SHARP A_2 INEQUALITY FOR HAAR SHIFT OPERATORS

MICHAEL T. LACEY*, STEFANIE PETERMICHL, AND MARIA CARMEN REGUERA*

Abstract. As a Corollary to the main result of the paper we give a new proof of the inequality

$$\|T f\|_{L^2(w)} \lesssim \|w\|_{A_2} \|f\|_{L^2(w)} ,$$

where T is either the Hilbert transform [12], a Riesz transform [13], or the Beurling operator [14]. The weight w is non-negative, and the linear growth in the A_2 characteristic on the right is sharp. Prior proofs relied strongly on Haar shift operators [11] and Bellman function techniques. The new proof uses Haar shifts, and then uses an elegant 'two weight T 1 theorem' of Nazarov-Treil-Volberg [10] to immediately identify relevant Carleson measure estimates, which are in turn verified using an appropriate corona decomposition of the weight w.

1. Introduction

We are interested in weighted estimates for singular integral operators, and cognate operators, with a focus on sharp estimates in terms of the A_p characteristic of the weight. In particular we give a new proof of the estimate of Petermichl [12]

$$\|H f\|_{L^2(w)} \lesssim \|w\|_{A_2} \|f\|_{L^2(w)} ,$$

where $H f(x) = \text{p.v.} \int f(x-y)\, dy/y$ is the Hilbert transform. Petermichl's proof, as well as corresponding inequalities for the Beurling operator [14] and the Riesz transforms [13] have relied upon a Bellman function approach to the estimate for the corresponding Haar shift. We also analyze the Haar shifts, but instead use a deep two-weight inequality of Nazarov-Treil-Volberg [10] as a way to quickly reduce the question to certain Carleson measure estimates. The latter estimates are proved by using the usual Haar functions together with appropriate corona decomposition. The linear growth in terms of the A_2 characteristic is neatly explained by this decomposition.

Let us precede to the definitions.

*Research supported in part by NSF grant 0456611.

1.1. **Definition.** For w a positive function (a weight) on $\mathbb{R}^d$ we define the A_p characteristic of w to be

$$\|w\|_{A_p} := \sup_Q |Q|^{-1} \int_Q w \, dx \cdot \Big[|Q|^{-1} \int_Q w^{-1/(p-1)} \, dx \Big]^{p-1}, \qquad 1 < p < \infty ,$$

where the supremum is over all cubes in $\mathbb{R}^d$.

The relevant conjecture concerning the behavior of singular integral operators on the spaces $L^p(w)$ is

1.2. **Conjecture.** *For a smooth singular integral operator* T *which is bounded on* $L^2(dx)$ *we have the estimate*

$$\|\mathrm{T}\, f\|_{L^p(w)} \lesssim \|w\|_{A_p}^{\alpha(p)} \|f\|_{L^p(w)} , \qquad \alpha(p) = \max\{1, 1/(p-1)\} . \tag{1.3}$$

An extrapolation estimate [6, 14] shows that it suffices to prove this estimate for $p = 2$, which is the case we consider in the remainder of this paper. Currently this estimate is known for the Hilbert transform, Riesz transforms and the Beurling operator, with the proof using in an essential way the so-called Haar shift operators. This proof will do so as well, but handle all Haar shifts at the same time.

1.4. **Definition.** By a *Haar function* h_Q on cube $Q \subset \mathbb{R}^d$, we mean any function which satisfies

(1) h_Q is a function supported on Q, and is constant on each dyadic subcubes of Q. (That is, h_Q is in the linear span of the indicators of the 'children' of Q.)
(2) $\|h_Q\|_\infty \le |Q|^{-1/2}$. (So $\|h_Q\|_2 \le 1$.)
(3) $\int_Q h_Q(x) \, dx = 0$.

1.5. **Definition.** We say that T is a *Haar shift operator of index* τ iff

$$\mathrm{T}\, f = \sum_{Q \in \mathcal{Q}} \sum_{\substack{Q', Q'' \subset Q \\ 2^{-\tau d}|Q| \le |Q'|, |Q''|}} a_{Q',Q'} \langle f, h_{Q'} \rangle h_{Q''} ,$$

$$|a_{Q',Q''}| \le \left[\frac{|Q'|}{|Q|} \cdot \frac{|Q''|}{|Q|} \right]^{1/2} .$$

The point of the conditions in the definition is that T be not only an $L^2(dx)$ bounded operator, but that it also be a Calderón-Zygmund operator. In particular, it should admit a weak-$L^1(dx)$ bound that depends only on the index τ. See Proposition 3.11.

1.6. **Theorem.** *Let* T *be a Haar shift operator of index* τ*, and let* w *be an* A_2 *weight. We have the inequality*

$$\|\mathrm{T}\|_{L^2(\omega)} \lesssim \|w\|_{A_2} \tag{1.7}$$

The implied constant depends only dimension d *and the index* τ *of the operator.*

We have this Corollary:

1.8. **Corollary.** *The inequalities* (1.3) *holds for the Hilbert transform, the Riesz transforms in any dimension* d*, and the Beurling operator on the plane.*

As is well-known, these singular integral operators are obtained by appropriate averaging of the Haar shifts, an argument invented in [11]. We also derive, as a corollary, the sharp A_2 bound for Haar square functions. We leave the details of this to the reader.

The starting point of our proof is a beautiful 'two weight $T1$ Theorem for Haar shifts' due to Nazarov-Treil-Volberg [10]. We recall a version of this Theorem in § 2. This Theorem supplies necessary and sufficient conditions for an individual Haar shift to satisfy a two-weight L^2 inequality, with the conditions being expressed in language of the $T\,1$ Theorem. In particular, it neatly identifies three estimates that need to be proved, with two related to paraproduct estimates. In fact, this step is well-known, and is taken up immediately in e. g. [12]. We then check the paraproduct bounds for A_2 weights in § 3 and § 4, which is the main new step in this paper.

The question of bounds for singular integral operators on $L^p(w)$ that are sharp with respect to the A_p characteristic was identified in an influential paper of Buckley, [3]. It took many years to find the first proofs of such estimates. We refer the reader to cite [12] for some of this history, and point to the central role of the work of Nazarov-Treil-Volberg [9] in shaping much of the work cited here. The prior proofs of Corollary 1.8 have all relied upon Bellman function techniques. And indeed, this technique will supply a proof of the results in this paper. The Beurling operator is the most easily available, since this operator can be seen as the average of the simplest of Haar shifts, namely martingale transforms, see [7]. The A_2 bound was derived for Martingale transforms by J. Wittwer [16]. The paraproduct structure is much more central to the problem if one works with Haar shifts that pair a 'parent' Haar with a 'child' Haar. If one considers Square Functions, sharp results were obtained in L^2 by Wittwer [17], and Hukovic-Treil-Volberg [8]. Recently, Beznosova [2], has proved the linear bound for discrete paraproduct operators, again using the Bellman function method. It would be of interest to prove her Theorem with techniques closer to those of this paper.

Acknowledgment. The authors are participants in a research program at the Centre de Recerca Matemática, at the Universitat Autònoma Barcelona, Spain. We thank the Centre for their hospitality, and very supportive environment. Xavier Tolsa pointed out some relevant references to us.

2. The Characterization of Nazarov-Treil-Volberg

The success of this approach is based upon a beautiful characterization of two weight inequalities. Indeed, this characterization is true for *individual* two-weight inequalities. This Theorem can be thought of as a 'Two Weight $T1$ Theorem.'

We are stating only a sub-case of their Theorem, which does not assume that the operators satisfy an $L^2(dx)$ bound.

2.1. **Theorem.** *[Nazarov-Treil-Volberg [10]] Let* T *be a Haar shift operator of index* τ*, as in Definition 1.5, and* σ, μ *two positive measures. The* L^2 *inequality*

$$\|T(\sigma f)\|_{L^2(\mu)} \lesssim \|f\|_{L^2(\sigma)}$$

holds iff the following three conditions hold. For all cubes Q, Q' *with* $Q \subset (Q')^{(\tau)}$ *or* $Q' \subset Q^{(\tau)}$,

$$\Big|\int_{Q'} T(\sigma \mathbf{1}_Q)\ \mu(dx)\Big| \le C_{\mathrm{WB}} \sqrt{\sigma(Q)\mu(Q')} \qquad \text{(Weak Bnded)} \tag{2.2}$$

$$\|T(\sigma \mathbf{1}_Q)\|_{L^2(Q,\mu)} \le C_{T1} \sqrt{\sigma(Q)} \qquad (T1 \in BMO) \tag{2.3}$$

$$\|T^*(\mu \mathbf{1}_Q)\|_{L^2(Q,\sigma)} \le C_{T^*1} \sqrt{\mu(Q)} \qquad (T^*1 \in BMO)$$

Moreover, we have the inequality

$$\|T(\sigma\cdot)\|_{L^2(\sigma)\to L^2(\mu)} \lesssim C_{\mathrm{WB}} + C_{T1} + C_{T^*1}\,. \tag{2.4}$$

This Theorem is contained in [10, Theorem 1.4], aside from the claim (2.4). But this inequality can be seen from the proof in their paper. Indeed, their proof is in close analogy to the T1 Theorem. Briefly, the proof is as follows. The operator $T(\sigma\cdot)$ is expanded in 'Haar basis', but as had been done prior papers, such as [7, 12, 16], the Haar bases are adapted to the two measures σ and μ. Expressing the bilinear form $\int T(\sigma f)\cdot g\ \mu$ as a matrix in these two bases, the matrix is split into three parts. Those terms 'close to the diagonal' are controlled by the 'weak boundedness' condition (2.2). Those terms below and above the diagonal are recognized as paraproducts. One of these is of the form

$$P(f) := \sum_Q \sigma(Q)^{-1} \int_Q f\sigma\ dy \cdot \Delta_Q^w(T(\sigma 1)) \tag{2.5}$$

Here the first term is an average of f with respect to the measure σ, and the second is a martingale difference of $T(\sigma 1)$ with respect to the measure w. In particular, $\Delta_Q^w(T(\sigma 1))$ are w-orthogonal functions in Q. Thus, one has the equality

$$\|P(f)\|_{L^2(w)}^2 = \sum_Q \Big|\sigma(Q)^{-1} \int_Q f\sigma\ dy\Big|^2 \cdot \|\Delta_Q^w(T(\sigma 1))\|_{L^2(w)}^2\,.$$

The inequality $\|P(f)\|_{L^2(w)} \lesssim \|f\|_{L^2(\sigma)}$ is a weighted Carleson embedding inequality that is implied by the '$T\,1 \in BMO$' condition (2.3). The other paraproduct is dual the one in (2.5).

3. Initial Considerations

We collect together a potpourri of facts that will be useful to us, and are of somewhat general nature. We begin with a somewhat complicated definition that we will use in order to organize the proof of our main estimate.

3.1. **Definition.** Let $\mathcal{Q}' \subset \mathcal{Q}$ be any collection of dyadic cubes. Call $(\mathcal{L} : \mathcal{Q}'(L))$ a *corona decomposition of* $\mathcal{Q}'$ *relative to measure* w if these conditions are met. For all $L \subsetneq L' \in \mathcal{L}$ and $L \subset Q \subset L'$ for $Q \in \mathcal{Q}'$ we have

$$\begin{aligned} \frac{w(L)}{|L|} &> 4\frac{w(L')}{|L'|}\,, \\ 4\frac{w(L')}{|L'|} &\geq \frac{w(Q)}{|Q|}\,. \end{aligned} \tag{3.2}$$

Define $\lambda : \mathcal{Q}' \to \mathcal{L}$ by requiring that $\lambda(Q)$ be the minimal element of $\mathcal{L}$ that contains Q. Then we set $\mathcal{Q}'(L) := \{Q \in \mathcal{Q}' : \lambda(Q) = L\}$. The collections $\mathcal{Q}'(L)$ partition $\mathcal{Q}'$.

Decompositions of this type appear in a variety of questions. We are using terminology which goes back to (at least) David and Semmes [4, 5]. A subtle corona decomposition is central to [15], and the paper [1] includes several examples in the context of dyadic analysis.

A basic fact is this.

$$\Bigl|\bigcup_{\substack{L' \in \mathcal{L} \\ L' \subsetneq L}} L'\Bigr| \leq \tfrac{1}{4}|L|\,, \qquad L \in \mathcal{L}\,.$$

This follows from (3.2), which says that the intervals $L' \subset L$ have much more than their fair share of the mass of w, hence the L' have to be smaller intervals. And this easily implies

$$\Bigl\|\sum_{\substack{L' \in \mathcal{L} \\ L' \subset L}} \mathbf{1}_{L'}\Bigr\|_2 \lesssim |L|^{1/2}\,. \tag{3.3}$$

We have the following (known) Lemma, but we detail it as it is one way that the A_2 condition enters in the proof.

3.4. **Lemma.** *Let* $\mathcal{L}$ *be a ssociated with corona decomposition for a weight* w. *For any cube* Q *we have*

$$\sum_{\substack{L \in \mathcal{L} \\ L \subset Q}} w(L) \leq \tfrac{16}{9}\|w\|_{A_2} w(Q)\,. \tag{3.5}$$

Proof. It suffices to show this: For $L \in \mathcal{L}$

$$w\bigl(\bigcup\{L' \in \mathcal{L} : L' \subsetneq L\}\bigr) \leq (1 - c\|w\|_{A_2}^{-1})w(L)\,, \qquad c = \tfrac{9}{16}\,. \tag{3.6}$$

We begin with a calculation related to A_∞. Let E be a measurable subset of Q. Then,

$$
\begin{aligned}
\frac{|E|}{|Q|} &= |Q|^{-1} \int_E w^{1/2} \cdot w^{-1/2} \, dx \\
&\le \left[\frac{w(E)}{|Q|} \cdot \frac{w^{-1}(Q)}{|Q|} \right]^{1/2} \\
&\le \left[\|w\|_{A_2} \frac{w(E)}{w(Q)} \right]^{1/2} . \qquad (3.7)
\end{aligned}
$$

Apply this with $L - E = \bigcup \{L' \in \mathcal{L} \,:\, L' \subsetneq L\}$. Then, by (3.3), $|L - E| < \frac{1}{4}|L|$, so that $|E| \ge \frac{3}{4}|L|$. It follows that we then have

$$
\frac{9}{16\|w\|_{A_2}} \cdot w(L) \le w(E) .
$$

Whence, we see that (3.6) holds. Our proof is complete. □

Concerning the Haar shift operators T, we make the following definition.

3.8. **Definition.** We say that T is a *simple Haar shift operator of index* τ iff

$$
\mathrm{T} f = \sum_{Q \in \mathcal{Q}} \langle f, g_Q \rangle \gamma_Q ,
$$

$$
g_Q, \gamma_Q \in \operatorname{span}(h_{Q'} \,:\, Q' \subset Q \,, 2^{-\tau d}|Q| \le |Q'|) , \tag{3.9}
$$

$$
\|g_Q\|_\infty \,, \, \|\gamma_Q\|_\infty \le |Q|^{-1/2} . \tag{3.10}
$$

Below, we will only consider simple Haar shift operators. The important property they satisfy is

3.11. **Proposition.** *A simple Haar shift operator* T *with index* τ *maps* $L^2(dx)$ *into itself with norm at most* $\lesssim \tau$. *It maps* $L^1(dx)$ *into* $L^{1,\infty}(dx)$ *with norm* $\lesssim 2^{\tau d}$.

The point is that these bounds only depend upon the index τ.

Proof. The proof is well-known, but we present it as some similar difficulties appear later in the proof; see the discussion following (5.3). Set

$$
\mathrm{T}_s f := \sum_{\substack{Q \in \mathcal{Q} \\ |Q| = 2^{sd}}} \langle f, g_Q \rangle \gamma_Q ,
$$

which is the operator at scale 2^s. The 'size condition' (3.10) implies that $\|\mathrm{T}_s\|_{L^2(dx)} \le 1$. The 'cancellation condition' (3.9) then implies that

$$
\mathrm{T}_s \mathrm{T}_{s'}^* = \mathrm{T}_s^* \mathrm{T}_{s'} = 0 , \qquad |s - s'| > \tau .
$$

So we see that $\|\mathrm{T}\|_{L^2(dx)} \le \tau + 1$.

Concerning the weak $L^1(dx)$ inequality, we use the usual proof. Fix $f \in L^1(dx)$. Apply the dyadic Calderón-Zygmund Decomposition to f at height λ. Thus, $f = g + b$ where $\|g\|_2 \lesssim \sqrt{\lambda}\|f\|_{L^1(dx)}^{1/2}$, and b is supported on a union of disjoint dyadic cubes $Q \in \mathcal{B}$ with

$$\int_Q b \; dx = 0\,, \qquad Q \in \mathcal{B}\,, \tag{3.12}$$

$$\sum_{Q \in \mathcal{B}} |Q| \lesssim \lambda^{-1}\|f\|_1\,. \tag{3.13}$$

For the 'good' function g, using the $L^2(dx)$ estimate we have

$$\begin{aligned} |\{T\, g > \tau\lambda\}| &\leq (\tau\lambda)^{-2}\|T\, g\|_{L^2(dx)}^2 \\ &\lesssim \lambda^{-2}\|g\|_2^2 \lesssim \tau^2\lambda^{-1}\|f\|_{L^1(dx)}\,. \end{aligned}$$

For the 'bad' function, we modify the usual argument. For a dyadic cube Q, and integer t, let $Q^{(t)}$ denote it's t-fold parent. Thus, $Q^{(1)}$ is the minimal dyadic cube that strictly contains Q, and inductively, $Q^{(t+1)} = (Q^{(t)})^{(1)}$. Observe that (3.13) implies

$$\Big|\bigcup\{Q^{(\tau)} \;:\; Q \in \mathcal{B}\}\Big| \lesssim 2^{\tau d}\lambda^{-1}\|f\|_1\,.$$

And, the 'cancellation condition' (3.9), with (3.12), imply that for $Q \in \mathcal{B}$, and $x \notin Q^{(\tau)}$, we have

$$T(\mathbf{1}_Q b)(x) = \sum_{Q' \,:\, Q^{(\tau)} \subsetneq Q'} \langle \mathbf{1}_Q b, g_{Q'}\rangle \gamma_{Q'}(x) = 0$$

since $g_{Q'}$ will be constant on the cube Q.

Hence, we have

$$|\{T(b) > \lambda\}| \leq \Big|\bigcup\{Q^{(\tau)} \;:\; Q \in \mathcal{B}\}\Big| \lesssim 2^{\tau d}\lambda^{-1}\|f\|_1\,.$$

This completes the proof. □

We need a version of the John-Nirenberg inequality, which says that a 'uniform L^0 condition implies exponential integrability.'

3.14. **Lemma.** *This holds for all integers τ. Let $\{\phi_Q \;:\; Q \in \mathcal{Q}\}$ be functions so that for all dyadic cubes Q we have*

1. ϕ_Q *is supported on* Q *and is constant on each sub-cube* $Q' \subset Q$ *with* $|Q'| = 2^{-\tau d}|Q|$*;*
2. $\|\phi_Q\|_\infty \leq 1$*;*
3. *for all dyadic cubes* Q*, we have*

$$\Bigg|\Bigg\{\Big|\sum_{Q' \,:\, Q' \subset Q} \phi_Q\Big| > 1\Bigg\}\Bigg| \leq 2^{-\tau d - 1}|Q|\,.$$

It then follows that we have the estimate

$$\Big|\Big\{\Big|\sum_{Q'\,:\,Q'\subset Q}\phi_Q\Big| > 2\tau t\Big\}\Big| \le \tau 2^{-t+1}|Q|\,, \qquad t>1\,.$$

4. The Main Argument

We begin the main line of argument to prove (1.7). We no longer try to keep track of the dependence on τ in our estimates. (It is, in any case, exponential in τ.) Accordingly, we assume that we work with a subset $\mathcal{Q}_\tau$ of dyadic cubes with 'scales separated by τ.' That is, we assume that for $Q' \subsetneq Q$ and $Q', Q \in \mathcal{Q}$ we have $|Q'| \le 2^{-d\tau}|Q|$, where d is dimension.

It is well-known that (1.7) is equivalent to showing that

$$\|\mathrm{T}(fw)\|_{L^2(w^{-1})} \lesssim \|w\|_{A_2}\|f\|_{L^2(w)}\,.$$

Here we are using the dual-measure formulation, so that the measure w appears on both sides of the inequality, as in Theorem 2.1.

By Theorem 2.1, and the symmetry of the A_2 condition, it is sufficient to check that the two inequalities below hold for all simple Haar shift operators T of index τ:

$$|\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_R\rangle| \lesssim \|w\|_{A_2}\sqrt{w(Q)w^{-1}(R)}\,, \tag{4.1}$$

$$\int_Q |\mathrm{T}(w\mathbf{1}_Q)|^2\, w^{-1}dx \lesssim \|w\|_{A_2}^2 w(Q)\,. \tag{4.2}$$

These should hold for all dyadic cubes Q, and in (4.1), we have $2^{-\tau d}|Q| \le |R| \le 2^{\tau d}|R|$.

In the present circumstance, the 'weak boundedness' inequality (4.1) can be derived from the 'T1' inequality (4.2). We can assume that $|Q| \le |R|$ by passing to the dual operator and replacing w by w^{-1}. If $|Q| = |R|$, the inner product is zero unless $Q = R$. But then we just appeal to (4.2).

$$\begin{aligned}|\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_Q\rangle| &\le \sqrt{w^{-1}(Q)}\cdot\|\mathbf{1}_Q\,\mathrm{T}(w\mathbf{1}_Q)\|_{L^2(w^{-1})}\\ &\lesssim \|w\|_{A_2}\sqrt{w(Q)\cdot w^{-1}(Q)}\,.\end{aligned}$$

If $|Q| < |R|$, we can assume that $Q \subset R$, and write

$$|\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_R\rangle| \le |\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_Q\rangle| + |\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_{R-Q}\rangle|\,.$$

The first term on the right is handled just as in the previous case. In the second case, we use the fact that $2^{-\tau d}|R| \le |Q| < |R|$, so that there is a difference in scales between the two rectangles of only at most τ scales. That, with the size conditions on T lead to

$$|\langle \mathrm{T}(w\mathbf{1}_Q), w^{-1}\mathbf{1}_{R-Q}\rangle| \lesssim \frac{w(Q)w^{-1}(R)}{|R|}$$

$$\lesssim \|w\|_{A_2} \sqrt{w(Q) \cdot w^{-1}(R)} \,.$$

The last inequality follows since

$$\sqrt{\frac{w(Q)w^{-1}(R)}{|R|^2}} \leq \sqrt{\frac{w(R)w^{-1}(R)}{|R|^2}} \leq \sqrt{\|w\|_{A_2}} \leq \|w\|_{A_2} \,.$$

Indeed, we always have $1 \leq \|w\|_{A_2}$.

To verify (4.2), we first treat the 'large scales'.

$$\begin{aligned} \Big\| \mathbf{1}_{Q_0} \sum_{Q \,:\, Q \supsetneq Q_0} \langle w \mathbf{1}_{Q_0}, g_Q \rangle \gamma_Q \Big\|_{L^2(w^{-1})} &\lesssim \frac{w(Q_0) w^{-1}(Q_0)^{1/2}}{|Q_0|} \\ &\lesssim \sqrt{w(Q_0)} \cdot \|w\|_{A_2} \end{aligned}$$

Therefore, it suffices to prove

$$\Big\| \sum_{Q \,:\, Q \subset Q_0} \langle w, g_Q \rangle \gamma_Q \Big\|_{L^2(w^{-1})} \lesssim \|w\|_{A_2} \sqrt{w(Q_0)} \,. \tag{4.3}$$

Let us define for dyadic cubes Q_0 and collections of dyadic cubes $\mathcal{Q}'$,

$$H(Q_0, \mathcal{Q}') \coloneqq \sum_{\substack{Q \subset Q_0 \\ Q \in \mathcal{Q}'}} \langle w, g_Q \rangle \gamma_Q \,,$$

$$\mathbf{H}(\mathcal{Q}') \coloneqq \sup_{Q_0} \frac{\|H(Q_0, \mathcal{Q}')\|_{L^2(w^{-1})}}{\sqrt{w(Q)}} \,. \tag{4.4}$$

It is a useful remark that in estimating $\mathbf{H}(\mathcal{Q}')$ we can restrict the supremum to cubes $Q_0 \in \mathcal{Q}'$. Of course, we are seeking to prove $\mathbf{H}(\mathcal{Q}) \lesssim \|w\|_{A_2}$.

The first important definition here is

$$\mathcal{Q}_n \coloneqq \left\{ Q \in \mathcal{Q} \;:\; 2^{n-1} < \frac{w(Q)}{|Q|} \cdot \frac{w^{-1}(Q)}{|Q|} \leq 2^n \right\} . \tag{4.5}$$

We show that

$$\mathbf{H}(\mathcal{Q}_n) \lesssim 2^{n/2} \|w\|_{A_2}^{1/2} \,. \tag{4.6}$$

Since $2^n \leq \|w\|_{A_2}$, this estimate is summable in n to prove (4.3).

Now fix a $Q_0 in \mathcal{Q}_n$ for which we are to test the supremum in (4.4). Let $\mathcal{P} = \{Q \in \mathcal{Q}_n \;:\; Q \subset Q_0\}$. Let $(\mathcal{L} \;:\; \mathcal{P}(L))$ be a corona decomposition of $\mathcal{P}$ relative to measure w. (The reader is advised to recall the definition Definition 3.1.) Let $\mathcal{P}(L) = \{Q \in \mathcal{P} \;:\; \lambda(Q) = L\}$. (So L is the minimal element of $\mathcal{L}$ that contains each element of $\mathcal{P}$.)

The essence of the matter is contained in the following Lemma.

4.7. **Lemma.** *We have these distributional estimates, uniform over* $L \in \mathcal{L}$*:*

$$\big|\{x \in L \,:\, |H(L, \mathcal{P}(L))(x)| > Kt\tfrac{w(L)}{|L|}\}\big| \lesssim e^{-t}|L|\,, \tag{4.8}$$

$$w^{-1}\big(\{x \in L \,:\, |H(L, \mathcal{P}(L))(x)| > Kt\tfrac{w(L)}{|L|}\}\big) \lesssim e^{-t} w^{-1}(L)\,. \tag{4.9}$$

Let us complete the proof of our Theorem based upon this Lemma. Let us estimate

$$\begin{aligned} \|H(Q_0, \mathcal{Q}_n)\|_{L^2(w^{-1})}^2 &\le \Big\| \sum_{L \in \mathcal{L}} |H(L, \mathcal{P}(L))| \Big\|_{L^2(w^{-1})}^2 \\ &= A + B = A + \sum_{L \in \mathcal{L}} B(L)\,, \end{aligned} \tag{4.10}$$

$$A \coloneqq \sum_{L \in \mathcal{L}} \|H(L, \mathcal{P}(L))\|_{L^2(w^{-1})}^2$$

$$B(L) \coloneqq \sum_{\substack{L' \in \mathcal{L} \\ L' \subsetneq L}} \int |H(L, \mathcal{Q}_n) \cdot H(L', \mathcal{Q}_n)|\, w^{-1}\,. \tag{4.11}$$

Note that these estimates show that all cancellation necessary for the truth of theorem is already captured in the corona decomposition.

The estimate of A is straight forward. By (4.9), we see that the A_2 estimate reveals itself.

$$\begin{aligned} \|H(L, \mathcal{P}(L))\|_{L^2(w^{-1})}^2 &\lesssim \Big[\frac{w(L)}{|L|}\Big]^2 w^{-1}(L) \\ &\lesssim w(L) \frac{w(L)}{|L|} \cdot \frac{w^{-1}(L)}{|L|} \\ &\lesssim 2^n w(L)\,. \end{aligned}$$

Therefore, by (3.5)

$$A \lesssim 2^n \sum_{L \in \mathcal{L}} w(L) \lesssim 2^n \|w\|_{A_2} w(Q_0)\,. \tag{4.12}$$

In the expression (4.11), the integral is not as complicated as it immediately appears. We have assumed that 'scales are separated by τ' at the beginning of this section, so that as L' is strictly contained in L, $H(L, \mathcal{Q}_n)$ takes a single value on all of L', which we denote by $H(L, \mathcal{Q}_n)(L')$. This observation simplifies our task of estimating the integral.

For $L' \subsetneq L$ we use (4.9) and (4.5) to see that

$$\begin{aligned} \int |H(L, \mathcal{Q}_n) \cdot H(L', \mathcal{Q}_n)|\, w^{-1} &\lesssim |H(L, \mathcal{Q}_n)(L')| \frac{w(L')}{|L'|} \cdot w^{-1}(L') \\ &\lesssim 2^n |H(L, \mathcal{Q}_n)(L')| \cdot |L'|\,. \end{aligned}$$

Note that the A_2 characteristic has entered in. And the presence of $|L'|$ indicates that there is an integral against Lebesgue measure here.

Employ this observation with Cauchy-Schwartz, *both* distributional estimates (4.8) and (4.9) and (3.3) to estimate

$$\begin{aligned}
B(L) &:= \sum_{\substack{L'\in\mathcal{L}\\ L'\subsetneq L}} \int |H(L,\mathcal{Q}_n)\cdot H(L',\mathcal{Q}_n)|\, w^{-1} \\
&\lesssim 2^n |H(L,\mathcal{Q}_n)(L')| \sum_{\substack{L'\in\mathcal{L}\\ L'\subsetneq L}} |L'| && \text{(by (4.9))} \\
&= 2^n \int |H(L,\mathcal{Q}_n)(L')| \cdot \sum_{\substack{L'\in\mathcal{L}\\ L'\subset L}} \mathbf{1}_{L'}\, dx && \text{(by defn.)} \\
&\le 2^n \|H(L,\mathcal{Q}_n)\|_{L^2(dx)} \Big\| \sum_{\substack{L'\in\mathcal{L}\\ L'\subset L}} \mathbf{1}_{L'} \Big\|_{L^2(dx)} && \text{(Cauchy-Schwartz)} \\
&\lesssim 2^n w(L)\,. && \text{(by (4.8) and (3.3))}
\end{aligned}$$

Therefore, by (3.5) again,

$$B \lesssim 2^n \sum_{L\in\mathcal{L}} w(L) \lesssim 2^n \|w\|_{A_2} w(Q_0)\,.$$

Combining this estimate with (4.10) and (4.12) completes the proof of (4.6), and so our Theorem, assuming Lemma 4.7.

5. The essence of the matter.

We prove Lemma 4.7. In this situation, both a cube Q_0 and cube $L \in \mathcal{L}$ are given. It is an important point that all the relevant cubes that we sum over are in the collection $\mathcal{Q}_n$, as defined in (4.6).

One more class of dyadic cubes are needed. For integers $\alpha \ge 0$ define $\mathcal{P}_\alpha(L)$ to be those $Q \in \mathcal{P}(L)$ such that

$$2^{-\alpha-1}\frac{w(L)}{|L|} \le \frac{w(Q)}{|Q|} < 2^{-\alpha}\frac{w(L)}{|L|}\,. \tag{5.1}$$

Recall that the condition in (4.6) is also in force.

The essential observation is this: By Proposition 3.11, T maps $L^1(dx)$ into weak-$L^1(dx)$, with norm depending only on the index τ of the operator. Hence,

$$\Big\| \sum_{\substack{Q\subset Q_1\\ Q\in\mathcal{P}_\alpha(L)}} \langle w, g_Q\rangle \gamma_Q \Big\|_{L^{1,\infty}(dx)} \lesssim w(Q_1)\,.$$

This is a uniform statement in Q_1. If in addition $Q_1 \in \mathcal{P}_\alpha(L)$, we have

$$\Big\| \sum_{\substack{Q \in Q_1 \\ Q \in \mathcal{P}_\alpha(L)}} \langle w, g_Q \rangle \gamma_Q \Big\|_{L^{1,\infty}(dx)} \lesssim 2^{-\alpha} \frac{w(L)}{|L|} \cdot |Q_1| \,.$$

Due to the functions g_Q and γ_Q are supported on Q, we see that this estimate also holds uniformly in Q_1.

Note that we have by the definition of Haar functions Definition 1.4, and a simple Haar shift, Definition 3.8,

$$|\langle w, g_Q \rangle \gamma_Q(x)| \le \frac{w(Q)}{|Q|} \lesssim 2^{-\alpha} \frac{w(L)}{|L|} \,.$$

Hence, Lemma 3.14 applies. Let us fix a maximal $Q^* \in \mathcal{P}_\alpha(L)$, and define

$$E_\alpha(t, Q^*) := \Big\{ \sum_{\substack{Q \in Q^* \\ Q \in \mathcal{P}_\alpha}} \langle w, g_Q \rangle \gamma_Q > Kt 2^{-\alpha} \frac{w(L)}{|L|} \Big\} \,, \qquad t \ge 1 \,. \tag{5.2}$$

Here, K is the constant from Lemma 3.14. We have the exponential inequality

$$|E_\alpha(t, Q^*)| \lesssim e^{-t} |Q^*| \,.$$

This is one of our two claims, the distributional estimate in Lebesgue measure (4.8), for the collection $\mathcal{P}_\alpha(L)$, not the collection $\mathcal{P}(L)$. But with the term $2^{-\alpha}$ appearing in (5.2), it is easy to supply (4.8) as written.

We want the corresponding inequality in w^{-1}-measure. But note that $E_\alpha(t, Q^*)$ is a union of disjoint dyadic cubes in a collection $\mathcal{E}_\alpha(t, Q^*)$, where for each $Q \in \mathcal{E}_\alpha(t, Q^*)$, we can choose dyadic $\phi(Q) \in \mathcal{P}_\alpha$ with $Q \subset \phi(Q) \subset Q^*$, and $|Q| \ge 2^{-\tau d} |\phi(Q)|$. This follows from the definition of a simple Haar shift. It follows that we have

$$\Big| \bigcup \{ \phi(Q) \ : \ Q \in \mathcal{E}_\alpha(t, Q^*) \} \Big| \lesssim e^{-t} |Q^*| \,. \tag{5.3}$$

(Recall that there is a similar difficulty in Proposition 3.11.) The point of these considerations is this: For each $Q \in \mathcal{P}_\alpha$, we have both the equivalences (4.5) and (5.1). Hence, $w^{-1}(Q) \simeq \rho |Q|$ where ρ is a fixed quantity (which we can compute, but is irrelevant to our conclusion). Thus, we can conclude from (5.3) the same inequality in w^{-1}-measure:

$$w^{-1}(E_{\alpha(t,Q^*)}) \le w^{-1}\big(\bigcup \{ \phi(Q) \ : \ Q \in \mathcal{E}_\alpha(t, Q^*) \} \big) \lesssim e^{-t} w^{-1}(Q^*) \,.$$

Sum this last estimate over maximal $Q^* \in \mathcal{P}_\alpha$ to conclude that

$$w^{-1} \Big(\sum_{Q \in \mathcal{P}(L)} \langle w, g_Q \rangle \gamma_Q > Kt 2^{-\alpha} \frac{w(L)}{|L|} \Big) \lesssim e^{-t} w^{-1}(L) \,,$$

since all relevant cubes are contained in cube L. This supplies (4.9), and so the proof is complete.

6. Sufficient Conditions for a Two Weight Inequality

There are a great many sufficient conditions for a two-weight inequality. To these results, let us add this one, for it's elegance.

6.1. **Theorem.** *Let α, β be positive functions on $\mathbb{R}^d$. For the inequality below to hold for all Haar shift operators* T

$$\|\mathrm{T}(f\alpha)\|_{L^2(\beta)} \lesssim \|f\|_{L^2(\alpha)}$$

It is sufficient that $\alpha, \beta \in A_\infty$ and the following 'two-weight A_2' hold:

$$\sup_Q \frac{\alpha(Q)}{|Q|} \cdot \frac{\beta(Q)}{|Q|} < \infty .$$

Of course these conditions are not necessary, for example one can take $\alpha = \beta = \mathbf{1}_E$, for any measurable subset E of $\mathbb{R}^d$. By $\alpha \in A_\infty$ we mean the measures α and β satisfy a variant of the estimate in (3.7).

6.2. **Definition.** We say that measure $\alpha \in A_\infty$ if this condition holds. For all $0 < \epsilon < 1$ there is a $0 < \eta < 1$ so that for all cubes Q and sets $E \subset Q$ with $|E| < \epsilon|Q|$, then $\alpha(E) < \beta\alpha(Q)$.

The proof of theorem is a modification of what we have already presented, so we do not give the details.

References

[1] P. Auscher, S. Hofmann, C. Muscalu, T. Tao, and C. Thiele, *Carleson measures, trees, extrapolation, and* $T(b)$ *theorems*, Publ. Mat. **46** (2002), no. 2, 257–325. MR **1934198 (2003f:**42019) ↑5

[2] Oleksandra V. Beznosova, *Linear bound for the dyadic paraproduct on weighted Lebesgue space* $L_2(w)$, J. Funct. Anal. **255** (2008), no. 4, 994–1007. MR 2433959 ↑3

[3] Stephen M. Buckley, *Estimates for operator norms on weighted spaces and reverse Jensen inequalities*, Trans. Amer. Math. Soc. **340** (1993), no. 1, 253–272. MR **1124164 (94a:**42011) ↑3

[4] Guy David and Stephen Semmes, *Analysis of and on uniformly rectifiable sets*, Mathematical Surveys and Monographs, vol. 38, American Mathematical Society, Providence, RI, 1993. MR **1251061 (94i:**28003) ↑5

[5] G. David and S. Semmes, *Singular integrals and rectifiable sets in* $\mathbf{R}^n$*: Beyond Lipschitz graphs*, Astérisque (1991), no. 193, 152 (English, with French summary). MR **1113517 (92j:**42016) ↑5

[6] Oliver Dragičević, Loukas Grafakos, María Cristina Pereyra, and Stefanie Petermichl, *Extrapolation and sharp norm estimates for classical operators on weighted Lebesgue spaces*, Publ. Mat. **49** (2005), no. 1, 73–91. MR **2140200 (2006d:**42019) ↑2

[7] Oliver Dragičević and Alexander Volberg, *Sharp estimate of the Ahlfors-Beurling operator via averaging martingale transforms*, Michigan Math. J. **51** (2003), no. 2, 415–435. MR **1992955 (2004c:**42030) ↑3, 4

[8] S. Hukovic, S. Treil, and A. Volberg, *The Bellman functions and sharp weighted inequalities for square functions*, Complex analysis, operators, and related topics, Oper. Theory Adv. Appl., vol. 113, Birkhäuser, Basel, 2000, pp. 97–113. MR **1771755 (2001j:**42012) ↑3

[9] F. Nazarov, S. Treil, and A. Volberg, *The Bellman functions and two-weight inequalities for Haar multipliers*, J. Amer. Math. Soc. **12** (1999), no. 4, 909–928. MR **1685781 (2000k:**42009) ↑3

[10] ———, *Two weight inequalities for individual Haar multipliers and other well localized operators*, Math. Res. Lett. **15** (2008), no. 3, 583–597. MR 2407233 ↑1, 3, 4

[11] Stefanie Petermichl, *Dyadic shifts and a logarithmic estimate for Hankel operators with matrix symbol*, C. R. Acad. Sci. Paris Sér. I Math. **330** (2000), no. 6, 455–460 (English, with English and French summaries). MR **1756958 (2000m:**42016) ↑1, 3

[12] S. Petermichl, *The sharp bound for the Hilbert transform on weighted Lebesgue spaces in terms of the classical* A_p *characteristic*, Amer. J. Math. **129** (2007), no. 5, 1355–1375. MR **2354322 (2008k:**42066) ↑1, 3, 4

[13] Stefanie Petermichl, *The sharp weighted bound for the Riesz transforms*, Proc. Amer. Math. Soc. **136** (2008), no. 4, 1237–1249. MR **2367098 (2009c:**42034) ↑1

[14] Stefanie Petermichl and Alexander Volberg, *Heating of the Ahlfors-Beurling operator: weakly quasiregular maps on the plane are quasiregular*, Duke Math. J. **112** (2002), no. 2, 281–305. MR **1894362 (2003d:**42025) ↑1, 2

[15] Xavier Tolsa, *Bilipschitz maps, analytic capacity, and the Cauchy integral*, Ann. of Math. (2) **162** (2005), no. 3, 1243–1304. MR **2179730 (2006g:**30033) ↑5

[16] Janine Wittwer, *A sharp estimate on the norm of the martingale transform*, Math. Res. Lett. **7** (2000), no. 1, 1–12. MR **1748283 (2001e:**42022) ↑3, 4

[17] ———, *A sharp estimate on the norm of the continuous square function*, Proc. Amer. Math. Soc. **130** (2002), no. 8, 2335–2342 (electronic). MR **1897458 (2003j:**42009) ↑3

Michael T. Lacey
School of Mathematics
Georgia Institute of Technology
Atlanta GA 30332, USA
E-mail address: `lacey@math.gatech.edu`

Stefanie Petermichl
Department of Mathematics
Universite de Bordeaux 1, France
E-mail address: `stefanie@math.u-bordeaux1.fr`

Maria Carmen Reguera
School of Mathematics
Georgia Institute of Technology
Atlanta GA 30332, USA
E-mail address: `mreguera@math.gatech.edu`