Publ. Mat. **53** (2009), 111–118

CUSP ALGEBRAS

Jim Agler* and John E. McCarthy†

Abstract

A *cusp* is the image of the unit disk under a proper holomorphic map into $\mathbb{C}^n$ that is one-to-one and whose derivative vanishes at exactly one point. It is *simple* if not all the second derivatives vanish. We characterize when two simple cusps are isomorphic, and show that they can all be realized in $\mathbb{C}^2$.

1. Introduction

By a *cusp* V we shall mean the image of the unit disk $\mathbb{D}$ under a bounded injective holomorphic map h into $\mathbb{C}^n$ whose derivative vanishes at exactly one point. The simplest example is the Neil parabola, given by $h(\zeta) = (\zeta^2, \zeta^3)$. See [**3**], [**4**] for background and theory on the Neil parabola, which is pictured in Figure 1.

A generalization of a cusp is a *petal*. A petal is the image of the unit disk $\mathbb{D}$ under a proper holomorphic map h from $\mathbb{D}$ into some bounded open set Ω, where, except for a finite set E_h, h is one-to-one and non-singular. If E_h is a singleton, then V is a cusp. Since the automorphism group of $\mathbb{D}$ is transitive, we may assume then that $E_h = \{0\}$.

The function $h\colon \mathbb{D} \to \Omega$ induces a finite codimensional subalgebra A_h of the algebra $O(\mathbb{D})$ of all analytic functions on $\mathbb{D}$, namely

$$A_h = \{F \circ h : F \in O(V)\},$$

where $O(V)$ is the algebra of functions on V that, in the neighborhood of every point P of V, coincides with the restriction to V of a function holomorphic in a neighborhood of $\mathbb{C}^n$ containing P. When V is a cusp, the algebra A_h has the property that it contains $z^m O(\mathbb{D})$ for some $m \geq 2$. Conversely, if a cofinite subalgebra of $O(\mathbb{D})$ contains $z^m O(\mathbb{D})$, then it arises in this way.

2000 *Mathematics Subject Classification.* 14H10, 30F20.
Key words. Petal, cusp, holomap.
*Partially supported by National Science Foundation Grant DMS 0400826.
†Partially supported by National Science Foundation Grant DMS 0501079.

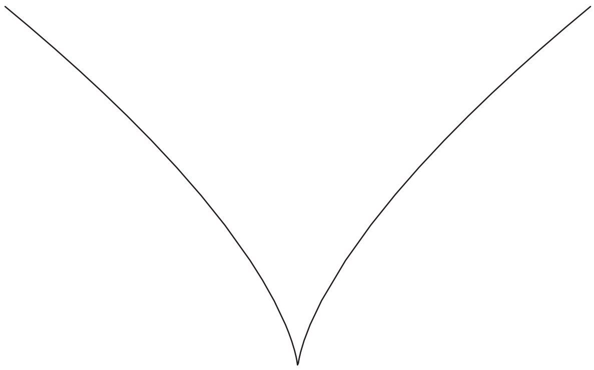

FIGURE 1. A simple cusp - the Neil parabola $z^3 = w^2$.

In light of these remarks, let us agree to say that a unital algebra $A \subseteq O(\mathbb{D})$ is a cusp algebra if

(i) $\dim (O(\mathbb{D})/A)$ is finite, and

(ii) for some $m \geq 2$, $z^m O(\mathbb{D}) \subseteq A$.

We shall say a map $h\colon \mathbb{D} \to \Omega \subsetneq \mathbb{C}^n$ is a *holomap* if it is a proper map that is one-to-one and non-singular except on a finite set. We shall say that a holomap h is a *holization* of the cofinite algebra A if $A = A_h$.

If A is a cusp algebra, then we define three integers attached to A. The *codimension* of A is defined by

$$\mathrm{cod}(A) = \dim (O(\mathbb{D})/A)\,.$$

The *order* of A is defined by

$$\mathrm{ord}(A) = \min\{k : z^{k+1} O(\mathbb{D}) \subseteq A\}.$$

The *contact* of A is defined by

$$\mathrm{con}(A) = \max\{n : f^{(j)}(0) = 0,\ \forall\, f \in A,\ 1 \leq j \leq n\}.$$

If V is a cusp, then we define $\mathrm{cod}(V)$, $\mathrm{ord}(V)$ and $\mathrm{con}(V)$ as the quantities for the corresponding algebra A_h.

By a *simple* cusp algebra, we shall mean a cusp algebra A such that $\mathrm{con}(A) = 1$. There is a unique cusp algebra of codimension 1, which corresponds via a holization to the holomorphic functions on the Neil parabola $\{(z, w) \in \mathbb{D}^2 : z^3 = w^2\}$ (this result can be seen directly, or will follow from Section 4). This can be thought of as a Riemann mapping theorem for cusps of codimension 1. In Section 4, we generalize this "Riemann mapping theorem" to arbitrary simple cusps. We show that locally the only invariant of simple cusps is the codimension, but globally cusps of codimension $n+1$ have a $2n-1$ real parameter moduli space.

This paper is a continuation of the authors' earlier work in [**1**], where the ideas are put in a more general context. A principal concern in that paper is when a given finite codimensional subalgebra can be holized in 2 dimensions, *i.e.* by a map into $\mathbb{C}^2$. In Section 5, we show that all simple cusp algebras can be holized in $\mathbb{C}^2$.

2. Preliminaries

Throughout this section, we shall assume that A is a simple cusp algebra, *i.e.* of contact one. By a *primitive* for A we mean a function π that lies in A, and satisfies $\pi(0) = 0$, $\pi''(0) = 2$; so π has a Taylor expansion $\pi(z) = z^2 + \cdots$.

For each n, let A_n be the space

$$A_n = \left[A \cap z^{2n} O(\mathbb{D})\right],$$

and let E_n be the quotient

$$E_n = A_n / A_{n+1}.$$

Observe that multiplication by any primitive π is always one-to-one from E_n to E_{n+1}. Therefore, there is some integer n_0 such that $\dim(E_n)$ is 1 for $n \leq n_0$, and 2 for $n > n_0$. Since $\dim(E_n) = 2$ if and only if both z^{2n} and z^{2n+1} are in A, it follows that

$$\operatorname{ord}(A) = 2n_0 + 1.$$

As π^n is in A_n for each n, if f is in A_n for some $n \leq n_0$, then there is a constant c_n such that $f - c_n\pi^n$ is in A_{n+1}. Therefore every function f in A has the representation

$$f = c_0 + c_1\pi + \cdots + c_{n_0}\pi^{n_0} + z^{2n_0+2}g, \tag{2.1}$$

where $c_0, \ldots, c_{n_0}$ are in $\mathbb{C}$ and $g \in O(\mathbb{D})$. From (2.1), we see that $\operatorname{cod}(A) = n_0 + 1$. Summarizing, we have:

Proposition 2.2. *If A is a simple cusp algebra, then* $\operatorname{ord}(A) = 2\operatorname{cod}(A) - 1$. *Furthermore, if we set $n_0 = \operatorname{cod}(A) - 1$, and π is a primitive for A, then every function f in A has a unique representation*

$$f(z) = p \circ \pi(z) + z^{2n_0+2}g(z) \tag{2.3}$$

for some polynomial p of degree at most n_0 and some $g \in O(\mathbb{D})$.

3. Connections and local theory

Let U be an open set in $\mathbb{C}$. A linear functional on $O(U)$ is called local if it comes from a finitely supported distribution, *i.e.* is of the form

$$\Lambda(f) = \sum_{i=1}^{m} \sum_{j=0}^{n_i} a_{ij} f^{(j)}(\alpha_i).$$

Definition 3.1. A connection on $\{\alpha_1, \ldots, \alpha_m\} \subset U$ is a finite dimensional set Γ of local functionals Λ supported by $\{\alpha_1, \ldots, \alpha_m\}$. We say Γ is algebraic if $\Gamma^\perp := \{f \in O(U) : \Lambda(f) = 0,\ \forall\, \Lambda \in \Gamma\}$ is an algebra.

It was proved by T. W. Gamelin [**2**] that every finite codimensional subalgebra A of $O(\mathbb{D})$ is $\Gamma^\perp$ for some algebraic connection on $\mathbb{D}$. Moreover, A will be a cusp algebra iff the support of Γ is $\{0\}$.

We shall say that a point P in a petal V is a *cusp point* if there is a one-to-one proper map h from $\mathbb{D}$ onto a neigborhood U of P in V such that the derivative of h vanishes only at the pre-image of P. The algebra A_h is then a cusp algebra. We wish to show that its codimension, order and contact do not depend on the choice of U.

Suppose that P_1 and P_2 are cusp points in cusps V_1 and V_2, and there is an injective holomorphic map $\phi\colon V_1 \to V_2$. For $r = 1, 2$ there are holizations $h_r\colon \mathbb{D} \to V_r$ with $h_r(0) = P_r$. By [**1**, Theorem 3.2], the map ϕ from V_1 to V_2 induces a one-to-one map $\psi\colon \mathbb{D} \to \mathbb{D}$ such that $h_2 \circ \psi = \phi \circ h_1$, and the connections Γ_1 and Γ_2 induced by h_1 and h_2 are related by

$$\begin{aligned}\Gamma_2 &\supseteq \psi_* \Gamma_1 \\ &:= \{f \mapsto \Lambda_1 (f \circ \psi) : \Lambda_1 \in \Gamma_1\}.\end{aligned}$$

If

$$\Lambda_1\colon f \mapsto \sum_{j=0}^{n} a_j f^{(j)}(0), \tag{3.2}$$

then

$$\psi_* \Lambda_1\colon g \mapsto \sum_{j=0}^{n} a_j (g \circ \psi)^{(j)}(0). \tag{3.3}$$

We can use Faá di Bruno's formula to evaluate the derivatives of $g \circ \psi$ in terms of those of g and ψ. As $\psi(0) = 0$ and $\psi'(0) \neq 0$, note in particular that the coefficient of $g^{(n)}(0)$ on the right-hand side of (3.3) is $a_n[\psi'(0)]^n$. Therefore ψ_* is injective, so $\mathrm{cod}(A_1) \leq \mathrm{cod}(A_2)$.

If $\mathrm{con}(A_1) = k$, it means that for each $1 \leq j \leq k$, the functional $f \mapsto f^{(j)}(0)$ is in Γ_1. Therefore each functional $g \mapsto g^{(j)}(0)$ is in Γ_2, and

$\mathrm{con}(A_1) \le \mathrm{con}(A_2)$. Finally if $\mathrm{ord}(A_1) = n$, it means there is some Λ_1 in Γ_1 of the form (3.2) with $a_n \neq 0$; it follows that $\mathrm{ord}(A_1) \le \mathrm{ord}(A_2)$.

Summarizing, we have shown:

Proposition 3.4. *If* $\phi\colon V_1 \to V_2$ *is an injective holomorphic map from one cusp to another, then* $\mathrm{cod}(V_1) \le \mathrm{cod}(V_2)$, $\mathrm{con}(V_1) \le \mathrm{con}(V_2)$ *and* $\mathrm{ord}(V_1) \le \mathrm{ord}(V_2)$.

Now, if P is a cusp point in a petal V_1, and one surrounds P by a decreasing sequence of open sets U_n, then with each U_n there is a cusp algebra with a codimension, contact and order, and there is an inclusion map from each U_{n+1} to U_n. By Proposition 3.4, the positive integers $\mathrm{cod}(U_n)$, $\mathrm{con}(U_n)$ and $\mathrm{ord}(U_n)$ must decrease as U_n shrinks. Therefore, at some point they must stabilize, and for all sufficiently small neighborhoods of P, the cusp algebras have the same codimension, contact and order. We shall call these $(\mathrm{cod})(P)$, $(\mathrm{con})(P)$ and $(\mathrm{ord})(P)$ respectively, and say P is a simple cusp point if $\mathrm{con}(P) = 1$.

4. Equivalence of simple cusps

We shall say that two points P_1 and P_2 in the petals V_1 and V_2 are *locally equivalent* if there is a neighborhood U_1 of P_1 in V_1, and a neighborhood U_2 of P_2 in V_2, and a biholomorphic homeomorphism F from U_1 onto U_2 that maps P_1 to P_2.

Theorem 4.1. *Two simple cusp points are locally equivalent if and only they have the same codimension.*

Proof: Necessity follows from Proposition 3.4. To prove sufficiency, let π_1 be a primitive of the first algebra, and π_2 a primitive of the second algebra. By Proposition 2.2, it is sufficient to prove that there are neighborhoods W_1 and W_2 of the origin, and a univalent map $\phi\colon W_1 \to W_2$ that maps 0 to 0 and such that

$$\pi_2 \circ \phi = \pi_1. \tag{4.2}$$

Each π_r has a square root χ_r in a neighborhood of 0, and each χ_r is locally univalent (because π_r is of order 2). So define

$$\phi := \chi_2^{-1} \circ \chi_1$$

on a suitable neighborhood W_1, and (4.2) is satisfied. □

How can we globally parametrize simple cusp algebras? We shall show that there is an essentially unique primitive with all its even Taylor coefficients (except for the second) zero. By $\hat{\pi}(k)$ we mean the k^{th} Taylor coefficient at 0.

Lemma 4.3. *Every simple cusp algebra of codimension $n+1$ has a primitive π such that $\hat{\pi}(2k) = 0$ for all $2 \le k \le n$. Moreover, π is unique up to $O(z^{2n+2})$.*

Proof: (Existence): Let χ_1 be any primitive. Define

$$\chi_2 := \chi_1 - \hat{\chi}_1(4)(\chi_1)^2.$$

Proceed inductively, with

$$\chi_k := \chi_{k-1} - \hat{\chi}_{k-1}(2k)(\chi_{k-1})^k.$$

Then let $\pi = \chi_n$.

(Uniqueness): Suppose π and χ are both primitives with their even Taylor coefficients, starting at 4, vanishing. By Proposition 2.2, we have

$$\chi = c_1\pi + c_2\pi^2 + \cdots + c_n\pi^n + O(z^{2n+2}).$$

By looking at the coefficient of z^2 we see $c_1 = 1$. Now looking at the coefficients of $z^4, z^6, \dots, z^{2n}$ in order, we see that $c_2 = 0 = c_4 = \cdots = c_n$. $\square$

So every algebraic connection of dimension $n+1$ supported at the origin is the annihilator of an algebra generated locally by a unique primitive

$$\pi(z) = z^2 + \alpha_1 z^3 + \alpha_2 z^5 + \cdots + \alpha_n z^{2n+1}. \tag{4.4}$$

Let us denote the codimension $n+1$ subalgebra of $O(\mathbb{D})$ with primitive (4.4) by $A(\alpha_1, \dots, \alpha_n)$, and let $V(\alpha_1, \dots, \alpha_n)$ denote the corresponding petal $h(\mathbb{D})$, where h holizes $A(\alpha_1, \dots, \alpha_n)$.

Two cusps V_1 and V_2 are globally equivalent if there is a biholomorphic homeomorphism ϕ from V_1 onto V_2. If each V_r is holized by $h_r \colon \mathbb{D} \to V_r$, then by [**1**, Theorem 3.1], this occurs if and only if there is a map $\psi \colon \mathbb{D} \to \mathbb{D}$ such that ψ_* maps the first connection to the second. As the only self-maps of the disk that leave the origin invariant are rotations, this is rather restrictive.

Theorem 4.5. *The cusps $V(\alpha_1, \dots, \alpha_n)$ and $V(\beta_1, \dots, \beta_n)$ are isomorphic if and only if there is a unimodular constant τ such that $\beta_j = \tau^{2j-1}\alpha_j$ for $1 \le j \le n$.*

As a corollary we have

Corollary 4.6. *The moduli space of all simple cusps of codimension $n+1$ is $\mathbb{R}^+ \times \mathbb{C}^{n-1}$.*

5. Embedding

The purpose of this section is to prove that every simple cusp algebra A contains a pair of functions h_1, h_2 such that the pair holizes the algebra (which is equivalent to saying that polynomials in h_1 and h_2 are dense in A in the topology of uniform convergence on compacta).

For the rest of this section, A will be a fixed simple cusp algebra of codimension $n+1$.

Lemma 5.1. *For every α on $\mathbb{D} \setminus \{0\}$, there is a function ψ_α in A that has a single simple zero at α, and no other zeroes on $\mathbb{D}$.*

Proof: Consider functions of the form $(z-\alpha)e^{h(z)}$. In order to be in A, h must satisfy $n+1$ equations on its first $2n+1$ derivatives at 0. This system is triangular, so can be solved with h a polynomial. □

Lemma 5.2. *If f is in A and f has no zeroes on $\mathbb{D}$, then $1/f$ is in A.*

Proof: Consider the vector space obtained by adjoining $1/f, 1/f^2, \ldots, 1/f^k$ to A. Since A is of finite codimension in $O(\mathbb{D})$, for some k there is a linear relation

$$1/f^k = \sum_{j=1}^{k-1} c_j 1/f^j + A.$$

Multiplying both sides by f^{k-1}, we get that $1/f$ is in A. □

Lemma 5.3. *If $f \in A$ and $f(\alpha) = 0$ for some $\alpha \in \mathbb{D} \setminus \{0\}$, then f/ψ_α is in A.*

Proof: Let Γ be the connection at 0 such that $A = \Gamma^\perp$. Consider the algebra

$$A' := \left\{ f \in O\left(\frac{|\alpha|}{2}\mathbb{D}\right) : \Lambda(f) = 0, \ \forall \Lambda \in \Gamma \right\}.$$

Then ψ_α is in A', so by Lemma 5.2, $1/\psi_\alpha$ is in A'. Therefore f/ψ_α is in A', and is therefore annihilated by every functional in Γ. But f/ψ_α is also in $O(\mathbb{D})$, so it is in A as required. □

Theorem 5.4. *There exists a map $h \colon \mathbb{D} \to \mathbb{C}^2$ that holizes A.*

Proof: By applying Lemma 5.3 repeatedly, one can find a primitive h_1 of A that has no zeroes in $\mathbb{D} \setminus \{0\}$. Define

$$h_2(z) := z\,(h_1(z))^{n+1}.$$

Consider the algebra A' that is the closure (in the topology of uniform convergence on compacta) of polynomials in h_1 and h_2. By [**1**, Theorem 4.2], the algebra A' is of finite codimension, and it is a cusp algebra, because $h(z) = (h_1(z), h_2(z))$ is one-to-one away from the origin. If $A' \neq A$, then it is contained in a maximal proper subalgebra A'' of A, which must have codimension $n+2$, and therefore order $2n+3$. But A'' contains h_2 and h_1^{n+1}, and these functions have order $2n+2$ and $2n+3$ respectively at the origin. So there can be no linear relation in A'' between the derivatives at the origin of order $(2n+2)$ and $(2n+3)$, and so the order of A'' must actually be at most $2n+1$. □

References

[1] J. Agler and J. E. McCarthy, Hyperbolic algebraic and analytic curves, *Indiana Univ. Math. J.* **56(6)** (2007), 2899–2933.

[2] T. W. Gamelin, Embedding Riemann surfaces in maximal ideal spaces, *J. Functional Analysis* **2** (1968), 123–146.

[3] G. Knese, Function theory on the Neil parabola, *Michigan Math. J.* **55(1)** (2007), 139–154.

[4] V. I. Paulsen an D. Singh, Modules over subalgebras of the disk algebra, *Indiana Univ. Math. J.* **55(5)** (2006), 1751–1766.

Jim Agler:
Department of Mathematics
University of California, San Diego (UCSD)
La Jolla, California 92093-0112
USA
E-mail address: jagler@ucsd.edu

John E. McCarthy:
Mathematics Department
Washington University
St. Louis, Missouri 63130
USA
E-mail address: mccarthy@math.wustl.edu

Rebut el 14 de setembre de 2007.