experiment—the portrayal issue

Although the considerations above are both suggestive and informative, it turns out that they are somewhat off track. One subtlety has to do with the probabilistic portrayal of the event that occurs in each of the three configurations of the experiment. Consider the configuration C_1 where both slits are open. The event at the screen is portrayed disjunctively as (AvB), meaning that when both slits are open the photon goes through either one slit or the other or both. However, it could instead be portrayed conjunctively as (A&B), meaning that it always goes through both slits when both slits are open. Similar considerations apply to C_2 and C_3. In configuration C_2 the event at the screen is portrayed simply as A, but it could be portrayed conjuctively as (A&¬B) to indicate the relevance of both its success in passing through c_1 and its failure to pass through c_2 in characterizing that event. The same goes for C_3 with regards to B and (¬A&B).

In light of the considerations above, there are four distinct portrayals of the two-slit experiment, depending on whether the configurations in which only one slit is open is characterized simply or conjunctively, and whether the configuration in which both slits are open is characterized disjunctively or conjunctively. The four portrayals are characterized in Table 2.

Configuration	Portrayal 1	Portrayal 2	Portrayal 3	Portrayal 4
C_1	A	A	A&¬B	A&¬B
C_2	B	B	¬A&B	¬A&B
C_3	AvB	A&B	AvB	A&B

Table 2: Four portrayals of the events at the screen in each configuration of the two-slit experiment.

In the previous section Portrayal 1, the typical portrayal of the two-slit experiment, was used. It turns out that only Portrayal 4 will do. The other three lead to divergent conditional probabilities, unlike the fourth, as shown in (Kronz 2006) and (Kronz 2007) using analyzer-loop experiments that bear marked resemblance to the two-slit experiment that is under consideration—see Diagram 3.[7]

7. Conditional probabilities have the form P(X|Y), which means the probability of X given that Y. See footnote 2 above for more details.

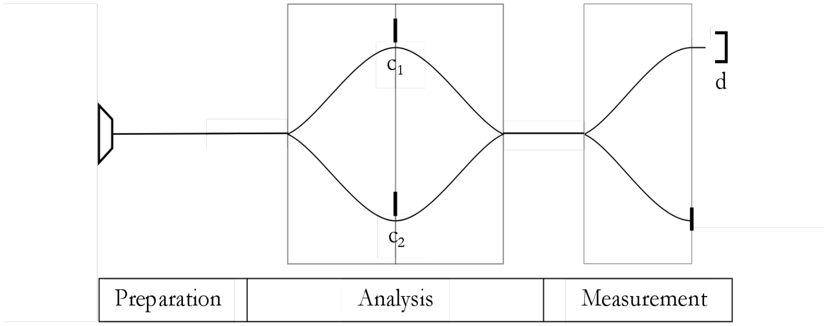

Diagram 3. The illustration above portrays a loop experiment for a two-state quantum system. There are two channels in the loop each with a moveable stop, so the experiment has three relevant configurations.

Those effects occur in recombined-beam experiments. Such experiments use an analyzer loop—two linearly arranged analyzers that are oppositely oriented.[8] The set-up is very similar to the two-slit experiment in that there are two channels c_1 and c_2 that can be stopped, and a detection point d. The key difference is that the formula that captures the probability values for the three key events is much simpler in the analyzer loop experiment than it is in the two-slit experiment, so the key result that only the fourth portrayal will work due to divergent conditional probabilities in the other three is more easily shown. The emphasis in this presentation is more qualitative than that in the works cited above, and it focuses more on interpretive matters concerning the potential/actual distinction; the two-slit experiment is best for such discussions due to its familiarity.

5. Another subtle aspect of the two-slit experiment—the conditionalization issue

Aside from the portrayal issue (Section 4), there is another issue that was left out of the preliminary account of the two-slit experiment (Section 3). Conditional probabilities are again involved as in the portrayal issue, but in a completely different way. When the conditional character of the operational probabilities is made explicit, the relevance of the potential/actual distinction comes to the fore.

8. One experiment of this type involves photons, in which case the key observable is photon polarization. The analyzers of the loop and the measuring device are calcite crystals. Calcite is bi-linearly refringent, meaning that it bends light into two distinct channels. In one channel the photons are polarized along the orientation of the crystal and in the other they are polarized in the direction orthogonal to both the crystal orientation and the direction of motion.

In presenting the preliminary account of constructive interference in the experiment, it was shown that if the probabilities associated with its three configurations are considered together via *general additivity*, a suggestive role for negative probabilities then emerges. That insight disappears, if the role of the associated configuration is made explicit using conditional probabilities; that is, it disappears. if one uses $P((AvB)\&D|C_1)$, $P(A\&D|C_2)$, and $P(B\&D|C_3)$ instead of $P((AvB)\&D)$, $P(A\&D)$, and $P(B\&D)$, respectively. One way to get the insight back is by introducing virtual probabilities, $P(A\&D|C_1)$ and $P(B\&D|C_1)$, and then by identifying them (meaning their respective values) with the corresponding actual probabilities, $P(A\&D|C_2)$ and $P(B\&D|C_3)$. One then has

$$P((A \vee B)\&D \mid C_1) > P(A\&D \mid C_1) + P(B\&D \mid C_1).$$

It appears much as before that constructive interference might be accounted for by way of *general additivity*, if $P(A\&B\&D|C_1)$ can be less than zero. In the preliminary account, $P(A\&B\&D|C_1)$ would be another virtual probability. However, when the portrayal issue is addressed in the manner indicated above, it would be interpreted as an actual probability. Some other probability would be interpreted as virtual and take on the explanatory role played by $P(A\&B\&D|C_1)$ in the preliminary account. Also, some other virtual probabilities would have to be involved in the account, and they would have to be identified with the actual probabilities of configurations 2 and 3 of the experiment.

6. Putting the two lessons above together

In Section 4, it was shown that each of the three configurations of the two-slit experiment should be portrayed conjunctively. For C_1 the probability is $P(A\&B\&D)$, for C_2 it is $P(A\&\neg B\&D)$, and for C_3 it is $P(\neg A\&B\&D)$. In Section 5, it was shown that these actual probabilities should be characterized conditionally with respect to the configuration, meaning as $P(A\&B\&D \mid C_1)$, $P(A\&\neg B\&D \mid C_2)$, and $P(\neg A\&B\&D \mid C_3)$, respectively. Using the developments in that section as a guide, the second and third should be associated with the virtual probabilities, $P(A\&\neg B\&D \mid C_1)$ and $P(\neg A\&B\&D \mid C_1)$, respectively. However, at this point more sophisticated considerations are need to obtain the counterpart to the key insight of the preliminary account.

First, one notices that *general additivity* cannot as before be used to generate the key insight. The form ascribed to the actual (operational) probabilities of the preliminary account suggested the relevance of that axiom since it relates those probabilities together. In the informed account, the form ascribed to the operational probabilities may be related by a theorem of the canonical theory (as well as of the new theory). A simple version of that theorem is the following:

$$P(A \vee B) = P(A\&B) + P(A\&\neg B) + P(\neg A\&B).$$

The key to deriving it is that $(A \vee B)$ is logically equivalent to $((A\&B) \vee (A\&\neg B) \vee (\vee A\&B))$. The relevant theorem has the following form:

$$P((A \vee B)\&D \mid C_1) = P(A\&B\&D \mid C_1) + P(A\&\neg B\&D \mid C_1) + P(\neg A\&B\&D \mid C_1).$$

The difficulty is that the above theorem does not formally manifest a role for negative probabilities, unlike general additivity. The route to that conclusion is less direct. To get there, it is necessary to use quantitative values, and it is relatively simple to obtain revealing values in loop experiments, the simplest being loop experiments for two-state quantum systems—see Diagram 3.

There are configurations of such experiments that yield revealing sets of values, such as those where $P(A\&B\&D \mid C_1) = 1$ and $P(A\&\neg B\&D \mid C_1) + P(\neg A\&B\&D \mid C_1) = ½$. Given these values and the above theorem, it follows that $P((A \vee B)\&D \mid C_1) > 1$, the upshot being that one can account for constructive quantum interference, if probability values greater than one are allowed. It turns out that this is effectively the same as the lesson drawn in the preliminary account, since hyper-unitary values and negative values go hand in hand in the new theory; the canonical theory's negation theorem, which says that $P(\neg A) = 1 - P(A)$ for any sentence A, is also a theorem of the new theory. Finally, it turns out that the maximal value of $P((A \vee B)\&D \mid C_1)$ is 2 over all possible loop-experiment configurations for two-state quantum systems, which entails (together with the negation theorem) that the range of probability values (for two state systems) is between –1 and 2.

7. Concluding remarks — some interpretive, formal, and practical considerations

Constructive quantum interference is very peculiar feature of quantum mechanics that deserves special interpretive consideration to provide a richer and fuller understanding of its character. Its peculiar character is understood by reference to what happens in the other two configurations where one of the paths is blocked. What happens in those configurations is relevant to what happens in the interference configuration, since the former is used to show why the latter is unexpected or peculiar. However, they must be understood differently when they are transported to that context since they are not operational in that context. The proposal here is that they are virtual probabilities having values that correspond to the actual ones.

Since the focus now is on the configuration in which both paths are open, the conjoined term D, reference to the associated configuration, and the interpretive move identifying a virtual probability value with the associated actual probability value may be suppressed for the sake of simplicity. The three measurement events of special interest for understanding constructive quantum interference may be characterized as $(A\&B)$, $(A\&\neg B)$, and $(\neg A\&B)$.

A fourth may be included that corresponds to (¬A&¬B), meaning the trivial experiment in which both passages are closed, in which case the probability for detection at d is zero. Probability values may be associated with all four, which are interpreted as virtual (in the configuration under discussion) with the exception of that corresponding to (A&B), which is interpreted as actual.

It turns out that a complete system of virtual probabilities involving only A and B (and D) can be generated from these four that includes all disjunctive, complementary, conditional, and marginal probabilities. Values can be obtained for all of them once one has probability values for the actual event (A&B) and the three virtual events (A&¬B), (¬A&B), and (¬A&¬B). It is difficult to say what those probabilities mean, nevertheless they must be regarded as meaningful since extent $P(A\&\neg B\&D \mid C_1)$ and $P(\neg A\&B\&D \mid C_1)$ are so regarded. That they are systematized via the non-standard theory of probability characterized above must also be regarded as significant.

For interpretive matters, there is a formal feature that each of four operational events (now including the trivial one) have in common that may be pertinent to the interpretation of virtual probabilities, namely that each of them is conjunctive. This suggests that the key operational events must be understood as being essentially and irreducibly non-local. Moreover, local events such as a photon's passing through c_1 can only be ascribed probabilities derivatively in terms of operational probabilities ascribed to the essentially non-local events, and these derivative probabilities are virtual and they are characterized by the axioms of the new probability theory.

Although non-local events are more fundamental from an epistemic (or operational) point of view, they need not be so regarded from an ontic point of view. Indeed, one may be inclined to regard local (but virtual) events as being more fundamental, ontologically speaking. In that case, interpretive considerations would move in the reverse direction. That is to say, virtual events that obey a non-standard theory of probability would be regarded as giving rise to operational events that obey the axioms of the standard theory. Such speculations are a bit premature, since it is important to determine the nature of virtual events in the generalized case (meaning for *n*-state systems, for any counting number *n*) and for compound systems consisting of two or more components. Some progress has been made in the generalized case, and it turns out that the non-standard probability theory must be suitably modified so that the range of probabilities becomes a function of *n* (Kronz, forthcoming). One interesting development that has been obtained for compound systems is that conditional probabilities do not stay within the standard range, as they do for single systems (Kronz, unpublished manuscript).[9]

Are there any reasons for thinking that the new theory of probability could have applicability beyond the quantum domain? Indeed, there are since there are already

9. For inquiries about this manuscript, please contact the author via email at kronz@mail.utexas.edu.

a number of non-quantum situations in which negative probabilities arise. They are discussed in the economics, queuing theory, and psychology literatures for example. A very brief sketch follows.

In economics, some market models involve negative probabilities, and their existence is typically seen as opening the door to arbitrage (risk-free profits arising from an asset price differential in two or more markets). The typical response is the introduction of a "no arbitrage" requirement that is satisfied by incorporating some ad hoc device into the model such as the substitution (at each stage of the process) of each negative probability with zero or some small fraction above zero (and similarly any hyper-unitary probabilities are replaced with one or some small fraction below one); for example, see (Rubenstein 1994) and (Kim and Park 2006). Very recently, an alternative perspective has been adopted towards negative probabilities and arbitrage possibilities, one that is more favorably disposed towards these notions (Khrennikov 2007).

In queuing theory (a part of operations research that is particularly useful for modeling computer systems and communications networks), there are models in which negative probabilities arise. Usually the strategy is to regard such probabilities as undesirable and eliminate them by substitution (Chandy and Sauer 1980), as in the previous case involving arbitrage. However, some queuing theorists have recently adopted a more favorable attitude (Tijms 2007).

Finally, in psychology negative probabilities arise in subjective probability judgment studies. A typical strategy is to set negative probabilities to zero, and it is regarded as an effective way of "removing inconsistencies" having to do with the possibility of creating a "Dutch book" (a betting scenario where the subject always loses); for example, see (Clemen and Ulu, forthcoming), which has additional references. However, if probabilities outside the normal range can be interpreted as virtual probabilities that correspond to virtual events, meaning that the occurrence or non-occurrence of those events cannot be operationally verified, then there will be no conflict with the rational constraints that are typically brought to bear in justifying the axioms of the standard theory of probability—see (Roberts, unpublished manuscript).[10] That is to say, Dutch books are effectively precluded. The range of applicability of the notion of negative probability now clearly extends beyond the quantum domain. Moreover, that notion is now being regarded as a potentially positive feature rather than a pathological one. The considerations introduced above and in the related works (Kronz 2006, 2007, forthcoming) serve to show that this notion can be formalized, systematically applied, and meaningfully interpreted in the quantum domain. There is substantial room for development on all of these fronts. Nevertheless, those considerations also hold promise for similar developments in the broader range of applicability.

10. For inquiries about this manuscript, please contact John Roberts via email at jtrosap@email.unc.edu.

References

Chandy, K. Mani and Sauer, Charles H. (1980). "Computational Algorithms for Product Form Queueing Networks," *Communications of the Association for Computing Machinery* 23: 573-583.

Clemen, Robert T. and Ulu, Canan (forthcoming). "Interior Additivity and Subjective Probability Assessment of Continuous Variables," *Management Science*.

Feynman, R. (1987) "Negative Probabilities," in *Quantum Implications*, editors B. J. Hiley and F. D. Peat, New York: Routledge.

Gnedenko, B. V. and Khinchin, A. Ya. (1961). *An Elementary Introduction to the Theory of Probability*. (San Francisco: W. H. Freeman and Co.).

Khrennikov, Andrei "Why so Negative about Negative Probabilities?" in Espen G. Haug (ed.), *Derivatives Models on Models*. San Francisco: John Wiley & Sons.

Kim, Joon and Park, Gun Youb (2006), "An Empirical Comparison of Implied Tree Models for KOSPI 200 Index Options," *International Review of Economics & Finance* 15: 52-71.

Kronz, F. M. (2006), "A Non-Monontonic Probability Theory for Spin-1/2 Systems." *International Journal of Theoretical Physics* 44: 1963-1976.

______, (2007), "A Non-Monontonic Probability Theory for Photon Polarization States." *Journal for Philosophical Logic* 36: 449-472.

______, (forthcoming), "Non-Monotonic Probability Theory for n-State Quantum Systems." *Studies in the History and Philosophy of Modern Physics*.

______, (unpublished manuscript), "Non-Monontonic Probabilities for Correlated Photon Pairs."

Muckenheim, W. (1986) "A Review of Extended Probabilities," *Physics Reports* 133: 337-401.

Roberts, J. (unpublished manuscript), "Non-Monotonic Probabilities as a Guide to Life?"

Rothman, T. and Sudarshan, G. (2000), "Hidden Variables or Positive Probabilities?" *International Journal of Theoretical Physics* 40: 1525-1543.

Rubinstein, Mark (1994). "Implied Binomial Trees," *The Journal of Finance* 49: 771-818.

Tijms, Henk (2007). "Negative Probabilities at Work in the M/D/1 Queue," *Probability in the Engineering and Informational Sciences* 21: 67–76.