Publ. Mat. **53** (2009), 439–456

ON THE PRODUCT OF TWO π-DECOMPOSABLE SOLUBLE GROUPS

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Abstract

Let the group $G = AB$ be a product of two π-decomposable subgroups $A = O_\pi(A) \times O_{\pi'}(A)$ and $B = O_\pi(B) \times O_{\pi'}(B)$ where π is a set of primes. The authors conjecture that $O_\pi(A)O_\pi(B) = O_\pi(B)O_\pi(A)$ if π is a set of odd primes. In this paper it is proved that the conjecture is true if A and B are soluble. A similar result with certain additional restrictions holds in the case $2 \in \pi$. Moreover, it is shown that the conjecture holds if $O_{\pi'}(A)$ and $O_{\pi'}(B)$ have coprime orders.

1. Notation and Preliminaries

All groups considered are finite.

The aim of this paper is to study groups $G = AB$ which are factorized as the product of π-*decomposable* subgroups A and B, for a set of primes π. A group X is said to be π-*decomposable* if $X = X_\pi \times X_{\pi'}$ is the direct product of a π-subgroup and a π'-subgroup, where π' stands for the complementary of π in the set of all prime numbers. Moreover, we always use X_π to denote a Hall π-subgroup of any group X.

More precisely we take further the study that was started in [**12**]. The main result in that paper states the following:

Theorem 1. *Let π be a set of odd primes. Let the group $G = AB$ be the product of a π-decomposable subgroup A and a π-subgroup B. Then $A_\pi = O_\pi(A) \leq O_\pi(G)$.*

It is worth recalling the following result, which is Lemma 1 in [**12**] and provides an equivalent statement to this theorem.

2000 *Mathematics Subject Classification.* 20D20, 20D40.
Key words. Products of groups, π-decomposable groups, Hall subgroups.

Lemma 1. *Let the group $G = AB$ be the product of a π-decomposable subgroup $A = A_\pi \times A_{\pi'}$ and a π-subgroup B. Then the following statements are equivalent:*

(i) $A_\pi \leq O_\pi(G)$;

(ii) *G contains Hall π-subgroups and $A_\pi B = BA_\pi$ is a Hall π-subgroup of G.*

The starting point for our work is the theorem of Kegel and Wielandt which states the solubility of a group which is the product of two nilpotent subgroups.

For the proof of this theorem Kegel found a very useful criterion for the non-simplicity of a finite group in terms of some suitable permutability conditions on subgroups ([**13**, Satz 3]). It was improved by Wielandt in [**15**, Satz 1]. (See also [**1**, Lemmas 2.4.1, 2.5.1].) We state here a reformulation of these results which is convenient for our purposes.

Lemma 2. *Let the group $G = AB$ be the product of the subgroups A and B and let A_0 and B_0 be normal subgroups of A and B, respectively. If $A_0B_0 = B_0A_0$, then $A_0^gB_0 = B_0A_0^g$ for all $g \in G$.*

Assume in addition that A_0 and B_0 are π-groups for a set of primes π. If $O_\pi(G) = 1$, then $[A_0^G, B_0^G] = 1$.

(We note that this result is applicable in particular if $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ are π-decomposable and considering $A_0 = A_\pi$ and $B_0 = B_\pi$.)

Proof: Let $g \in G$ and consider $g = ab$ with $a \in A$ and $b \in B$. Since A_0 and B_0 are normal subgroups of A and B, respectively, and they permute, we have:

$$A_0^gB_0 = A_0^{ab}B_0 = (A_0B_0)^b = (B_0A_0)^b = B_0A_0^{ab} = B_0A_0^g.$$

Now the final assertion follows from [**1**, Lemma 2.5.1]. □

If $G = AB$ is the product of nilpotent subgroups A and B, then the hypotheses of this result for $A_0 = A_p$ and $B_0 = B_p$, the Sylow p-subgroups of A and B, respectively, and for any prime p, hold. This fact is in the core of the solubility of the group G.

Our aim is to find a more general structure involving π-decomposable groups for which these hypotheses also hold. Then, together with Lemma 2, our results also provide non-simplicity criteria for a group G.

Precisely we conjecture the following:

Conjecture. *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G.*

Theorem 1 provides already a first approach to this conjecture. We state next another case for which the conjecture holds and that follows from Theorem 1. For notation, we set $\pi(G)$ for the set of prime divisors of $|G|$, the order of the group G.

Proposition 1. *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Assume in addition that $(|A_{\pi'}|, |B_{\pi'}|) = 1$. Then $A_\pi B_\pi = B_\pi A_\pi$.*

Proof: Since $2 \in \pi'$ and $(|A_{\pi'}|, |B_{\pi'}|) = 1$ we may assume w.l.o.g. that $2 \notin \pi(B)$. Now we consider the set of odd primes $\sigma := \pi(B) \cup \pi(A_\pi)$. Then G is the product of the σ-decomposable subgroup A and the σ-subgroup B. From Theorem 1 it follows that B and $A_\sigma = A_\pi$ permutes. Considering now the group BA_π, we can deduce that B_π permutes with A_π as desired. □

It is worthwhile emphasizing that the conjectured result holds in the significant case when $(|A|, |B|) = 1$. In particular, our results extend previous ones of Berkovič [**4**], Arad and Chillag [**3**], Rowley [**14**] and Kazarin [**9**], where products of a 2-decomposable group and a group of odd order, with coprime orders, were considered.

In this paper we will study as a first step the structure of a minimal counterexample to our conjecture. Afterwards we will prove it under the additional hypotheses that A and B are soluble groups. In the case of soluble factors, we will consider also the analogous problem when π is a set of primes containing the prime 2. As a consequence of these results we deduce in Corollary 1 a criterion of π-separability for a group which is the product of π-decomposable soluble factors, for an arbitrary set of primes π.

First we state some more notation. If n is an integer and p a prime number, we denote by n_p the largest power of p dividing n. A group G satisfies the C_π-property if G possesses a unique conjugacy class of Hall π-subgroups. Moreover G satisfies the D_π-property if it satisfies the C_π-property and every π-subgroup of G is contained in some Hall π-subgroup of G. We recall that a π-separable group satisfies the D_π-property.

We need specifically the following result (see [**1**, Corollary 1.3.3]).

Lemma 3. *Let the group $G = AB$ be the product of the subgroups A and B. Then for each prime p there exist Sylow p-subgroups A_p of A and B_p of B such that A_pB_p is a Sylow p-subgroup of G.*

For products of soluble subgroups the following lemma will be also used.

Lemma 4. *Let $G = AB = AN = BN$ be a group with A and B soluble subgroups of G and with a unique minimal normal subgroup N, which is non-abelian. Let $N = N_1 \times \cdots \times N_r$ with $N_1 \cong N_i$ be a non-abelian simple group, $i = 1, \ldots, r$. Then:*

(i) *A and B act transitively by conjugacy on the set $\Omega = \{N_1, \ldots, N_r\}$ of direct factors of N. Moreover, $N \cap A = \times_{i=1}^{r}(N_i \cap A)$ and $N \cap B = \times_{i=1}^{r}(N_i \cap B)$.*

(ii) *$|N_1|$ divides $|\mathrm{Out}(N_1)||N_1 \cap A||N_1 \cap B|$.*

Proof: See Lemmas 2.3 and 2.5 of [**10**]. □

2. The minimal counterexample

Proposition 2. *Let π be a set of odd primes. Assume that the group $G = AB$ is the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$, and G is a counterexample of minimal order to the assertion $A_\pi B_\pi = B_\pi A_\pi$.*

Then G has a unique minimal normal subgroup $N = N_1 \times \cdots \times N_r$, which is a direct product of isomorphic non-abelian simple groups $N_1, \ldots, N_r$. Moreover $G = AN = BN = AB$, $(|A_{\pi'}|, |B_{\pi'}|) \neq 1$ and $A_{\pi'} \cap B_{\pi'} = 1$.

Proof: First note that $A_\pi \neq 1$ and $B_\pi \neq 1$. Moreover, $|\pi(G) \cap \pi| > 1$, because of Lemma 3, and also $(|A_{\pi'}|, |B_{\pi'}|) \neq 1$ by Proposition 1; in particular, $A_{\pi'} \neq 1$ and $B_{\pi'} \neq 1$. We split the proof into the following steps:

Step 1. The group G has a unique minimal normal subgroup N, which is neither a π-group nor a π'-group. In particular, N is not soluble. Consequently, $N = N_1 \times \cdots \times N_r$ with $N_1 \cong N_i$ a non-abelian simple group, $i = 1, \ldots, r$.

Let N be a minimal normal subgroup of G and assume that there exists $M \neq N$ another minimal normal subgroup of G. The choice of G implies that $A_\pi B_\pi N/N$ is a subgroup of G/N and $A_\pi B_\pi M/M$ is a subgroup of G/M. Then

$$O^\pi(\langle A_\pi, B_\pi \rangle) \leq N \cap M = 1.$$

This implies that $\langle A_\pi, B_\pi \rangle$ is a π-group and, consequently, $\langle A_\pi, B_\pi \rangle = A_\pi B_\pi$, a contradiction.

If N is a π-group, then $\langle A_\pi, B_\pi \rangle \leq A_\pi B_\pi N$ is a π-group which implies the contradiction $\langle A_\pi, B_\pi \rangle = A_\pi B_\pi$, as $|A_\pi B_\pi| = |G|_\pi$ is the largest π-number dividing $|G|$.

Assume now that N is a π'-group. Note that

$$|A_\pi(B_\pi N)| = \frac{|A_\pi||B_\pi||N|}{|A_\pi \cap B_\pi N|}$$

and so $|A_\pi B_\pi N/N|$ is a π-number. Consequently, $X := A_\pi B_\pi N$ is a π-separable group and, in particular, it satisfies the D_π-property. We deduce now that there exists a Hall π-subgroup X_π of X and an element $x \in X$ such that $A_\pi B_\pi^x \subseteq \langle A_\pi, B_\pi^x \rangle \leq X_\pi$. But $|A_\pi B_\pi^x| = |G|_\pi$ which implies in particular that $A_\pi B_\pi^x = X_\pi$ is a subgroup of G. Since $G = AB$ and A_π and B_π are normal subgroups of A and B respectively, it follows that $A_\pi B_\pi$ is a subgroup of G.

Put now $H = \langle A_\pi, B_\pi \rangle$. Then the following properties hold:

Step 2. $N \leq H \trianglelefteq G$.

From [**1**, Lemma 1.2.2] we have that $N_G(H) = N_A(H)N_B(H)$. If $N_G(H)$ is a proper subgroup of G, then $A_\pi B_\pi$ is a subgroup of G by the choice of G, which is a contradiction. Hence H is a normal subgroup of G and so $N \leq H$.

Step 3. $G = AH = BH = AB$.

Observe that $AH = A(AH \cap B)$. If AH is a proper subgroup of G, then the choice of G implies again the contradiction $A_\pi B_\pi = B_\pi A_\pi$. Therefore $G = AH$ and, analogously, $G = BH$.

Step 4. $H = A_\pi B_\pi N$.

This is clear since $A_\pi B_\pi N$ is a subgroup of G and $N \leq H \leq A_\pi B_\pi N \leq H$.

Step 5. $A_{\pi'} N = B_{\pi'} N = A_{\pi'} B_{\pi'} N$.

Since $G = AH = AB_\pi N$, we deduce that

$$\begin{aligned} B &= B_\pi(B \cap AN) = B_\pi((B_\pi \cap AN) \times (B_{\pi'} \cap AN)) \\ &= B_\pi(B_{\pi'} \cap AN) = B_\pi B_{\pi'}. \end{aligned}$$

Then $B_{\pi'} = B_{\pi'} \cap AN$, that is, $B_{\pi'} \leq AN$ and, consequently, $B_{\pi'} \leq A_{\pi'} N$.

Analogously the equality $G = BH = BA_\pi N$ implies that $A_{\pi'} \leq B_{\pi'} N$.

Therefore $A_{\pi'} N = B_{\pi'} N = A_{\pi'} B_{\pi'} N$.

Step 6. $G/N = O_{\pi'}(G/N) \times O_\pi(G/N)$.

Note first that $H/N = A_\pi B_\pi N/N \in \mathrm{Hall}_\pi(G/N)$ and $H/N \trianglelefteq G/N$. On the other hand, we deduce from Step 5 that $A_{\pi'} N/N = B_{\pi'} N/N$ is a Hall π'-subgroup of G/N normalized by AN/N and by BN/N, that is, it is normal in G/N, and the assertion follows.

Step 7. $A_{\pi'} \cap B_{\pi'} = 1$.

If $L = A_{\pi'} \cap B_{\pi'}$, then $N \leq \langle A_\pi, B_\pi \rangle \leq C_G(L)$, and so $L \leq C_G(N) = 1$.

Step 8. Assume that $1 \neq M \trianglelefteq G$ and $K := AM \neq G$. Then $O_\pi(K) = 1$, $A_\pi \tilde{B}_\pi \in \mathrm{Hall}_\pi(K)$ and $[A_\pi^K, \tilde{B}_\pi^K] = 1$, where $\tilde{B}_\pi := B_\pi \cap AM = B_\pi \cap A_\pi M$. Moreover, $\tilde{B}_\pi \neq 1$ and $B_\pi \cap M = \tilde{B}_\pi \cap M = 1$.

First observe that $[O_\pi(K), N] \leq O_\pi(K) \cap N = 1$, which implies $O_\pi(K) \leq C_G(N) = 1$. Moreover, since $K = AM = A(AM \cap B) < G$, the choice of G implies that $T := A_\pi \tilde{B}_\pi = \tilde{B}_\pi A_\pi \in \mathrm{Hall}_\pi(K)$. Hence, from Lemma 2, it follows that $[A_\pi^K, \tilde{B}_\pi^K] = 1$.

Suppose now that $\tilde{B}_\pi = 1$. Then $T = A_\pi \in \mathrm{Hall}_\pi(K)$ and $A_\pi \cap M \in \mathrm{Hall}_\pi(M)$. Note that $A_\pi \cap M \neq 1$ because otherwise M would be a π'-group, which contradicts Step 1. Since π is a set of odd primes, then M satisfies the C_π-property by [**8**, Theorem A] and so, by the Frattini argument, we conclude that $G = MN_G(A_\pi \cap M)$. Hence

$$|G : N_G(A_\pi \cap M)| = |M : N_M(A_\pi \cap M)|$$

is a π'-number, since $A_\pi \cap M \in \mathrm{Hall}_\pi(N_M(A_\pi \cap M))$, and so $|G|_\pi = |N_G(A_\pi \cap M)|_\pi$. Note also that $N_G(A_\pi \cap M) \neq G$, by Step 1. Then, by the choice of G, $N_G(A_\pi \cap M) = A(B_\pi \cap N_G(A_\pi \cap M)) \times (B_{\pi'} \cap N_G(A_\pi \cap M))$ satisfies the theorem, that is,

$$A_\pi(B_\pi \cap N_G(A_\pi \cap M)) \in \mathrm{Hall}_\pi(N_G(A_\pi \cap M)).$$

But $|A_\pi(B_\pi \cap N_G(A_\pi \cap M))| = |N_G(A_\pi \cap M)|_\pi = |G|_\pi = |A_\pi B_\pi|$ implies that $B_\pi \cap N_G(A_\pi \cap M) = B_\pi$ and so $A_\pi B_\pi$ is a subgroup, a contradiction. This proves that $\tilde{B}_\pi \neq 1$.

Finally note that $B_\pi \cap M = \tilde{B}_\pi \cap M$ is normalized by both B_π and A_π because $[A_\pi, \tilde{B}_\pi] = 1$. Hence $N \leq \langle A_\pi, B_\pi \rangle$ normalizes $B_\pi \cap M$ and so $[B_\pi \cap M, N] \leq B_\pi \cap M \cap N = B_\pi \cap N = 1$, since this is a π-group normalized by N. Therefore $B_\pi \cap M \leq C_G(N) = 1$ and the last assertion follows.

Step 9. A acts transitively on the set $\Omega = \{N_1, \ldots, N_r\}$.

Assume that this is not true and take $R := \cap_{i=1}^r N_G(N_i) \trianglelefteq G$. Then $AR < G$ and we can apply Step 8 with $M = R$. In particular, from the facts that $\tilde{B}_\pi = B_\pi \cap AR \neq 1$ and $B_\pi \cap R = \tilde{B}_\pi \cap R = 1$ we deduce that $\tilde{B}_\pi \not\leq R$. Then there exists $1 \neq b \in \tilde{B}_\pi \setminus R$. Without loss of generality we may assume that $b \notin N_G(N_1)$, and so $|\Omega_{\langle b \rangle}(N_1)| \geq 2$, where $\Omega_{\langle b \rangle}(N_1)$ denotes the orbit of N_1 under the action of b on $\Omega = \{N_1, \ldots, N_r\}$. On the other hand, since $\tilde{B}_\pi \leq RA_\pi$, then $b = ca$ for some $c \in R$ and $a \in A_\pi$. Since R normalizes each N_i, we have $\Omega_{\langle b \rangle}(N_1) = \Omega_{\langle a \rangle}(N_1)$. Now note that $[N_1, \langle b \rangle] = N_{i_1} \times \cdots \times N_{i_k}$, where $\Omega_{\langle b \rangle}(N_1) = \{N_1 =$

$N_{i_1}, \dots, N_{i_k}\} \subseteq \Omega$. Analogously, $[N_1, \langle a\rangle] = N_{i_1} \times \cdots \times N_{i_k} = [N_1, \langle b\rangle]$. Therefore $[N_1, \langle a\rangle] = [N_1, \langle b\rangle] \le [N_1, \tilde{B}_\pi] \cap [N_1, A_\pi]$. Now from Step 8 we have that

$$[[N_1, \tilde{B}_\pi], [N_1, A_\pi]] \le [A_\pi^K, \tilde{B}_\pi^K] = 1$$

and so $N_1, N_{i_2}, \dots, N_{i_k}$ are abelian, which is a contradiction. The assertion is now proved.

Step 10. $G = AN = BN = AB$.

Assume that this is not true and, for instance, $AN < G$. Then we can apply Step 8 with $M = N$. In particular, $[A_\pi^K, \tilde{B}_\pi^K] = 1$, where $K = AN$, $\tilde{B}_\pi = B_\pi \cap AN = B_\pi \cap A_\pi N$ and $\tilde{B}_\pi \neq 1$. Since $C_G(N) = 1$ we may assume that there exists $1 \neq b \in \tilde{B}_\pi$ such that $[N_1, \langle b\rangle] \neq 1$. But this means that $N_1 \le [N_1, \langle b\rangle]$ and A_π centralizes this subgroup. Since A acts transitively on $\Omega = \{N_1, \dots, N_r\}$ and $A_\pi \trianglelefteq A$, it follows that A_π centralizes each N_i, for $i = 1, \dots, r$, and so $A_\pi \le C_G(N) = 1$, a contradiction which proves that $AN = G$.

By the symmetry between A and B we can also prove $G = BN$ and we are done. □

3. The soluble case with π a set of odd primes

Theorem 2. *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable soluble subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G.*

Proof: Assume the result is not true and let G be a counterexample of minimal order. We know by Proposition 2 that G has a unique minimal normal subgroup $N = N_1 \times \cdots \times N_r$, which is a direct product of isomorphic non-abelian simple groups $N_1, \dots, N_r$. Moreover, $G = AB = AN = BN$ and so, by Lemma 4, A and B act transitively on the set $\Omega = \{N_1, \dots, N_r\}$ and $|N_1|$ divides $|\mathrm{Out}(N_1)||N_1 \cap A||N_1 \cap B|$. Clearly $A_\pi \neq 1$, $B_\pi \neq 1$, and, moreover, $A_{\pi'} \neq 1$, $B_{\pi'} \neq 1$. Recall also that $A_{\pi'} \cap B_{\pi'} = 1$.

From [**10**] we know that N_i should be isomorphic to one of the groups in the set:

$$\mathfrak{M} = \{L_2(q),\ q > 3;\ L_3(q),\ q < 9;\ L_4(2), M_{11}, \mathrm{PS}p_4(3), U_3(8)\}.$$

We claim first that $N = N_1$ is a simple group.

We note that either $N_1 \cap A \neq 1$ or $N_1 \cap B \neq 1$ because $|N_1|$ does not divide $|\mathrm{Out}(N_1)|$. We set $\{\sigma, \sigma'\} = \{\pi, \pi'\}$. We may assume that $N_1 \cap A_\sigma \neq 1$. Then $A_{\sigma'}$ normalizes N_1. This holds also for $B_{\sigma'}$ because $A_{\sigma'}N = B_{\sigma'}N$ since $G = AN = BN$. If in addition $N_1 \cap A_{\sigma'} \neq 1$ we have also that A_σ normalizes N_1 and consequently $N = N_1$ is simple, since $G = AN$, and the claim is proved. We get analogously to the same conclusion if $N_1 \cap B_{\sigma'} \neq 1$. Let us assume now that $N_1 \cap A_{\sigma'} = 1 = N_1 \cap B_{\sigma'}$. In particular, $N_1 \cap A$ and $N_1 \cap B$ are σ-groups. On the other hand, we recall that N is not a σ-group. Hence $1 \neq |N_1|_{\sigma'}$ divides $|\mathrm{Out}(N_1)|$. We discard next this case by checking the different possibilities for N_1:

• $N_1 \in \mathfrak{M}$, $N_1 \not\cong M_{11}$, $N_1 \not\cong L_2(q)$, $q = p^n$. If r is a prime dividing $|\mathrm{Out}(N_1)|$, then $r \in \{2, 3\}$. But in all the considered cases $|N_1|_r > |\mathrm{Out}(N_1)|_r$ and so these are not possible cases for N_1.

• $N_1 \cong M_{11}$. This case cannot occur since $\mathrm{Out}(M_{11}) = 1$.

• $N_1 \cong L_2(q)$, $q = p^n$. From Lemma 4 we have that $N \cap A = \times_{i=1}^{r}(N_i \cap A)$, and so $N \cap A_{\sigma'} = \times_{i=1}^{r}(N_i \cap A_{\sigma'}) = 1$. Moreover, since $A_{\sigma'}$ normalizes N_1, it normalizes N_i for any $i = 1, \ldots, r$, because A acts transitively on the set $\Omega = \{N_1, \ldots, N_r\}$. Therefore $A_{\sigma'} \cong A_{\sigma'}N/N$ is a subgroup of $\mathrm{Out}(N_1) \times \cdots \times \mathrm{Out}(N_r)$. Analogously $B_{\sigma'} \cong B_{\sigma'}N/N$. Moreover $A_{\sigma'}N/N = B_{\sigma'}N/N$. By the structure of $\mathrm{Out}(L_2(q))$ we deduce that there exists a prime $r \in \sigma'$ such that A and B have normal Sylow r-subgroups. From Lemmas 3 and 2 we deduce that N is abelian, which is a contradiction.

Therefore our claim follows and N is a simple group.

We recall that $G = AN = BN = AB$ and so we deduce that $|N||A \cap B| = |N \cap A||N \cap B||G/N|$. In particular, if X, Y are maximal soluble subgroups of N such that $N \cap A \leq X$ and $N \cap B \leq Y$, then $|N|$ divides $|X||Y||\mathrm{Out}(N)|$. Then we will use the fact that the orders of X and Y are known from the proof of [**2**, Lemma 2.5].

We recall also that $A_\pi \neq 1$, $B_\pi \neq 1$, $A_{\pi'} \neq 1$, $B_{\pi'} \neq 1$. Moreover, we have that $|\pi(G) \cap \pi| > 1$ and $|\pi(G) \cap \pi'| > 1$ because of Lemmas 3 and 2, as N is non-abelian.

We check next that each of the possibilities for the group N leads to a contradiction.

• $N \cong L_3(3)$ and $N \cong \mathrm{PS}p_4(3)$. In both cases $|G|$ would be divided only by three distinct primes which is a contradiction.

• $N \cong M_{11}$. In this case $\mathrm{Out}(N) = 1$ and so $G = N$ is simple. Since all subgroups of the group M_{11} are known, it is easily deduced that this case cannot occur.

• $N \cong L_3(4)$ or $N \cong L_3(7)$. These cases can be excluded since, as proved in [**2**, Lemma 2.5], for these groups it is not possible that $|N|$ divides $|X||Y||\mathrm{Out}(N)|$, for soluble subgroups X and Y of N.

• $N \cong L_3(5)$. In this case $|N| = 2^5 \cdot 3 \cdot 5^3 \cdot 31$ and $|\mathrm{Out}(N)| = 2$. By [**2**, Lemma 2.5] we may suppose w.l.o.g. that $|N \cap A|$ divides $31 \cdot 3$ and $|N \cap B|$ divides $2^4 \cdot 5^3$. Hence the case $G = N$ cannot occur by order arguments. So $|G/N| = 2$ and $G \cong \mathrm{Aut}(N)$. This means that $|N \cap A| = 31 \cdot 3$ and $|N \cap B| = 2^4 \cdot 5^3$. Since B is neither a π-group nor a π'-group and $2 \in \pi'$ it should be $5 \in \pi$. This fact forces the primes 3 and 31 to be in different sets of primes. But this also leads to a contradiction, since a Sylow 31-subgroup of N is self-centralizing.

• $N \cong L_3(8)$. In this case $|N| = 2^9 \cdot 3^2 \cdot 7^2 \cdot 73$ and by [**2**, Lemma 2.5] we may assume that $|N \cap A|$ divides $73 \cdot 3$ and $|N \cap B|$ divides $2^9 \cdot 7^2$. Since $|\mathrm{Out}(N)| = 2 \cdot 3$ and $|N|$ divides $|G/N||N \cap A||N \cap B|$, the cases $G = N$ and $|G/N| = 2$ are not possible by order arguments.

If either $|G/N| = 3$ or $|G/N| = 2 \cdot 3$, it follows that $|N \cap A| = 73 \cdot 3$. Since a Sylow 73-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$, we can deduce that A is either a π-group or a π'-group, a contradiction.

• $N \cong L_4(2) \cong A_8$. In this case, there is no factorization $G = AB$ with A, B soluble subgroups.

• $N \cong U_3(8)$. Then $|N| = 2^9 \cdot 3^4 \cdot 7 \cdot 19$ and $|\mathrm{Out}(N)| = 2 \cdot 3^2$. By [**2**, Lemma 2.5], we may assume that $|N \cap A|$ divides $3 \cdot 19$ and $|N \cap B|$ divides $2^9 \cdot 7 \cdot 3$. Hence by order arguments it follows that $|G| \geq |N| \cdot 3^2$. Note also that since $\mathrm{Out}(N)$ is not a direct product of a 2-group and a 3-group, G/N should be a π-group or a π'-group. By [**2**, Lemma 2.5], we may assume that $|N \cap A|$ divides $3 \cdot 19$ and $|N \cap B|$ divides $2^9 \cdot 7 \cdot 3$.

If $|G/N| = 3^2$, then $|N \cap A| = 3 \cdot 19$ and $|N \cap B| = 2^9 \cdot 7 \cdot 3$. Now the fact that a Sylow 19-subgroup of N is self-centralizing in N forces 3 and 19 to belong to the same set of primes, that is, $\pi \cap \pi(G) = \{3, 19\}$ and $\pi' \cap \pi(G) = \{2, 7\}$. But then A would be a π-group, a contradiction.

Now assume that $|G/N| = 2 \cdot 3^2$, that is, $G \cong \mathrm{Aut}(N)$. Then $|N \cap A| = 3 \cdot 19$, $|N \cap B| = 2^8 \cdot 7 \cdot 3$ and $2, 3$ are in the same set of primes, that is, $\pi' \cap \pi(G) = \{2, 3\}$ and $\pi \cap \pi(G) = \{7, 19\}$. But this cannot occur again because a Sylow 19-subgroup of N is self-centralizing.

• $N \cong L_2(q)$, $q = p^n$.

Recall that, in this case, $|N| = \epsilon q(q^2 - 1)$, $\epsilon = (p - 1, 2)^{-1}$, and $\mathrm{Out}(N)$ is a cyclic group of order $\epsilon^{-1} n$. From [**2**, Lemma 2.5] it follows that, apart from some exceptional cases with $q \in \{5, 7, 11, 23\}$ that we will study later, the maximal soluble subgroups X and Y of N satisfies the condition $\{X, Y\} = \{N_N(N_p), D_{\nu(q+1)}\}$, with $N_p \in \mathrm{Syl}_p(N)$, $|N_N(N_p)| = \epsilon q(q-1)$ and $D_{\nu(q+1)}$ a dihedral group of order $\nu(q+1)$ with $\nu = (2, p)$.

We claim that p does not divide $(|N \cap A|, |N \cap B|)$. Assume first that $p \in \pi$. If p would divide $(|N \cap A|, |N \cap B|)$, then $A_{\pi'} \cap N = 1 = B_{\pi'} \cap N$, since the centralizer of any element of order p in N is a p-group. Therefore $A_{\pi'} \cong A_{\pi'}N/N$ is a subgroup of $\mathrm{Out}(N)$ and, analogously, $B_{\pi'} \cong B_{\pi'}N/N$. Moreover, $A_{\pi'}N/N = B_{\pi'}N/N$. By the structure of $\mathrm{Out}(N)$ we deduce that there exists a prime $r \in \pi'$ such that A and B have normal Sylow r-subgroups. Again from Lemmas 3 and 2 we get the contradiction that N is abelian. Note that the same conclusion follows if $p \in \pi'$.

Assume, therefore, w.l.o.g. that p does not divide $|N \cap A|$. Hence we can deduce that $|N \cap B|$ divides $|N_N(N_p)| = q(q-1)/(2, q-1)$ and $|N \cap A|$ divides $|D_{\nu(q+1)}| = \nu(q+1)$. In particular, it follows that $N \cap B$ is either a π-group or a π'-group, since the centralizer of any element of order p in N is a p-group.

We claim now that p divides $|G/N|$ and, in particular, $n > 1$. Since $|N|$ divides $|G/N||N \cap A||N \cap B|$, if p does not divide $|G/N|$, it follows that $|N|_p = |N \cap B|_p$. Then a Sylow p-subgroup of $N \cap B$ is a Sylow p-subgroup of N contained in B. Hence B must be a π-group or a π'-group, because the centralizer in $\mathrm{Aut}(N)$ of any Sylow p-subgroup of N is a p-group by [**11**, 1.17], which is a contradiction.

We have that $G/N = BN/N$ and also that $|N|_p$ divides $|G/N|_p|N \cap B|_p$. Since $B_\pi \neq 1$, $B_{\pi'} \neq 1$ and $n > 1$, it is clear that there exists some outer automorphism ϕ centralizing a Sylow p-subgroup of $N \cap B$. Then it follows that $|C_N(\phi)|_p \geq |N \cap B|_p \geq q/n$. But $|C_N(\phi)|_p \leq q^{1/2}$ (see, for instance, [**5**, Chapter 12]). Hence $q \leq q^{1/2}n$, that is, $q = p^n \leq n^2$. This leads to a contradiction, except for the cases $p = 2$ and $n \leq 4$.

The case $(p, n) = (2, 3)$ can be easily excluded, since the group $L_2(2^3) = L_2(8)$ has order divisible only by three distinct primes. Finally, the case $(p, n) = (2, 4)$ is also excluded, because in this case B would be a π'-group, which is not possible.

For $q \in \{5, 7, 11, 23\}$ there exists another possibility for the maximal soluble subgroups X and Y (see [**2**, Lemma 2.5]). But note that in all these cases $G = N$ and one of the subgroups $A = N \cap A$ or $B = N \cap B$ is contained in $N_N(N_p)$ for some $N_p \in \mathrm{Syl}_p(N)$. Then A or

B should be either a π-group or a π'-group, which provides the final contradiction. $\square$

4. The soluble case with $2 \in \pi$

Theorem 3. *Let π be a set of primes with $2 \in \pi$. Let the group $G = AB$ be the product of two soluble π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Assume that the following simple groups are not involved in G:*

(i) *$L_2(2^n)$, $n \geq 2$, except if either $n = 3$ or $q = 2^n + 1 > 5$ is a Fermat prime,*

(ii) *$L_2(q)$, $q > 3$ odd, except if q is a Mersenne prime.*

Then $A_\pi B_\pi = B_\pi A_\pi$ and this is a Hall π-subgroup of G.

Proof: Assume the result is not true and let G be a couterexample of minimal order. Obviously $A_\pi \neq 1$ and $B_\pi \neq 1$. Moreover $|\pi(G) \cap \pi| > 1$ because of Lemma 3.

We can argue as in Step 1 of Proposition 2 to deduce that G has a unique minimal normal subgroup N, which is neither a π-group nor a π'-group. We note that $N = N_1 \times \cdots \times N_r$, where N_i are isomorphic non-abelian simple groups for $i = 1, \ldots, r$, $C_G(N) = 1$ and $N \trianglelefteq G \leq \mathrm{Aut}(N)$.

On the other hand, we have by Theorem 2 that $A_{\pi'} B_{\pi'}$ is a Hall π'-subgroup of G. Consequently, if $A_{\pi'} \neq 1$ and $B_{\pi'} \neq 1$, it would follow from Lemma 2 the contradiction $[N, N] \leq [A_{\pi'}^G, B_{\pi'}^G] = 1$. Therefore, w.l.o.g. we may assume that $B_{\pi'} = 1$, i.e., $B = B_\pi$, and $A_{\pi'} \neq 1$. We recall that now Lemma 1 implies that the conditions $A_\pi B_\pi = B_\pi A_\pi$ and $A_\pi \leq O_\pi(G)$ are equivalent.

We claim first that $G = A_\pi N$ and N is a simple group.

The choice of G implies that $A_\pi N/N \leq T/N := O_\pi(G/N)(BN/N)$. In particular, $N \leq T = A_\pi(T \cap A_{\pi'})B$. If T were a proper subgroup of G, then $A_\pi \leq O_\pi(T) \leq C_G(N) = 1$, which is a contradiction. Consequently G/N is a π-group and, in particular, $A_{\pi'} \leq N$. Then $X := A_\pi N = A(B \cap X)$. If X were a proper subgroup of G, we would argue as above to conclude the contradiction $A_\pi \leq O_\pi(X) = 1$. Therefore $X = A_\pi N = G$.

We can deduce now that $A_{\pi'} = (N_1 \cap A_{\pi'}) \times \cdots \times (N_r \cap A_{\pi'})$ is a Hall π'-subgroup of N and A_π acts transitively by conjugacy on the components $N_1, \ldots, N_r$ of N. This implies $r = 1$, that is, N is a simple group and the claim is proved.

We prove next that $G = BN$.

Assume that $NB < G$. We claim that $N = BA_{\pi'}$, $N \cap A_\pi = 1$ and $|A_\pi| = t$ for some prime t.

Let us consider $M := NB = B(NB \cap A) = BA_{\pi'}(NB \cap A_\pi)$. If we denote $R = NB \cap A_\pi$, we deduce by the choice of G that $R \leq O_\pi(M) = 1$ and, in particular, $N \cap A_\pi = 1$. Since $G = NA_\pi = (NB)A_\pi$, we deduce that $|N| = |NB|$ and so $B \leq N = BA_{\pi'}$.

Now let C be a subgroup of A_π of order t, for some prime t, and assume that $X := NC = BA_{\pi'}C$ is a proper subgroup of G. Again we deduce that $C \leq O_\pi(X) = 1$, a contradiction. Therefore, $|A_\pi| = t$ for some prime t.

Since N is a non-abelian simple group factorized as the product of two soluble subgroups of coprime orders, we have from [**10**] and [**7**, Theorem 1.1] that N should be isomorphic to one of the following: M_{11}, $L_3(3)$, $L_2(q)$ with $q > 3$ odd and $q \equiv -1(4)$, $L_2(8)$ and $L_2(2^n)$ with $2^n + 1 > 5$ a Fermat prime. (Recall that the remainder cases for $L_2(2^n)$, $n \geq 2$, are excluded by hypothesis.) We discard next all these possibilities for the group N which will show that $G = NB$.

- $N \cong M_{11}$.

We have that $A_\pi \neq 1$ is a isomorphic to a subgroup of $\mathrm{Out}(M_{11}) = 1$, a contradiction.

- $N \cong L_3(3)$.

In this case $\pi \cap \pi(G) = \{2, 3\}$ and $\pi' \cap \pi(G) = \{13\}$. Moreover the outer automorphism of order 2 of N should centralize a Sylow 13-subgroup of N but this is not true.

- $N \cong L_2(q), q > 3$ a Mersenne prime.

In this case $|\mathrm{Out}(N)| = 2$, so A_π has order 2.

The possible factorizations for N can be found in [**7**]. So we have that $\{B, A_{\pi'}\}$ should be a pair of subgroups of N among pairs of subgroups of N of type $\{N_N(N_q), D_{q+1}\}$, with $N_q \in \mathrm{Syl}_q(N)$ and D_{q+1} a dihedral group of order $q+1$. Moreover the subgroups in these pairs are maximal subgroups of N. Since $2 \in \pi$ and 2 divides $q+1$ we have $B = D_{q+1}$ and $A_{\pi'} = N_N(N_q)$; in particular $q \in \pi'$. But then it is not possible that A_π centralizes $A_{\pi'} = N_N(N_q)$, since $C_{\mathrm{Aut}(N)}(N_q)$ is a q-group by [**11**, 1.17].

- $N \cong L_2(2^n)$, for either $n = 3$ or $2^n + 1 > 5$ is a Fermat prime.

The only factorizations of $L_2(q)$, $q = 2^n$, as product of soluble subgroups of coprime orders should be among pairs of subgroups of N of type $\{N_N(N_2), C_{q+1}\}$, with C_{q+1} a cyclic group of order $q + 1$ and $N_2 \in \mathrm{Syl}_2(N)$ (see for instance [**7**]). Since $2 \in \pi$ we have $B = N_N(N_2)$ and $A_{\pi'} = C_{q+1}$. But then there exists an outer automorphism of order t in A_π centralizing the subgroup $A_{\pi'} = C_{q+1}$ which is not the case.

Now we have proved that $G = AN = BN = AB$ and so $|N||A\cap B| = |N\cap A||N\cap B||G/N|$. From now on X and Y will denote maximal soluble subgroups of N such that $N\cap A \le X$ and $N\cap B \le Y$, respectively, and we will use [**2**, Lemma 2.5]. We check next that each of the possibilities for the group N leads to a contradiction which will conclude the proof. Recall that we have excluded the cases $L_2(2^n)$, $n \ge 2$, except if either $n = 3$ or $r = 2^n + 1 > 5$ is a Fermat prime, and the cases $L_2(q)$, q odd, except if q is a Mersenne prime.

• $N \cong L_3(3)$. In this case $|N| = 3^3 \cdot 2^4 \cdot 13$ and $|\mathrm{Out}(N)| = 2$. Moreover, X and Y should satisfy $\{|X|, |Y|\} = \{13\cdot 3, 3^3 \cdot 2^4\}$. By order arguments $2^3 \cdot 3^3$ divides either $|N\cap A|$ or $|N\cap B|$. Then, since a Sylow 3-subgroup of N is self-centralizing, we have $\pi\cap\pi(G) = \{2, 3\}$ and $\pi'\cap\pi(G) = \{13\}$. Moreover, since a Sylow 13-subgroup of N is also self-centralizing, the case $|N \cap A| = 13 \cdot 3$ is not possible and so $|N \cap A| = 13$. Hence the case $G = N$ cannot occur and it follows $G \cong \mathrm{Aut}(G)$. But in this case, there would exist an automorphism of N of order 2 centralizing a Sylow 13-subgroup of N, which is not possible (see [**6**]).

• $N \cong \mathrm{PS}p_4(3)$. In this case $|N| = 2^6 \cdot 3^4 \cdot 5$ and $|\mathrm{Out}(N)| = 2$. From [**2**, Lemma 2.5] it follows that $\{|X|, |Y|\} = \{2^5 \cdot 5, 3^4 \cdot 2^4\}$. By order arguments we have that 2 and 5 divides either $|N \cap A|$ or $|N \cap B|$ and 3^4 divides the other. Then $5 \in \pi$, because there are no 2-elements in N centralizing a Sylow 5-subgroup of N. Also $3 \in \pi$, since a Sylow 3-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$. Consequently, G is a π-group, which is a contradiction.

• $N \cong M_{11}$. In this case $G = N$ is simple and $\{|A|, |B|\} = \{55, 2^4 \cdot 3^2\}$, which gives a contradiction with the fact that $A_\pi \ne 1$ and $A_{\pi'} \ne 1$.

• $N \cong L_3(4)$ or $N \cong L_3(7)$. These cases can be excluded as said in the proof of Theorem 2.

• $N \cong L_3(5)$. By [**2**, Lemma 2.5], one of the numbers $|N \cap A|$ and $|N\cap B|$ divides $31\cdot 3$ and the other divides $2^4\cdot 5^3$. Hence the case $G = N$ cannot occur by order arguments. So we may deduce that $G \cong \mathrm{Aut}(N)$ and $|G/N| = 2$. Since a Sylow 5-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$, this forces the primes 2 and 5 to be in the same set of primes. Recall also that $2 \in \pi$ and B is a π-group, so we have $|N \cap B| = 2^4 \cdot 5^3$ and $|N\cap A| = 31\cdot 3$. Since a Sylow 31-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$ (see [**6**]), we deduce that A should be a π-group, which is a contradiction.

• $N \cong L_3(8)$. Now $|N| = 2^9 \cdot 3^2 \cdot 7^2 \cdot 73$, $|\mathrm{Out}(N)| = 2 \cdot 3$ and from [**2**, Lemma 2.5] it follows that one of the numbers $|N \cap A|$ and $|N \cap B|$ divides $73 \cdot 3$, and the other divides $2^9 \cdot 7^2$. The cases $G = N$ and $|G/N| = 2$ cannot occur by order arguments. Moreover, since G/N is a π-group, we have $\{2, 3\} \subseteq \pi$. The fact that B is a π-group and a Sylow 73-subgroup of N is self-centralizing forces that $\pi = \{2, 3, 73\}$ and $\pi' = \{7\}$. The case $|G/N| = 3$ and $|N \cap A| = 2^9 \cdot 7^2$ cannot occur since a Sylow 2-subgroup of N is self-centralizing. So, $|G/N| = 2 \cdot 3$ and $|N \cap A| = 2^8 \cdot 7^2$. But in this case $N \cap A$ would be a normal subgroup of a Borel subgroup of N containing a central subgroup of order 7^2 which is a contradiction.

• $N \cong L_4(2) \cong A_8$. This case is not possible because there is no factorization of G with soluble factors.

• $N \cong U_3(8)$. Recall that $|N| = 2^9 \cdot 3^4 \cdot 7 \cdot 19$, $|\mathrm{Out}(N)| = 2 \cdot 3^2$ and by [**2**, Lemma 2.5], it should be $|G| \geq |N| \cdot 3^2$. Moreover, G/N is a π-group and $\{2, 3\} \subseteq \pi$.

If $|G/N| = 3^2$, then $\{|N \cap A|, |N \cap B|\} = \{3 \cdot 19, 2^9 \cdot 7 \cdot 3\}$, and so the fact that a Sylow 19-subgroup is self-centralizing in N leads to $\pi \cap \pi(G) = \{2, 3, 19\}$. But if $\pi' \cap \pi(G) = \{7\}$, there would be an element of order 7 in N centralizing a Sylow 2-subgroup of N, a contradiction.

Now assume that $|G/N| = 2 \cdot 3^2$ and so $\{|N \cap A|, |N \cap B|\} = \{3 \cdot 19, 2^8 \cdot 7 \cdot 3\}$ or $\{|N \cap A|, |N \cap B|\} = \{3 \cdot 19, 2^9 \cdot 7 \cdot 3\}$. In any case it follows $19 \in \pi$, since a Sylow 19-subgroup of N is self-centralizing. But $\pi' \cap \pi(G) = \{7\}$ cannot occur again because this would mean in both cases that a Borel subgroup of N would have a subgroup of order 7 centralizing a subgroup of order 2^8, which is not possible.

• $N \cong L_2(q)$, $q > 3$ a Mersenne prime.

In this case, we know from [**2**, Lemma 2.5] that $|\mathrm{Out}(N)| = 2$ and $\{X, Y\} = \{N_N(N_q), D_{q+1}\}$, with $N_q \in \mathrm{Syl}_q(N)$ and D_{q+1} a dihedral group of order $q + 1 = 2^n$, for some $n \geq 2$. (For $q = 2^3 - 1 = 7$ there exist another factorization which will be considered later.)

Since D_{q+1} is a 2-group, it follows that $N \cap A \subseteq N_N(N_q)$. Now by order arguments q divides $|N \cap A|$. Since a Sylow q-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$, we deduce that A is either a π-group or a π'-group which is a contradiction.

If $q = 7$, it might be also possible that $\{X, Y\} = \{N_N(N_q), S_4\}$ with $N_q \in \mathrm{Syl}_q(N)$ and S_4 the symmetric group of degree 4. Since N_q is self-centralizing in $\mathrm{Aut}(N)$, we deduce that $N \cap B \subseteq N_N(N_q)$ and $N \cap A \subseteq S_4$. Then the factorization $A = A_\pi \times A_{\pi'}$ with $A_{\pi'} \neq 1$ and $A_\pi \neq 1$ is not possible.

• $N \cong L_2(2^n)$, for either $n = 3$ or $2^n + 1 > 5$ a Fermat prime.

Set $q = 2^n$. Recall that, in this case, $|N| = q(q^2 - 1)$, and $\mathrm{Out}(N)$ is a cyclic group of order n. From [**2**, Lemma 2.5] it follows that $\{X, Y\} = \{N_N(N_2), D_{2(q+1)}\}$, with $N_2 \in \mathrm{Syl}_2(N)$, $|N_N(N_2)| = q(q - 1)$ and $D_{2(q+1)}$ a dihedral group of order $2(q + 1)$. Since the subgroups of prime order $q + 1$ in N are self-centralizing in $\mathrm{Aut}(N)$ and $q + 1$ does not divide $|\mathrm{Out}(N)|$, we deduce that $N \cap A \not\leq D_{2(q+1)}$. Hence $N \cap A \leq N_N(N_2)$. But again the fact that a Sylow 2-subgroup of N is self-centralizing in $\mathrm{Aut}(N)$ provides the final contradiction. □

Remark. In [**12**, Final examples, 3] it has been shown that the conclusion of Theorem 3 is not true for the groups $L_2(2^n)$, $n \geq 2$, except if either $n = 3$ or $2^n + 1$ is a Fermat prime.

Next we show that Theorem 3 is also false for groups involving $L_2(q)$, $q > 3$ odd, except if q is a Mersenne prime. (We note that $L_2(4) \cong L_2(5)$.) To see this we consider the group $G = PGL_2(q)$, q odd. Note that $|G : L_2(q)| = 2$. Thus $|G| = q(q^2 - 1)$ and it is known that this group has cyclic subgroups of orders $(q - 1)$ and $(q + 1)$. Then $G = AB$ where $A \cong C_{q+1}$ is a cyclic group of order $q + 1$ and $B = N_G(G_p)$, $G_p \in \mathrm{Syl}_p(G)$, is a subgroup of order $q(q-1)$. Clearly $\pi(A) \cap \pi(B) = \{2\}$. Set $\pi = \pi(N_G(G_p))$ and note that $2 \in \pi$. Then $A = O_\pi(A) \times O_{\pi'}(A)$ is a π-decomposable group and B is a π-group, but $O_\pi(A)B$ is not a subgroup, except if $q + 1$ is a power of 2, that is, q is a Mersenne prime, in which case G is a π-group.

As a consequence of Theorems 2 and 3 we deduce the following result for an arbitrary set of primes π.

Corollary 1. *Let π be a set of primes. Let the group $G = AB$ be the product of two soluble π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Assume that the following simple groups are not involved in G:*

(i) *$L_2(2^n)$, $n \geq 2$, except if either $n = 3$ or $q = 2^n + 1 > 5$ is a Fermat prime,*

(ii) *$L_2(q)$, q odd, except if q is a Mersenne prime.*

Then the composition factors of G belong to one of the following types:

(1) *π-groups,*

(2) *π'-groups,*

(3) *the following groups in the list of Fisman* [**7**, Theorem 1.1]*:*

(i) *$L_2(2^n)$, $n \geq 2$, with either $n = 3$ or $q = 2^n + 1 > 5$ is a Fermat prime,*

(ii) *$L_2(q)$ with $q > 3$ and q is a Mersenne prime,*

(iii) *$L_3(3)$,*

(iv) *M_{11}.*

In particular, let the group $G = AB$ be the product of the two soluble π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ and assume that the simple groups $L_2(q)$, $q > 3$, $L_3(3)$ and M_{11} are not involved in G. Then the group G is π-separable.

Proof: The last statement of the corollary follows directly from the first part. Assume that this one is not true and let G be a counterexample of minimal order. Since G/M satisfies the corresponding hypotheses for each normal subgroup M, we may assume that G has a unique minimal normal subgroup, say N. We can also deduce that $O_{\pi'}(G) = O_\pi(G) = 1$, and so N is non-abelian. Assume, for instance, that $2 \in \pi'$. From Theorem 2 we have that $A_\pi B_\pi = B_\pi A_\pi$ and, by Lemma 2, we deduce that $[A_\pi^G, B_\pi^G] = 1$, which is a contradiction to the fact that N is non-abelian, unless either $A_\pi = 1$ or $B_\pi = 1$. Now applying Theorem 3 in a similar way we deduce that either $A_{\pi'} = 1$ or $B_{\pi'} = 1$. Then, in any of the cases, G would be the product of a π-group and a π'-group and the conclusion follows from [**7**, Theorem 1.1]. $\square$

Acknowledgements. The second and third author have been supported by Proyecto MTM2007-68010-C03-03, Ministerio de Educación y Ciencia and FEDER, Spain. The first author would like to thank the Universitat de València and the Universidad Politécnica de Valencia for their warm hospitality during the preparation of this paper. They are also grateful to B. Amberg for interesting suggestions during the visit of the first author to Mainz (Germany).

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Primera versió rebuda el 28 d'abril de 2008,
darrera versió rebuda el 16 de desembre de 2008.