Publ. Mat. **53** (2009), 179–196

BCR ALGORITHM AND THE $T(b)$ THEOREM

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Abstract

We show using the Beylkin-Coifman-Rokhlin algorithm in the Haar basis that any singular integral operator can be written as the sum of a bounded operator on L^p, $1 < p < \infty$, and of a perfect dyadic singular integral operator. This allows to deduce a local $T(b)$ theorem for singular integral operators from the one for perfect dyadic singular integral operators obtained by Hofmann, Muscalu, Tao, Thiele and the first author.

1. Introduction

The purpose of this note is to fill in a gap of [**AHMTT**] concerning a local $T(b)$ theorem for singular integrals with a method that could be of interest elsewhere.

In [**C**], M. Christ proves a local $T(b)$ theorem for singular integral operators on a space of homogeneous type, the motivation being the potential application to several questions related to analytic capacity. It lead to the solution of the Vitushkin's conjecture by G. David [**D**] or to a proof of the semiadditivity of analytic capacity (Painlevé problem) by X. Tolsa [**T**]. Those solutions required similar $T(b)$ theorems but in non-homogeneous spaces as developed by G. David [**D**], and F. Nazarov, S. Treil and A. Volberg [**NTV1**], [**NTV2**], [**V**].

Let us explain Christ's theorem and the word "local". He introduces the notion of accretive systems (b_Q) consisting of functions supported on the corresponding cube Q, bounded, non-degenerate (i.e. of mean 1). He requires that each b_Q is mapped through the operator to a bounded function on Q (and a similar hypothesis for the adjoint with a different accretive system if need be). He designs globally defined para-accretive

2000 *Mathematics Subject Classification.* 42B20, 42C40.
Key words. Singular integral operators, Haar basis.
Research supported by NNSF of China No. 10001027, the innovation funds of Wuhan University and the subject construction funds of Mathematic and Statistic School, Wuhan University.

functions b and b^* adapted to the operator and its adjoint, and applies the David-Journé-Semmes' $T(b)$ theorem [**DJS**] to obtain the L^2 boundedness of the operator.

A generalization of Christ's result is proposed in [**AHMTT**] in Euclidean space for a model situation.

Theorem 1.1 ([**AHMTT**]). *Assume that T is a perfect dyadic singular integral operator. Assume that there exists a constant $C \geq 0$ such that for each dyadic cube Q, one can find functions b_Q^1, b_Q^2 supported in Q with*

$$\int_Q b_Q^1 = |Q| = \int_Q b_Q^2, \tag{1.1}$$

$$\int_Q |b_Q^1|^2 + |b_Q^2|^2 \leq C|Q|, \tag{1.2}$$

$$\int_Q |Tb_Q^1|^2 + |T^* b_Q^2|^2 \leq C|Q|. \tag{1.3}$$

Then T is bounded on $L^2(\mathbb{R}^n)$.

The idea of proof is different from Christ's argument (in fact, it is not clear how to adapt it): it amounts to verify the hypotheses of a variant of the $T(1)$ theorem of David-Journé [**DJ**], namely a local $T(1)$ theorem. Perfect dyadic means essentially that the regularity is adapted to the dyadic grid: any function supported in a dyadic cube with mean 0 is mapped to a function supported in the same cube. This property kills most tail terms that would appear with standard singular integrals.

The following natural extension is announced in [**AHMTT**].

Theorem 1.2. *Assume that T is singular integral operator with locally bounded kernel on $\mathbb{R}^n \times \mathbb{R}^n$. Assume that there exists a constant $C \geq 0$ such that for each dyadic cube Q, one can find functions b_Q^1, b_Q^2 supported on Q with*

$$\int_Q b_Q^1 = |Q| = \int_Q b_Q^2, \tag{1.4}$$

$$\int_Q |b_Q^1|^2 + |b_Q^2|^2 \leq C|Q|, \tag{1.5}$$

$$\int_Q |Tb_Q^1|^2 + |T^* b_Q^2|^2 \leq C|Q|. \tag{1.6}$$

Then T is bounded on $L^2(\mathbb{R}^n)$.

It looks like a straightforward exercise to adapt the proof in the model case by handling the tails as error terms. This is actually said in [**AHMTT**] but, on second thoughts, it may have been too optimistic*. The far away tails are indeed easy to handle, that is integrals $\int_{Q\times R} g(x)K(x,y)f(y)\,dx\,dy$ when $R\cap 3Q\neq\varnothing$ or $Q\cap 3R\neq\varnothing$ with f or g having mean value 0. But the same integrals on adjacents cubes of different sizes seem a problem. The reader can be convinced by reading the proof of Theorem 8.6 of [**AAAHK**] in [**H**] where the hypothesis (1.5) has been strengthened to an $L^{2+\varepsilon}$ condition to work out the transposition of the [**AHMTT**] argument.

It would be interesting to have a direct proof of this result but we have not succeeded. Our idea is to reduce to the model case via the following result, interesting on its own.

Theorem 1.3. *Assume that T is a singular integral operator with locally bounded kernel on $\mathbb{R}^n\times\mathbb{R}^n$. Then there exists a perfect dyadic singular integral operator $\mathbb{T}$ such that $T-\mathbb{T}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1<p<\infty$.*

This is done using the Beylkin-Coifman-Rokhlin algorithm in the Haar basis and ideas from the PhD thesis of one of us [**Y1**].

Let us say that the extension of Christ's result for singular integrals is not just an academic exercise. Such a generalization found recently an application in [**AAAHK**] towards the L^2 boundedness of boundary layer potentials for some PDE's. Other potential applications require a similar theorem with L^2 conditions on the accretive system replaced by L^p conditions for $p>1$†. For perfect dyadic models, it is remarked in [**AHMTT**] that the L^2 conditions can be replaced by L^p conditions for any $1<p\leq\infty$‡. See Theorems 3.3 and 3.4. At present, none of the arguments for standard singular integrals in [**H**] or here work with L^p conditions for $p<2$. We leave this question open.

2. From an operator to a perfect dyadic operator

Here is a formal approach. We begin with the BCR algorithm in the Haar basis. Consider the Haar wavelets in $\mathbb{R}^n$ given by

$$\psi_{j,k}(x)=2^{nj/2}\psi(2^jx-k),\quad j\in\mathbb{Z},\ k\in\mathbb{Z}^n,\ \psi\in E \tag{2.1}$$

*The first author feels responsible for that.

†Personal communication of Steve Hofmann.

‡The inequality $1\leq p$ is written but this is obviously a typo as the whole argument depends on the stopping time argument in Lemma 6.5, which does not give anything for $p=1$.

where E is a set of cardinal $2^n - 1$. Recall that $\psi_{j,k}$ has support in the dyadic cube $Q = Q_{j,k} = 2^{-j}k + 2^{-j}[0,1)^n$, that $\int \psi_{j,k} = 0$ and that $\{\psi_{j,k}\}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$. Define also

$$\phi_{j,k}(x) = 2^{nj/2}\phi(2^j x - k), \quad j \in \mathbb{Z},\, k \in \mathbb{Z}^n,\, \phi = 1_{[0,1)^n}.$$

We also use the notation ψ_Q and ϕ_Q when more convenient. It is understood that that the Haar functions ψ describe the set E and we forget from now on to mention this as it plays no role.

For $j \in \mathbb{Z}$, we let V_j be the closed subspace of L^2 generated by the orthonormal system $\phi_{j,k}$, $k \in \mathbb{Z}^n$ and W_j the closed subspace of L^2 generated by the orthonormal system $\psi_{j,k}$, $k \in \mathbb{Z}^n$. It is well-known that V_j and W_j are orthogonal spaces and $L^2(\mathbb{R}^n) = \oplus W_j$. Furthermore, one has $P_{j+1} = P_j + Q_j$ where P_j and Q_j are the orthogonal projections onto V_j and W_j. In what follows, $\langle\ ,\ \rangle$ denotes the bilinear duality bracket and the adjoint of an operator T for this duality is denoted by T^*.

Consider an operator T for which one can define the coefficients for all $j \in \mathbb{Z}$,

$$\langle \phi_Q, T\phi_R\rangle, \qquad Q = Q_{j,k}, \quad R = Q_{j,\ell}, \tag{2.2}$$

$$\langle \psi_Q, T\psi_R\rangle = a_{Q,R} = a^j_{k,\ell}, \quad Q = Q_{j,k}, \quad R = Q_{j,\ell}, \tag{2.3}$$

$$\langle \psi_Q, T\phi_R\rangle = b_{Q,R} = b^j_{k,\ell}, \quad Q = Q_{j,k}, \quad R = Q_{j,\ell}, \tag{2.4}$$

$$\langle \phi_Q, T\psi_R\rangle = c_{Q,R} = c^j_{k,\ell}, \quad Q = Q_{j,k}, \quad R = Q_{j,\ell}, \tag{2.5}$$

and such that for f, g in some appropriate vector space(s) of measurable functions,

$$\lim_{j\to+\infty} \langle P_j g, TP_j f\rangle = \langle g, Tf\rangle \tag{2.6}$$

and

$$\lim_{j\to-\infty} \langle P_j g, TP_j f\rangle = 0. \tag{2.7}$$

Note that $\langle P_j g, TP_j f\rangle$ is defined using the first set of coefficients in (2.2). Then, one can expand formally

$$\langle g, Tf\rangle = \langle g, Uf\rangle + \langle g, Vf\rangle + \langle g, Wf\rangle \tag{2.8}$$

where

$$\langle g, Uf\rangle = \sum_{j=-\infty}^{+\infty} \langle Q_j g, TQ_j f\rangle \tag{2.9}$$

$$\langle g, Vf\rangle = \sum_{j=-\infty}^{+\infty} \langle Q_j g, TP_j f\rangle \tag{2.10}$$

$$\langle g, Wf\rangle = \sum_{j=-\infty}^{+\infty} \langle P_j g, TQ_j f\rangle. \tag{2.11}$$

Expanding on the bases of V_j and W_j, one finds

$$\langle g, Uf\rangle = \sum_{j=-\infty}^{+\infty} \sum_{k,\ell\in\mathbb{Z}^n} \langle g, \psi_{j,k}\rangle a^j_{k,\ell}\langle \psi_{j,\ell}, f\rangle \tag{2.12}$$

$$\langle g, Vf\rangle = \sum_{j=-\infty}^{+\infty} \sum_{k,\ell\in\mathbb{Z}^n} \langle g, \psi_{j,k}\rangle b^j_{k,\ell}\langle \phi_{j,\ell}, f\rangle \tag{2.13}$$

$$\langle g, Wf\rangle = \sum_{j=-\infty}^{+\infty} \sum_{k,\ell\in\mathbb{Z}^n} \langle g, \phi_{j,k}\rangle c^j_{k,\ell}\langle \psi_{j,\ell}, f\rangle. \tag{2.14}$$

This is the so-called BCR algorithm in the Haar basis. The operator U is diagonal in the decomposition of L^2 given by the W_j. The operator V is some sort of paraproduct and W is like V^*. This decomposition can be used to prove the $T(1)$ theorem.

Let us go further and modify formally U, V, W. Set

$$\alpha^j_{k,\ell} = \begin{cases} a^j_{k,\ell}, & \text{if } k \neq \ell, \\ 0, & \text{if } k = \ell, \end{cases} \tag{2.15}$$

$$\beta^j_{k,\ell} = \begin{cases} b^j_{k,\ell}, & \text{if } k \neq \ell, \\ -\sum_{m\neq 0} b^j_{k,k+m}, & \text{if } k = \ell, \end{cases} \tag{2.16}$$

$$\gamma^j_{k,\ell} = \begin{cases} c^j_{k,\ell}, & \text{if } k \neq \ell, \\ -\sum_{m\neq 0} c^j_{k+m,k}, & \text{if } k = \ell, \end{cases} \tag{2.17}$$

and $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ the operators associated with the family of coefficients α, β, γ as U, V, W with the family of coefficients a, b, c. The α, β, γ are designed so that $\mathcal{V}(1)=\mathcal{V}^*(1)=0$ and $\mathcal{W}(1)=\mathcal{W}^*(1)=0$ (of course, this has only a formal meaning) so that the main result is the following.

Theorem 2.1. *Assume that for some $s>0$ and $C>0$ one has for all j, k, ℓ with $k\neq\ell$,*

$$|a^j_{k,\ell}|+|b^j_{k,\ell}|+|c^j_{k,\ell}|\leq C(1+|k-\ell|)^{-n-s}. \tag{2.18}$$

Then $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ are bounded operators on $L^p(\mathbb{R}^n)$, $1<p<\infty$ and also from $H^1_d(\mathbb{R}^n)$ into $L^1(\mathbb{R}^n)$ and from $L^\infty(\mathbb{R}^n)$ into $\mathrm{BMO}_d(\mathbb{R}^n)$.

We set $\mathcal{T}=\mathcal{U}+\mathcal{V}+\mathcal{W}$. Here, $H^1_d(\mathbb{R}^n)$ and $\mathrm{BMO}_d(\mathbb{R}^n)$ are the dyadic Hardy and BMO spaces. The proof is in Section 4. In fact, a decay in $|k-\ell|^{-n}\ln^{-2-\varepsilon}(1+|k-\ell|)$ with $\varepsilon>0$ suffices.

We remark that the point of this statement is to avoid use of the "diagonal coefficients" in the families a, b, c as this would require some sort of weak boundedness property which we do not want to assume.

This theorem has its origin in [**Y1**] where the Haar functions are replaced by smooth compactly supported wavelets. But the point was different. The operator T was supposed bounded on L^2 and the objective was to obtain the rate of approximation of T by some truncated T_m in the non-standard representation defined by the BCR algorithm. Here, we do not assume that the original T is bounded. See also [**Y2**], [**Y3**], [**DYY**] for related ideas.

Let $\mathbb{U}$, $\mathbb{V}$, $\mathbb{W}$ be the differences $U-\mathcal{U}$, $V-\mathcal{V}$, $W-\mathcal{W}$ and $\mathbb{T}=\mathbb{U}+\mathbb{V}+\mathbb{W}$. Thus the boundedness of T on L^2 is equivalent to that of $\mathbb{T}$. Note that

$$\langle g,\mathbb{U}f\rangle=\sum_{j=-\infty}^{+\infty}\sum_{k\in\mathbb{Z}^n}\langle g,\psi_{j,k}\rangle\mathbf{a}^j_k\langle\psi_{j,k},f\rangle \tag{2.19}$$

$$\langle g,\mathbb{V}f\rangle=\sum_{j=-\infty}^{+\infty}\sum_{k\in\mathbb{Z}^n}\langle g,\psi_{j,k}\rangle\mathbf{b}^j_k\langle\phi_{j,k},f\rangle \tag{2.20}$$

$$\langle g,\mathbb{W}f\rangle=\sum_{j=-\infty}^{+\infty}\sum_{k\in\mathbb{Z}^n}\langle g,\phi_{j,k}\rangle\mathbf{c}^j_k\langle\psi_{j,k},f\rangle \tag{2.21}$$

for some family of complex coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. The only use of these formulae is in the following (formal) observation.

Lemma 2.2. *If f is supported in a dyadic cube and has mean 0, then $\mathbb{T}f$ is supported in the same cube in the sense that $\langle g,\mathbb{T}f\rangle=0$ if g is supported away from Q.*

Let Q be the dyadic cube supporting f. The coefficients $\langle \psi_{j,k}, f\rangle$ and $\langle \phi_{j,k}, f\rangle$ are 0 if $Q_{j,k} \cap Q = \varnothing$ and also if $Q \subsetneq Q_{j,k}$ since f has mean 0. Hence the sums reduce to couples (j,k) such that $Q_{j,k} \subset Q$. Thus, if g is supported away from Q, we have $\langle g, \mathbb{T}f\rangle = 0$.

We are now ready to apply all this to singular integral operators.

3. Application to singular integral operators

Assume that T is a singular integral operator, that is a linear continuous operator from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$ whose distributional kernel $K(x,y)$ satisfies the Calderón-Zygmund estimates, that is the size condition

$$|K(x,y)| \leq C|x-y|^{-n}, \tag{3.1}$$

for all x, y with $x \neq y$ and the regularity condition for some $0 < s < 1$

$$|K(x,y) - K(x',y)| + |K(y,x) - K(y,x')| \leq C\frac{|x-x'|^s}{|x-y|^{n+s}}, \tag{3.2}$$

for all x, x', y with $|x-x'| \leq \frac{1}{2}|x-y|$.

Assume also that K is locally bounded on $\mathbb{R}^n \times \mathbb{R}^n$. The local boundedness of K guarantees that one can start the BCR algorithm with T and obtain operators $\mathcal{T}$ and $\mathbb{T}$. More precisely, we first extend $\langle g, Tf\rangle$ a priori defined for $f, g \in \mathcal{D}(\mathbb{R}^n)$ to $f, g \in L^1_c(\mathbb{R}^n)$, the space of compactly supported integrable functions, by

$$\langle g, Tf\rangle = \iint_{\mathbb{R}^n \times \mathbb{R}^n} g(x)K(x,y)f(y)\,dx\,dy.$$

Hence all the coefficients a, b, c in (2.3), (2.4), (2.5) can be computed and the limits in (2.6) and (2.7) hold for $f, g \in L^1_c(\mathbb{R}^n)$. Moreover, the Calderón-Zygmund conditions on the kernel and standard estimates insure that (2.18) holds so that Theorem 2.1 applies. Thus, the operator $\mathcal{T}$ is bounded on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Furthermore, one has

Proposition 3.1. *The distribution kernel of $\mathcal{T}$ satisfies the size condition* (3.1).

This is also a standard computation from (2.18). Hence by difference and incorporating Lemma 2.2, $\mathbb{T}$ has the following properties:

(1) $\mathbb{T}$ is a linear continuous operator from $\mathcal{D}(\mathbb{R}^n)$ to $\mathcal{D}'(\mathbb{R}^n)$.
(2) $\mathbb{T}$ has a kernel satisfying the size condition (3.1).
(3) $\langle g, \mathbb{T}f\rangle$ is well-defined for pairs of functions $(f,g) \in L^p_c(\mathbb{R}^n) \times L^{p'}_c(\mathbb{R}^n)$ for $1 < p < \infty$ and if, furthermore, they are integrable

with support on disjoint cubes (up to a set of measure 0)

$$\langle g, \mathbb{T}f\rangle = \iint_{\mathbb{R}^n\times\mathbb{R}^n} g(x)\mathbb{K}(x,y)f(y)\,dx\,dy.$$

(4) For all (f,g) as above, if f has support in a dyadic cube and mean 0, then $\langle g, \mathbb{T}f\rangle = 0$ when the support of g does not meet Q (up to a set of measure 0).

We say that an operator satisfying the above four properties is a perfect dyadic singular integral operator. We note that this is not exactly the definition in [**AHMTT**], which is concerned with a dyadic and finite model, where the operator is defined on a finite dimensional subspace of the one generated by the ψ_Q and the ϕ_Q. But this is a superficial difference. Let us summarize the main result.

Theorem 3.2. *Assume that T is a singular integral operator with locally bounded kernel on $\mathbb{R}^n\times\mathbb{R}^n$. Then there exists a perfect dyadic singular integral operator $\mathbb{T}$ such that $T-\mathbb{T}$ is bounded on $L^p(\mathbb{R}^n)$ for all $1<p<\infty$.*

The criterion for L^2 boundedness of perfect dyadic singular integral operators in [**AHMTT**] is (see Theorem 6.8 there when $p=q=2$ and a remark after the proof for general p,q).

Theorem 3.3. *Assume that T is a perfect dyadic singular integral operator. Let $1<p,q\le\infty$ with dual exponents p', q'. Assume that there exists a constant $C\ge 0$ such that for each dyadic cube Q, one can find functions b^1_Q, b^2_Q supported in Q with*

$$\int_Q b^1_Q = |Q| = \int_Q b^2_Q, \tag{3.3}$$

$$\int_Q |b^1_Q|^p + |b^2_Q|^q \le C|Q|, \tag{3.4}$$

$$\int_Q |Tb^1_Q|^{q'} + |T^*b^2_Q|^{p'} \le C|Q|. \tag{3.5}$$

Then T is bounded on $L^2(\mathbb{R}^n)$.

Although we have a different definition of perfect dyadic operators, the proof there can be copied *in extenso* in our case. The non trivial part is to prove first

$$\int_Q |T\mathbf{1}_Q| + |T^*\mathbf{1}_Q| \le C'|Q|. \tag{3.6}$$

Then, one deduces L^2 boundedness by a version of the $T(1)$ theorem for dyadic perfect operators.

The conclusion of this discussion is the following local $T(b)$ theorem for singular integral operators.

Theorem 3.4. *Assume that T is a singular integral operator. Let $1 < p, q \le \infty$ with dual exponents p', q' be such that $1/p + 1/q \le 1$. Assume that there exists a constant $C \ge 0$ such that for each dyadic cube Q, one can find functions b_Q^1, b_Q^2 supported in Q with*

$$\int_Q b_Q^1 = |Q| = \int_Q b_Q^2, \tag{3.7}$$

$$\int_Q |b_Q^1|^p + |b_Q^2|^q \le C|Q|, \tag{3.8}$$

$$\int_Q |Tb_Q^1|^{q'} + |T^* b_Q^2|^{p'} \le C|Q|. \tag{3.9}$$

Then T is bounded on $L^2(\mathbb{R}^n)$.

Here is the proof. Write $T = \mathcal{T} + \mathbb{T}$. Since $\mathcal{T}$ is bounded on L^p and $q' \le p$ by (3.8), we have

$$\begin{aligned}\left(\frac{1}{|Q|}\int_Q |\mathcal{T} b_Q^1|^{q'}\right)^{1/q'} &\le \left(\frac{1}{|Q|}\int_Q |\mathcal{T} b_Q^1|^{p}\right)^{1/p} \\ &\le \|\mathcal{T}\|_{p,p}\left(\frac{1}{|Q|}\int_Q |b_Q^1|^{p}\right)^{1/p} \le C\|\mathcal{T}\|_{p,p}.\end{aligned}$$

Thus the same conclusion holds for $\mathbb{T}b_Q^1$ by (3.9) with constant $C\|\mathcal{T}\|_{p,p} + C$. Similarly

$$\left(\frac{1}{|Q|}\int_Q |\mathbb{T}^* b_Q^2|^{p'}\right)^{1/p'} \le C\|\mathcal{T}^*\|_{q,q} + C.$$

Hence we can apply Theorem 3.3 to $\mathbb{T}$ and conclude that $\mathbb{T}$, hence T, is bounded on $L^2(\mathbb{R}^n)$.

Remark 3.5. We do not know how to drop the constraint $1/p + 1/q \le 1$. It is satisfied if $p = q = 2$, which proves Theorem 1.2.

Remark 3.6. If one does not want to develop the $T(1)$ theory for perfect dyadic operators, here is a direct way: first, prove (3.6) for $\mathbb{T}$ following [**AHMTT**], then observe that this yields back the same conclusion for T. This classically implies the L^2 boundedness of T by the $T(1)$ theorem for singular integral operators.

Remark 3.7. Actually the Calderón-Zygmund conditions on the kernel of T can be weakened. It suffices that for Q, R distinct dyadic cubes with same sizes

$$\int_Q \int_R |K(x,y)|\,dx\,dy \le C|Q| \tag{3.10}$$

whenever Q and R are adjacent (i.e. $d(Q,R)=0$) and

$$\int_Q \int_R |K(x,y)-K(x,y_R)|\,dx\,dy \le \frac{C}{d(Q,R)^n} \ln^{-2-\varepsilon}\left(2+\frac{d(Q,R)}{|Q|^{1/n}}\right)$$

where y_R is the center of R, otherwise (i.e., $d(Q,R)>0$), and similarly for $K(y,x)$. It is easy to adapt Theorem 3.2 with such hypotheses. In such a case the kernel of $\mathcal{T}$ satisfies (3.1) and the kernel of $\mathbb{T}$, (3.10). Next, the proof of Theorem 3.3 easily adapts under (3.10) by changing the conclusion of Corollary 6.10 in [**AHMTT**] to, with the notation there, $|\langle T(b_P^1\chi_{I_Q}),\chi_{2I_Q}\rangle| \lesssim K|I_Q|$, as this suffices to run the argument.

4. Proof of Theorem 2.1

The case of $\mathcal{U}$ is the easiest one. In fact, it is bounded on all L^p, $1<p<\infty$, on H^1_d and on BMO_d. This is classical but we include a proof for convenience. Let us see the L^2 boundedness first. Set

$$A = \sup_{j,k} \left\{ \sum_\ell |\alpha^j_{k,\ell}| + |\alpha^j_{\ell,k}| \right\}.$$

Recall that $\alpha^j_{k,k}=0$ so that by (2.18), $A<\infty$. Write $f=\sum_j f_j$ with $f_j = Q_j f$. Then, by Schur's lemma and using the orthonormal basis property of the Haar functions,

$$\left| \sum_{k,\ell\in\mathbb{Z}^n} \langle g,\psi_{j,k}\rangle \alpha^j_{k,\ell} \langle \psi_{j,\ell}, f\rangle \right| \le A\|g_j\|_2\|f_j\|_2.$$

Hence

$$|\langle g,\mathcal{U}f\rangle| \le A \sum_{j=-\infty}^{\infty} \|g_j\|_2\|f_j\|_2 \le A\|g\|_2\|f\|_2.$$

It remains to prove the H^1_d boundedness of $\mathcal{U}$ as the boundedness on BMO_d is obtained by duality and the L^p boundedness by interpolation. To do that, we pick an L^2 dyadic atom a: it is supported in a dyadic cube Q, its L^2 norm is bounded by $1/|Q|^{1/2}$ and it is of mean 0. By scale and translation invariance, it suffices to assume that $Q=Q_{0,0}$.

Write $a = \sum_{Q_{j,\ell}\subset Q_{0,0}} \langle a, \psi_{j,\ell}\rangle \psi_{j,\ell}$ so that $\|a\|_2^2 = \sum_{Q_{j,\ell}\subset Q_{0,0}} |\langle a, \psi_{j,\ell}\rangle|^2$. We have

$$\begin{aligned}
\mathcal{U}a &= \sum_{j=-\infty}^{\infty} \sum_{k,\ell} \langle a, \psi_{j,\ell}\rangle \alpha^j_{k,\ell} \psi_{j,k} \\
&= \sum_{j=0}^{\infty} \sum_{k} \sum_{\ell; Q_{j,\ell}\subset Q_{0,0}} \langle a, \psi_{j,\ell}\rangle \alpha^j_{k,\ell} \psi_{j,k} \\
&= \sum_{m\in\mathbb{Z}^n} a_m
\end{aligned}$$

with

$$a_m = \sum_{j=0}^{\infty} \sum_{k;\, Q_{j,k}\subset Q_{0,m}} \left\{ \sum_{\ell; Q_{j,\ell}\subset Q_{0,0}} \langle a, \psi_{j,\ell}\rangle \alpha^j_{k,\ell} \right\} \psi_{j,k}.$$

We have that a_m is supported in $Q_{0,m}$ and has mean 0. Thus $\|a_m\|_2^{-1} a_m$ is an L^2 dyadic atom. It suffices to show that $B = \sum \|a_m\|_2 < \infty$ to conclude that $\mathcal{U}a \in H^1_d$ with norm not exceeding B. By (2.18), we have

$$\sup_{j\geq 0} \sup_{k;\, Q_{j,k}\subset Q_{0,m}} \left\{ \sum_{\ell;\, Q_{j,\ell}\subset Q_{0,0}} |\alpha^j_{k,\ell}| \right\} \leq C(1+|m|)^{-(n+s)}$$

and similarly exchanging the roles of k and ℓ. Using Cauchy-Schwarz inequality,

$$\begin{aligned}
\|a_m\|_2^2 &\leq \sum_{j=0}^{\infty} \sum_{k;\, Q_{j,k}\subset Q_{0,m}} \left\{ \sum_{\ell;\, Q_{j,\ell}\subset Q_{0,0}} |\langle a, \psi_{j,\ell}\rangle|^2 |\alpha^j_{k,\ell}| \right\} \left\{ \sum_{\ell;\, Q_{j,\ell}\subset Q_{0,0}} |\alpha^j_{k,\ell}| \right\} \\
&\leq C(1+|m|)^{-(n+s)} \sum_{j=0}^{\infty} \sum_{\ell;\, Q_{j,\ell}\subset Q_{0,0}} |\langle a, \psi_{j,\ell}\rangle|^2 \sum_{k;\, Q_{j,k}\subset Q_{0,m}} |\alpha^j_{k,\ell}| \\
&\leq C^2(1+|m|)^{-2(n+s)} \sum_{j=0}^{\infty} \sum_{\ell;\, Q_{j,\ell}\subset Q_{0,0}} |\langle a, \psi_{j,\ell}\rangle|^2
\end{aligned}$$

and we are done provided one has a definition of $\mathcal{U}$ on H^1_d. Let $\mathcal{U}_{J,N}$ be a partial sum obtained truncating the sum defining $\mathcal{U}$ with $|j| \leq J$ and $|k-\ell| \leq 2^N$. It is immediate to define the action of $\mathcal{U}_{J,N}$ on all H^1_d and we have $\|\mathcal{U}_{J,N} f\|_{H^1_d} \leq C\|f\|_{H^1_d}$ for all $f \in H^1_d$ thanks to the previous calculations with C independent of J, N and f. Next, by tedious but not difficult calculations refining the above estimates, one shows that $\|\mathcal{U}_{J,N}f - \mathcal{U}f\|_{H^1_d} \to 0$ whenever f is a finite linear combination of L^2 dyadic atoms as $J, N \to \infty$. Thus, we obtain the boundedness of $\mathcal{U}$ on a dense subspace of H^1_d and we conclude by a density argument.

We next concentrate on $\mathcal{W}$. Once this is done, $\mathcal{V}$ is handled by observing that $\mathcal{V}^*$ is of the same type as $\mathcal{W}$. Recall that

$$\langle g, \mathcal{W}f\rangle = \sum_{j=-\infty}^{+\infty} \sum_{k,\ell\in\mathbb{Z}^n} \langle g, \phi_{j,k}\rangle \gamma^j_{k,\ell} \langle \psi_{j,\ell}, f\rangle \tag{4.1}$$

with $\gamma^j_{k,\ell} = c^j_{k,\ell}$ if $k \neq \ell$ and $\gamma^j_{\ell,\ell} = -\sum_{k\neq\ell} c^j_{k,\ell}$. We decompose further $\mathcal{W}$ as

$$\mathcal{W} = \sum_{R\in\mathbb{N}^*} \mathcal{W}_R$$

here

$$\langle g, \mathcal{W}_R f\rangle = \sum_{j=-\infty}^{+\infty} \sum_{k,\ell\in\mathbb{Z}^n} \langle g, \phi_{j,k}\rangle \gamma^{j,R}_{k,\ell} \langle \psi_{j,\ell}, f\rangle \tag{4.2}$$

$$\gamma^{j,R}_{k,\ell} = \begin{cases} c^j_{k,\ell}, & \text{if } 2^{R-1} \le |k-\ell| < 2^R, \\ -\sum_{2^{R-1}\le|m|<2^R} c^j_{k+m,k}, & \text{if } k = \ell, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Here, for $x, y \in \mathbb{R}^n$, $|x-y| = \sup(|x_1 - y_1], \dots, |x_n - y_n|)$. Let

$$\Gamma(R) = \sup_{j,k} \left\{ \sum_\ell |\gamma^{j,R}_{k,\ell}| + |\gamma^{j,R}_{\ell,k}| \right\}.$$

We notice that under (2.18), we have $\Gamma(R) = O(2^{-Rs})$.

Lemma 4.1. *For $R \ge 1$, we have:*

$$\|\mathcal{W}_R\|_{L^2\to L^2} \le CR^{\frac{1}{2}}\Gamma(R). \tag{4.4}$$

$$\|\mathcal{W}_R\|_{H^1_d\to L^1} \le CR\,\Gamma(R). \tag{4.5}$$

$$\|\mathcal{W}_R\|_{L^\infty\to \mathrm{BMO}_d} \le CR\,\Gamma(R). \tag{4.6}$$

Hence, for $1 < p < \infty$,

$$\|\mathcal{W}_R\|_{L^p\to L^p} \le CR\,\Gamma(R).$$

It is clear that Theorem 2.1 for $\mathcal{W}$ follows at once from this lemma.

Let us begin the proof of this lemma by proving the L^2 boundedness. Write $f = \sum_j f_j$ with $f_j = Q_j f$. Then,

$$\|\mathcal{W}_R f\|_2^2 \leq \sum_{j,j'} |\langle \mathcal{W}_R f_j, \mathcal{W}_R f_{j'}\rangle|.$$

First, for each j, by expanding f_j on the $\psi_{j,\ell}$, $\ell \in \mathbb{Z}^n$, and $\mathcal{W}_R f_j$ on the $\phi_{j,k}$, $k \in \mathbb{Z}^n$, Schur's lemma yields

$$\|\mathcal{W}_R f_j\|_2 \leq \Gamma(R)\|f_j\|_2.$$

Thus, using Cauchy-Schwarz inequality and $\|f\|^2 = \sum_j \|f_j\|_2^2$, we have

$$\sum_{|j-j'|\leq R+2} |\langle \mathcal{W}_R f_j, \mathcal{W}_R f_{j'}\rangle| \leq (2R+5)\Gamma(R)^2\|f\|_2^2. \tag{4.7}$$

It remains to handle the sum where $|j-j'| > R+2$. It is enough to assume $j - j' > R+2$ and to show that

$$|\langle \mathcal{W}_R f_j, \mathcal{W}_R f_{j'}\rangle| \leq C\,\Gamma(R)^2\, 2^{\frac{j'-j+R}{2}}\|f_j\|_2\|f_j'\|_2. \tag{4.8}$$

By dyadic scale invariance, assume also $j = 0$, hence $-j' > R+2$. We have

$$\begin{aligned}
&\langle \mathcal{W}_R f_0, \mathcal{W}_R f_{j'}\rangle \\
&\quad = \left\langle \sum_{k,\ell}\langle f, \psi_{0,\ell}\rangle \gamma_{k,\ell}^{0,R}\phi_{0,k}, \sum_{k',\ell'}\langle f, \psi_{j',\ell'}\rangle \gamma_{k',\ell'}^{j',R}\phi_{j',k'}\right\rangle \\
&\quad = \sum_{k'}\left\{\sum_{\ell}\langle f, \psi_{0,\ell}\rangle\left\langle \sum_k \gamma_{k,\ell}^{0,R}\phi_{0,k}, \phi_{j',k'}\right\rangle\right\}\left\{\sum_{\ell'}\langle f, \psi_{j',\ell'}\rangle\gamma_{k',\ell'}^{j',R}\right\}.
\end{aligned}$$

Now the support of $\phi_{j',k'}$ is the cube $Q_{j',k'} = 2^{-j'}(k' + [0,1)^n)$ and for fixed ℓ, the support of $\gamma_{k,\ell}^{0,R}$ is in the set of $k \in \mathbb{Z}^n$ such that $|k-\ell| < 2^R$. Thus, if $d(\ell, Q_{j',k'}) > 2^{R+2}$ then $\sum_k \gamma_{k,\ell}^{0,R}\phi_{0,k} = 0$ identically on $Q_{j',k'}$. Next, if $\ell \in Q_{j',k'}$ and $d(\ell, \mathbb{R}^n\backslash Q_{j',k'}) > 2^{R+2}$ then all the $\phi_{0,k}$ concerned in the sum have support inside $Q_{j',k'}$. Thus

$$\left\langle \sum_k \gamma_{k,\ell}^{0,R}\phi_{0,k}, \phi_{j',k'}\right\rangle = 2^{\frac{nj'}{2}}\sum_{k\in\mathbb{Z}^n}\gamma_{k,\ell}^{0,R} = 0$$

by construction of the γ's. Thus, for the sum in ℓ, we have contribution only for $\ell \in E_{j',k'}$ defined as the set of those $\ell \in \mathbb{Z}^n$ with $d(\ell, \partial Q_{j',k'}) \leq$

2^{R+2} (here, ∂Q is the boundary of the cube Q) and the sum in k inside the brackets reduces to those $k \in Q_{j',k'}$. Hence, we have

$$|\langle \mathcal{W}_R f_0, \mathcal{W}_R f_{j'}\rangle| \leq \sum_{k'} \left\{ \sum_{\ell \in E_{j',k'}} |\langle f, \psi_{0,\ell}\rangle| \left(\sum_{k \in Q_{j',k'}} |\gamma^{0,R}_{k,\ell}| 2^{\frac{nj'}{2}} \right) \right\} \times \left\{ \sum_{\ell'} |\langle f, \psi_{j',\ell'}\rangle| |\gamma^{j',R}_{k',\ell'}| \right\}.$$

By Cauchy-Schwarz inequality, it suffices to estimate

$$I = \left(\sum_{k'} \left\{ \sum_{\ell'} |\langle f, \psi_{j',\ell'}\rangle| |\gamma^{j',R}_{k',\ell'}| \right\}^2 \right)^{1/2}$$

and

$$II = \left(\sum_{k'} \left\{ \sum_{\ell \in E_{j',k'}} |\langle f, \psi_{0,\ell}\rangle| \left(\sum_{k \in Q_{j',k'}} |\gamma^{0,R}_{k,\ell}| 2^{\frac{nj'}{2}} \right) \right\}^2 \right)^{1/2}.$$

By Schur's lemma, we have

$$I \leq \Gamma(R) \left(\sum_{\ell'} |\langle f, \psi_{j',\ell'}\rangle|^2 \right)^{1/2} = \Gamma(R) \|f_{j'}\|_2.$$

Next,

$$\begin{aligned} II &= 2^{\frac{nj'}{2}} \left(\sum_{k'} \left\{ \sum_{k \in Q_{j',k'}} \sum_{\ell \in E_{j',k'}} |\langle f, \psi_{0,\ell}\rangle| |\gamma^{0,R}_{k,\ell}| \right\}^2 \right)^{1/2} \\ &\leq 2^{\frac{nj'}{2}} \left(\sum_{k'} \left\{ \sum_{k \in Q_{j',k'}} \sum_{\ell \in E_{j',k'}} |\gamma^{0,R}_{k,\ell}| \right\} \right. \\ &\qquad \left. \times \left\{ \sum_{k \in Q_{j',k'}} \sum_{\ell \in E_{j',k'}} |\langle f, \psi_{0,\ell}\rangle|^2 |\gamma^{0,R}_{k,\ell}| \right\} \right)^{1/2}. \end{aligned}$$

But, for fixed k', since the cardinal of $E_{j',k'}$ is $O(2^{-j'(n-1)+R})$,

$$\sum_{k \in Q_{j',k'}} \sum_{\ell \in E_{j',k'}} |\gamma^{0,R}_{k,\ell}| \leq C\, \Gamma(R)\, 2^{-j'(n-1)+R}.$$

Also

$$\sum_{k'}\sum_{k\in Q_{j',k'}}\sum_{\ell\in E_{j',k'}}|\langle f,\psi_{0,\ell}\rangle|^2|\gamma^{0,R}_{k,\ell}| \le \Gamma(R)\sum_{\ell\in\mathbb{Z}^n}|\langle f,\psi_{0,\ell}\rangle|^2\left\{\sum_{k'\in\mathbb{Z}^n}\mathbf{1}_{E_{j',k'}}(\ell)\right\}$$

$$\le \Gamma(R)\sum_{\ell\in\mathbb{Z}^n}|\langle f,\psi_{0,\ell}\rangle|^2\,2^n$$

$$= 2^n\,\Gamma(R)\|f_0\|_2^2.$$

All together

$$II \le C\,\Gamma(R)\,2^{\frac{j'+R}{2}}\|f_0\|_2,$$

and (4.8) is proved.

Next, we prove that $\mathcal{W}_R$ is bounded from H^1_d to L^1 with norm $O(R\,\Gamma(R))$. To do that, we pick an L^∞ dyadic atom a: it is supported in a dyadic cube Q, is bounded by $1/|Q|$ and is of mean 0. By scale and translation invariance, it suffices to assume that $Q = Q_{0,0}$. Write

$$a = \sum_{j'=0}^{\infty}\sum_{\ell;\,Q_{j',\ell'}\subset Q_{0,0}}\langle a,\psi_{j',\ell'}\rangle\psi_{j',\ell'}$$

and set

$$a_1 = \sum_{j'=R+1}^{\infty}\sum_{\ell';\,Q_{j',\ell'}\subset Q_{0,0}}\langle a,\psi_{j',\ell'}\rangle\psi_{j',\ell'}.$$

Observe that $\|a_1\|_2 \le \|a\|_2 \le \|a\|_\infty \le 1$. We have

$$\mathcal{W}_R\,a_1 = \sum_{j=-\infty}^{\infty}\sum_{k,\ell}\langle a_1,\psi_{j,\ell}\rangle\gamma^{j,R}_{k,\ell}\phi_{j,k} = \sum_{j=R+1}^{\infty}\sum_{k}\sum_{\ell;Q_{j,\ell}\subset Q_{0,0}}\langle a,\psi_{j,\ell}\rangle\gamma^{j,R}_{k,\ell}\phi_{j,k}$$

and because $Q_{j,\ell}\subset[0,1]^n$, $|k-\ell| < 2^R$ and $j \ge R+1$, we have $Q_{j,k}\subset[-1,2]^n$ for all (j,k) in the summation. Hence $\mathcal{W}_R\,a_1$ is supported in $[-1,2]^n$. Thus, the L^2 estimate yields

$$\|\mathcal{W}_R\,a_1\|_1 \le C\|\mathcal{W}_R\,a_1\|_2 \le CR^{1/2}\,\Gamma(R)\|a_1\|_2 \le CR^{1/2}\,\Gamma(R).$$

Set $a_2 = a - a_1$. Then, a straightforward estimate yields

$$\|\mathcal{W}_R\,a_2\|_1 \le \sum_{j=0}^{R}\sum_{k}\sum_{\ell;\,Q_{j,\ell}\subset Q_{0,0}}|\gamma^{j,R}_{k,\ell}|2^{-nj}\|a\|_\infty \le (R+1)\,\Gamma(R)$$

by summing first in k, then in ℓ and in j. A truncation procedure with respect to the sum over j as for $\mathcal{U}$ allows to fully justify the boundedness of $\mathcal{W}_R$ from H^1_d to L^1. We skip details which are easy.

Our last task is prove that $\mathcal{W}_R$ is bounded from L^∞ to BMO_d with norm $O(R\,\Gamma(R))$. Modulo a truncation procedure as above which is left to the reader, it suffices to show that $\mathcal{W}_R^*$ is bounded from H^1_d to L^1. So we pick again an L^∞ dyadic atom a and assume that it is supported in $Q = Q_{0,0}$. We have

$$\mathcal{W}_R^*\, a = \sum_{j=-\infty}^{\infty} \sum_{k,\ell\in\mathbb{Z}^n} \langle a, \phi_{j,k}\rangle \gamma^{j,R}_{k,\ell} \psi_{j,\ell} = \sum_{j=0}^{\infty} \sum_{k;\, Q_{j,k}\subset Q_{0,0}} \sum_{\ell} \langle a, \phi_{j,k}\rangle \gamma^{j,R}_{k,\ell} \psi_{j,\ell}$$

where we used that a has support in $Q_{0,0}$ and mean 0. We split the sum as $b_1 + b_2$ according to $j \ge R+1$ or $j \le R$. In the first case, we have as before, that $Q_{j,k} \subset Q_{0,0}$, $j \ge R+1$ and $|k-\ell| < 2^R$ imply that $Q_{j,\ell} \subset [-1,2]^n$ for all (j,ℓ) concerned by the summation. Also b_1 can be written as $\widetilde{\mathcal{V}}_R(a)$ where $\widetilde{\mathcal{V}}_R$ is an operator of the same type as $\mathcal{V}_R$ with "truncated" coefficients (note that the L^2 bounds depends on a size estimate of the coefficients and on the nullity of the sum of the coefficients with respect to ℓ with j and k fixed). Thus, it is bounded on L^2 with bound $O(R^{1/2}\,\Gamma(R))$. Hence

$$\|b_1\|_1 \le C\|b_1\|_2 \le CR^{1/2}\,\Gamma(R)\|a\|_2 \le CR^{1/2}\,\Gamma(R).$$

For the b_2 part, a straightforward estimate yields a bound

$$\begin{aligned} \|b_2\|_1 &\le \sum_{j=0}^{R} \sum_{k;\, Q_{j,k}\subset Q_{0,0}} \sum_{\ell\in\mathbb{Z}^n} \|a\|_\infty 2^{-nj} |\gamma^{j,R}_{\ell,k}| \\ &\le (R+1)\,\Gamma(R) \end{aligned}$$

where $\Gamma(R)$ occurs by taking the sum in ℓ first.

Remark 4.2. It can be shown that $\mathcal{W}_R$ is bounded on BMO_d with bound $O(R2^{-Rs})$. Also $\mathcal{W}_R$ is bounded from H^1_d into H^1, the Hardy space on $\mathbb{R}^n$ with a similar bound. The proofs are a little more involved. However, it may not be bounded on H^1_d. The counterexample is the following: if $n=1$, set

$$\langle g, \mathcal{W} f\rangle = (\langle g, \phi_{0,0}\rangle - \langle g, \phi_{0,-1}\rangle)\langle \psi_{0,0}, f\rangle.$$

Then, observe that $\mathcal{W}_1 = \mathcal{W}$ and $\mathcal{W}_1(\psi_{0,0})(x) = \phi_{0,0}(x) - \phi_{0,-1}(x) = \phi(x) - \phi(x+1) \notin H^1_d$ since it does not vanish on $\mathbb{R}^+$ and $\mathbb{R}^-$ which is necessary.

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Primera versió rebuda el 3 d'octubre de 2007,
darrera versió rebuda el 29 de febrer de 2008.