The Infinite between the Inexhaustible and the Negation

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Resumen. *El Infinito entre lo inagotable y la negación*

Después de haber analizado las razones que indujeron a las antiguas matemáticas griegas y de que Aristóteles sólo admitiera una débil forma de lo infinito (el potencial), se explora una ampliación de este concepto más allá de sus referencias numéricas y geométricas. El infinito puede expresar la "inagotable" riqueza ontológica de los atributos de las entidades individuales o, en otro sentido, el infinito puede ser entendido como aquello "ilimitado". En este segundo sentido la "negación" (de las limitaciones) se presenta como una fuerza positiva en la formación del sentido ontológico de lo infinito.

Palabras clave: ontología infinita, Aristóteles, inagotable, entidades, limitada, ilimitada.

Abstract

After having analyzed the reasons that induced ancient Greek mathematics and Aristotle to admit only a weak form of the infinite (the potential one) a broadening of this notion beyond its numerical and geometrical references is explored. The infinite can express the "inexhaustible" ontological richness of the attributes of individual entities or, in another sense, the infinite can be understood as the "unlimited". In this second sense the "negation" (of limitations) appears as a positive force in the shaping of the ontological meaning of the infinite

Keywords: infinite, Aristotle, inexhaustible, ontology, entities, limited, unlimited.

From mathematics to ontology

The conviction that whatever *really* or *actually* exists is finite is by no means an Aristotelian tenet, but a very spontaneous idea of human mind. When we use the adjective "infinite" in common speech, we implicitly give it an epistemic meaning, amounting more or less to the *empirical* notion of "uncountable". When we say, for instance, that the grains of sand of Sahara are infinite, we do not mean that there is no finite natural number corresponding to their actual amount, but simply that none could "count" them concretely. Therefore, "infinite" is rather synonymous with "indefinite". As a matter of fact, the term *ápeiron* used by the early Greek philosophers is translated sometimes by "indefinite" and sometimes by "infinite", and this is not an unavoidable ambiguity, but rather mirrors the double sense (epistemic and ontological) of this notion. Remaining within the boundaries of common sense it seems that finiteness is the ontological mark of what really exists, both in the sense of being finite in number and magnitude, and of being determined (whatever exists must be "what it is" and in this sense something definite). This fact does not exclude that we may be unable to *know* by means of an affordable procedure the exact number of elements of a given collection of entities (or the exact size of a given magnitude), or to determine the exact nature of a given entity.

The creation of *rational mathematics* by the Greek scholars did not get rid of this spontaneous conviction. When Pythagoras made of numbers the ultimate essence of reality, he was thinking in terms of a *discrete* universal arithmetic, and the discovery of incommensurability, that challenged this basic view, was perceived as such a scandal of reason that (according to tradition) it was kept as a strict secret within his school. We shall come back to this issue in a moment, but it is interesting to note that a few centuries later, mathematics was used as a tool for proving the finiteness of the concrete world: Archimedes proposed a proof to show that the total amount of matter in the world is finite. In *The Sandreckoner* he proposes a number system capable of expressing numbers up to 8.10^{63}in modern notation. He argues in this work that this number is large enough to count the number of grains of sand which could be fitted into the universe. To sustain his argument he obviously has to give the dimensions of the universe to be able to count the number of grains of sand which it could contain, and he. states that Aristarchus has proposed a system with the sun at the centre and the planets, including the Earth, revolving round it. In quoting results on the dimensions of this system he states results due to Eudoxus, Phidias (his father), and to Aristarchus himself. Nothing could be a better evidence of the strong belief in the ontological finiteness of reality than this skillful cooperation of astrophysical theories and mathematical expertise.

This brief story helps us in understanding what Aristotle says in his *Physics*, after having developed a detailed series of arguments against actual infinity: "Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it" (*Phys,* III 7, 207b 28-30*)*. This statement, however, allows for the suspicion that not all mathematicians were really convinced that they could avoid actual infinity, and we can show certain reasons for that. When the Pythagoreans discovered the incommensurability of the side and the diagonal of the square, they were also obliged to recognize that the *point* cannot be the ultimate indivisible part of any figure, because in this case it should be the minimal *unit* whose multiples render all segments commensurable (because any segment must contain a "very great" number of points, but in any case a *finite* number of points). The conclusion was that the number of points in a segment is *infinite*. Is it actually infinite? If the diagonal is brought to lie on the same line as the side by a simple rotation, the endpoint of the diagonal coincides with a point on this line that *actually exists* in the line though not being attainable from the origin by any segment commensurable with the side. It is not arbitrary to see in this simple reasoning a *constructive* argument anticipating the diagonal method by which Cantor proved that the "infinity" of the continuum is greater than the "infinity" of the rational numbers. One could expect that, after this result, the Greek mathematicians were ready to admit actual infinity in mathematics, but unfortunately they had to meet a serious difficulty. A simple theorem of elementary geometry states that, if we join the middle points of an isosceles triangle , the segment so obtained is parallel to the base and equal to half of it. But if we project from the vertex any point of this segment we meet a single point on the base *and vice versa.* This means that there are *as many points* in the segment as in the base, but through a simple translation, this segment can be made identical with *a part* of the basis and it follows that this part contains as many points as the *whole* base. This method is again *constructive* and is nothing but the procedure of *biunivocal correspondence* that has been later introduced for the comparison of cardinalities in set theory. One of the most solid principles of ontology, however, was that the whole is always greater than its parts and the only way to safeguard this principle was to consider actual infinity as not legitimate. This was really a conscious *decision*, whose evidence we find in Euclid's axioms and postulates. The second postulate is often called the postulate "of the infinity of the straight line", but it actually sounds, literally, "To produce a finite straight line continuously in a straight line", which could be translated more explicitly, "Any straight line segment can be extended indefinitely in a straight line.", and this obviously expresses a potential infinity. (When Euclid speaks of line he always means a segment of line). The fifth "axiom", that is, proposition which is valid in general and not just for geometry, sounds "The whole is greater than the part". Of course, this "decision" implied heroic efforts for developing geometry in a "finitary" way, and Euclid's *Elements* are really such a masterpiece (let us simply mention the splendid construction of the theory of proportions that permitted to treat the comparison of magnitudes without resorting to commensurability).

The Aristotelian doctrine

The foregoing summary of the historical vicissitudes of the infinite in ancient Greek mathematics is useful in order to understand and appreciate the extremely complex and not always transparent discussion on the infinite we find in Aristotle, and especially in his *Physics*. We can note, first, that the tacit reference to mathematical discussions is present in the background of his doctrine, as it appears from the fact that, despite of often speaking in terms of pure ontological concepts such as substance, principle, attribute, the real referents of the concept of infinity remain of a mathematical nature, that is, number and magnitude ("how can the infinite be itself any thing, unless both number and magnitude, of which it is an essential attribute, exist in that way?", *Phys.,*III 5, 204a 18). Moreover, precise allusions to mathematical disputes are made when the logical difficulties regarding the infinite are mentioned (see, e.g., "Thus the view of those who speak after the manner of the Pythagoreans is absurd. With the same breath they treat the infinite as substance, and divide it into parts.", *Ibid.* – "Now, as we have seen, magnitude is not actually infinite. But by division it is infinite. (There is no difficulty in refuting the theory of indivisible lines.) The alternative then remains that the infinite has a potential existence.", *Phys.,*III 6, 206a 15-16). The second quotation is significant because it indicates two things, that Aristotle wanted to admit the existence of the infinite, and that mathematics was the main reason for this admission.

Indeed a rather common idea is that Aristotle was contrary to the admission of the infinite and that, at most, he downplayed it to the status of a pure "potentiality" that, in turn, is conceived as a kind of mental projection and nothing more. A not superficial reading of his considerations in the third Book of *Physics*, however (i.e., without broadening the analysis to other parts of his work) clearly shows that, on the contrary, he was convinced that the infinite *exists,* and made a considerable effort in order to find out a *kind of existence* suitable for it:

> But on the other hand to suppose that the infinite does not exist in any way leads obviously to many impossible consequences: there will be a beginning and an end of time, a magnitude will not be divisible into magnitudes, number will not be infinite. If, then, in view of the above considerations, neither alternative seems possible, an arbiter must be called in; and clearly there is a sense in which the infinite exists and another in which it does not. (*Phys.,* III 6, 206a 9-14)

At this point ontology comes into play, and more precisely the well known doctrine of the "analogy of being" that constitutes one of the major breakthroughs of the Stagyrite's philosophy. No wonder that this happens, since Aristotle - after having criticized in a "dialectical" way (in several parts of his *Metaphysics* and of his *Topics* and *Categories*) the doctrine of the "principles" developed by Plato and his followers - had presented in the first Book of *Physics* his own theory of the "principles" (matter, form, privation) and the notions of potentiality and act that enabled him to explain that the *substance* exists in the most proper and radical sense, while *accidents* exist but not independently of substances (they exist as properties *in* a substance, or as relations *between* substances). In such a way he was able to overcome the doctrine of the "univocity" of being maintained by the Eleatic school (that Plato had inaugurated in the dialogues *Sophist* and *Parmenides* in a still inadequate manner) and, in particular, could make fully intelligible the primordial *empirical evidence* of multiplicity and change while, at the same time, explaining that numbers are not substances.

Even after all these clarifications the issue of the infinite was not easily settled, because there seems to be a sense in which it has a real existence in mathematics, but in order to make this sense precise one should elaborate a satisfactory ontology of the mathematical entities, something that is difficult to find in Aristotle (and remains very problematic also today), and in the texts of *Physics* that we are considering he tries, so to speak, to evade the difficulty:

> This discussion, however, involves the more general question whether the infinite can be present in mathematical objects and things which are intelligible and do not have extension, as well as among sensible objects. Our inquiry (as physicists) is limited to its special subject-matter, the objects of sense, and we have to ask whether there is or is not among them a body which is infinite in the direction of increase. (*Phys.,* III 5, 204a35-b3)

These lines indicate that the question of the infinite in mathematics was left open by Aristotle, and that his doctrine of the infinite was restricted (at least for thematic reasons as he says here) to the realm of physical realities, of "bodies". Yet, though the articulated discussion he then develops has a flavor of generality, it is undeniable that the concepts he employs, such as addition, division, magnitude, number, are of a mathematical kind, so that the conclusion he reaches, that is, that the infinite exists only in the sense of potentiality, seems to be something like a reflection of a mathematical consideration of physical reality that brings with it the admission of a "weaker" form of that infinity that, perhaps, admits of a "strong" form in the domain of pure mathematics and in "things which are intelligible and do not have extension". We are not going to venture, however, in the exploration of this issue, and shall add only one more consideration.

Aristotle admits that the infinite exists only potentially, but is obliged to spend a lot of (not very persuasive) clarifications in order to explain that this potential existence differs from the conception of potentiality he has elaborated as one of the fundamental principles of his ontology. In fact potentiality in this fundamental sense means the orientation of change in the direction of the achievement of a goal, no potentiality can be defined unless it is referred to an *act*, but in the case of the infinite we cannot admit this: a potential infinity can never end up with an actual infinity, because an actual infinity cannot exist at all (at least in the domain of physical reality). It is not our intention to deepen the Aristotelian discussion here, and we shall simply submit that a plausible translation of the concept of potential infinity is that of an endless *process*, that is in itself well defined, but is neither a substance nor an accident and must be distinguished from its products. The number of the products increases but remains always finite, yet we can say that, in a certain sense, we encompass their *totality* because we know their "law of production" and we know that, by applying this law, we would be able to "catch" *anyone* of them. This means that we have to do with a *distributive universal* to which an *infinite class* corresponds because the members of this class belong to it not in virtue of the fact of being *actually* produced by the process, but only because *anyone* of them *could* be produced by the process.

We stop here and we leave the discussion of Aristotle in order to focus on a characteristic of the infinite that is implicit in the different presentations he makes of this concept: infinite is what is *inexhaustible*. We shall also consider the role that *negation* plays in the shaping of the notion of infinity.

The inexhaustible

From the above discussion it is rather clear that the concept of the infinite that the Greek philosophers had in mind was bound to the idea of *quantity* and to its mathematical expressions, that is, numbers and magnitudes, the latter being essentially understood along the patterns of geometrical intuition. This is why, on the one hand, they could not admit that the extension of a magnitude be infinite, neither actually nor potentially, because any given geometrical figure (plane or solid) must have a definite shape and is never such that 'it cannot be gone through' (as Aristotle says to define infinity of this kind) because it can always be crossed in any direction, and this remains true even if we enlarge it by "addition" of new parts (the result is always a finite figure). On the other hand, potential infinity is admitted only when it is the result of the indefinite division of a finite magnitude because, obviously, there is no increase of the *extension* neither of the magnitude nor of its parts, but only an indefinite increase of the *number* of such finite and smaller parts.

This is already an indication that, contrary to a first impression, the idea of endless repetition or indefinite reiteration is not sufficient to adequately characterize infinity. A better notion seems to be that of *inexhaustible* by which we mean a "plenitude", a "richness" that is *actually* given and at the same time contains an uncountable number of aspects or properties that can be made explicit through an endless exploration. We find here something adumbrated in the Aristotelian idea of potential infinity, but we do not really need to make a difference between potentiality and actuality, because these properties are *actually* present in the given entity, though we do not and possibly cannot know all of them at a glance. We find here the distinction between the epistemic and the ontological sense of infinity that we mentioned at the beginning.

This notion seems very abstract and possibly applicable only to god, but this is not the case. Let us simply put the problem of giving a full characterization of a concrete individual, for example, of Peter. We can start by saying tat he is a human being, and this already entails the fact of having a very large set of properties (he is a living being, an animal, a mammal, and each of these properties entails several other characteristics). Yet there re millions of individuals sharing the property of being human, and we can go on listing other properties of Peter, such as his weight, eyes color, nationality, profession, religious faith, birthday, and so on. By such a procedure we reduce the "intersection" of the classes to which Peter belongs, but we never attain a class to which only Peter *can* belong. Note that our problem is not that of "identifying" him, but to "fully characterize" him. For the first purpose a couple of "indexical" properties could be enough, for example an extremely exact determination of the place and time of his birth, but this information would say practically nothing of "what he really is". The full characterization of Peter, on the contrary, would need an "infinite" list of characteristics for at least two reasons. First, that among the properties of Peter there are also those depending on his being included in some relation (e.g. the fact of having visited London on the 3rd of July 1975), and these are really "uncountable". Second, the fact that many properties depend on the "point of view" from which Peter is considered, every point of view expressing an *attribute* that Peter can possess or not possess (but also the non-possession of a given attribute is a property) and the proliferation of such points of view is obviously unlimited. Our reasoning is not that bizarre, and reflects the awareness of the medieval logicians who said that the individual is *ineffable*, that is, impossible to characterize univocally by means of an explicit discourse, and this because any explicit discourse contains a finite number of predicates, each predicate expressing one *general* attribute and the conjunction of a finite number of general attributes falls short of characterizing a single entity.

Of course, several questions could be asked regarding our sketch. The first could concern the "reality" of the properties of Peter, and open up something similar to the old dispute on the "universals". We are not much concerned with this issue, not only because we have learned already from Aristotle that there are many senses of existence, that allow for affirming a certain "kind of existence" also for properties and relations, but also because if it is *true* to say of Peter, e.g., that he is a human being, an Italian, a mathematician, that he speaks English, German, Italian and French, that he loves the music of Mozart, and so on, this plainly means that all these properties and relations *really* belong to Peter, that they are part of the "richness" of his personality.

Another question could be the following: though, according to the above discourse, the number of properties and relations that can be attributed to an individual is extremely large and even impossible to determine precisely, it remains in any case *finite* and not infinite. This seemingly obvious remark is far from self-evident and would be in need of some argument. It is rather an unconscious generalization of the common sense conviction (shared by

Aristotle and supported by arguments by him) that no actual infinity of substances exists in the physical world, but this does not imply that the infinity may exist in the world of properties and relations. This, however, is not what really matters, since the example we are considering precisely shows how poor it is to restrict the notion of infinity to infinity *in number*. When we say that the properties and relations characterizing an individual are infinite, we mean that they constitute an *inexhaustible* richness that really "cannot be gone through" by any proposed procedure, owing also to their indefinite *variety.* Let us note that a pale image of this conception of infinity is adumbrated in the Aristotelian admission of potential infinity when this is the result of a process of successive subdivision of a finite magnitude in smaller and smaller parts. This process, so to speak, brings to light the "internal richness" of the original finite magnitude, though this richness only concerns the existence of smaller parts, and their number can be considered as "potentially infinite" simply because it is "inexhaustible" by this process.

Negation and infinity

We have considered so far infinity, from an ontological point of view, as expression of richness, of the abundance of being that, as such, prevails over negation, over non-being. Yet negation plays a not negligible role in ontology and, in ontologies of a monistic kind, it even serves to define the individual simply as a "negation" of the unique substance, according to the famous statement *omnis determinatio est negatio* (every determination is a negation) expressed by Spinoza, taken up by Hegel and other monistic philosophers. We are not interested, however, in this issue and simply note that, when Plato abandoned the absoluteness of the Parmenidean dichotomy between "being" and "non-being", and stressed that when something is "not P" does not dissolve into the "non-being", but simply remains into "being" and is simply *different* from P, he opened the path to a fruitful use of negation in ontology and epistemology, inaugurating that method of "dichotomy" that was a powerful tool for conceptual analysis.

The link of negation with infinity appears in any statement consisting in a *pure negation*, such as, for instance, "X is not a dog". One cannot say that this statement says absolutely "nothing" about X, but in any case it leaves open an *infinity* of possibilities regarding the nature of X (it could be a horse, a car, a theorem, a dream, a symphony, etc.). Hence, pure negation, so to speak, "infinitizes", but then this infinite space of possibilities can and must be reduced by subsequent steps that cannot consist only of negations, but must necessarily include the mention of *positive* features that determine a more restricted domain in which sometimes negation can play a useful role by excluding one among a definite number of possibilities. All this means that negation, though "opening" in a certain sense toward infinity, is unable to "fill" this infinity and this in particular shows that the "richness" and "abundance" of determinations of which we have spoken above cannot be constructed via negation (though, as we shall see later, some significant properties are so defined).

This reflection gives us the opportunity of pointing out a misunderstanding of Hegel when in his *Logic* he comes to identify pure being with nothing, along the following reasoning. If we consider en entity X and begin to deprive it of its properties one after the other, at the end the only property that remains is simple existence or pure being, which is so totally empty that is equal to nothing. This reasoning presupposes that properties are something that is *added* to being, that are in a certain sense external to it, whereas the right view is that we find the properties by "digging" into being, by sounding the inexhaustible richness of "what it is". Of course, this is not the whole story about Hegel's philosophy, but we have mentioned it only to stress the inadequacy of conceiving infinity according to the pattern of "addition".

The domain in which negation plays a direct role in the definition of infinity is when the infinite is conceived as the *unlimited*, that is, when the *negation of limits* becomes the core of the notion. This point of view is alien to the classical Greek philosophers because (as we have already noted) they remain under a certain influence of geometrical intuition and in their geometry all figures have definite shape and boundaries, that is, they are limited. The presence of a limit (in this sense) is a condition for being something *determined* and the absence of a limit amounts to *incompleteness*, to being *unfinished* and therefore also *imperfect*. Let us note, by the way, that this idea is not a simple projection of a geometric intuition, but expresses a point of view that has remained in our tradition and is expressed in the famous Latin statement, *finis coronat opus* (the end crowns the work), that means that an unfinished work is as such imperfect, is unaccomplished, and we still share to some extent this idea. This explains in particular why the Greek term *ápeiron* can be translated by "indefinite" as well as by "infinite": for us there is a difference between the meanings of these two words, for the Greeks they were practically synonymous. Therefore we can understand why Aristotle, in the texts of *Physics* that we have considered, makes two statements that sound paradoxical to us. On the one hand he says that the Whole is that which has nothing outside it (and according to our modern understanding this should mean that it is infinite), but on the other hand he says that the Whole is finite, precisely because nothing is wanting in it, it is hence complete, and "nothing is complete (*téleion*) which has no end (*télos*); and the end is a limit." (*Phys.,* III 6, 207a 10-15). In conclusion, we can say that in ancient philosophy the infinite is a *negative concept*, in the sense that it expresses the idea of a process that cannot be completed, that is "inexhaustible" not in the positive sense of superabundance of which we have spoken above, but in the sense of never attaining its full achievement. It remains so conceived not only in Aristotle and, say, Plotinus, but still in Kant and in mathematics even after having been admitted in it within the technical notion of *limit*. Only with the creation of the Cantorian set theory, as is well known, the use of actually infinite classes and of "infinite magnitudes" (transfinite cardinals and ordinals) was admitted in mathematics. We are not going to consider this story and simply note that it was not just an expression of intellectual boldness, but relied upon a *positive characterization* of the infinite This characteristic property is precisely the *paradox* that already in antiquity had led people to reject the infinite: *in an infinite collection the part can be equivalent to the whole.*

The classical meaning according to which "limit" indicates the defining border that contains an entity complete and perfect in itself (and which gives rise to the adjective "delimited") became secondary with respect to a less positive meaning that expresses the idea of only partial realization, of incomplete fulfillment, of imperfection (and which gives rise to the adjective "limited"). The absence of limits in this second sense has been adopted as a definition of infinity especially in the theological contexts during the whole Western Christian tradition, but its origins can be found in Plotinus, who distinguished two kinds of infinity, the infinity of number, that is characterized as "inexhaustibility", and the infinity of the One, that is "the non-limitation of power" (*En,* VI, 9, 6). It is obvious that for any religion for which God is unique and not material, neither the characteristic of number nor that of magnitude could be predicated of him; at the same time God was conceived as the supreme Being, endowed with all possible perfections, and qualified as *infinite* in this new sense, that is, the sense of the *negation of any limit* in its nature, for He transcends every possible degree of perfection. This conception of God as infinite, actually as the only real infinite, inspires two of the best known "ways" for proving the existence of God and knowing him, the *via eminantiae* (the way of eminence) and the *via negationis* (the way of negation). The first argues, for example, on the reduction from the diminished to the perfect good, or on the comparison of the degrees of truth and, in general, derives knowledge of God from predicating of God the creature's perfections in the most supreme fashion or degree. The second derives knowledge of God from removing the "imperfections" of the perfections of creatures. They are obviously very similar and, in the last analysis, can be seen as different forms of the *negation of limits*.

In modern philosophy (that is, outside theology) the conception of the infinite as non-limitation of power is explicitly developed by Fichte and finds its best expression in Hegel. He distinguishes the "false infinity" from the "true infinity". The first is the infinity of mathematics, in which there is certainly a progress "beyond" the finite but it never attains this "behind" and remains therefore at the status of an "ought to be" and never of a "being". The true infinite, on the contrary, is reality in its conscious actualization, the infinite is "the force of the existence". This conception has deeply influenced the romantic philosophy.