Publ. Mat. **54** (2010), 25–52

STABILIZATION IN $H^\infty_{\mathbb{R}}(\mathbb{D})$

Brett D. Wick

Abstract

It is shown that for $H^\infty_{\mathbb{R}}(\mathbb{D})$ functions f_1 and f_2 with

$$\inf_{z\in\mathbb{D}}(|f_1(z)|+|f_2(z)|)\geq\delta>0$$

and f_1 being positive on the real zeros of f_2, then there exists $H^\infty_{\mathbb{R}}(\mathbb{D})$ functions g_2 and g_1, g_1^{-1} with norm controlled by a constant depending only on δ and

$$g_1f_1+g_2f_2=1\quad\forall\, z\in\mathbb{D}.$$

These results are connected to the computation of the stable rank of the algebra $H^\infty_{\mathbb{R}}(\mathbb{D})$ and to results in Control Theory.

Contents

2000 *Mathematics Subject Classification.* Primary: 46E25, 46J10.
Key words. Banach Algebras, Control Theory, Corona Theorem, Stable Rank.
Research supported in part by a National Science Foundation DMS Grant # 0752703.

Notation

$:=$	equal by definition;
$\mathbb{C}$	complex plane;
$\operatorname{Re} z$	real part of $z \in \mathbb{C}$;
$\operatorname{Im} z$	imaginary part of $z \in \mathbb{C}$;
$\mathbb{D}$	the unit disc, $\mathbb{D} := \{z \in \mathbb{C} : \vert z\vert < 1\}$;
$\mathbb{C}_+$	the upper half-plane, $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$;
$\mathbb{T}$	the unit circle, $\mathbb{T} := \partial\mathbb{D}$;
$\mathbb{R}$	the real line, $\mathbb{R} := \partial\mathbb{C}_+$;
H^∞	the algebra of all bounded analytic functions on either $\mathbb{C}_+$ ($H^\infty(\mathbb{C}_+)$) or $\mathbb{D}$ ($H^\infty(\mathbb{D})$);
$H^\infty_{\mathbb{R}}$	the algebra of all bounded analytic functions with real Fourier coefficients on either $\mathbb{C}_+$ ($H^\infty_{\mathbb{R}}(\mathbb{C}_+)$) or $\mathbb{D}$ ($H^\infty_{\mathbb{R}}(\mathbb{D})$);

0. Introduction and main results

The stable rank of a ring (also called the Bass stable rank) was introduced by H. Bass in [**1**] to assist in computations of algebraic K-Theory. We recall the definition of the stable rank.

Let A be an algebra with a unit e. An n-tuple $a \in A^n$ is called *unimodular* if there exists an n-tuple $b \in A^n$ such that $\sum_{j=1}^n a_j b_j = e$. An n-tuple a is called *stable* or *reducible* if there exists an $(n-1)$-tuple x such that the $(n-1)$-tuple $(a_1+x_1a_n, \dots, a_{n-1}+x_{n-1}a_n)$ is unimodular. The *stable rank* (also called $bsr(A)$ in the literature) of the algebra A is the least integer n such that every unimodular $(n+1)$-tuple is reducible.

The stable rank is a purely algebraic concept but can be combined with analysis when studying commutative Banach algebras of functions. In this context, the stable rank is related to the zero sets of ideals, and the spectrum of the Banach algebra. The stable rank for different algebras of analytic functions have been considered by many authors. The computation of the stable rank of the disc algebra $A(\mathbb{D})$ was shown to be one by Jones, Marshall and Wolff [**7**]. The computation was done for sub-algebras of the disk algebra $A(\mathbb{D})$ by Corach-Suárez [**4**], and Rupp [**10**].

For the Banach algebra $H^\infty(\mathbb{D})$, the classification of its unimodular elements and its stable rank are well understood. Carleson's Corona Theorem, see [**3**], can be phrased as an n-tuple $(f_1, \dots, f_n) \in H^\infty(\mathbb{D})^n$

is unimodular if and only if it satisfies the Corona condition,

$$\inf_{z\in\mathbb{D}} (|f_1(z)| + \cdots + |f_n(z)|) = \delta > 0.$$

The stable rank of $H^\infty(\mathbb{D})$ was computed by S. Treil and is one of the motivations for this paper. Treil's result is the following theorem:

Theorem 0.1 (S. Treil, [**12**]). *Let $f_1, f_2 \in H^\infty(\mathbb{D})$ be such that $\inf_{z\in\mathbb{D}}(|f_1(z)| + |f_2(z)|) = \delta > 0$. Then there exists $g_1, g_2, g_1^{-1} \in H^\infty(\mathbb{D})$ with $\|g_1\|_\infty$, $\|g_2\|_\infty$ and $\|g_1^{-1}\|_\infty$ controlled by $C(\delta)$, a constant only depending on δ, and*

$$1 = f_1(z)g_1(z) + f_2(z)g_2(z) \quad \forall\, z \in \mathbb{D}.$$

It is immediately apparent that Theorem 0.1 implies the stable rank of $H^\infty(\mathbb{D})$ is one. Questions about the stable rank of some sub-algebras of $H^\infty(\mathbb{D})$ have been studied by Mortini [**8**].

It is possible to phrase Treil's result [**12**] in the language of Control Theory. In this language, the result can be viewed as saying that it is possible to stabilize (*in the sense given above*) a linear system (*the Corona data, viewed as a rational function*) via a stable (*analytic*) controller. But, in applications of Control Theory, the linear systems and transfer functions have real coefficients, so in this context Treil's result is physically meaningless. From the point of view of Control Theory, it is important to know if results like Theorem 0.1 hold, but for a more physically meaningful algebra, and serves as the main motivation for this paper. This paper is interested in questions related to the stable rank of a natural sub-algebra of $H^\infty(\mathbb{D})$, the real Banach algebra $H^\infty_{\mathbb{R}}(\mathbb{D})$. In particular, does some variant of Theorem 0.1 hold for this algebra?

First, recall that $H^\infty_{\mathbb{R}}(\mathbb{D})$ is the subset of $H^\infty(\mathbb{D})$ with the additional property that the Fourier coefficients of an element of $H^\infty_{\mathbb{R}}(\mathbb{D})$ must be real. This property can be captured by the following symmetry condition:

$$f(z) = \overline{f(\overline{z})} \quad \forall\, z \in \mathbb{D}.$$

When we translate between $\mathbb{D}$ and $\mathbb{C}_+$ this condition takes the following form,

$$f(z) = \overline{f(-\overline{z})} \quad \forall\, z \in \mathbb{C}_+.$$

This condition is implying that the functions in $H^\infty_{\mathbb{R}}(\mathbb{D})$ possess a symmetry that is not present for general $H^\infty(\mathbb{D})$ functions.

Carleson's Corona result is inherited by the algebra $H^\infty_{\mathbb{R}}(\mathbb{D})$. More precisely, it is an immediate application of the usual Corona Theorem and the symmetry properties of $H^\infty_{\mathbb{R}}(\mathbb{D})$ to show that an n-tuple

$(f_1, \dots, f_n) \in H^\infty_{\mathbb{R}}(\mathbb{D})^n$ is unimodular if and only if it satisfies the Corona condition,

$$\inf_{z\in\mathbb{D}} (|f_1(z)| + \cdots + |f_n(z)|) = \delta > 0.$$

Indeed, one direction is immediate, and in the other direction, if we know that

$$\inf_{z\in\mathbb{D}} (|f_1(z)| + \cdots + |f_n(z)|) = \delta > 0,$$

then we can find a solution $(g_1, \dots, g_n) \in H^\infty(\mathbb{D})^n$. We then symmetrize the g_j via the operation

$$\tilde{g}_j(z) := \frac{g_j(z) + \overline{g_j(\overline{z})}}{2}.$$

The $\tilde{g}_j \in H^\infty_{\mathbb{R}}(\mathbb{D})$ and will then be the $H^\infty_{\mathbb{R}}(\mathbb{D})$ Corona solution we are seeking.

This leads to the main question considered in this paper. Is Theorem 0.1 true for the algebra $H^\infty_{\mathbb{R}}(\mathbb{D})$? Namely, given Corona data f_1 and f_2 in $H^\infty_{\mathbb{R}}(\mathbb{D})$, is there a solution g_1 and g_2 to the Corona problem with g_1 invertible in $H^\infty_{\mathbb{R}}(\mathbb{D})$? This can also be phrased as attempting to show that the stable rank of $H^\infty_{\mathbb{R}}(\mathbb{D})$ is one.

It is easy to see that there is an additional necessary condition. Suppose that Theorem 0.1 were true for $H^\infty_{\mathbb{R}}$ functions, then we shall see that the real zeros of f_1 and f_2 must intertwine correctly. Indeed, let λ_1 and λ_2 be real zeros of f_2. Then we have

$$\begin{aligned} f_1(\lambda_1)g_1(\lambda_1) &= 1 \\ f_1(\lambda_2)g_1(\lambda_2) &= 1. \end{aligned}$$

Now $f_1(\lambda_1)$ and $f_1(\lambda_2)$ must have the same sign at these zeros. If this were not true, then without loss of generality, suppose that $f_1(\lambda_1) > 0 > f_1(\lambda_2)$. Then $g_1(\lambda_1) > 0 > g_1(\lambda_2)$. By continuity there will exist a point λ_{12}, between λ_1 and λ_2, with $g_1(\lambda_{12}) = 0$. But this contradicts the fact that $g_1^{-1} \in H^\infty_{\mathbb{R}}(\mathbb{D})$. So f_1 must have the same sign at real zeros of f_2. Abusing notation slightly, we will say that f_1 is *positive on the real zeros* of f_2, if f_1 has the same sign at all real zeros of f_2.

This is also an intertwining condition of the zeros of f_1 and f_2. More precisely, the function f_1 is positive on the real zeros of f_2 if, and only if, between every real zero of f_2 there must be an even number of real zeros of f_1. In Control Theory, this condition is called the *parity interlacing property* and appears in conditions for the stabilization of a linear system.

The main result of this paper is the following theorem.

Theorem 0.2. *Suppose that $f_1, f_2 \in H^\infty_{\mathbb{R}}(\mathbb{D})$, $\|f_1\|_\infty, \|f_2\|_\infty \leq 1$, f_1 is positive on the real zeros of f_2 and*

$$\inf_{z\in\mathbb{D}} (|f_1(z)| + |f_2(z)|) = \delta > 0.$$

Then there exists $g_1, g_1^{-1}, g_2 \in H^\infty_{\mathbb{R}}(\mathbb{D})$ with $\|g_1\|_\infty, \|g_2\|_\infty, \|g_1^{-1}\|_\infty \leq C(\delta)$ and

$$f_1(z)g_1(z) + f_2(z)g_2(z) = 1 \quad \forall\, z \in \mathbb{D}.$$

The reasoning leading to this theorem in turn implies that the stable rank of $H^\infty_{\mathbb{R}}(\mathbb{D})$ is at least two because the additional condition of f_1 being positive on the real zeros of f_2 is necessary to find Corona solutions with one of them invertible.

We remark that these results transfer immediately to analogous statements $H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ via the standard conformal mapping between $\mathbb{C}_+$ and $\mathbb{D}$.

Throughout the paper, the adjective *real symmetric* is used to indicate that the function in question satisfies the symmetry condition

$$f(z) = \overline{f(\overline{z})} \quad \forall\, z \in \mathbb{D} \quad \text{or} \quad f(z) = \overline{f(-\overline{z})} \quad \forall\, z \in \mathbb{C}_+$$

which will be clear from the context. We also will work with either the upper half-plane or the disc, and will transfer the problem to either domain, depending upon where the problem is easiest to work with.

Acknowledgements. The author would like to thank S. Treil for suggesting this problem to him, the many useful discussions that we had about the problem, and the insightful comments he made on an early draft of this paper. The author would also like to thank the referee for several useful remarks about the paper.

1. Idea of the proof

The method of proof is inspired by Treil's proof in [**12**], but must be suitably modified. A key component of the modifications is to exploit the symmetry properties of $H^\infty_{\mathbb{R}}(\mathbb{D})$ functions. Additionally, it is important to include the condition about f_1 being positive on the real zeros of f_2 in an appropriate way.

It is straightforward to demonstrate that it is enough to prove Theorem 0.2 only in the case of real symmetric rational functions whose zeros satisfy the additional condition about intertwining zeros. While not immediately clear, it is also true that one can show that it is sufficient to prove Theorem 0.2 in the case of real symmetric finite Blaschke products possessing this condition as well. This can be seen from the appropriate modifications of the original proof by Carleson and the proof by Treil [**12**]. Thus, we specialize to the situation where we have real symmetric simple Blaschke products with the intertwining zero property.

To prove Theorem 0.2 we begin by solving an interpolation problem. Suppose that f_1 and f_2 are real symmetric finite simple Blaschke products, which satisfy the condition that f_1 is positive on the real zeros of f_2 and

$$\inf_{z\in\mathbb{C}_+}(|f_1(z)|+|f_2(z)|)=\delta>0.$$

Note that the function $f_1(z)$ satisfies $|f_1(z)|\geq\frac{\delta}{2}$ on the set $\{z\in\mathbb{C}_+ : |f_2(z)|<\frac{\delta}{2}\}$. Further, observe that each component of this set is simply connected. With this in hand, we now use the following proposition. Because of these properties, since f_1 is positive on the real zeros of f_2 and the Corona condition holds there is a well defined branch of $\log f_1$ with respect to the symmetric set $\{z\in\mathbb{C}_+ : |f_2(z)|<\frac{\delta}{2}\}$ that can be chosen with the additional property that $\log f_1(z)=\overline{\log f_1(-\overline{z})}$.

Proposition 1.1. *Suppose that f_1 and f_2 are real symmetric finite simple Blaschke products which satisfies the condition that f_1 is positive on the real zeros of f_2 and*

$$\inf_{z\in\mathbb{C}_+}(|f_1(z)+|f_2(z)|)=\delta>0.$$

On the set $\{z : |f_2(z)|<\frac{\delta}{2}\}$ if $\log f_1$ is a bounded analytic real symmetric function, then there exists a function $h\in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ such that

$$\log f_1(z)=h(z)\quad \textit{for all } z \textit{ in the zero set of } f_2$$

and $\|h\|_\infty\leq C(\delta)$.

The proof of Proposition 1.1 is a variation of a standard result found first in [**3**] and further explained in [**5**]. But we must incorporate the condition about the zeros in an appropriate manner. To do this we use results from [**13**]. The condition that f_1 is positive on the real zeros of f_2 implies that $\log f_1$ is well defined on the set where f_2 is small. Thus, it is possible to solve the above interpolation problem, namely, finding the function h which takes the values of $\log f_1$. One then need only choose a function with minimal norm. This is accomplished by mimicking the proof found in [**3**] or [**5**] using Carleson's Lemma constructing Carleson contours about the zeros of the Blaschke product f_2.

Since f_1 is rational, we have a bounded branch of the logarithm $\log f_1$ on the set $\{z : |f_2|<\frac{\delta}{2}\}$. Additionally, we have f_1 is positive on the real zeros of f_2 so we apply Proposition 1.1 to f_1 and f_2. This gives rise to a function $h\in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ with $\|h\|_\infty\leq C\|\log f_1\|_\infty$, and

$$e^{h(z)}=f_1(z)\quad\text{for all } z \text{ in the zero set of } f_2.$$

The function e^h is invertible in $H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ and there is a function $G \in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ with $e^h = f_1 + f_2 G$.

This is almost enough to conclude the proof of the theorem. If $\log f_1$ were bounded on $\{z \in \mathbb{C}_+ : |f_2(z)| < \frac{\delta}{2}\}$ by a constant only depending on δ and not on the degrees of f_1 and f_2, we would be done. However, this is not generally true, so we need a method to overcome this difficulty. To do this we will find an analytic function κ that is real symmetric and will "correct" the function f_1.

To find the correcting function, we will prove the following propositions.

Proposition 1.2. *Let $p, q \in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ be finite simple real symmetric Blaschke products with $\inf_{z\in\mathbb{C}_+}(|p(z)| + |q(z)|) = \delta > 0$. Then there exists an analytic function κ with the following properties:*

(i) $|\mathrm{Re}\, \kappa(z)| \le C(\delta) \quad \forall\, z \in \mathbb{C}_+$;

(ii) $|\log p(z) - \kappa(z)| \le C(\delta)$ *for all z in $\{z \in \mathbb{C}_+ : |q(z)| < \delta'\}$ for some $0 < \delta' \le \delta$ and an appropriate branch of* $\log p$ *on the set* $\{z \in \mathbb{C}_+ : |q(z)| < \delta'\}$;

(iii) $\kappa(z) = \overline{\kappa(-\overline{z})} \quad \forall\, z \in \mathbb{C}_+$.

To find κ we will construct an auxiliary function V.

Proposition 1.3. *Let $p, q \in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ be finite simple real symmetric Blaschke products with $\inf_{z\in\mathbb{C}_+}(|p(z)| + |q(z)|) = \delta > 0$. Then there exists a function V with the following properties:*

(i) $|\mathrm{Re}\, V(z)| \le C(\delta) \quad \forall\, z \in \mathbb{C}_+$;

(ii) $|\log p(z) - V(z)| \le C(\delta)$ *for all z in $\{z \in \mathbb{C}_+ : |q(z)| < \delta'\}$ for some $0 < \delta' \le \delta$ and an appropriate branch of* $\log p$ *on the set* $\{z \in \mathbb{C}_+ : |q(z)| < \delta'\}$;

(iii) $V(z) = \overline{V(-\overline{z})} \quad \forall\, z \in \mathbb{C}_+$;

(iv) *some conditions to guarantee the existence of a bounded solution v on the entire upper half-plane $\mathbb{C}_+$ of the equation $\bar\partial v = \bar\partial V$, in particular:*

   (a) $|\Delta V(z)|\, \mathrm{Im}\, z\, dx\, dy$ *is a Carleson measure with intensity* $C(\delta)$;

   (b) $|\partial V(z)|\, dx\, dy$ *is a Carleson measure with intensity* $C(\delta)$;

   (c) $|\Delta V(z)| \le \frac{C(\delta)}{(\mathrm{Im}\, z)^2} \quad \forall\, z \in \mathbb{C}_+$.

Proposition 1.3 immediately implies Proposition 1.2. To see this, once we have constructed V, set $\kappa = V - v$. Trivially, we have that κ is analytic because the $\bar\partial$-derivatives of V and v agree. In Section 2 we show that it is possible to force the solution v to also possess the property $v(z) =$

$\overline{v(-\overline{z})}$, so κ will have the symmetry property. Condition (i) on κ then follows from the corresponding condition on V and the boundedness of the solution v (condition (iv) above). Finally, we have

$$|\log p - \kappa| = |\log p - V + v| \leq |\log p - V| + |v|.$$

So, the boundedness of v and condition (ii) on V imply the corresponding condition on κ. We will prove Proposition 1.3 in Sections 3 and 4.

Now, consider the function $e^{-\kappa} f_1$. Conditions (i) and (iii) of Proposition 1.2 imply that $e^{\kappa} \in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$. Condition (ii) implies that $e^{-\kappa} f_1$ has a bounded branch of logarithm on $\{z \in \mathbb{C}_+ : |f_2(z)| < \delta'\}$. Applying Proposition 1.1 and the argument that followed, we obtain $e^h = f_1 e^{-\kappa} + f_2 G_1$ with $h, G_1 \in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ with $\|h\|_\infty \leq C(\delta)$. Set $g_2 := G_1 e^{-h}$ and $g_1 := e^{-(\kappa+h)}$. Then we have that $g_1, g_2, g_1^{-1} \in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ such that $\|g_1\|_\infty$, $\|g_2\|_\infty$, $\|g_1^{-1}\|_\infty$ is controlled by $C(\delta)$ and

$$f_1(z) g_1(z) + f_2(z) g_2(z) = 1 \quad \forall\, z \in \mathbb{C}_+.$$

This argument then shows that to prove Theorem 0.2, we need to establish Proposition 1.3.

2. Construction of bounded solutions to the $\bar{\partial}$-equation with $H^\infty_{\mathbb{R}}(\mathbb{D})$ data

As is well known, solutions to the $\bar{\partial}$-equation on the disc are intimately connected with solutions to the Corona Problem because of connections between $\bar{\partial}$-equations and Carleson measures. We now recall the definition of Carleson measures. Let I be an interval in $\mathbb{R}$ and form the Carleson square $Q = Q(I)$ over I,

$$Q(I) := \{z \in \mathbb{C}_+ : \operatorname{Re} z \in I, \operatorname{Im} z \leq |I|\}.$$

Then we say a non-negative measure μ in the upper half-plane $\mathbb{C}_+$ is a Carleson measure if

$$\sup_I \frac{\mu(Q(I))}{|I|} := K < \infty,$$

with the supremum taken over all intervals I in $\mathbb{R}$. The constant K will be called the *intensity* of the Carleson measure. It is immediate to transfer these notions to the disc $\mathbb{D}$.

We have the following well known theorem, which can be found in [**12**].

Theorem 2.1. *Let V be a $\mathcal{C}^2$ function on the unit disc $\mathbb{D}$ which is continuous up to the boundary $\mathbb{T}$. Suppose that*

(1) $|\bar{\partial} V(z)|\,dx\,dy$ *is a Carleson measure with intensity* K_1*;*
(2) $|\Delta V(z)|(1-|z|^2)\,dx\,dy$ *is a Carleson measure with intensity* K_2*;*
(3) $|\Delta V(z)| \leq \frac{K_3}{(1-|z|^2)^2}$.

Then the equation

$$\bar{\partial} v = \bar{\partial} V$$

has a bounded solution v on all of $\mathbb{D}$ (not only the boundary $\mathbb{T}$) with

$$|v(z)| \leq C(K_1, K_2, K_3) \quad \forall\, z \in \mathbb{D}.$$

Now we want to show that if the function V has the property that $V(z) = \overline{V(\overline{z})}$ for all $z \in \mathbb{D}$, then this property is inherited by the solution v. This leads to the following theorem.

Theorem 2.2. *Let V be a $\mathcal{C}^2$ function on the unit disc $\mathbb{D}$ which is continuous up to the boundary $\mathbb{T}$. Suppose that*

(1) $V(z) = \overline{V(\overline{z})}$ *for all* $z \in \mathbb{D}$*;*
(2) $|\bar{\partial} V(z)|\,dx\,dy$ *is a Carleson measure with intensity* K_1*;*
(3) $|\Delta V(z)|(1-|z|^2)\,dx\,dy$ *is a Carleson measure with intensity* K_2*;*
(4) $|\Delta V(z) \leq \frac{K_3}{(1-|z|^2)^2}$.

Then the equation

$$\bar{\partial} v = \bar{\partial} V$$

has a bounded solution v on all of $\mathbb{D}$ (not only the boundary $\mathbb{T}$) with

$$|v(z)| \leq C(K_1, K_2, K_3) \text{ and } v(z) = \overline{v(\overline{z})} \quad \forall\, z \in \mathbb{D}.$$

Proof of Theorem 2.2: We first apply Theorem 2.1 to find a solution v which is bounded on all of $\mathbb{D}$. Then we replace it with the following function

$$\tilde{v}(z) := \frac{v(z) + \overline{v(\overline{z})}}{2}.$$

Note that we have $\tilde{v}(z) = \overline{\tilde{v}(\overline{z})}$ and that $\|\tilde{v}\|_{H^\infty(\mathbb{D})} \leq \|v\|_{H^\infty(\mathbb{D})} \leq C(K_1, K_2, K_3)$. We only need that $\bar{\partial}\tilde{v} = \bar{\partial} V$ for all $z \in \mathbb{D}$. But, this follows from direct application of the chain rule. Indeed,

$$\begin{aligned}\bar{\partial}V(z) &= \frac{1}{2}(\bar{\partial}(V)(z)+\bar{\partial}(V)(z))\\ &= \frac{1}{2}(\bar{\partial}(V)(z)+\bar{\partial}(\overline{V(\overline{z})}))\\ &= \frac{1}{2}(\bar{\partial}(v)(z)+\bar{\partial}(\overline{v(\overline{z})}))\\ &= \bar{\partial}\tilde{v}(z).\end{aligned}$$ □

Using the conformal equivalence between $\mathbb{D}$ and $\mathbb{C}_+$, it is possible to translate the above theorem, leading to the following.

Theorem 2.3. *Let V be a $\mathcal{C}^2$ function on the upper half-plane $\mathbb{C}_+$ which is continuous up to the boundary $\mathbb{R}$ and at the point $z=\infty$. Suppose further that*

(1) $V(z)=\overline{V(-\overline{z})}$ *for all* $z\in\mathbb{C}_+$;

(2) $|\bar{\partial}V(z)|\,dx\,dy$ *is a Carleson measure with intensity* K_1;

(3) $|\Delta V(z)|\operatorname{Im} z\,dx\,dy$ *is a Carleson measure with intensity* K_2;

(4) $|\Delta V(z)|\leq\frac{K_3}{(\operatorname{Im} z)^2}$.

Then the equation

$$\bar{\partial}v=\bar{\partial}V$$

has a bounded solution v on all of $\mathbb{C}_+$ (not only the boundary $\mathbb{R}$) with

$$|v(z)|\leq C(K_1,K_2,K_3) \text{ and } v(z)=\overline{v(-\overline{z})} \quad \forall\, z\in\mathbb{C}_+.$$

These theorems will be used to find $H^\infty_{\mathbb{R}}(\mathbb{D})$ solutions to certain $\bar{\partial}$-equations.

3. Main construction

Recall that it only remains to prove Proposition 1.3.

Proposition 1.3. *Let $p,q\in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ be finite simple real symmetric Blaschke products with $\inf_{z\in\mathbb{C}_+}(|p(z)|+|q(z)|)=\delta>0$. Then there exists a function V with the following properties:*

(i) $|\operatorname{Re} V(z)|\leq C(\delta) \quad \forall\, z\in\mathbb{C}_+$;

(ii) $|\log p(z)-V(z)|\leq C(\delta)$ *for all z in $\{z\in\mathbb{C}_+ : |q(z)|<\delta'\}$ for some $0<\delta'\leq\delta$ and an appropriate branch of $\log p$ on the set $\{z\in\mathbb{C}_+ : |q(z)<\delta'\}$;*

(iii) $V(z)=\overline{V(-\overline{z})} \quad \forall\, z\in\mathbb{C}_+$;

(iv) *some conditions to guarantee the existence of a bounded solution v on the entire upper half-plane $\mathbb{C}_+$ of the equation $\bar\partial v = \bar\partial V$, in particular:*
   (a) $|\Delta V(z)| \operatorname{Im} z \, dx\, dy$ *is a Carleson measure with intensity $C(\delta)$;*
   (b) $|\partial V(z)| \, dx\, dy$ *is a Carleson measure with intensity $C(\delta)$;*
   (c) $|\Delta V(z)| \leq \frac{C(\delta)}{(\operatorname{Im} z)^2} \quad \forall\, z \in \mathbb{C}_+$.

The construction of V is inspired by the construction given by S. Treil in [**12**], however we need to appropriately modify it to take advantage of the symmetry that $H^\infty_{\mathbb{R}}(\mathbb{D})$ functions possess. The main approach to this proposition is the construction of a Carleson contour. We use the modification developed by Bourgain in [**2**] and exploited by Treil in [**12**]. We further modify the method to force symmetry into the Carleson regions, which is possible since we are working with the algebra $H^\infty_{\mathbb{R}}(\mathbb{C}_+)$. This is an essential point in the argument.

We let $b_a(z)$ denote the elementary Blaschke factor in $H^\infty(\mathbb{C}_+)$ with zero at $a \in \mathbb{C}_+$, i.e., $b_a(z) := \frac{z-a}{z-\overline{a}}$. The following lemmas will be of use.

Lemma 3.1. *Let $B = \prod_{a\in\sigma} b_a$ be a finite Blaschke product with simple zeros. Suppose that for a given $z \in \mathbb{C}_+$ and $\gamma > 0$ we have*

$$|b_a(z)| \geq \gamma \quad \forall\, a \in \sigma.$$

Then

$$\sum_{a\in\sigma} \frac{2 \operatorname{Im} z \operatorname{Im} a}{|z-\overline{a}|^2} \leq \log \frac{1}{|B(z)|} \leq \frac{1}{\gamma} \sum_{a\in\sigma} \frac{2 \operatorname{Im} z \operatorname{Im} a}{|z-\overline{a}|^2}.$$

We include the proof of this lemma.

Proof:

$$\begin{aligned} \log|B(z)| &= \frac{1}{2}\log(|B(z)|^2) \\ &= \frac{1}{2}\sum_{a\in\sigma}\log(|b_a(z)|^2) \\ &= \frac{1}{2}\sum_{a\in\sigma}\log\left(1 - \frac{4\operatorname{Im} z \operatorname{Im} a}{|z-\overline{a}|^2}\right). \end{aligned}$$

The proof is then finished by observing that $-\frac{t}{\gamma} \leq \log(1-t) \leq -t$ for $0 \leq t \leq 1-\gamma$. □

The next lemma will be used to construct the Carleson regions appropriately adapted to our functions.

Lemma 3.2. *Let B be a finite Blaschke product with simple zeros with σ denoting its zero set. Let $Q = Q(I)$ be a square with the base I and suppose that there is a point z_0 in the top half of Q with $|B(z_0)| \geq \epsilon > 0$. Then, given $M < \infty$, there exists a collection of disjoint closed subinterval $\{I_k\}$ of I with the following properties:*

(i) $\sum |I_k| \leq 20 \log \frac{1}{\epsilon} M^{-1} |I|$;

(ii) $\sum_{a \in \sigma \cap Q(3I_k)} \operatorname{Im} a \geq M|I_k| \quad \forall\, k$;

(iii) *If $z \in Q \setminus \cup_k Q(I_k)$, then $\sum_{a\in\sigma} \frac{\operatorname{Im} z \operatorname{Im} a}{|z-\overline{a}|^2} \leq C(M + \log \frac{1}{\epsilon})$ with C an absolute constant;*

(iv) *The measure $\mu := \sum_{a \in \sigma \cap Q \setminus \cup_k Q(I_k)} \operatorname{Im} a\ \delta_a$ is a Carleson measure with intensity at most $5M$.*

The proof of this lemma is a stopping time argument. See [**2**] for a version of this lemma or Treil [**12**] for the version indicated above. Since the functions we have possess additional symmetry, we will apply the above lemmas to "half" of our function. This is a key difference between the result found in [**12**]. With these lemmas, we now construct generations of closed intervals and regions in the following manner. First, note that for functions in $H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ we have the following symmetry property

$$f(z) = \overline{f(-\overline{z})} \quad \forall\, z \in \mathbb{C}_+.$$

Recall that a function is *real symmetric* if it satisfies this symmetry condition. Note that this symmetry is interchanging the left and right halves of $\mathbb{C}_+$. What is important in our case is that for finite Blaschke products with this symmetry property, a point a is a zero if and only if $-\overline{a}$ is a zero. We will use this symmetry in the selection of generations of intervals. Let

$$\sigma_R := \{a \in \sigma : \operatorname{Re} a \geq 0\} \quad \sigma_L := \{a \in \sigma : \operatorname{Re} a < 0\}.$$

This splits the zero set σ into its left and right halves. Choose an interval such that the real part of all zeros of the function are contained in this interval. Since we are working with $H^\infty_{\mathbb{R}}(\mathbb{D})$ functions, then the zeros will be symmetric and it will be possible to choose a symmetric interval, i.e. $I = (-L, L)$ for some L. Now, take a square $Q = Q(I)$ where $(I = [0, L))$, which contains the zero set σ_R and we have $|p(z_0)| \geq \delta$ for some point z_0 in the top half of Q and $|p(z)| \geq \delta$ for all $z \notin Q(I) \cup Q(-I)$. One should observe that a rectangle is all that is required to contain the zeros of the function.

Choose $M = M(\delta) > 2 \cdot 100 \log \frac{1}{\delta}$. We apply the discussion from the previous paragraph and Lemma 3.2 with the choice M, $\epsilon = \delta$ and $B = p$ and $Q = Q(I) \supset \sigma_R$. We thus obtain a sequence of disjoint closed sub-intervals of I, $\{I_k\}$, such that:

(i) $\sum |I_k| \leq \frac{1}{2 \cdot 5}|I|$;
(ii) $\sum_{a\in\sigma\cap Q(3I_k)} \operatorname{Im} a \geq 2 \cdot 100 \log \frac{1}{\delta}|I_k| \quad \forall\, k$;
(iii) If $z \in Q \setminus \cup_k Q(I_k)$, then $\sum_{a\in\sigma} \frac{\operatorname{Im} z \operatorname{Im} a}{|z-\overline{a}|^2} \leq C \log \frac{1}{\delta}$ with C an absolute constant;
(iv) The measure $\mu := \sum_{a\in\sigma\cap Q\setminus\cup_k Q(I_k)} \operatorname{Im} a\, \delta_a$ is a Carleson measure with intensity at most $5M \geq 250 \log \frac{1}{\delta}$.

Given this collection of intervals, we now form a new collection of intervals in the following manner. It is here that we exploit the symmetry of $H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ functions. This is a point in the proof where the construction of Treil must be modified, and the symmetry allows this. Set $\mathcal{K}_1 := \{I_k, -I_k\} = \{I_k'\}$. Then these intervals are closed disjoint sub-intervals (after a possible union of two of them) of an interval of the form $(-L, L)$. These intervals also possess the property that $I_k', -I_k' \in \mathcal{K}_1$ (again there could be the possibility that $I_k' = -I_k'$ for some k). Set $\tilde{Q} := Q(I) \cup Q(-I)$. Furthermore, they have the property that:

(i) $\sum |I_k'| \leq \frac{1}{5}|I|$;
(ii) $\sum_{a\in\sigma\cap Q(3I_k')} \operatorname{Im} a \geq 2 \cdot 100 \log \frac{1}{\delta}|I_k'| \quad \forall\, k$;
(iii) If $z \in \tilde{Q} \setminus \cup_k Q(I_k')$, then $\sum_{a\in\sigma} \frac{\operatorname{Im} z \operatorname{Im} a}{|z-\overline{a}|^2} \leq C \log \frac{1}{\delta}$ with C an absolute constant;
(iv) The measure $\mu := \sum_{a\in\sigma\cap\tilde{Q}\setminus\cup_k Q(I_k')} \operatorname{Im} a\, \delta_a$ is a Carleson measure with intensity at most $10M \geq 500 \log \frac{1}{\delta}$.

This is a straightforward application of the symmetry that $H^\infty_{\mathbb{R}}$ functions possess. We indicate some of this now. Since property (i) holds for the collection $\{I_k\}$, by disjointness and symmetry it will hold for $\{-I_k\}$ and hence for $\mathcal{K}_1$. Property (ii) also holds by symmetry. Since $\mathcal{K}_1 = \{I_k'\} = \{I_k, -I_k\}$, and we know that property (ii) holds for the collection $\{I_k\}$ by reflection in the y-axis and the symmetry of the zero set of $H^\infty_{\mathbb{R}}$ functions, property (ii) holds for $\{-I_k\}$ as well. Property (iii) and property (iv) also follow immediately by the symmetry of $H^\infty_{\mathbb{R}}$ functions.

Note that for any interval $J \in \mathcal{K}_1$, we have $|p(z)| < \delta$ for any z in the top half of $Q(J)$. This follows by Lemma 3.1 and the above construction.

We have

$$\log \frac{1}{|p(z)|} \geq \sum_{a \in \sigma} \frac{2 \operatorname{Im} a \operatorname{Im} z}{|z-\overline{a}|^2} \geq \sum_{a \in \sigma \cap Q(3J)} \frac{2 \operatorname{Im} a \operatorname{Im} z}{|z-\overline{a}|^2}.$$

But, for z in the top half of $Q(J)$ and $z \in Q(3J)$, we have $\operatorname{Im} z \geq \frac{|J|}{2}$ and $|z-\overline{a}| \leq 2\sqrt{2}|J|$. So by the construction of the intervals in $\mathcal{K}_1$ and the properties that they possess, we have

$$\log \frac{1}{|p(z)|} \geq \sum_{a \in \sigma \cap Q(3J)} \frac{1}{8|J|} \operatorname{Im} a \geq \frac{1}{8} M > \log \frac{1}{\delta}.$$

We iterate the above construction of construct generations of intervals and corresponding Carleson regions. Fix an interval $J \in \mathcal{K}_1$ and let $\mathcal{D}(J)$ be the maximal dyadic sub-intervals $J' \subset J$ such that the top half of each $Q(J')$ contains a point z_0 where $|p(z_0)| > \delta$. Note that, by the symmetry of the function p, we will obtain a symmetric selection of intervals. Since p is a finite Blaschke product, then $\mathcal{D}(J)$ is finite as well. Moreover,

$$J = \bigcup_{J' \in \mathcal{D}(J)} J'.$$

For each $J \in \mathcal{K}_1$, we set $\mathcal{U}(J) := \operatorname{clos}\big(Q(J) \setminus \cup_{J' \in \mathcal{D}(J)} Q(J')\big)$ and we set $\mathcal{R}_1 := \{\mathcal{U}(J) : J \in \mathcal{K}_1\}$. The set $\mathcal{R}_1$ is the first generation of Carleson regions. We should note that by symmetry, since $J, -J \in \mathcal{K}_1$, then $\mathcal{U}(J)$ and $\mathcal{U}(-J)$ are symmetric with respect to reflection in the imaginary axis. Also, it is easy to see that the sets $\mathcal{D}(J)$ and $\mathcal{D}(-J)$ will be symmetric in this manner as well.

For each $J' \in \mathcal{D}(J)$ with $J \in \mathcal{K}_1$, we apply the above construction to obtain a second generation of intervals $\mathcal{K}_2$. Note that we only need to perform the construction for half the intervals; the other half is obtained by symmetry, i.e., reflection in the imaginary axis. For each $J \in \mathcal{K}_2$, we form $\mathcal{U}(J)$ and $\mathcal{R}_2 := \{\mathcal{U}(J) : J \in \mathcal{K}_2\}$.

Finally, we define $\mathfrak{U} := \cup_{j \geq 1} \cup_{J \in \mathcal{K}_j} \mathcal{U}(J)$. We also set $\sigma_1 = \sigma \setminus \mathfrak{U}$. Since we have preserved symmetry throughout the construction, we will have the property that $J, -J \in \mathcal{K}_j$. Also, $\mathcal{U}(J)$ and $\mathcal{U}(-J)$ will be symmetric with respect to reflection in the imaginary axis. This implies that the set σ_1 will also be symmetric, i.e., $a \in \sigma_1$ if and only if $-\overline{a} \in \sigma_1$. Additionally, by construction, we have the property that if $z \in \mathfrak{U}$ then $|p(z)| \leq \delta$.

Letting $l_{\partial\mathfrak{U}}$ denote the arc length on the boundary of $\partial\mathfrak{U}$ of the region $\mathfrak{U}$, and letting δ_a denote the unit mass at the point a, the following result is straightforward.

Proposition 3.3. *Let $\mathfrak{U}$ and σ_1 be as above. Then*

(i) *The measure $l_{\partial\mathfrak{U}}$ is a Carleson measure with intensity at most $C = C(\delta, M) = C(\delta)$;*

(ii) *The measure $\sum_{a\in\sigma_1} \operatorname{Im} a\ \delta_a$ is a Carleson measure with intensity at most $C = C(\delta, M)$ (which in our case is $C(\delta)$ by the selection of M).*

Using the regions $\mathfrak{U}$ and σ_1, we will construct the function V. The function V is constructed as a finite sum of summands of two types, with each type having two sub-types. Here again, the construction of Treil must be appropriately modified. This is necessary because we need to make sure that the function V possesses the symmetry property $V(z) = \overline{V(-\overline{z})}$ for all $z \in \mathbb{C}_+$.

3.1. Summands of the first type.

In this subsection, we do the construction of the summands corresponding to the zeros in σ_1. Recall that $b_a(z) := \frac{z-a}{z-\overline{a}}$ is the simple Blaschke factor with zero at the point $a \in \mathbb{C}_+$. One immediately observes that $b_a \in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$ if and only if $a = -\overline{a}$. We also comment that $\overline{b_a(-\overline{z})} = b_{-\overline{a}}(z)$. This can be applied to a product of terms, and one sees $b_a b_{-\overline{a}} \in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$. These will be the elementary building blocks used.

We now make the distinction of when the point a is "close" or "far" from the the imaginary axis. We thus further split the points in σ_1 into the classes for which $|\operatorname{Re} a| \leq \frac{\delta}{2} \operatorname{Im} a$ and $|\operatorname{Re} a| > \frac{\delta}{2} \operatorname{Im} a$. We first deal with the case where $|\operatorname{Re} a| > \frac{\delta}{2} \operatorname{Im} a$, i.e., when the zeros are "far" from the the imaginary axis. Let D_a denote the disc with center at the point a of radius $\frac{\delta}{2} \operatorname{Im} a$, and let $T_a = \partial D_a$. Let I_a be the vertical slit which connects the circle T_a with the real axis at the point $\operatorname{Re} a$, i.e.,

$$I_a = \left\{ z \in \mathbb{C}_+ : \operatorname{Re} z = \operatorname{Re} a,\ 0 \leq \operatorname{Im} z \leq \left(1 - \frac{\delta}{2}\right) \operatorname{Im} a \right\}.$$

By construction, we have that the symmetric point $-\overline{a} \in \sigma_1$. We also construct the corresponding disc $D_{-\overline{a}}$ and $T_{-\overline{a}}$ and the corresponding slit $I_{-\overline{a}}$. Note that in the "far" case the discs D_a and $D_{-\overline{a}}$ are disjoint. The above construction is explained in Figure 1.

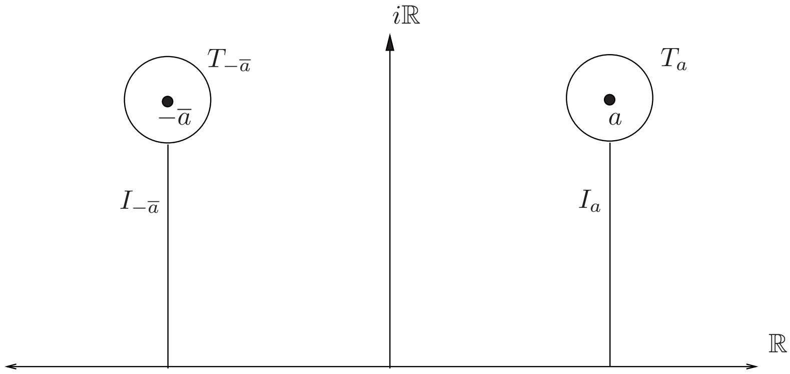

FIGURE 1

Then we have that $|b_a b_{-\overline{a}}| \leq \delta$ on the set $D_a \cup D_{-\overline{a}}$ and $|b_a b_{-\overline{a}}| \geq \frac{\delta^2}{36}$ on the compliment in $\mathbb{C}_+$. We then define a function $\varphi : \mathbb{C}_+ \setminus (T_a \cup I_a \cup T_{-\overline{a}} \cup I_{-\overline{a}})$ in the following manner

$$\varphi(z) := \begin{cases} 0 & : z \in D_a \cup D_{-\overline{a}} \\ \log(b_a b_{-\overline{a}}) & : \text{otherwise} \end{cases}.$$

Here we use the principal branch of the logarithm. Since the term $b_a b_{-\overline{a}} \in H^\infty_{\mathbb{R}}(\mathbb{C}_+)$, we have that φ is real symmetric on the set $\mathbb{C}_+ \setminus (T_a \cup I_a \cup T_{-\overline{a}} \cup I_{-\overline{a}})$. We change φ in the $\frac{\delta}{100}\operatorname{Im} a$-neighborhood of $T_a \cup I_a \cup T_{-\overline{a}} \cup I_{-\overline{a}}$ to obtain a smooth function V_a on $\mathbb{C}_+$ such that

(i) $|\bar{\partial} V_a(z)| \leq \frac{C(\delta)}{\operatorname{Im} a}$;

(ii) $|\Delta V_a(z)| \leq \frac{C(\delta)}{(\operatorname{Im} a)^2}$;

(iii) $V_a(z) = \varphi(z)$ if $\operatorname{dist}(z, T_a \cup I_a \cup T_{-\overline{a}} \cup I_{-\overline{a}}) > \frac{\delta}{100}\operatorname{Im} a$;

(iv) $V_a(z) = \overline{V_a(-\overline{z})}$.

The function V_a is obtained by the convolution of φ with a smooth kernel possessing the symmetry property. Properties (i) and (ii) follow from well known estimates for H^∞ functions. Property (iii) and (iv) are a simple construction and verification that, if you convolve with a function that possesses the real symmetry property then the function V_a will possess this property as well. The construction of the mollifying function is straightforward.

The construction in the case when $a \in \sigma_1$ is close to the imaginary axis is similar. Recall that in this situation, we are supposing that $|\operatorname{Re} a| \leq \frac{\delta}{2}\operatorname{Im} a$. In this situation, if we used the above construction the two circles would intersect, so instead we take one circle about the two points. The construction is almost identical then. We let $D_{a,-\overline{a}}$ denote the disc with center $\operatorname{Im} ai$ and radius $\frac{\delta}{2}\operatorname{Im} a$. Since $|\operatorname{Re} a| \leq \frac{\delta}{2}\operatorname{Im} a$, $D_{a,-\overline{a}}$ contains the points a and $-\overline{a}$. We let $T_{a,-\overline{a}} := \partial D_{a,-\overline{a}}$ denote the boundary of the circle. We let $I_{a,-\overline{a}}$ denote the vertical slit which connects the boundary $T_{a,-\overline{a}}$ with the real line. Namely, $I_{a,-\overline{a}} = \{z \in \mathbb{C}_+ : \operatorname{Re} z = 0, 0 \leq \operatorname{Im} z \leq (1-\frac{\delta}{2})\operatorname{Im} a\}$. The construction is explained in Figure 2.

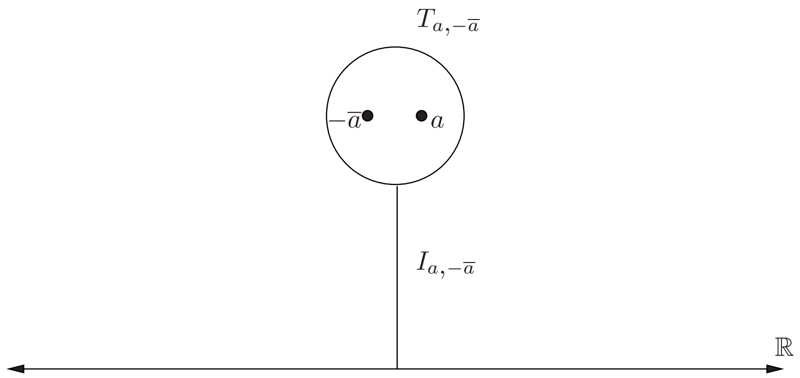

FIGURE 2

Then we will have that $|b_a b_{-\overline{a}}| \leq \delta$ on $D_{a,-\overline{a}}$ and $|b_a b_{-\overline{a}}| \geq \frac{\delta}{6}$ for points in the compliment. We then define φ in the following manner,

$$\varphi(z) := \begin{cases} 0 & : z \in D_{a,-\overline{a}} \\ \log(b_a b_{-\overline{a}}) & : \text{otherwise} \end{cases}.$$

We again smooth φ to find our function V. Namely, we change φ in a $\frac{\delta}{100}\operatorname{Im} a$ neighborhood of $T_{a,-\overline{a}} \cup I_{a,-\overline{a}}$ to obtain a smooth function on $\mathbb{C}_+$ such that:

(i) $|\bar{\partial} V_a(z)| \leq \frac{C(\delta)}{\operatorname{Im} a}$;
(ii) $|\Delta V_a(z)| \leq \frac{C(\delta)}{(\operatorname{Im} a)^2}$;
(iii) $V_a(z) = \varphi(z)$ if $\operatorname{dist}(z, T_{a,-\overline{a}} \cup I_{a,-\overline{a}}) > \frac{\delta}{100}\operatorname{Im} a$;
(iv) $V_a(z) = \overline{V_a(-\overline{z})}$.

This concludes this case.

3.2. Summands of the second type. The construction of these summands is slightly more involved. Let $\mathcal{R}$ be a connected component of $\mathfrak{U}$. Let

$$\mathcal{R}_\delta := \left\{ z \in \mathbb{C}_+ : \inf_{a \in \mathcal{R}} |b_a(z)| < \frac{\delta}{100} \right\}.$$

Set $B_{\mathcal{R}} := \prod_{a \in \sigma \cap \mathcal{R}} b_a$. By Lemma 3.2 for any $z \in \partial\mathcal{R}$ we have

$$\sum_{a \in \sigma \cap \mathcal{R}} \frac{\operatorname{Im} a \operatorname{Im} z}{|z - \overline{a}|^2} \leq C \left(M + \log \frac{1}{\delta} \right).$$

It is straightforward to demonstrate that for any $z \in \partial\mathcal{R}_\delta$, we have an identical estimate only with a larger constant. So, by Lemma 3.1 we have $|B_{\mathcal{R}}(z)| \geq \epsilon = \epsilon(\delta)$, which by the maximum modulus principle holds for all $z \notin \partial\mathcal{R}_\delta$.

Recall that we used the symmetry of $H^\infty_{\mathbb{R}}$ functions in the construction of the Carleson regions. By construction we have that if $\mathcal{R}$ is a connected component of $\mathfrak{U}$, then so is $-\mathcal{R}$. We again have two sub-cases, which distinguish proximity to the imaginary axis.

If $\mathcal{R} = -\mathcal{R}$, then $B_{\mathcal{R}} \in H^\infty_{\mathbb{R}}$. So we define

$$\varphi(z) := \begin{cases} 0 & : z \in \mathcal{R}_\delta \\ \log(B_{\mathcal{R}}) & : \mathbb{C}_+ \setminus \mathcal{R}_\delta \end{cases},$$

for an appropriate branch of the logarithm. To be precise, we will split the set $\mathbb{C}_+ \setminus \mathcal{R}_\delta$ into connected domains, and then define φ in each such domain as an appropriate branch of logarithm. The function φ will be real symmetric, and we again smooth φ by convolving with an appropriate real symmetric function. This takes care of the "close" case.

If $\mathcal{R} \neq -\mathcal{R}$, then $B_{\mathcal{R}}$ is not in $H^\infty_{\mathbb{R}}$, but $B_{\mathcal{R}} B_{-\mathcal{R}}$ is. We define

$$\varphi(z) := \begin{cases} 0 & : z \in \mathcal{R}_\delta \cup -\mathcal{R}_\delta \\ \log(B_{\mathcal{R}} B_{-\mathcal{R}}) & : \mathbb{C}_+ \setminus (\mathcal{R}_\delta \cup -\mathcal{R}_\delta) \end{cases},$$

for some branch of the logarithm. We also split the set $\mathbb{C}_+ \setminus (\mathcal{R}_\delta \cup -\mathcal{R}_\delta)$ into connected components and technically define φ as an appropriate branch of logarithm on each such component. Again, the function φ is real symmetric. We arrive at a smooth function by convolving φ with an appropriate real symmetric function. In either the "close" or "far" case, we let $V_{\mathcal{R}}$ be the mollification of the function φ constructed using information from the region $\mathcal{R}$.

The splitting is obtained in a similar manner to what appears in Treil's construction. We recall the splitting that is used in [**12**]. Each component $\mathcal{R}$ is a union of Carleson regions $\mathcal{U}$. Each region was constructed as

$$\mathcal{U} = \operatorname{clos}\left(Q(I) \setminus \bigcup_{J\in\mathcal{D}(I)} Q(J)\right)$$

where $\mathcal{D}(I)$ was a family of dyadic subintervals of I with $I = \cup_{J\in\mathcal{D}(I)} J$. For each such dyadic sub-interval J with center c at the point c, we draw a vertical interval (slit) $[c, c+i|J|]$. In addition to these vertical slits, we also need Γ-slits. To construct the Γ-slits, consider a vertical sub-interval of $\partial\mathcal{R}$ which is maximal with respect to inclusion. Let $I = [a+ib, a+ic]$, $a \in \mathbb{R}$, $c > b > 0$. For any integer k, $k \geq 1$, we do nothing if $c2^{-k-2} \leq b$. If $c2^{-k-2} > b$ we draw in $\mathbb{C}_+ \setminus \mathcal{R}$ a horizontal interval of length $2 \cdot 2^{-k-1}$ with the endpoint $a + ic2^{-k}$. Then we draw the vertical interval which connects the other endpoint to the real axis. Note that we can do this construction for one region $\mathcal{R}$ and then symmetry will deal with the corresponding region.

We need the following proposition from Treil's paper.

Proposition 3.4. *All slits (vertical and* Γ*-slits) corresponding to a component* $\mathcal{R}$ *are disjoint and the origin of each slit is the only point of its intersection with the component of* $\mathcal{R}$*. Moreover, if we consider for each slit* S *of altitude* d*, its* $\frac{\delta}{100}$*–neighborhood (usual, not hyperbolic)* S_δ*, all* S_δ *are also disjoint.*

The proof of this proposition is a direct repeat of what appears in Treil's work [**12**], so we omit it. We simply remark that we have forced additional symmetry upon our Carleson regions so that we can arrive at functions with certain symmetry properties.

The slits constructed divide $\partial\mathcal{R}$ (or, equivalently, the boundary $\partial\mathcal{R}_\delta$ of $\mathcal{R}_\delta$) into arcs with hyperbolic length bounded by some constant, depending only on an absolute constant. Next, recall that for H^∞ functions we have $|f'(z)| \leq \frac{\|f\|_\infty}{\operatorname{Im} z}$, and since $(\log B)' = \frac{B'}{B}$, we will have

$$|\log B(z) - \log B(\xi)| \leq C = C(\delta)$$

for any $z, \xi \in \partial\mathcal{R}_\delta$. The slits split the set $\mathbb{C}_+ \setminus (\mathcal{R}_\delta \cup -\mathcal{R}_\delta)$ (or $\mathbb{C}_+ \setminus \mathcal{R}_\delta$ in the case $\mathcal{R} = -\mathcal{R}$) into connected components. We then define, in each such domain E, the function φ as a branch of $\log B$ for which $0 \leq \operatorname{Im} \log B(z) \leq C(\delta)$ if $z \in \operatorname{clos} E \cup \partial\mathcal{R}_\delta$. The jumps of φ on the slits, and on the boundary $\mathcal{R}_\delta$, are bounded by a constant depending only on δ. Let Γ_δ denote the hyperbolic $\frac{\delta}{100}$–neighborhood of $\partial\mathcal{R}_\delta$, i.e.,

$$\Gamma_\delta := \left\{ z \in \mathbb{C}_+ : \inf_{a\in\partial\mathcal{R}_\delta} |b_a(z)| < \frac{\delta}{100} \right\}.$$

Also, for a slit S of altitude d let S_δ be its $\frac{\delta}{100}$-neighborhood (usual *not* hyperbolic) of S, i.e.,

$$S_\delta := \left\{ z \in \mathbb{C}_+ : \operatorname{dist}(z, S) < \frac{\delta}{100} \right\}.$$

Because of the trivial estimate $|f'(z)| \leq \frac{\|f\|_\infty}{\operatorname{Im} z}$ for $f \in H^\infty(\mathbb{C}_+)$, and since $\varphi' = (\log B)'$, if $z \in \Gamma_\delta$ we have

$$|\varphi'(z)| \leq \frac{C(\delta)}{\operatorname{Im} z} \quad \text{for } z \in \Gamma_\delta \setminus \partial\mathcal{R}_\delta.$$

Since the Blaschke product $B = B_{\mathcal{R}} B_{-\mathcal{R}}$ has no zeros is $\mathbb{C}_+ \setminus (\mathcal{R} \cup -\mathcal{R})$, it is analytic on the set $\mathbb{C} \setminus (\overline{\mathcal{R}} \cup -\overline{\mathcal{R}})$ (the bar denotes complex conjugation), and therefore,

$$|B'(z)| \leq \frac{C}{\operatorname{dist}(z, \mathcal{R} \cup -\mathcal{R})}.$$

Hence, for any slit S we have

$$|\varphi'(z)| \leq \frac{C(\delta)}{d} \quad \text{for } z \in S_\delta \setminus S.$$

These estimates and the boundedness of the jumps of φ allow one to change the function φ on the set $\Gamma_\delta \cup \bigcup_{S \in \mathcal{S}} S_\delta$, where $\mathcal{S}$ denotes the collection of all slits for the component $\mathcal{R}$, to obtain a function $V_{\mathcal{R}}$ satisfying

$$V_{\mathcal{R}} = \varphi(z) \qquad \forall\, z \notin \Gamma_\delta \cup \left(\bigcup_{S \in \mathcal{S}} S_\delta \right),$$

$$|V'_{\mathcal{R}}(z)| \leq \frac{C(\delta)}{\operatorname{Im} z}, \quad |\Delta V_{\mathcal{R}}(z)| \leq \frac{C(\delta)}{(\operatorname{Im} z)^2} \quad \forall\, z \in \Gamma_\delta,$$

$$|V'_{\mathcal{R}}(z)| \leq \frac{C(\delta)}{d}, \quad |\Delta V_{\mathcal{R}}(z)| \leq \frac{C(\delta)}{d^2} \quad \forall\, z \in S_\delta.$$

The function $V_{\mathcal{R}}$ will be smooth by taking the convolution of φ with an appropriate smooth kernel.

The function V is then defined as the sum of all summands of the first kind and all summands of the second kind, i.e.,

$$V = \sum_{a \in \sigma_1} V_a + \sum_{\mathcal{R} \in \mathfrak{R}} V_{\mathcal{R}}$$

where $\mathfrak{R}$ is the set of all connected components of $\mathfrak{U}$. By construction, V will be real symmetric. Therefore, it only remains to show that V satisfies the required conditions (i), (ii) and (iv) from Section 1.

4. Verification of the properties of the function V

With V now constructed, we need only show that it possesses all the required properties. First, we need an auxiliary definition and some propositions. These lemmas and propositions are taken from Treil's paper [**12**]. We omit the proofs.

Let S be a slit corresponding to a component $\mathcal{R}$. A common point of S and $\partial\mathcal{R}$ will be called an *origin* of S. For a slit corresponding to a point $a \in \sigma_1$, we shall call the origin of S simply the point a.

The following lemma is a straightforward application of Proposition 3.3.

Lemma 4.1. *Let A denote the set of origins of all the slits constructed above. Then the measure $\sum_{a\in A} \operatorname{Im} a\, \delta_a$ is a Carleson measure with intensity at most $C = C(\delta)$.*

Let S be a slit constructed above, and let $d = d(S)$ be its altitude. An integer k will be called the *rank* of S and denoted $\operatorname{rk}(S)$ if $2^k \leq d < 2^{k+1}$. Of course the rank of a slit can be negative.

Lemma 4.2. *For a given $z \in \mathbb{C}_+$ and $k \in \mathbb{Z}$ the number of slits of rank k for which $z \in S_\delta$ is at most $C = C(\delta)$.*

Lemma 4.3. *For a given $z \in \mathbb{C}_+$, there exists at most $C = C(\delta)$ components $\mathcal{R} \in \mathfrak{R}$ such that the hyperbolic $\frac{\delta}{100}$-neighborhood $\Gamma^{\mathcal{R}}_{\delta}$ of $\partial\mathcal{R}_\delta$ contains the point z.*

Following Treil's notation for $a \in \sigma_1$, we constructed circles T_a (or a circle $T_{a,-\overline{a}}$) about the point a, $-\overline{a}$ with T_a having a radius $\frac{\delta}{2}\operatorname{Im} a$. We let T^{δ}_a ($T^{\delta}_{a,-\overline{a}}$) be its $\frac{\delta}{100}\operatorname{Im} a$ neighborhood. We have the following proposition.

Lemma 4.4. *For a given $z \in \mathbb{C}_+$ there exist at most $C = C(\delta)$ points $a \in \sigma_1$ such that $z \in T^{\delta}_a$ (or $T^{\delta}_{a,-\overline{a}}$ when appropriate).*

To show that the function V satisfies the inequality, we will use a result due to Treil [**11**].

Lemma 4.5. *Let $0 < \epsilon < 1$. Let Θ_n ($n \in \mathbb{N}$) be inner functions and suppose γ_n is a "semi-Carleson contour" for Θ_n, i.e., $\gamma_n = \partial\mathcal{V}_n$, where $\mathcal{V}_n$ is an open set $\mathcal{V}_n \supset \{z : |\Theta_n(z)| < \epsilon\}$. Moreover, the measure l_{γ_n} (arclength on γ_n) is a Carleson measure with intensity at most C, with C independent of n. Suppose that the measure $\sum_n l\gamma_n$ is a Carleson measure with intensity at most C_1. Then*

$$\sum_{n\in\mathbb{N}} \left(1 - |\Theta_n(z)|^2\right) \leq KC_1, \forall\, z \in \mathbb{C}_+, \text{ where } K = K(\epsilon, C_1).$$

This lemma is applied to the family of functions made from the union of the following two families.

$$\{B_{\mathcal{R}}B_{-\mathcal{R}} : \mathcal{R} \in \mathfrak{R},\, \mathcal{R} \neq -\mathcal{R}\} \cup \{B_{\mathcal{R}} : \mathcal{R} \in \mathfrak{R},\, \mathcal{R} = -\mathcal{R}\}$$

$$\left\{b_a b_{-\overline{a}} : a \in \sigma_1,\, |\operatorname{Re} a| > \frac{\delta}{2}\operatorname{Im} a\right\} \cup \left\{b_a b_{-\overline{a}} : a \in \sigma_1,\, |\operatorname{Re} a| \leq \operatorname{Im} a \frac{\delta}{2}\right\}.$$

For the case of the $B_{\mathcal{R}}B_{-\mathcal{R}}$, we take for the semi-Carleson contour the boundary $\partial\mathcal{R}_\delta \cup -\partial\mathcal{R}_\delta$. In the case $B_{\mathcal{R}}$ we simply take $\partial\mathcal{R}_\delta$. For the other cases, we take either $T_a \cup T_{-\overline{a}}$ or $T_{a,-\overline{a}}$. The assumption of Lemma 4.5 follows from Proposition 3.3. Therefore, we have

$$\begin{aligned}\sum_{\substack{\mathcal{R}\in\mathfrak{R}\\ \mathcal{R}\neq-\overline{\mathcal{R}}}} (1-|B_{\mathcal{R}}B_{-\overline{\mathcal{R}}}|^2) + \sum_{\substack{\mathcal{R}\in\mathfrak{R}\\ \mathcal{R}=-\overline{\mathcal{R}}}} (1-|B_{\mathcal{R}}|^2)& \\ + \sum_{\substack{a\in\sigma_1\\ |\operatorname{Re} a|>\frac{\delta}{2}\operatorname{Im} a}} (1-|b_a(z)b_{-\overline{a}}(z)|^2)& \\ + \sum_{\substack{a\in\sigma_1\\ |\operatorname{Re} a|\leq\frac{\delta}{2}\operatorname{Im} a}} (1-|b_a(z)b_{-\overline{a}}(z)|^2) &\leq C = C(\delta).\end{aligned}$$

But, by the construction of V in Section 3, we have that

$$|\operatorname{Re} V_{\mathcal{R}}(z)| \leq \min\left\{\log\frac{1}{\delta'}, \log|B_R|^{-1}\right\}$$

$$|\operatorname{Re} V_a(z)| \leq \min\left\{\log\frac{1}{\delta'}, \log|b_a(z)b_{-\overline{a}}(z)|^{-1}\right\}$$

for some constant $0 < \delta' \leq \delta$. Hence we have,

$$|\operatorname{Re} V(z)| \leq C = C(\delta).$$

We now prove that V satisfies the conditions necessary to guarantee the existence of solutions to the $\overline{\partial}$-equation. Namely, we prove that the Laplacian and the derivative of V gives rise to a Carleson measure.

The proof again is basically a repeat of what appears in [**12**]. We include it for the ease of the reader. Note that

$$\Delta V(z) = \sum_{\mathcal{R}\in\mathfrak{R}} \Delta V_{\mathcal{R}}(z) + \sum_{a\in\sigma_1} \Delta V_a(z)$$

and the summand $\Delta V_{\mathcal{R}}(z)$ is not equal to zero (respectively, $\Delta V_a(z) \neq 0$), only if either

(a) $z \in \Gamma^{\mathcal{R}}_\delta$ (respectively, $z \in T^\delta_a$), or

(b) $z \in S_\delta$ where S is a slit corresponding to the component $\mathcal{R}$ (respectively, to the point a).

By Lemmas 4.3 and 4.4, for each $z \in \mathbb{C}_+$, at most $K = K(\delta)$ summands satisfy condition (a). Therefore, given the estimates obtained in the construction of $V_{\mathcal{R}}$ and V_a, we have

$$\sum_{\mathcal{R}: z \in \Gamma^{\mathcal{R}}_\delta} |\Delta V_{\mathcal{R}}(z)| + \sum_{a: z \in T^\delta_a} |\Delta V_a(z)| \leq \frac{C}{(\operatorname{Im} z)^2}.$$

It remains to deal with those points that contribute arising from condition (b). Let $\mathcal{N}_k(z)$ be the collection of all components $\mathcal{R} \in \mathfrak{R}$ such that there exists a slit S of rank k corresponding to $\mathcal{R}$ for which $z \in S_\delta$. By symmetry, if $\mathcal{R}$ has this property then so will $-\overline{\mathcal{R}}$. Similarly, define $\mathcal{A}_k(z)$ to be the set of all zeros $a \in \sigma_1$ for which $z \in S_\delta$, $\operatorname{rk} S = k$, and S is the slit corresponding to the point a. Lemma 4.2 implies that $\operatorname{card} \mathcal{N}_k(z) + \operatorname{card} \mathcal{A}_k(z) \leq C(\delta)$ for all $z \in \mathbb{C}_+$, and the estimates from the construction $V_{\mathcal{R}}$ and V_a imply

$$\sum_{\mathcal{R} \in \mathcal{N}_k(z)} |\Delta V_{\mathcal{R}}(z)| + \sum_{a \in \mathcal{A}_k(z)} |\Delta V_a(z)| \leq \frac{C}{(2^k)^2}.$$

The sets $\mathcal{N}_k(z)$ and $\mathcal{A}_k(z)$ are non-empty when $2^k > \frac{1}{2} \operatorname{Im} z$. Hence,

$$\begin{aligned}
&\sum_{\text{condition (b)}} |\Delta V_{\mathcal{R}}(z)| + \sum_{\text{condition (b)}} |\Delta V_a(z)| \\
&\qquad = \sum_{\substack{k \\ 2^{k+1} > \operatorname{Im} z}} \left(\sum_{\mathcal{R} \in \mathcal{N}_k(z)} |\Delta V_{\mathcal{R}}(z)| + \sum_{a \in \mathcal{A}_k(z)} |\Delta V_a(z)| \right) \\
&\qquad \leq \sum_{\substack{k \\ 2^{k+1} > \operatorname{Im} z}} \frac{C(\delta)}{(2^k)^2} \leq \frac{C(\delta)}{(\operatorname{Im} z)^2}.
\end{aligned}$$

Thus, $|\Delta V(z)| \leq \frac{C(\delta)}{(\operatorname{Im} z)^2}$. The same logic and method of proof shows that $|\partial V| \leq \frac{C(\delta)}{\operatorname{Im} z}$.

We now turn to demonstrating that V gives rise to Carleson measures. Fix a square $Q = Q(I)$ with $|I| = d$. Let $\mathcal{N}_k(z)$ and $\mathcal{A}_k(z)$ be as above.

Let n be the integer such that $2^n \le d < 2^{n+1}$, and let

$$\mathcal{N}^+(z) := \bigcup_{k \ge n+4} \mathcal{N}_k(z), \quad \mathcal{N}^+ := \bigcup_{z \in Q} \mathcal{N}^+(z),$$

$$\mathcal{A}^+(z) := \bigcup_{k \ge n+4} \mathcal{A}_k(z), \quad \mathcal{A}^+ := \bigcup_{z \in Q} \mathcal{A}^+(z),$$

and let

$$\mathcal{N}^-(z) := \left(\bigcup_{k < n+4} \mathcal{N}_k(z) \right) \cup \{\mathcal{R} \in \mathfrak{R} : z \in \Gamma_\delta^{\mathcal{R}}\}, \quad \mathcal{N}^- := \bigcup_{z \in Q} \mathcal{N}^-(z),$$

$$\mathcal{A}^-(z) := \left(\bigcup_{k < n+4} \mathcal{A}_k(z) \right) \cup \{a \in \sigma_1 : z \in T_a^\delta\}, \quad \mathcal{A}^- := \bigcup_{z \in Q} \mathcal{A}^-(z).$$

By construction we have that if $S_\delta \cap Q \neq \emptyset$ for a slit S of $\operatorname{rk} S = k \ge n+4$, corresponding to a component $\mathcal{R}$, then $\Gamma_\delta^{\mathcal{R}} \cap Q = \emptyset$ and similarly, for all other slits S' corresponding to the component, we will have $S'_\delta \cap Q = \emptyset$. Thus, $\mathcal{N}^+ \cap \mathcal{N}^- = \emptyset$. It is also possible to show that $\mathcal{A}^+ \cap \mathcal{A}^- = \emptyset$ as well.

If $\Delta V_{\mathcal{R}}(z) \neq 0$ for a point $z \in Q$ (respectively $\Delta V_a(z) \neq 0$ for $z \in Q$) then $\mathcal{R} \in \mathcal{N}^+ \cup \mathcal{N}^-$ (respectively $a \in \mathcal{A}^+ \cup \mathcal{A}^-$). By the same logic as above, we can conclude that

$$\sum_{\mathcal{R} \in \mathcal{N}^+} |\Delta V_{\mathcal{R}}(z)| + \sum_{a \in \mathcal{A}^+} |\Delta V_a(z)| \le \frac{C}{d^2},$$

and so

$$\iint_Q \left(\sum_{\mathcal{R} \in \mathcal{N}^+} |\Delta V_{\mathcal{R}}(z)| + \sum_{a \in \mathcal{A}^+} |\Delta V_a(z)| \right) \operatorname{Im} z \, dx \, dy \le Cd = C|I|.$$

By the geometry of the construction, for any $\mathcal{R} \in \mathcal{N}^-$ and the estimates on the Laplacian of $V_{\mathcal{R}}$, we have

$$\iint_Q |\Delta V_{\mathcal{R}}| \operatorname{Im} z \, dx \, dy \le Cl(\partial \mathcal{R} \cap Q(20I))$$

with l the arc length. For $a \in \mathcal{A}^-$, $a \in Q(20I)$, and the estimates on V_a imply

$$\iint_Q |\Delta V_a| \operatorname{Im} z \, dx \, dy \le C \operatorname{Im} a.$$

Using the above estimates, and the fact that $\mu=\sum_{\mathcal{R}\in\mathfrak{R}} l_{\mathcal{R}}+\sum_{a\in\sigma_1}\operatorname{Im} a\,\delta_a$ is a Carleson measure, we can conclude

$$\iint_Q \left(\sum_{\mathcal{R}\in\mathcal{N}^-} |\Delta V_{\mathcal{R}}(z)| + \sum_{a\in\mathcal{A}^-} |\Delta V_a(z)|\right) \operatorname{Im} z\, dx\, dy \le Cd = C|I|.$$

This implies that $|V(z)|\operatorname{Im} z\, dx\, dy$ is a Carleson measure.

To complete the verifications of the properties of V, we must show that V is sufficiently close to an appropriate branch of $\log f_1$ on the set $\{z : |f_2(z)| < \delta'\}$ where $0 < \delta' \le \delta$. To accomplish this, we need the following proposition, which is a consequence of Hall's Lemma, see [**5**] or [**6**].

Let $f \in H^\infty$ $\|f\|_\infty \le 1$ and let Q be a square. Let ϵ be a constant $0 < \epsilon < 1$. Let

$$E_\epsilon := \{z \in Q : |f(z)| < \epsilon\}.$$

The sets E_ϵ^{Re} and E_ϵ^{Im} will denote the vertical and horizontal projections of the set E_ϵ.

Lemma 4.6. *For a given $0 < \delta < 1$ there exists a constant $\epsilon = \epsilon(\delta)$, $0 < \epsilon < \delta$ such that for any $f \in H^\infty$, $\|f\|_\infty \le 1$ satisfying*

$$\max\{|E_\epsilon^{\mathrm{Re}}|, |E_\epsilon^{\mathrm{Im}}|\} \ge \frac{1}{4}|I|$$

the inequality $|f(z)| < \delta$ holds for all z in the upper half of Q.

Let $\mathcal{O}$ be a connected component of the set $\{z \in \mathbb{C}_+ : |f_2(z)| < \epsilon\}$ where $\epsilon = \epsilon(\delta)$ from the above lemma. By the maximum modulus principle, the set $\mathcal{O}$ is simply connected. Let $n \in \mathbb{Z}$ be the smallest integer such that there exists a square $Q = Q(I)$ with base I, $|I| = 2^n$ for which $\mathcal{O} \subset Q$.

Lemma 4.7. *If, for a slit S, we have $S_\delta \cup \mathcal{O} \ne \emptyset$ then the rank of S is at least $n - 3$.*

Proof: Let $\mathcal{O}^{\mathrm{Re}}$ and $\mathcal{O}^{\mathrm{Im}}$ be the vertical and horizontal projections of $\mathcal{O}$. If

$$\max\{|\mathcal{O}^{\mathrm{Re}}|, |\mathcal{O}^{\mathrm{Im}}|\} < \frac{1}{4}|I| = 2^{n-2}$$

then by the definition of n and the fact that $\mathcal{O}$ is simply connected, we have for all $z \in \mathcal{O}$, $\operatorname{Im} z > \frac{1}{4}|I|$. Hence, if $\operatorname{rk} S < n - 3$ then $S_\delta \cap \mathcal{O} = \emptyset$.

Now suppose that

$$\max\{|\mathcal{O}^{\mathrm{Re}}|, |\mathcal{O}^{\mathrm{Im}}|\} \ge \frac{1}{4}|I| = 2^{n-2}.$$

Let S be a slit of rank k, with $k < n-3$, and $S_\delta \cap \mathcal{O} \neq \emptyset$. Let z_0 be the origin of the slit S, i.e., either a point in $\partial\mathcal{R}_\delta$ or σ_1. Let J be the interval in $\mathbb{R}$ with center at the point $\operatorname{Re} z_0$ or length $2\operatorname{Im} z_0$. Since $\mathcal{O}$ is connected, for the set $E := \mathcal{O} \cap Q(J)$ we have

$$\max\{|E^{\mathrm{Re}}|, |E^{\mathrm{Im}}|\} > \frac{1}{4}|J|.$$

By Proposition 4.6, $|f_2(z_0)| < \delta$. This leads to a contradiction, since we were working under the assumption that f_1 and f_2 were Corona Data, namely,

$$\inf_{z\in\mathbb{C}_+} (|f_1(z)| + |f_2(z)|) = 3\delta > 0$$

and by construction of the Carleson regions, $|f_1(z_0)| < \delta$. □

Fix a point $z_0 \in \mathcal{O}$. For each connected component $\mathcal{R}$, define on the set $\mathcal{O}$ a branch of $\log(\alpha_{\mathcal{R}} B_{\mathcal{R}}(z) B_{-\mathcal{R}}(z))$, $\alpha_R \in \mathbb{C}$ a unimodular number, such that $\log(\alpha_{\mathcal{R}} B_{\mathcal{R}}(z_0) B_{-\mathcal{R}}(z_0)) = V_{\mathcal{R}}(z_0)$. Analogously, for any $a \in \sigma_1$, define a branch of $\log(\alpha_a b_a(z) b_{-\overline{a}}(z))$. If, for a component $\mathcal{R}$, there is a slit corresponding to it such that $S_\delta \cap \mathcal{O} \neq \emptyset$, then

$$|V_{\mathcal{R}}(z) - \log(\alpha_{\mathcal{R}} B_{\mathcal{R}}(z) B_{-\mathcal{R}}(z))| \leq NK(\delta),$$

where N is the number of slits S corresponding to $\mathcal{R}$ for which $S_\delta \cap \mathcal{O} \neq \emptyset$. As above, we let n be the smallest integer for which $\mathcal{O} \subset Q$, with $Q = Q(I)$ such a square. Let $\mathfrak{R}_0$ be the set of all components $\mathcal{R} \in \mathfrak{R}$ such that there exist no slits , $\operatorname{rk} S \geq n+4$ corresponding to this component and satisfying $S_\delta \cap \mathcal{O} \neq \emptyset$. By Lemma 4.2, for any $k \geq n-3$ there exists at most $C(\delta)$ slits of rank k such that $S_\delta \cap \mathcal{O} \neq \emptyset$. By Lemma 4.7, there exists no slits S with $\operatorname{rk} S < n-3$ for which $S_\delta \cap \mathcal{O} \neq \emptyset$, and therefore at most $7C(\delta)$ slits S, with $\operatorname{rk} S < n+4$ satisfying $S_\delta \cap \mathcal{O} \neq \emptyset$. It then follows that

$$\sum_{\mathcal{R}\in\mathfrak{R}_0} |V_{\mathcal{R}}(z) - \log(\alpha_{\mathcal{R}} B_{\mathcal{R}}(z) B_{-\mathcal{R}}(z))| \leq 7C(\delta)K(\delta), \quad z \in \mathcal{O}.$$

For a component $\mathcal{R} \notin \mathfrak{R}_0$, it is possible to obtain a better estimate. Indeed, if for a component $\mathcal{R}$ there exists a slit S of rank $k \geq n+4$ satisfying $S_\delta \cap \mathcal{O} \neq \emptyset$, then for other slits S' corresponding to this component, $S'_\delta \cap \mathcal{O} = \emptyset$. Therefore,

$$\begin{aligned}
|V_{\mathcal{R}}(z) - {}&\log(\alpha_{\mathcal{R}} B_{\mathcal{R}}(z) B_{-\mathcal{R}}(z))| \\
&\leq \left(\sup_{z\in S_\delta} |V'_{\mathcal{R}}(z)| + \sup_{z\in S_\delta} |(B_{\mathcal{R}}(z)B_{-\mathcal{R}}(z))'|\right) \operatorname{diam}(S_\delta \cap \mathcal{O}) \\
&\leq C2^{-k} \operatorname{diam} Q \leq C2^{n-k}.
\end{aligned}$$

We sum this estimate over $n \geq n+4$ and can conclude

$$\left|\sum_{\mathcal{R}\in\mathfrak{R}} V_{\mathcal{R}}(z) - \log(\alpha_{\mathcal{R}} B_{\mathcal{R}}(z) B_{-\mathcal{R}}(z))\right| \leq C(\delta), \quad z \in \mathcal{O}.$$

Identical reasoning then implies the corresponding result for V_a. This, in turn, implies that the function V is sufficiently close to an appropriate branch of logarithm of p and hence V has all the necessary properties. This then proves Proposition 1.3.

5. Concluding remarks

The argument above shows that the stable rank of $H^\infty_{\mathbb{R}}(\mathbb{D})$ is at least two. A natural conjecture, first made by S. Treil, is

Conjecture 5.1. *The stable rank of $H^\infty_{\mathbb{R}}(\mathbb{D})$ is two.*

In recent work with R. Mortini, we have been able to demonstrate the validity of this conjecture. In [**9**] we show that, in fact, the stable rank of $H^\infty_{\mathbb{R}}(\mathbb{D})$ is two, but do so using the maximal ideal theory of $H^\infty(\mathbb{D})$. This results in a proof of the above conjecture that does not produce estimates for the solutions. Thus, a natural question to pose is the following:

Problem 5.2. *Let $f_1, f_2, f_3 \in H^\infty_{\mathbb{R}}(\mathbb{D})$ be such that $\inf_{z\in\mathbb{D}}(|f_1(z)| + |f_2(z)| + |f_3(z)|) = \delta > 0$. Does there exist $h_1, h_2, g_1, g_2 \in H^\infty_{\mathbb{R}}(\mathbb{D})$ such that*

$$1 = f_1(z)g_1(z) + f_2(z)g_2(z) + (h_1(z)g_1(z) + h_2(z)g_2(z))f_3(z) \quad \forall\, z \in \mathbb{D}$$

and such that $\|h_j\|_\infty, \|g_j\|_\infty \leq C(\delta)$ for $j = 1, 2$?

By the result in [**9**], we know that solutions g_j, h_j exist, and the real question is what happens in terms of the estimates.

References

[1] H. BASS, K-theory and stable algebra, *Inst. Hautes Études Sci. Publ. Math.* **22** (1964), 5–60.

[2] J. BOURGAIN, On finitely generated closed ideals in $H^\infty(D)$, *Ann. Inst. Fourier (Grenoble)* **35(4)** (1985), 163–174.

[3] L. CARLESON, Interpolations by bounded analytic functions and the corona problem, *Ann. of Math. (2)* **76** (1962), 547–559.

[4] G. CORACH AND F. D. SUÁREZ, Stable rank in holomorphic function algebras, *Illinois J. Math.* **29(4)** (1985), 627–639.

[5] P. L. DUREN, *"Theory of H^p spaces"*, Pure and Applied Mathematics **38**, Academic Press, New York-London, 1970.

[6] J. B. Garnett, *“Bounded analytic functions”*, Pure and Applied Mathematics **96**, Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[7] P. W. Jones, D. Marshall, and T. Wolff, Stable rank of the disc algebra, *Proc. Amer. Math. Soc.* **96(4)** (1986), 603–604.

[8] R. Mortini, An example of a subalgebra of H^∞ on the unit disk whose stable rank is not finite, *Studia Math.* **103(3)** (1992), 275–281.

[9] R. Mortini and B. D. Wick, The bass and topological stable ranks of $H^\infty_{\mathbb{R}}(\mathbb{D})$ and $A_{\mathbb{R}}(\mathbb{D})$, *J. Reine Angew. Math.* (to appear).

[10] R. Rupp, Stable rank of subalgebras of the disc algebra, *Proc. Amer. Math. Soc.* **108(1)** (1990), 137–142.

[11] S. Treil, Hankel operators, embedding theorems and bases of co-invariant subspaces of the multiple shift operator, (Russian), *Algebra i Analiz* **1(6)** (1989), 200–234; translation in: *Leningrad Math. J.* **1(6)** (1990), 1515–1548.

[12] S. Treil, The stable rank of the algebra H^∞ equals 1, *J. Funct. Anal.* **109(1)** (1992), 130–154.

[13] M. Vidyasagar, *“Control system synthesis”*, A factorization approach, MIT Press Series in Signal Processing, Optimization, and Control **7**, MIT Press, Cambridge, MA, 1985.

Department of Mathematics
University of South Carolina
LeConte College
1523 Greene Street
Columbia, SC 29208
USA
E-mail address: wick@math.sc.edu

Primera versió rebuda el 14 de maig de 2008,
darrera versió rebuda el 10 de setembre de 2008.