NEW ESTIMATES FOR THE MAXIMAL SINGULAR INTEGRAL

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Abstract. In this paper we pursue the study of the problem of controlling the maximal singular integral $T^* f$ by the singular integral Tf. Here T is a smooth homogeneous Calderón-Zygmund singular integral of convolution type. We consider two forms of control, namely, in the $L^2(\mathbb{R}^n)$ norm and via pointwise estimates of $T^* f$ by $M(Tf)$ or $M^2(Tf)$, where M is the Hardy-Littlewood maximal operator and $M^2 = M \circ M$ its iteration. It is known that the parity of the kernel plays an essential role in this question. In a previous article we considered the case of even kernels and here we deal with the odd case. Along the way, the question of estimating composition operators of the type $T^\star \circ T$ arises. It turns out that, again, there is a remarkable difference between even and odd kernels. For even kernels we obtain, quite unexpectedly, weak $(1,1)$ estimates, which are no longer true for odd kernels. For odd kernels we obtain sharp weaker inequalities involving a weak L^1 estimate for functions in $L\, LogL$.

1. Introduction: the model examples

In this paper we prove new estimates for the maximal singular integral associated with a singular integral of Calderón-Zygmund type. We start by considering two model examples, the Hilbert transform and the Beurling transform. The Hilbert transform is the linear operator defined for almost every $x \in \mathbb{R}$ by the principal value integral

$$Hf(x) = p.v. \int_{\mathbb{R}} \frac{f(y)}{y-x}\, dy\,,$$

where f is a function in some $L^p(\mathbb{R})\,, 1 \le p < \infty$, and the maximal Hilbert transform is

$$H^* f(x) = \sup_{\epsilon>0} \left| \int_{|y-x|>\epsilon} \frac{f(y)}{y-x}\, dy \right|, \quad x \in \mathbb{R}\,.$$

The Beurling transform is the one complex variable analog of the Hilbert transform, that is,

$$Bf(z) = p.v. \int_{\mathbb{C}} \frac{f(\omega)}{(\omega-z)^2}\, d\omega\,,$$

where f is in some $L^p(\mathbb{C})\,, 1 \le p < \infty$, and the maximal Beurling transform is

$$B^* f(z) = \sup_{\epsilon>0} \left| \int_{|\omega-z|>\epsilon} \frac{f(\omega)}{(\omega-z)^2}\, d\omega \right|, \quad z \in \mathbb{C}\,.$$

Our motivation comes from classical Cotlar's pointwise estimate

$$T^*(f)(x) \leq C\ (M(Tf)(x) + M(f)(x)) \tag{1}$$

where T is any Calderón-Zygmund singular operator, not necessarily of convolution type, and M is the standard Hardy-Littlewood maximal function. (See the original result in [Cot] and the modern treatment in [GrMF, p. 185]).

It has been shown in [MV] that it is possible, in some cases, to improve this estimate by removing $M(f)$ in the right hand side of (1). For example, for the Beurling transform one gets

$$B^*(f) \leq C\, M(Bf)\,. \tag{2}$$

It follows from the same paper that the analogous estimate for the Hilbert transform

$$H^*(f) \leq C\, M(Hf) \tag{3}$$

does not hold. In this paper we show that the right substitute for the inequality above is

$$H^*(f) \leq C\, M^2(Hf) \tag{4}$$

where $M^2 = M \circ M$ is the iterated Hardy-Littlewood maximal operator.

The crucial property to derive (2) is the even character of the kernel defining B. For further developments in this direction see [MOV], where one characterizes those even smooth homogeneous Calderón-Zygmund kernels for which the pointwise estimate (2) holds with B replaced by the convolution operator T associated with the kernel. One characterizing condition is the L^2 estimate,

$$\|T^* f\|_2 \leq C \|Tf\|_2\,, \quad f \in L^2(\mathbb{R}^n)\,, \tag{5}$$

an apparently weaker condition. Another description is expressed in terms of a purely algebraic condition involving the spherical harmonics expansion of the kernel. In particular, even higher order Riesz transforms T do satisfy (2), with B replaced by T.

The first purpose of this paper is to pursue this point of view in the case of odd smooth kernels, for which the model example provided by the Hilbert transform points towards pointwise inequalities of the type (4). The main result is given in Theorem 1 (see Section 2 below) where the pointwise inequality (4), with B replaced by T, is shown to be equivalent to the L^2 estimate (5) and, as in the even case, to a purely algebraic condition in terms of the spherical harmonics expansion of the kernel.

The second purpose of this paper is to gain a better understanding of why (3) fails and to provide appropriate sharp substitutes. The failure of (3) is related to the endpoint boundedness properties of the composition of the maximal singular integral operator and the singular integral operator itself. For instance, for the Hilbert transform we are referring to the operator of the form

$$f \to (H^* \circ H)(f) = H^*(Hf). \tag{6}$$

We show in Section 9 that, indeed, this operator is **not** of weak type $(1,1)$ and, as a consequence, (3) cannot hold. On the other hand, we show that $H^* \circ H$ satisfies an "$L \log L$" type estimate, namely, that there is a constant C such that

$$\left|\{x \in \mathbb{R} : H^*(Hf)(x) > t\}\right| \le C \int_{\mathbb{R}} \Phi\left(\frac{|f(x)|}{t}\right) dx, \quad t > 0 \tag{7}$$

where $\Phi(t) = t \log(e+t)$. This estimate seems to be the right replacement for (3), because of the presence of M^2 in (4) and because it is well known (see [P2]) that

$$\left|\{x \in \mathbb{R}^n : M^2 f(x) > t\}\right| \le C \int_{\mathbb{R}^n} \Phi\left(\frac{|f(x)|}{t}\right) dx\,. \tag{8}$$

We remark that, since $\|M^2\|_{L^{1,\infty}} = \infty$, the preceding inequality is sharp.

In fact, we show that the above "$L \log L$" phenomenon holds for arbitrary Calderón-Zygmund singular integral operators, not necessarily of convolution type. Specifically, if T_1 and T_2 are such operators, then $T_1^* \circ T_2$ and even $T_1^* \circ T_2^*$ satisfy inequalities similar to (7) (see Theorem 2 in Section 2 below). However, there are special situations, always associated to even kernels, in which one gets a weak type $(1,1)$ inequality. For example, for the Beurling transform B one has

$$\left|\{z \in \mathbb{C} : B^*(Bf)(z) > t\}\right| \le \frac{C}{t} \int_{\mathbb{C}} |f(z)|\, dA(z), \quad t > 0\,,$$

dA being two dimensional Lebesgue measure. The explanation is that even operators enjoy an extra cancellation property smoothing out the composition.

2. Main results

2.1. The pointwise estimate for odd kernels. Let T be a smooth homogeneous Calderón-Zygmund singular integral operator on $\mathbb{R}^n$ with kernel

$$K(x) = \frac{\Omega(x)}{|x|^n}, \quad x \in \mathbb{R}^n \setminus \{0\}, \tag{9}$$

where Ω is a (real valued) homogeneous function of degree 0 whose restriction to the unit sphere S^{n-1} is of class $C^\infty(S^{n-1})$ and satisfies the cancellation property

$$\int_{|x|=1} \Omega(x)\, d\sigma(x) = 0,$$

σ being the normalized surface measure on S^{n-1}. Recall that Tf is the principal value convolution operator

$$Tf(x) = p.v. \int f(x-y)\, K(y)\, dy \equiv \lim_{\epsilon \to 0} T^\epsilon f(x), \tag{10}$$

where T^ϵ is the truncation at level ϵ defined by

$$T^\epsilon f(x) = \int_{|y-x|>\epsilon} f(x-y)K(y)\,dy\,.$$

It is well known that the limit in (10) exists for almost all x for f in $L^p(\mathbb{R}^n)$, $1 \le p < \infty$.

The operator T is said to be odd (or even) if the kernel is odd (or even), that is, if $\Omega(-x) = -\Omega(x),\ x \in \mathbb{R}^n \setminus \{0\}$ (or $\Omega(-x) = \Omega(x),\ x \in \mathbb{R}^n \setminus \{0\}$).

Let T^* be the maximal singular integral

$$T^* f(x) = \sup_{\epsilon>0} |T^\epsilon f(x)|, \quad x \in \mathbb{R}^n.$$

Consider the problem of controlling T^*f by Tf. The most basic form of control one may think of is the L^2 estimate

$$\|T^*f\|_2 \le C\|Tf\|_2\,, \quad f \in L^2(\mathbb{R}^n)\,. \tag{11}$$

Another way of saying that T^*f is dominated by Tf, much stronger, is provided by the pointwise inequality

$$T^*f(x) \le C\,M(Tf)(x), \quad x \in \mathbb{R}^n\,, \tag{12}$$

where M denotes the Hardy-Littlewood maximal operator. A third form of control, weaker than (12), but which still implies the L^2 inequality (11), is given by the condition

$$T^*f(x) \le C\,M^2(Tf)(x), \quad x \in \mathbb{R}^n\,, \tag{13}$$

where $M^2 = M \circ M$ is the iterated Hardy-Littlewood maximal operator. It was shown in [MV] that the Hilbert transform does not satisfy (12), but does satisfy a pointwise inequality slightly weaker than (13) (see [MV, p. 959]).

In this paper we prove that if T is an odd higher order Riesz transform, then (13) holds. In [MOV] it was shown that even higher order Riesz transforms satisfy the stronger inequality (12). Recall that T is a higher order Riesz transform if its kernel is given by a function Ω of the form

$$\Omega(x) = \frac{P(x)}{|x|^d}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

with P a homogeneous harmonic polynomial of degree $d \ge 1$. If $P(x) = x_j$, then one obtains the j-th Riesz transform R_j. If the homogeneous polynomial P is not required to be harmonic, but has still zero integral on the unit sphere, then we call T a polynomial operator.

Condition (13) clearly implies the L^p inequality

$$\|T^*f\|_p \le C\|Tf\|_p\,, \quad f \in L^p(\mathbb{R}^n) \quad 1 < p \le \infty\,.$$

As we said before, the Hilbert Transform does not satisfy (12). Therefore the presence of the iterated Hardy-Littlewood maximal operator in the case of odd kernels is in the nature of the problem.

Our main result states that for odd operators inequalities (11) and (13) are equivalent to an algebraic condition involving the expansion of Ω in spherical harmonics. This condition may be very easily checked in practice and so, in particular, we can produce extremely simple examples of odd polynomial operators for which (11) and (13) fail. For these operators no alternative way of controlling T^*f by Tf is known. To state our main result we need to introduce a piece of notation.

Recall that Ω has an expansion in spherical harmonics, that is,

$$\Omega(x) = \sum_{j=1}^{\infty} P_j(x), \quad x \in S^{n-1}, \tag{14}$$

where P_j is a homogeneous harmonic polynomial of degree j. If Ω is odd, then only the P_j of odd degree j may be non-zero.

An important role in this paper will be played by the algebra A consisting of the bounded operators on $L^2(\mathbb{R}^n)$ of the form

$$\lambda I + S,$$

where λ is a real number and S a smooth homogeneous Calderón-Zygmund operator.

Our main result reads as follows.

Theorem 1. *Let T be an odd smooth homogeneous Calderón-Zygmund operator with kernel* (9) *and assume that Ω has the expansion* (14). *Then the following are equivalent.*

(i)
$$T^*f(x) \leq C\, M^2(Tf)(x)\,, \quad x \in \mathbb{R}^n\,.$$

(ii)
$$\|T^*f\|_2 \leq C\|Tf\|_2\,, \quad f \in L^2(\mathbb{R}^n)\,.$$

(iii) *The operator T can be factorized as $T = R \circ U$, where U is an invertible operator in the algebra A and R is an odd higher order Riesz transform associated to a harmonic homogeneous polynomial P which divides each P_j in the ring of polynomials in n variables with real coefficients .*

Two remarks are in order.

Remark 1. As in [MOV], (iii) can be reformulated in a more concrete fashion as follows. Assume that the expansion of Ω in spherical harmonics is

$$\Omega(x) = \sum_{j=j_o}^{\infty} P_{2j+1}(x)\,,\ P_{2j_0+1} \neq 0\,.$$

Then (iii) is equivalent to the following

(iv) *For each j there exists a homogeneous polynomial Q_{2j-2j_0} of degree $2j-2j_0$ such that $P_{2j+1} = P_{2j_0+1}\, Q_{2j-2j_0}$ and $\sum_{j=j_o}^{\infty} \gamma_{2j+1}\, Q_{2j-2j_0}(\xi) \neq 0\,, \quad \xi \in S^{n-1}$.*

Here for a positive integer j we have set

$$\gamma_j = i^{-j}\,\pi^{\frac{n}{2}}\,\frac{\Gamma(\frac{j}{2})}{\Gamma(\frac{n+j}{2})}\,. \tag{15}$$

The quantities γ_j appear in the computation of the Fourier multiplier of the higher order Riesz transform R with kernel given by a homogeneous harmonic polynomial P of degree j. One has (see [St, p. 73])

$$\widehat{Rf}(\xi) = \gamma_j\,\frac{P(\xi)}{|\xi|^j}\,\hat{f}(\xi), \quad f \in L^2(\mathbb{R}^n)\,.$$

Throughout this paper the Fourier transform of f is $\hat{f}(\xi)=\int f(x)e^{-ix\cdot\xi}dx,\ \xi\in\mathbb{R}^n$.

The proof that (iii) and (iv) are equivalent is exactly as in [MOV].

Remark 2. Condition (iii) is rather easy to check in practice. For instance, consider the polynomial of third degree

$$P(x) = x_1 + (n+1)\left(x_1^3 - 3x_1x_2^2\right)\,, \quad x \in S^{n-1}\,.$$

The polynomial operator associated with P does not satisfy (i) nor (ii), because the definition of P above is also the spherical harmonics expansion of P and, although x_1 divides the two terms, a calculation shows that $\gamma_1+\gamma_3\,(n+1)\,(\xi_1^2-3\,\xi_2^2)$ vanishes on the sphere. On the other hand, if $-1<\lambda<1$ the polynomial operator associated with the polynomial

$$P(x) = x_1 + \lambda\,(n+1)\left(x_1^3 - 3x_1x_2^2\right)\,, \quad x \in S^{n-1}\,,$$

does satisfy (i) and (ii). Thus, as in the even case, we conclude that the condition on Ω so that T satisfies (i) or (ii) is rather subtle.

For the proofs we will rely heavily on [MOV] and the reader will be assumed to have some familiarity with that paper. The strategy for the proof is essentially the same as in [MOV], but two main differences arise, which require some new ideas. In the even case, in the proof of "(iii) implies (i)" ("the sufficient condition") for polynomial operators associated with a homogeneous polynomial of degree $2N$, the differential operator Δ^N plays an essential role. In the odd case the natural substitute for Δ^N is a pseudo-differential operator, which is non local. Thus one loses the support of certain functions. This is a new difficulty which must be overcome. A second difference is that one cannot hope to have the subtle L^∞ estimates of [MOV], which have to be replaced by BMO estimates. This is, in some sense, favorable, because proofs are simpler at some points, just because an L^∞ estimate is not possible and must be replaced by a straight BMO estimate.

We devote Sections 4, 5 and 6 to the proof of the sufficient condition ((iii) implies (i)). In Section 4 we prove that the odd higher order Riesz transforms satisfy (i). Section 5 is devoted to the proof of the sufficient condition for polynomial operators. The drawback of the argument used is that we lose control on the dependence of the constants on the degree of the polynomial. The main difficulty we have to overcome in Section 6 to complete the proof of the sufficient condition in the general case, is to find a second approach to the polynomial case which gives some estimates with constants independent of the degree of the polynomial. This allows the use of a compactness argument to finish the proof. As in the even case, the approach in Section 4 cannot be dispensed with, because it provides certain properties which are vital for the final argument and do not follow otherwise.

In Section 7 we prove the necessary condition, that is, (ii) implies (iii). First we deal with the polynomial case. Analysing the inequality (ii) via Plancherel at the frequency side we obtain various inclusion relations among zero sets of certain polynomials. This requires a considerable computational effort, as in the even case. In a second step we solve the division problem which leads us to (iii) by a recurrent argument with some algebraic geometry ingredients . The question of independence on the degree of the polynomial appears again, this time related to the coefficients of certain expansions. Section 8 is devoted to the proof of the combinatorial lemmas used in the previous sections.

2.2. **Composing maximal singular integrals with singular integrals.** In the previous section we discussed several estimates for operators of the form $H^* \circ H$ or $B^* \circ B$. We now extend these results by considering general Calderón-Zygmund singular integral operators. Our point of view is strongly motivated by a celebrated result of R. Coifman and C. Fefferman from the seventies, Theorem 6 below. We recall that A_∞ is the class of weights $\cup_{p\geq 1} A_p$.

Theorem 2. *Let T_1 and T_2 be two the Calderón-Zygmund singular integral operators.*

a) If $0 < p < \infty$ and $w \in A_\infty$, then there is a constant C depending on the A_∞ constant of w such that

$$\int_{\mathbb{R}^n} (T_1^* \circ T_2)(f)(x)^p \, w(x)dx \leq C \int_{\mathbb{R}^n} (M^2(f)(x))^p \, w(x)dx, \tag{16}$$

and

$$\sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} \, w\left(\{x \in \mathbb{R}^n : (T_1^* \circ T_2)(f)(x) > t\}\right) \leq C \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w\left(\{x \in \mathbb{R}^n : M^2(f)(x) > t\}\right) \tag{17}$$

where $\Phi(t) = t\log(e+t)$ and f is a function for which the left hand side is finite.

b) The estimates (16) *and* (17) *hold with $T_1^* \circ T_2$ replaced by $T_1^* \circ T_2^*$ in the left hand side.*

Corollary 1. *Let T_1 and T_2 as above and let $w \in A_1$. Then there is a constant C depending on the A_1 constant of w such that*

$$w(\{x \in \mathbb{R} : T_1^* \circ T_2(f)(x) > t\}) \leq C \int_{\mathbb{R}} \Phi\left(\frac{|f(x)|}{t}\right) w(x)dx, \quad t > 0 \tag{18}$$

where $\Phi(t) = t\log(e+t)$.

The estimate (18) *holds with $T_1^* \circ T_2$ replaced by $T_1^* \circ T_2^*$ in the left hand side.*

As we mentioned before, for general Calderón-Zygmund singular integral operators T_1 and T_2 their composition $T_1 \circ T_2$ is not of weak type $(1,1)$. This should be compared with the case of Fourier multipliers T_m when the multiplier m satisfies the classical Mihlin condition. Indeed, by classical well known results, if T_{m_1} and T_{m_2} are two multipliers the composition operators $T_{m_1} \circ T_{m_2} = T_{m_1 m_2}$ is also of weak type $(1,1)$, and hence is an algebra [GrCF, 2.5.5]. Indeed, suppose that T_{m_1} and T_{m_2} are two multiplier operators such that each multiplier m_j, $j = 1,2$, is bounded, belongs to $C^{[\frac{n}{2}]+1}$ in the complement of the origin and satisfies the classical Mihlin condition,

$$(\partial^\alpha m_j)(\xi) \leq \frac{c}{|\xi|^\alpha} \quad \xi \neq 0$$

for any α such that $|\alpha| \leq [\frac{n}{2}]+1$. Now consider as above the composition operator $T_{m_1} \circ T_{m_2}$, which is another multiplier operator with multiplier $m_1 m_2$. Then since $m_1 m_2$ satisfies again Mihlin's condition by the Leibnitz rule, then $T_{m_1} \circ T_{m_2}$ is of weak type $(1,1)$. We recall here that in either the case of the Hilbert transform (Theorem 7) or the Riesz transform ([MV]), where the multipliers are smooth, T_{m_1} cannot be replaced by $T^*_{m_1}$ in the above result.

The Beurling transform, being an even smooth Calderón-Zygmund operator, should enjoy an extra cancellation property that allows for an improvement of Theorem 2. This can be readily verified for the operator $B^* \circ \overline{B}$, where $\overline{B}$ denotes the operator whose kernel is the complex conjugate of the kernel of the Beurling transform. Since $\overline{B}$ is precisely the inverse of B, the pointwise inequality (2) implies immediately that $B^* \circ \overline{B}$ is of weak type $(1,1)$.

It turns out that the operator $B^* \circ B$ is also of weak type $(1,1)$, in striking contrast with the fact that $H^* \circ H$ is not. This is more difficult to prove and follows from the pointwise inequality

$$B^*(B(f))(z) \leq C\,\left((B^2)^*(f)(z) + M(f)(z)\right), \quad z \in \mathbb{C}, \tag{19}$$

because $B^2 = B \circ B$ is again a smooth Calderón-Zygmund singular integral operator and hence its maximal operator is of weak type $(1,1)$. This will be shown in Section 10. Indeed, we prove there a more general result which reads as follows.

Theorem 3. *Let R be an even higher order Riesz transform and let T be an even smooth homogeneous Calderón-Zygmund operator. Then there exists a smooth homogeneous Calderón-Zygmund operator S such that*

$$R^*(T(f))(x) \leq C(S^*(f)(x) + M(f)(x)) \quad x \in \mathbb{R}^n.$$

The operator S is defined by the identity $R \circ T = S + cI$, where c is an appropriate constant.

In particular, the operator $R^* \circ T$ is of weak type $(1,1)$.

3. Some preliminaries

3.1. **Sharp maximal operators.** For $\delta > 0$, let M_δ be the maximal function

$$M_\delta f(x) = M(|f|^\delta)^{1/\delta}(x) = \left(\sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|^\delta \, dy\right)^{1/\delta}.$$

Also, let $M^\#$ be the usual sharp maximal function of Fefferman and Stein [FS],

$$M^\#(f)(x) = \sup_{Q \ni x} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c| \, dy \approx \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,$$

where as usual $f_Q = \frac{1}{|Q|} \int_Q f(y) \, dy$ denotes the average of f over Q.

We also consider the following useful variant of the above sharp maximal operator

$$M_\delta^\# f(x) = M^\#(|f|^\delta)(x)^{1/\delta}.$$

The main inequality between these operators to be used is a version of the classical one due to C. Fefferman and E. Stein (see [Jo] for a proof simpler than the original, or [GrMF, p. 148]).

Theorem 4. *Let w be an A_∞ weight and let $\delta > 0$.*
a) Let $0 < p < \infty$. Then there exists a positive constant C depending on the A_∞ condition of w and p such that

$$\int_{\mathbb{R}^n} (M_\delta f(x))^p \, w(x) dx \le C \int_{\mathbb{R}^n} (M_\delta^\# f(x))^p \, w(x) dx,$$

for every function f such that the left hand side is finite.

b) Let $\varphi\colon (0,\infty) \to (0,\infty)$ satisfy the doubling condition . Then, there exists a constant C depending upon the A_∞ condition of w and the doubling condition of φ such that

$$\sup_{t>0} \varphi(t)\, w\Big(\{y \in \mathbb{R}^n : M_\delta f(y) > t\}\Big) \le C \sup_{t>0} \varphi(t)\, w\Big(\{y \in \mathbb{R}^n : M_\delta^\# f(y) > t\}\Big)$$

for every function such that the left hand side is finite.

3.2. **Orlicz spaces and normalized measures.** We need some few facts from the theory of Orlicz spaces that we will state without proof. For more information about these spaces the reader may consult the recent book by Wilson [W] or [GrMF, p. 158]. Let $\Phi\colon [0,\infty) \to [0,\infty)$ be a Young function. The Φ-average of a function f over a cube Q is defined to be the $L_\Phi(\mu)$ norm of f with μ the normalized measure of the cube Q and it is denoted by $\|f\|_{\Phi,Q}$. That is,

$$\|f\|_{\Phi,Q} = \inf\{\lambda > 0 \, : \, \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1\}.$$

In this paper we will consider the Young functions $\Phi(t) = t\,(1+\log^+ t) \approx t\, \log(e+t)$ and $\Psi(t) = e^t - 1$. The corresponding averages will be denoted by $\| \cdot \|_{\Phi,Q} =$

$\|\cdot\|_{L(\log L),Q}$ and $\|\cdot\|_{\Psi,Q}=\|\cdot\|_{\exp L,Q}$ respectively. We will use the following well known generalized Hölder's inequality

$$\frac{1}{|Q|}\int_Q |f(x)\,g(x)|\,dx \le C\,\|f\|_{\exp L,Q}\,\|g\|_{L(\log L),Q}. \tag{20}$$

In particular, we obtain the following inequality, which will be used later on in this article,

$$\frac{1}{|Q|}\int_Q |b(y)-b_Q|\,f(y)\,dy \le C\|b\|_{BMO}\|f\|_{L(\log L),Q}\,, \tag{21}$$

for any function $b\in BMO$ and any non negative function f. This inequality follows from (20) combined with the classical John-Nirenberg inequality [JN] for BMO functions: there is a dimensional constant c such that

$$\frac{1}{|Q|}\int_Q \exp\Big(\frac{|b(y)-b_Q|}{c\|b\|_{BMO}}\Big)\,dy \le 2$$

which easily implies that

$$\|b-b_Q\|_{\exp L,Q}\le c\,\|b\|_{BMO}.$$

In view of this result and its applications it is natural to define as in [P2] a maximal operator

$$M_{L(\log L)}f(x)=\sup_{Q\ni x}\|f\|_{L(\log L),Q},$$

where the supremum is taken over all the cubes containing x. (Other equivalent definitions can be found in the literature.) We will also use the pointwise equivalence

$$M_{L(\log L)}f(x)\approx M^2f(x). \tag{22}$$

This equivalence was obtained in [P1] (see [CGMP] for a different argument) and it relationship with commutators of singular integrals and BMO functions was studied in [P2] and [P3]. The sharp endpoint modular inequality for M^2, already mentioned in (8), will play an important role.

Finally, we will employ several times the following simple Kolmogorov inequality. Let $0<p<q<\infty$, then there is a constant $C=C_{p,q}$ such that for any measurable function f

$$\|f\|_{L^p(Q,\frac{dx}{|Q|})}\le C\,\|f\|_{L^{q,\infty}(Q,\frac{dx}{|Q|})}.$$

3.3. **Cotlar's pointwise inequality for Calderón-Zygmund operators.** By a Calderón-Zygmund operator we mean a continuous linear operator $T\colon C_0^\infty(\mathbb{R}^n)\to\mathcal{D}'(\mathbb{R}^n)$ that extends to a bounded operator on $L^2(\mathbb{R}^n)$, and whose distributional kernel K coincides, away from the diagonal, with a function K satisfying the size estimate

$$|K(x,y)|\le\frac{c}{|x-y|^n}$$

and the regularity condition

$$|K(x,y)-K(z,y)|+|K(y,x)-K(y,z)|\le c\frac{|x-z|^{\epsilon}}{|x-y|^{n+\epsilon}},$$

for some $\epsilon>0$ and whenever $2|x-z|<|x-y|$. The kernel of T is K in the sense that

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy,$$

whenever $f\in C_0^\infty(\mathbb{R}^n)$ and $x\notin \operatorname{supp}(f)$. Let T^* be the maximal singular integral

$$T^*f(x)=\sup_{\epsilon>0}|T^\epsilon f(x)|,\quad x\in\mathbb{R}^n,$$

where T^ϵ is the truncation at level ϵ defined by

$$T^\epsilon f(x)=\int_{|y-x|>\epsilon}K(x,y)f(y)dy\,.$$

We refer to [GrMF, p. 175] for a complete account on these operators. In the same reference, p. 185, it can be found an improvement of Cotlar's inequality (1) that will be useful for our estimates. It reads as follows.

Theorem 5. *Let T and T^* as before and let $0<\delta<1$. Then there is a positive constant $C=C_\delta$ such that*

$$T^*(f)(x)\le C\,M_\delta(Tf)(x)+C\,Mf(x),\quad x\in\mathbb{R}^n. \tag{23}$$

Observe that by Jensen's inequality, (23) is an improvement of (1). Also, it should be mentioned that A. Lerner has improved this estimate in [Le2].

The ideas leading to Cotlar estimate (23) were crucial to derive the good-λ inequality relating T^* and M found by R. Coifman and C. Fefferman in [CoF]. In particular we will use in Section 10 the following result.

Theorem 6. *Let T be any Calderón-Zygmund operator. Then*

a) If $0<p<\infty$ and $w\in A_\infty$, then there exists a positive constant C depending upon the A_∞ condition of w such that

$$\int_{\mathbb{R}^n}|T^*f(x)|^p\,w(x)\,dx\le C\int_{\mathbb{R}^n}Mf(x)^p\,w(x)\,dx. \tag{24}$$

b) Let $\varphi\colon(0,\infty)\to(0,\infty)$ doubling. Then, there exists a positive constant C depending upon the A_∞ condition of w and the doubling condition of φ such that

$$\sup_{t>0}\varphi(t)w(\{y\in\mathbb{R}^n:|T^*f(x)|>t\})\le C\sup_{t>0}\varphi(t)w(\{y\in\mathbb{R}^n:Mf(x)>t\})$$

Also we will use a local versión of (24) in the proof of (26) in Lemma 2 below: if $0<p<\infty$, $w\in A_\infty$, and Q is an arbitrary cube, then there exists a constant C depending upon the A_∞ condition of w such that

$$\int_{2Q}|T^*f(x)|^p\,w(x)\,dx\le C\int_{2Q}Mf(x)^p\,w(x)\,dx, \tag{25}$$

for any function f supported in Q. The proof of this estimate is an adaptation of the proof in [CoF] by considering everything at local level. However, it should be mentioned that a different approach to the above theorem, which may be found in [AP], yields the local version. It is based on the combination of the well known good-λ inequality of Fefferman-Stein Theorem 4, which is much simpler, and the pointwise estimate (27) from next lemma which will be used in the paper. This procedure has been applied in [LOPTT] into the context of multilinear Calderón-Zygmund singular integral operators to derive sharp results.

Lemma 2. *If T is a Calderón-Zygmund singular operator, then*

$$M^{\#}(Tf)(x) \le C\, M^2(f)(x)\,, \tag{26}$$

and, for $0<\delta<1$,

$$M_\delta^{\#}(Tf)(x) \le C_\delta\, Mf(x)\,. \tag{27}$$

It is well known that inequality (26) holds with the right hand side replaced by the larger operator $M_p(f)$ with $p>1$ (see, for instance, [GrMF, p. 153]). However, this is not sharp enough for many purposes and an excellent alternative is given by (27) which can be found in [AP]. We sketch the proof of inequality (26) in Section 10.

4. Odd higher order Riesz transforms

In this section we prove that if T is an odd higher order Riesz transform, then

$$T^* f(x) \le C\, M^2(Tf)(x), \quad x \in \mathbb{R}^n\,. \tag{28}$$

By translating and dilating one reduces the proof of (28) to

$$|T^1 f(0)| \le C\, M^2(Tf)(0)\,,$$

where

$$T^1 f(0) = -\int_{|y|>1} f(y) K(y)\, dy$$

is the truncated integral at level 1. Recall that the kernel of our singular integral is

$$K(x) = \frac{\Omega(x)}{|x|^n} = \frac{P(x)}{|x|^{n+d}}\,,$$

where P is an odd homogeneous harmonic polynomial of degree $d \ge 1$. The argument proceeds along the lines of the even case, but, as we said above, two important differences arise. The first is that, for odd d, $(-\Delta)^{d/2}$ is not a differential operator and this complicates the situation. We will work with the pseudo-differential operator $(-\Delta)^{1/2}\Delta^N$, where $d = 2N+1$. The definition of $(-\Delta)^{1/2}$ on test functions Ψ is $(-\Delta)^{1/2}\Psi = \sum_{j=1}^n R_j(\partial_j \Psi)$, where the R_j are the Riesz transforms normalized so that $\widehat{R_j \Psi}(\xi) = -i\xi_j/|\xi|\, \Psi(\xi)$. The kernel of R_j is then $\rho\, x_j |x|^{-n-1}$, where ρ is a constant which depends only on the dimension n and whose concrete value is irrelevant in this paper. On the Fourier transform side we then have $\widehat{(-\Delta)^{1/2}\Psi}(\xi) = |\xi|\widehat{\Psi}(\xi)$. The idea is to obtain an identity of the form

$$K(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = T(b)(x)\,, \tag{29}$$

where B is the open ball of radius 1 centered at the origin and b is a certain function. To this end, consider a fundamental solution of $(-\Delta)^{1/2}\Delta^N$, that is, a function E such that $(-\Delta)^{1/2}\Delta^N E = \delta$, where δ is the Dirac delta at the origin. One can take E as a solution of $\Delta^N E = c_n/|x|^{n-1}$, where the constant c_n is chosen so that $\widehat{c_n/|x|^{n-1}}(\xi) = 1/|\xi|$. The formula $c_n = \dfrac{\Gamma(\frac{n-1}{2})}{2\pi^{n/2}\Gamma(\frac{1}{2})}$ wil be used in Section 8. Notice that E can always be taken to be radial (see Section 8 for a precise expression). Consider the function

$$\varphi(x) = E(x)\,\chi_{\mathbb{R}^n\setminus\overline{B}}(x) + (A_0 + A_1\,|x|^2 + \cdots + A_{2N}\,|x|^{4N})\,\chi_B(x)\,, \tag{30}$$

where the constants $A_0, A_1, \ldots, A_{2N}$ are chosen so that the derivatives of φ up to order $2N$ extend continuously to the boundary of B. Then, in computing the distributional derivatives of φ, one can apply $2N+1$ times Green-Stokes' Theorem and the boundary terms will vanish. This is most conveniently done by applying N times Corollary 1 and one time Lemma 1 in [MOV]. The conclusion is that, for some constants α_j and β_k,

(31)
$$\begin{aligned}
(-\Delta)^{1/2}\Delta^N\varphi &= (-\Delta)^{1/2}\left(\frac{c_n}{|x|^{n-1}}\chi_{B^c}(x) + (\alpha_0 + \alpha_1|x|^2 + \cdots + \alpha_N|x|^{2N})\chi_B(x)\right)\\
&= \sum_{j=1}^n R_j\left(c_n\,(1-n)\,\frac{x_j}{|x|^{n+1}}\chi_{B^c}(x)\right.\\
&\qquad\left. + (\beta_1 x_j + \beta_2 x_j|x|^2 + \cdots + \beta_N x_j|x|^{2N-2})\chi_B(x)\right)\\
&:= b(x),
\end{aligned}$$

where the last identity is a definition of b. Since

$$\varphi = E * (-\Delta)^{1/2}\triangle^N\varphi\,,$$

taking derivatives of both sides we obtain

$$P(\partial)\,\varphi = P(\partial)\,E * (-\Delta)^{1/2}\triangle^N\varphi\,.$$

To compute $P(\partial)E$ we take the Fourier transform

$$\widehat{P(\partial)E}(\xi) = P(i\xi)\,\hat{E}(\xi) = i\frac{P(\xi)}{|\xi|^d}\,.$$

On the other hand, as it is well known ([St, p. 73]),

$$\widehat{p.v.\frac{P(x)}{|x|^{n+d}}}(\xi) = \gamma_d \, \frac{P(\xi)}{|\xi|^d} \, .$$

See (15) for the precise value of γ_d, which is not important now. We conclude that, for some constant a_d depending on d,

$$P(\partial)E = a_d \; p.v.\frac{P(x)}{|x|^{n+d}} \, .$$

Thus

$$P(\partial)\varphi = a_d \; p.v.\frac{P(x)}{|x|^{n+d}} * (-\Delta)^{1/2} \triangle^N \varphi = a_d \; T(b) \, .$$

The only thing left is the computation of $P(\partial)\,\varphi$. We have, by Corollary 1 in [MOV],

$$\begin{aligned} P(\partial)\,\varphi &= a_d \; K(x)\,\chi_{\mathbb{R}^n\setminus\overline{B}} + P(\partial)(A_0 + A_1\,|x|^2 + \cdots + A_{d-1}\,|x|^{2d-2})(x)\,\chi_B(x) \\ &= a_d \; K(x)\,\chi_{\mathbb{R}^n\setminus\overline{B}}\,, \end{aligned}$$

where the last identity follows from the fact that, since P is harmonic,

$$P(\partial)(|x|^{2j}) = 0\,, \quad 1 \le j \le d-1. \tag{32}$$

The identity (32) is a special case of a formula of Lyons and Zumbrun [LZ] which will be discussed in the next section (see Lemma 2).

Once (29) is at our disposition we get, for f in some $L^p(\mathbb{R}^n)$, $1 \le p < \infty$,

$$\begin{aligned} T^1 f(0) &= -\int \chi_{\mathbb{R}^n\setminus\overline{B}}(y)\,K(y)\,f(y)\,dy \\ &= -\int T(b)(y)\,f(y)\,dy \\ &= \int b(y)\,Tf(y)\,dy \\ &= \int_{2B} Tf(y)b(y)dy \; + \int_{\mathbb{R}^n\setminus 2B} Tf(y)b(y)dy \\ &= \int_{2B} Tf(y)(b(y)-b_{2B})dy \; + b_{2B}\int_{2B} Tf(y)dy \; + \int_{\mathbb{R}^n\setminus 2B} Tf(y)b(y)dy \\ &= I + II + III. \end{aligned}$$

Notice that b_{2B} is a dimensional constant, because of the definition of b. In particular, it is independent of f. Hence

$$|II| \le C\,M(Tf)(0)\,.$$

To estimate the local term I we remark that $b \in BMO(\mathbb{R}^n)$. This follows from the fact that b is a sum on j of the j-th Riesz transform of a bounded function (depending on j). Hence we can apply (21) to get

$$\begin{aligned} |I| &\le C\|b\|_{BMO}\|Tf\|_{L(\log L),2B} \\ &\le C\,\|Tf\|_{L(\log L),2B} \\ &\le C\,M^2(Tf)(0), \end{aligned}$$

where we have used (22) in the last inequality.

To estimate the term III we first prove the decay estimate

$$|b(x)| \le \frac{C}{|x|^{n+1}}, \qquad |x| > 2. \tag{33}$$

From the decay of b we obtain

$$III \le C \int_{|x|>2} |Tf(x)| \frac{1}{|x|^{n+1}} dx \le CM(Tf)(0),$$

using a standard argument which consists in estimating the integral on the annuli $\{2^k \le |x| < 2^{k+1}\}$. Let us prove (33). From the definition of b we see that $b = b_1 + b_2$, where

$$b_1 = c_n(1-n)\sum_{j=1}^{n} R_j\left(\frac{x_j}{|x|^{n+1}}\chi_{B^c}(x)\right)$$

and

$$b_2 = \sum_{j=1}^{n} R_j(a_j)$$

each a_j being a bounded function supported on B with zero integral (indeed, a_j is odd).

If $|x| > 2$, then, since the kernel of R_j is $\rho\, x_j|x|^{-n-1}$, for some numerical constant ρ depending on n,

$$\begin{aligned} R_j(a_j)(x) &= \rho \int_{|y|<1} \frac{x_j - y_j}{|x-y|^{n+1}}\; a_j(y)\,dy \\ &= \rho \int_{|y|<1} \left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}}\right)\; a_j(y)\,dy\,. \end{aligned} \tag{34}$$

Thus

$$|R_j(a_j)(x)| \le \frac{C}{|x|^{n+1}}\,, \quad |x| > 2\,,$$

and hence b_2 satisfies the decay estimate (33) with b replaced by b_2. That this is also the case for b_1 was shown in [MV]. We repeat the argument here for completeness. One has

$$\sum_{j=1}^{n} R_j\left(\frac{\rho\, y_j}{|y|^{n+1}}\,\chi_{\mathbb{R}^n\setminus\overline{B}}\,(y)\right) = \sum_{j=1}^{n} R_j * R_j - \sum_{j=1}^{n} R_j\left(\frac{\rho\, y_j}{|y|^{n+1}}\,\chi_B\,(y)\right)$$

$$= \delta_0 - \sum_{j=1}^{n} R_j\left(\frac{\rho\, y_j}{|y|^{n+1}}\,\chi_B\,(y)\right),$$

where δ_0 is the Dirac delta at the origin. If $|x| > 2$, then

$$R_j\left(\frac{y_j}{|y|^{n+1}}\,\chi_B\,(y)\right)(x) = \rho \lim_{\epsilon\to 0}\int_{\epsilon<|y|<1} \frac{x_j - y_j}{|x-y|^{n+1}}\;\frac{y_j}{|y|^{n+1}}\,dy$$

$$= \rho \lim_{\epsilon\to 0}\int_{\epsilon<|y|<1} \left(\frac{x_j - y_j}{|x-y|^{n+1}} - \frac{x_j}{|x|^{n+1}}\right)\;\frac{y_j}{|y|^{n+1}}\,dy\,.$$

Hence

$$\left|R_j\left(\frac{y_j}{|y|^{n+1}}\,\chi_B\,(y)\right)(x)\right| \le \frac{C}{|x|^{n+1}}\int_{|y|<1}\frac{dy}{|y|^{n-1}} \le \frac{C}{|x|^{n+1}}\,,$$

which completes the proof of (33).

5. Proof of the sufficient condition: the polynomial case

This section does not differ substantially from its analogue for the even case. Nevertheless, we will present the argument in detail for the reader's sake, because it is technically sophisticated and we would like to describe clearly the changes that have to be made.

Let us assume that T is an odd polynomial operator. This amounts to say that for some odd integer $2N+1$, $N \ge 0$, the function $|x|^{2N+1}\,\Omega(x)$ is a homogeneous polynomial of degree $2N+1$. Such a polynomial may be written as [St, p. 69]

$$|x|^{2N+1}\,\Omega(x) = P_1(x)|x|^{2N} + \cdots + P_{2j+1}(x)|x|^{2N-2j} + \cdots + P_{2N+1}(x)\,,$$

where P_{2j+1} is a homogeneous harmonic polynomial of degree $2j+1$, $0 \le j \le N$. In other words, the expansion of $\Omega(x)$ in spherical harmonics is

$$\Omega(x) = P_1(x) + P_3(x) + \cdots + P_{2N+1}(x), \quad |x| = 1\,.$$

As in the previous section, we want to obtain an expression for the kernel $K(x)$ off the unit ball B. For this we need the differential operator $Q(\partial)$ defined by the polynomial

$$Q(x) = \gamma_1\,P_1(x)|x|^{2N} + \cdots + \gamma_{2j+1}\,P_{2j+1}(x)|x|^{2N-2j} + \cdots + \gamma_{2N+1}\,P_{2N+1}(x)\,.$$

If E is the standard fundamental solution of $(-\Delta)^{1/2}\Delta^N$, then

$$Q(\partial)E = i\,p.v.\;K(x)\,,$$

which may be easily verified by taking the Fourier transform of both sides (K is the kernel of T).

Take now the function φ of the previous section. We have $\varphi = E*(-\Delta)^{1/2}\triangle^N \varphi$ and thus

$$Q(\partial)\varphi = Q(\partial)E*(-\Delta)^{1/2}\triangle^N \varphi = p.v.\ K(x)*b = T(b)\,,$$

where b is defined as $i\ (-\Delta)^{1/2}\triangle^N \varphi$. On the other hand, by Corollary 2 of [MOV]

$$Q(\partial)\,\varphi = i\,K(x)\,\chi_{\mathbb{R}^n\setminus\overline{B}} + Q(\partial)(A_0 + A_1\,|x|^2 + \cdots + A_{2N}\,|x|^{4N})(x)\,\chi_B(x)\,. \tag{35}$$

Contrary to what happened in the previous section, the term

$$S(x) := -Q(\partial)(A_0 + A_1\,|x|^2 + ... + A_{2N}\,|x|^{4N})(x)$$

does not necessarily vanish, the reason being that now Q does not need to be harmonic.

Our goal is to find a function $\beta \in BMO(\mathbb{R}^n)$, satisfying the decay estimate

$$|\beta(x)| \leq \frac{C}{|x|^{n+1}}\,, \quad |x| \geq 2\,, \tag{36}$$

and

$$S(x)\chi_B(x) = T(\beta)(x)\,. \tag{37}$$

By (35), the definition of $S(x)$ and (37), we then get

$$i\ K(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = T(b)(x) + T(\beta)(x) = T(\gamma)(x)\,, \tag{38}$$

where $\gamma = b + \beta$ belongs to BMO and satisfies the decay estimate (36) with β replaced by γ. Once this is achieved the proof of (i) is just the argument presented in Section 4.

To construct β satisfying (36) and (37) we resort to our hypothesis, condition (iii) in the Theorem, which says that $T = R\circ U$, where U is an invertible operator in the algebra A, R is a higher order Riesz transform and the polynomial P which determines R divides P_{2j+1} , $0 \leq j \leq N$, in the ring of polynomials in n variables with real coefficients. The construction of β is performed in two steps.

The first step consists in proving that there exists a function β_1 in BMO, satisfying some additional properties, such that

$$S(x)\chi_B(x) = R(\beta_1)(x)\,. \tag{39}$$

It will become clear later what these *additional properties* are and how they are used. To prove (39) we need an explicit formula for $S(x)$ and for that we will make use of the following formula of Lyons and Zumbrun [LZ].

Lemma 3. *Let L be a homogeneous polynomial of degree l and let f be a smooth function of one variable. Then*

$$L(\partial)f(r) = \sum_{\nu\geq 0} \frac{1}{2^\nu\,\nu!}\ \Delta^\nu L(x)\ \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{l-\nu} f(r)\,, \quad r = |x|\,.$$

An immediate consequence of Lemma 1 is

Lemma 4. *Let P_{2j+1} a homogeneous harmonic polynomial of degree $2j+1$ and let k be a non-negative integer. Then*

$$P_{2j+1}(\partial)(|x|^{2k}) = 2^{2j+1} \frac{k!}{(k-2j-1)!} P_{2j+1}(x)\,|x|^{2(k-2j-1)} \quad \textit{if} \quad 2j+1 \le k\,,$$

and

$$P_{2j+1}(\partial)(|x|^{2k}) = 0\,, \qquad k < 2j+1\,.$$

On the other hand, a routine computation gives

$$\triangle^j(|x|^{2k}) = 4^j \frac{j!\,k!}{(k-j)!} \binom{\frac{n}{2}+k-1}{j} |x|^{2(k-j)}\,, \quad j \le k\,, \tag{40}$$

and

$$\triangle^j(|x|^{2k}) = 0\,, \quad k < j\,. \tag{41}$$

By Lemma 2, (40) and (41) we get that for some constants $c_{k,j}$ one has, in view of the definitions of $Q(x)$ and $S(x)$,

$$S(x) = \sum_{j=0}^{N-1} \sum_{k=0}^{N-1-j} c_{k,j}\; P_{2j+1}(x)\; |x|^{2k}\,. \tag{42}$$

Therefore it suffices to prove (39) with $S(x)$ replaced by $P_{2j+1}(x)\,|x|^{2k}$, for $0 \le j \le N-1$ and each non-negative integer $k \le N-1-j$. The idea is to look for an appropriate function ψ such that

$$P(\partial)\psi(x) = P_{2j+1}(x)\,|x|^{2k}\,\chi_B(x)\,. \tag{43}$$

Let $2d+1$ be the degree of P. Assume for the moment that (43) holds and ψ is good enough. Then

$$\psi = E * (-\Delta)^{1/2}\Delta^d\psi\,,$$

where E is the fundamental solution of $(-\Delta)^{1/2}\triangle^d$. Hence

$$P(\partial)\psi = P(\partial)E * (-\Delta)^{1/2}\Delta^d\psi \;= c\,p.v. \frac{P(x)}{|x|^{n+2d+1}} * (-\Delta)^{1/2}\Delta^d\psi \;= R(\beta_1)\,,$$

where β_1 is defined as $c\,(-\Delta)^{1/2}\Delta^d\psi = c\displaystyle\sum_{i=1}^n R_i\partial_i(\Delta^d\psi)$. The conclusion is that we have to solve (43) in such a way that $\partial_i\Delta^d\psi$ is a bounded function supported on B with zero integral, $1 \le i \le n$. If this is the case, then β_1 is in BMO and satisfies the decay estimate $|\beta_1(x)| \le C|x|^{-n-1}$ if $|x| > 2$, as we proved before (see (34)).

Taking Fourier transforms in (43) we get

$$i(-1)^d P(\xi)\,\widehat{\psi}(\xi) = i(-1)^{j+k}\, P_{2j+1}(\partial)\,\triangle^k\,(\widehat{\chi_B}(\xi))\;. \tag{44}$$

Recall that for $m = n/2$ one has [GrCF, B.5]

$$\widehat{\chi_B}(\xi) = (2\pi)^m \, \frac{J_m(|\xi|)}{|\xi|^m} \,, \quad \xi \in \mathbb{R}^n \,,$$

where J_m is the Bessel function of order m. Set

$$G_\lambda(\xi) = \frac{J_\lambda(|\xi|)}{|\xi|^\lambda} \,, \quad \xi \in \mathbb{R}^n \,, \quad \lambda > 0 \,.$$

In computing the right hand side of (44) we apply Lemma 3 to $L(x) = P_{2j+1}(x)\,|x|^{2k}$ and $f(r) = G_m(r)$ and we get

$$P(\xi)\,\widehat{\psi}(\xi) = (2\pi)^m (-1)^{j+k+d} \sum_{\nu \geq 0} \frac{(-1)^{\nu+1}}{2^\nu \, \nu!} \triangle^\nu \left(P_{2j+1}(\xi)\,|\xi|^{2k} \right) G_{m+2j+1+2k-\nu}(|\xi|) \,,$$

owing to the well known formula (e.g. [GrCF, B.2])

$$\frac{1}{r}\frac{d}{dr}\, G_\lambda(r) = -G_{\lambda+1}(r) \,, \quad r > 0 \,, \quad \lambda > 0 \,.$$

Since $P_{2j+1}(\xi)$ is homogeneous of degree $2j+1$, $\nabla P_{2j+1}(\xi) \cdot \xi = (2j+1)\, P_{2j+1}(\xi)$, and hence one may readily show by an inductive argument that

$$\triangle^\nu \left(P_{2j+1}(\xi)\,|\xi|^{2k} \right) = a_{jk\nu}\, P_{2j+1}(\xi)\,|\xi|^{2(k-\nu)} \,,$$

for some constants $a_{jk\nu}$. Thus, for some other constants $a_{jk\nu}$, we get

$$P(\xi)\,\widehat{\psi}(\xi) = \sum_{\nu \geq 0} a_{jk\nu}\, P_{2j+1}(\xi)\,|\xi|^{2(k-\nu)}\, G_{m+2j+1+2k-\nu}(\xi) \,. \tag{45}$$

By hypothesis P divides P_{2j+1} in the ring of polynomials in n variables and so

$$P_{2j+1}(\xi) = P(\xi)\, Q_{2j-2d}(\xi) \,,$$

for some homogeneous polynomial Q_{2j-2d} of degree $2j - 2d$. Cancelling out the factor $P(\xi)$ in (45) we conclude that

$$\widehat{\psi}(\xi) = Q_{2j-2d}(\xi) \, \sum_{\nu=0}^{k} a_{jk\nu}\, |\xi|^{2(k-\nu)}\, G_{m+2j+1+2k-\nu}(|\xi|) \,.$$

Since [GrCF, B.5]

$$(\widehat{(1-|x|^2)^\lambda \chi_B(x)})(\xi) = c_\lambda\, G_{m+\lambda}(|\xi|) \,,$$

we finally obtain

$$\psi(x) = Q_{2j-2d}(\partial) \, \sum_{\nu=0}^{k} a_{jk\nu} \triangle^{k-\nu} \left((1-|x|^2)^{2j+1+2k-\nu} \, \chi_B(x) \right) \,,$$

for other constants $a_{jk\nu}$. Observe that ψ restricted to B is a polynomial which vanishes on ∂B up to order $2d+1$ and ψ is zero off B. Therefore, $\partial_i \triangle^d \psi$ has zero integral, is supported on B and its restriction to B is a polynomial, $1 \leq i \leq n$. This completes the first step of the construction of β.

The second step proceeds as follows. Since by hypothesis $T = R \circ U$, with U invertible in the algebra A, we have

$$R(\beta_1) = R \circ U(U^{-1}\beta_1) = T(U^{-1}\beta_1)\,.$$

Setting

$$\beta = U^{-1}\beta_1\,, \tag{46}$$

we are only left with the task of showing that

$$\beta \in BMO(\mathbb{R}^n)$$

and that, for some positive constant C,

$$|\beta(x)| \le \frac{C}{|x|^{n+1}}\,, \quad |x| \ge 2\,. \tag{47}$$

Since $U^{-1} \in A$, for some real number λ and some smooth homogeneous Calderón-Zygmund operator V,

$$U^{-1} = \lambda\, I + V\,.$$

Thus

$$\beta = \lambda\, \beta_1 + V(\beta_1)\,.$$

By construction, $\beta_1 = \sum_{i=1}^{n} R_i a_i$, where each a_i is a bounded function supported on B with zero integral. By (34) β_1 satisfies (47) with β replaced by β_1. Clearly, $\beta \in BMO$ and so we only have to get the decay estimate (47) with $V(\beta_1)$ in place of β. In fact,

$$V(\beta_1) = \sum_{i=1}^{n} V\, R_i a_i = \sum_{i=1}^{n} \lambda_i a_i + \sum_{i=1}^{n} V_i a_i,$$

because each $V\, R_i \in A$ and thus $V\, R_i = \lambda_i I + V_i$ for some real number λ_i and some smooth homogeneous Calderón-Zygmund operator V_i. Following the argument we used in (34) with V_i in place of R_i we finally get (47).

6. Proof of the sufficient condition: the general case

In [MOV] several facts about the convergence of the expansion (14) of Ω in spherical harmonics were established. In particular, since Ω is infinitely differentiable on the unit sphere and has the spherical harmonics expansion

$$\Omega(x) = \sum_{j\ge 0}^{\infty} P_{2j+1}(x)\,. \tag{48}$$

One has that, for each positive integer M,

$$\sum_{j\ge 1} (2j+1)^M \, \|P_{2j+1}\|_\infty < \infty\,,$$

where the supremum norm is taken on S^{n-1}.

By hypothesis there is a homogeneous harmonic polynomial P of degree $2d+1$ such that $P_{2j+1} = P\,Q_{2j-2d}$, where Q_{2j-2d} is a homogeneous polynomial of degree $2j-2d$. As in [MOV], one shows that the series $\sum_j Q_{2j-2d}(x)$ is convergent in $C^\infty(S^{n-1})$, that is, that for each positive integer M

$$\sum_{j\geq d} j^M \, \|Q_{2j-2d}\|_\infty < \infty\,. \tag{49}$$

The scheme for the proof of the sufficient condition in the general case is as in [MOV]. Nevertheless, we will have to overcome several new difficulties, which are not substantial but still require significant work.

Taking a large partial sum of the series (48) we pass to a polynomial operator T_N (associated to a polynomial of degree $2N+1$), which still satisfies the hypothesis *(iii)* of the Theorem. Then we may apply the construction of Section 5 to T_N and get functions b_N and β_N. Unfortunately what was done in Section 5 does not give any uniform estimate in N, which is precisely what we need to try a compactness argument. The rest of the section is devoted to get the appropriate uniform estimates and to describe the final compactness argument.

By hypothesis, $T = R \circ U$, where R is the higher order Riesz transform associated to the harmonic polynomial P of degree $2d+1$ that divides all P_{2j+1}, and U is invertible in the algebra A. The Fourier multiplier of T is

$$\sum_{j=d}^{\infty} \gamma_{2j+1} \frac{P_{2j+1}(\xi)}{|\xi|^{2j+1}} = \gamma_{2d+1} \frac{P(\xi)}{|\xi|^{2d+1}} \sum_{j\geq d} \frac{\gamma_{2j+1}}{\gamma_{2d+1}} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}\,.$$

Therefore the Fourier multiplier of U is

$$\mu(\xi) = \gamma_{2d+1}^{-1} \sum_{j\geq d} \gamma_{2j+1} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}\,, \tag{50}$$

and the series is convergent in $C^\infty(S^{n-1})$ because $\gamma_{2j+1} \simeq (2j+1)^{-n/2}$ [SW, p. 226]. Set, for $N \geq d$,

$$\mu_N(\xi) = \gamma_{2d+1}^{-1} \sum_{j=d}^{N} \gamma_{2j+1} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}\,.$$

If

$$K_N(x) = \sum_{j=d}^{N} \frac{P_{2j+1}(x)}{|x|^{2j+1+n}}, \quad x \in \mathbb{R}^n \setminus \{0\}\,,$$

and T_N is the polynomial operator with kernel K_N, then $T_N = R \circ U_N$, where U_N is the operator in the algebra A with Fourier multiplier $\mu_N(\xi)$. From now on N is assumed to be big enough so that $\mu_N(\xi)$ does not vanish on S^{n-1}. In fact, we will need later on the inequality

$$|\partial^\alpha \mu_N^{-1}(\xi)| \leq C, \quad |\xi| = 1, \quad 0 \leq |\alpha| \leq 2(n+3)\,, \tag{51}$$

which may be taken for granted owing to the convergence in $C^\infty(S^{n-1})$ of the series (50). In (51) C is a positive constant depending only on the dimension n and on μ.

Notice that T_N satisfies condition (iii) in the Theorem (with T replaced by T_N), because $\mu_N(\xi) \neq 0$, $|\xi| = 1$, and so we can apply the results of Section 5. In particular,

$$\imath\, K_N(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = T_N(b_N)(x) + T_N(\beta_N)(x)\,,$$

where b_N and β_N are respectively the functions b and β appearing in (38). It is important to remark that b_N does not depend on T. As (31) shows, the function b_N depends on N only through the fundamental solution of the operator $(-\Delta)^{1/2}\triangle^N$. The uniform estimate we need on b_N is given by part (i) of the next lemma. The polynomial estimates in N of (ii) and (iii) are also central for the compactness argument we are looking for, and they were not present in the corresponding lemma for the even case (Lemma 7 in [MOV]).

Lemma 5. *There exist a constant C depending only on n such that*

(i)
$$|\widehat{b_N}(\xi)| \le C, \quad \xi \in \mathbb{R}^n\,,$$

(ii)
$$\|b_N\|_{BMO} \le C\,(2N+1)^{2n}\,,$$

and

(iii)
$$\|b_N\|_2 \le C\,(2N+1)^{2n}\,,$$

where $\|\cdot\|_{BMO}$ and $\|\cdot\|_2$ denote respectively the BMO and L^2 norms on $\mathbb{R}^n$.

Proof. We first prove (i). Let $h_1, \dots, h_d$ be an orthonormal basis of the subspace of $L^2(d\sigma)$ consisting of all homogeneous harmonic polynomials of degree $2N+1$. As in the proof of Lemma 6 in [MOV] we have $h_1^2 + \cdots + h_d^2 = d$, on S^{n-1}. Set

$$H_j(x) = \frac{1}{\gamma_{2N+1}\sqrt{d}}\, h_j(x), \quad x \in \mathbb{R}^n\,,$$

and let S_j be the higher order Riesz transform with kernel $K_j(x) = H_j(x)/|x|^{2N+1+n}$. The Fourier multiplier of S_j^2 is

$$\frac{1}{d}\,\frac{h_j(\xi)^2}{|\xi|^{4N+2}}, \quad 0 \neq \xi \in \mathbb{R}^n\,,$$

and thus

$$\sum_{j=1}^{d} S_j^2 = I\,.$$

By (29), we get

$$K_j(x)\,\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = S_j(b_N)(x), \quad x \in \mathbb{R}^n, \quad 1 \le j \le d\,,$$

and so

$$b_N = \sum_{j=1}^{d} S_j \left(K_j(x)\, \chi_{\mathbb{R}^n \setminus \overline{B}}(x) \right) . \tag{52}$$

We now appeal to a lemma of Calderón and Zygmund ([CZ]) and we readily get (i) (see [MOV]).

We now turn to the proof of (ii) in Lemma 5. In view of the expression (52) for b_N, we obtain, by the standard $L^\infty - BMO$ estimate,

$$\|b_N\|_{BMO} \le C\, d \max_{1\le j\le d} \|K_j\|_{CZ}\, \|K_j\|_{L^\infty(\mathbb{R}^n\setminus\overline{B})} .$$

Recall that the constant of the kernel $K(x) = \Omega(x)/|x|^n$ of the smooth homogeneous Calderón-Zygmund operator T is

$$\|T\|_{CZ} \equiv \|K\|_{CZ} = \|\Omega\|_\infty + \| |x|\, \nabla\Omega(x)\|_\infty .$$

As it is well known, $d \simeq (2N+1)^{n-2}$ [SW, p. 140]. On the other hand

$$\|K_j\|_{CZ} \le \|H_j\|_\infty + \|\nabla H_j\|_\infty ,$$

where the supremum norms are taken on S^{n-1}. Clearly

$$\|H_j\|_\infty = \frac{1}{\gamma_{2N+1}} \| \frac{h_j}{\sqrt{d}} \|_\infty \le \frac{1}{\gamma_{2N+1}} \simeq (2N+1)^{n/2} .$$

For the estimate of the gradient of H_j we use the inequality [St, p. 276]

$$\|\nabla H_j\|_\infty \le C\,(2N+1)^{n/2+1}\, \|H_j\|_2 ,$$

where the L^2 norm is taken with respect to $d\sigma$. Since the h_j are an orthonormal system,

$$\|H_j\|_2 = \frac{1}{\sqrt{d}\,\gamma_{2N+1}} \simeq \frac{(2N+1)^{n/2}}{(2N+1)^{(n-2)/2}} \simeq 2N+1 .$$

Gathering the above inequalities we get

$$\|K_j\|_{CZ} \le C\,(2N+1)^{n/2+2} .$$

On the other hand, $\|K_j\|_{L^\infty(\mathbb{R}^n\setminus\overline{B})} \le (2N+1)^{n/2}$ and therefore

$$\|b_N\|_{BMO} \le C\,(2N+1)^{n-2}\,(2N+1)^{n/2+2}\,(2N+1)^{n/2} = C\,(2N+1)^{2n} .$$

The estimate (iii) in Lemma 5 follows from $\|b_N\|_2 \le C\, d \max_{1\le j\le d} \|K_j\|_{CZ}\, \|K_j\|_{L^2(\mathbb{R}^n\setminus\overline{B})}$, $\|K_j\|_{L^2(\mathbb{R}^n\setminus B)} \le C\; \|H_j\|_\infty$ and the previous inequalities. □

Our goal is now to show that under condition (iii) of the Theorem we can find a function γ in $BMO(\mathbb{R}^n)$ such that

$$\imath\, K(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = T(\gamma)(x), \quad x \in \mathbb{R}^n , \tag{53}$$

and having a decay as in (36) with β replaced by γ. If T is a polynomial operator this was proven in the preceding section for a γ of the form $b+\beta$ (see (38)). The plan is to produce a different approach to this result, which has the advantage that, when applied to T_N, gives a uniform BMO bound on $\gamma_N = b_N + \beta_N$.

Since Ω has the expansion (48) in spherical harmonics, we have

$$\begin{aligned} K(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) &= \sum_{j\geq 0} \frac{P_{2j+1}(x)}{|x|^{2j+1+n}}\,\chi_{\mathbb{R}^n\setminus\overline{B}}(x) \\ &= \sum_{j\geq 0} T_j(b_j)(x)\,, \end{aligned}$$

where T_j is the higher order Riesz transform with kernel $P_{2j+1}(x)/|x|^{2j+1+n}$ and b_j is the function constructed in Section 4 (see (29) and (31)). The Fourier multiplier of T_j is

$$\gamma_{2j+1}\,\frac{P_{2j+1}(\xi)}{|\xi|^{2j+1}} = \gamma_{2d+1}\,\frac{P(\xi)}{|\xi|^{2d+1}}\,\frac{\gamma_{2j+1}}{\gamma_{2d+1}}\,\frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n\setminus\{0\}\,.$$

Let S_j be the operator whose Fourier multiplier is

$$\frac{\gamma_{2j+1}}{\gamma_{2d+1}}\,\frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}}, \quad \xi \in \mathbb{R}^n\setminus\{0\}\,, \tag{54}$$

so that $T_j = R\circ S_j$. Then

$$\begin{aligned} K(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) &= \sum_{j\geq d} (R\circ S_j)(b_j) \\ &= \sum_{j\geq d} T\left((U^{-1}\circ S_j)(b_j)\right) \\ &= T\left(\sum_{j\geq d} (U^{-1}\circ S_j)(b_j)\right). \end{aligned}$$

The latest identity is justified by the absolute convergence of the series $\sum_{j\geq d}(U^{-1}\circ S_j)(b_j)$ in $L^2(\mathbb{R}^n)$, which follows, using (iii) in Lemma 5, from the estimate

$$\begin{aligned} \sum_{j\geq d} \|(U^{-1}\circ S_j)(b_j)\|_2 &\leq C \sum_{j\geq d} \|Q_{2j-2d}\|_\infty\, \|b_j\|_{L^2(\mathbb{R}^n)} \\ &\leq C \sum_{j\geq d} \|Q_{2j-2d}\|_\infty\, (2j+1)^{2n} < \infty\,. \end{aligned}$$

We claim now that the series $\sum_{j\geq d}(U^{-1}\circ S_j)(b_j)$ converges in $BMO(\mathbb{R}^n)$ to a function $-\imath\,\gamma$, which will prove (53) . Observe that the operator $U^{-1}\circ S_j \in A$ is not necessarily a Calderón-Zygmund operator because the integral on the sphere of its multiplier does not need to vanish. However it can be written as $U^{-1}\circ S_j =$

$c_j I + V_j$, where

$$c_j = \frac{\gamma_{2j}}{\gamma_{2d}} \int_{S^{n-1}} \mu(\xi)^{-1}\, Q_{2j-2d}(\xi)\, d\sigma(\xi)$$

and V_j is the Calderón-Zygmund operator with multiplier

$$\mu(\xi)^{-1} \frac{\gamma_{2j}}{\gamma_{2d}} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}} - c_j\,. \tag{55}$$

Now

$$\sum_{j\geq d} (U^{-1}\circ S_j)(b_j) = \sum_{j\geq d} c_j\, b_j + \sum_{j\geq d} V_j(b_j)$$

and the first series offers no difficulties because, by Lemma 5 (ii) and (49)

$$\sum_{j\geq d} |c_j|\, \|b_j\|_{BMO} \leq C \sum_{j\geq d} (2j+1)^{-n/2}(2j+1)^{2n} \|Q_{2j-2d}\|_\infty < \infty\,.$$

The second series is more difficult to treat. By Lemma 5 and Lemma 5 (ii) and (iii),

$$\begin{aligned} \|V_j(b_j)\|_{BMO} &\leq C\, \|V_j\|_{CZ}\, \|b_j\|_{BMO} \\ &\leq C\, (2j+1)^{2n}\, \|V_j\|_{CZ}\,. \end{aligned}$$

Estimating the Calderón-Zygmund constant of the kernel of the operator V_j is not an easy task, because we do not have an explicit expression for the kernel. We do know, however, the multiplier (55) of V_j. We need a way of estimating the constant of the kernel in terms of the multiplier and this is what Lemma 9 of [MOV] achieves. The final outcome is

$$\|V_j\|_{CZ} \leq C\, j^M\, \|P_{2j+1}\|_2\,,$$

for some positive integer M depending only on n and the polynomial P. Thus

$$\|V_j(b_j)\|_{BMO} \leq C\, j^M\, \|P_{2j+1}\|_2\,,$$

where again $M = M(n,P)$ is a positive integer. Hence the series $\sum_{j\geq d}(U^{-1}\circ S_j)(b_j)$ converges in $BMO(\mathbb{R}^n)$ and the proof of (53) is complete.

We are now ready for the discussion of the final compactness argument that will complete the proof of the sufficient condition. We know from Section 5 (see (38)) that

$$\imath\, K_N(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = T_N(b_N)(x) + T_N(\beta_N)(x)\,. \tag{56}$$

On the other hand, by the construction of the function γ we have just described, we also have

$$\imath\, K_N(x)\chi_{\mathbb{R}^n\setminus\overline{B}}(x) = T_N(\gamma_N)(x), \quad \gamma_N = \sum_{j\geq d}^{N} (U_N^{-1}\circ S_j)(b_j)\,. \tag{57}$$

Notice that (51) guaranties that Lemma 9 of [MOV] may be applied to the operator T_N and so the estimate of the BMO norm of γ_N is uniform in N. Since T_N is injective, (56) and (57) imply

$$b_N + \beta_N = \gamma_N \tag{58}$$

and, in particular, we conclude that the functions $b_N+\beta_N$ are uniformly bounded in $BMO(\mathbb{R}^n)$, a fact that cannot be derived from the work done in Section 5. On the other hand, Section 5 tells us that γ_N satisfies the decay estimate (36) with β replaced by γ_N, which we cannot infer from the preceding construction of γ. The advantages of both approaches will be combined now to get both the boundedness in BMO and the decay property for γ.

In view of (57) and the expressions of the multipliers of U_N and S_j (see (54)),

$$\widehat{\gamma_N}(\xi) = \sum_{j=d}^{N} \frac{1}{\mu_N(\xi)} \frac{\gamma_{2j+1}}{\gamma_{2d+1}} \frac{Q_{2j-2d}(\xi)}{|\xi|^{2j-2d}} \widehat{b_j}(\xi)\,,$$

which yields, by Lemma 5 (i) and (49) for $M=0$,

$$\begin{aligned}\|\widehat{\gamma_N}\|_{L^\infty(\mathbb{R}^n)} &\le C \sum_{j=d}^{N} \|Q_{2j-2d}\|_\infty \\ &\le C \sum_{j=d}^{\infty} \|Q_{2j-2d}\|_\infty \\ &\le C\,,\end{aligned}$$

where C does not depend on N. Recall that, from (46) in Section 5, we have

$$\beta_N = U_N^{-1}(\beta_{1,N})\,,$$

with $\beta_{1,N} = \sum_{i=1}^{n} R_i \partial_i(f_N)$, where f_N is a C^1 function supported on B. Since

$$\widehat{\beta_{1,N}} = \mu_N\, \widehat{\beta_N} = \mu_N\,(\widehat{\gamma_N} - \widehat{b_N})\,,$$

we have, again by Lemma 5 (i),

$$\|\widehat{\beta_{1,N}}\|_{L^\infty(\mathbb{R}^n)} \le C\,.$$

Therefore, passing to a subsequence, we may assume that, as N goes to ∞,

$$\widehat{b_N} \longrightarrow a_0 \qquad \text{and} \qquad \widehat{\beta_{1,N}} \longrightarrow a_1\,, \tag{59}$$

weak $*$ in $L^\infty(\mathbb{R}^n)$. Hence

$$b_N \longrightarrow \Phi_0 = \mathcal{F}^{-1} a_0 \qquad \text{and} \qquad \beta_{1,N} \longrightarrow \Phi_1 = \mathcal{F}^{-1} a_1\,,$$

in the weak $*$ topology of tempered distributions, $\mathcal{F}^{-1}$ being the inverse Fourier transform.

We would like now to understand the convergence properties of the sequence of the β_N's . Since

$$\widehat{\beta_N}(\xi) = \mu_N^{-1}(\xi)\, \widehat{\beta_{1,N}}(\xi)\,,$$

and we have pointwise bounded convergence of $\mu_N^{-1}(\xi)$ towards $\mu^{-1}(\xi)$ on $\mathbb{R}^n \setminus \{0\}$, we get that $\widehat{\beta_N} \to \mu^{-1}\, a_1$, in the weak $*$ topology of $L^\infty(\mathbb{R}^n)$. Thus $\beta_N \to U^{-1}(\Phi_1)$ in the weak $*$ topology of tempered distributions. Letting $N \to \infty$ in (58) we obtain

$$\begin{aligned} \gamma &= \Phi_0 + U^{-1}(\Phi_1) \\ &= \Phi_0 + \lambda \Phi_1 + V(\Phi_1)\,, \end{aligned}$$

where λ is a real number and V a smooth homogeneous Calderón-Zygmund operator.

We come now to the last delicate point of the proof, namely, that one has the decay estimate

$$|\gamma(x)| \le \frac{C}{|x|^{n+1}}, \quad |x| \ge 2\,. \tag{60}$$

We claim that, as tempered distributions,

$$\Phi_0 = \sum_{i=1}^{n} R_i \partial_i (S_0) + c\delta_0 \qquad \text{and} \qquad \Phi_1 = \sum_{i=1}^{n} R_i \partial_i (S_1)\,, \tag{61}$$

where S_0 and S_1 are distributions supported on $\overline{B}$ and c is a constant depending only on n. Recall that $\beta_{1,N} = \sum_{i=1}^n R_i \partial_i (f_N)$, where f_N is a C^1 function supported on $\overline{B}$, and, by (31),

$$b_N = \sum_{i=1}^{n} R_i \partial_i \left(\frac{c_n}{|x|^{n-1}} \chi_{B^c} + P_N \chi_B \right),$$

where P_N is a polynomial. Set

$$\alpha_N = \frac{c_n}{|x|^{n-1}} \chi_{B^c} + P_N \chi_B.$$

Hence

$$\widehat{\beta_{1,N}}(\xi) = |\xi|\, \widehat{f_N}(\xi) \qquad \text{and} \qquad \widehat{b_N}(\xi) = |\xi|\, \widehat{\alpha_N}(\xi).$$

By (59), since $a_0, a_1 \in L^\infty$ and $\dfrac{1}{|\xi|}$ is locally integrable in $\mathbb{R}^n$ (because we may assume $n \ge 2$),

$$\widehat{\alpha_N} \longrightarrow \frac{a_0}{|\xi|} \qquad \text{and} \qquad \widehat{f_N} \longrightarrow \frac{a_1}{|\xi|}\,,$$

in the weak $*$ topology of tempered distributions. Hence

$$\alpha_N \longrightarrow \alpha := \mathcal{F}^{-1}\left(\frac{a_0}{|\xi|}\right) \qquad \text{and} \qquad f_N \longrightarrow S_1 := \mathcal{F}^{-1}\left(\frac{a_1}{|\xi|}\right),$$

in the weak $*$ topology of tempered distributions. Since each f_N is supported on B we get that S_1 is also supported on $\overline{B}$ and we obtain (61) for Φ_1. On the other hand, observe that

$$P_N \chi_B = \alpha_N - \frac{c_n}{|x|^{n-1}} \chi_{B^c} \longrightarrow \alpha - \frac{c_n}{|x|^{n-1}} \chi_{B^c} := \alpha' ,$$

with α' a tempered distribution supported on $\overline{B}$. Set

$$S_0 = \alpha' - \frac{c_n}{|x|^{n-1}} \chi_B .$$

The claim now follows from the chain of identities

$$\begin{aligned}
\Phi_0 &= \sum_{i=1}^n R_i \partial_i(\alpha) \\
&= \sum_{i=1}^n R_i \partial_i \left(\alpha' + \frac{c_n}{|x|^{n-1}} \chi_{B^c} \right) \\
&= \sum_{i=1}^n R_i \partial_i \left(\alpha' - \frac{c_n}{|x|^{n-1}} \chi_B \right) + \sum_{i=1}^n R_i \partial_i \left(\frac{c_n}{|x|^{n-1}} \right) \\
&= \sum_{i=1}^n R_i \partial_i(S_0) + c \sum_{i=1}^n R_i * R_i \\
&= \sum_{i=1}^n R_i \partial_i(S_0) + c\delta_0 .
\end{aligned}$$

Therefore,

$$\gamma = \sum_{i=1}^n R_i \partial_i(S_0) + c\delta_0 + \lambda \sum_{i=1}^n R_i \partial_i(S_1) + V \left(\sum_{i=1}^n R_i \partial_i(S_1) \right) .$$

Write, for each i, $V \circ R_i = \lambda_i\, I + V_i$ for some real number λ_i and some homogeneous smooth Calderón-Zygmund operator V_i. Thus to get (60) it is enough to show that

$$|V(\partial_i S)(x)| \le \frac{C}{|x|^{n+1}} , \quad |x| \ge 2 ,$$

where V is a homogeneous smooth Calderón-Zygmund operator and S a distribution supported on $\overline{B}$. Regularizing S one checks that, for a fixed x with $|x| \ge 2$,

$$\begin{aligned}
V(\partial_i S)(x) &= \langle \partial_i S,\, L(x-y) \rangle \\
&= -\langle S,\, \frac{\partial}{\partial y_i} L(x-y) \rangle ,
\end{aligned} \tag{62}$$

Since S is a distribution supported on $\overline{B}$ there exists a positive integer ν and a constant C such that

$$|\langle S, \varphi \rangle| \leq C \sup_{|\alpha|\leq \nu} \sup_{|y|\leq 3/2} |\partial^\alpha \varphi(y)| \,, \tag{63}$$

for each infinitely differentiable function φ on $\mathbb{R}^n$. The kernel L satisfies

$$|\frac{\partial^\alpha}{\partial y^\alpha} \frac{\partial}{\partial y_i} L(x-y)| \leq \frac{C_\alpha}{|x|^{n+1+|\alpha|}}, \quad |y| \leq 3/2 \,,$$

and hence, by (62) and (63),

$$|V(\partial_i S)(x)| \leq \frac{C}{|x|^{n+1}}, \quad |x| \geq 2 \,,$$

which proves (60) and then completes the proof of the sufficient condition in the general case.

7. Proof of the necessary condition

The proof of the necessary condition is completely analogue to the even case. We will just start the argument to help the reader in capturing the context.

We first assume that T is a polynomial operator with kernel

$$K(x) = \frac{\Omega(x)}{|x|^n} = \frac{P_1(x)}{|x|^{1+n}} + \frac{P_3(x)}{|x|^{3+n}} + \cdots + \frac{P_{2N+1}(x)}{|x|^{2N+1+n}} \,, \quad x \neq 0 \,,$$

where P_{2j+1} is a homogeneous harmonic polynomial of degree $2j+1$. Let Q be the homogeneous polynomial of degree $2N+1$ defined by

$$Q(x) = \left(\gamma_1 \, P_1(x)|x|^{2N} + \cdots + \gamma_{2j+1} \, P_{2j+1}(x)|x|^{2N-2j} + \cdots + \gamma_{2N+1} \, P_{2N+1}(x)\right) \,.$$

Then

$$\widehat{p.v.K}(\xi) = \frac{Q(\xi)}{|\xi|^{2N+1}} \,.$$

Our assumption is now the $L^2(\mathbb{R}^n)$ control of T^*f by Tf (i.e., (*ii*) in the statement of the Theorem). Since the truncated operator T^1 at level 1 is obviously dominated by T^*, we have

$$\int (T^1 f)^2(x)\, dx \leq \int (T^* f)^2(x)\, dx \leq C \int (Tf)^2(x)\, dx \,.$$

The kernel of T^1 is (see (35))

$$K(x)\, \chi_{\mathbb{R}^n \setminus \overline{B}}(x) = T(b)(x) + S(x)\, \chi_B(x) \,, \tag{64}$$

where b is given in equation (31) and

$$-S(x) = Q(\partial)(A_0 + A_1 \, |x|^2 + \cdots + A_{2N-1} \, |x|^{4N})(x) \,, \quad x \in \mathbb{R}^n \,.$$

The reader may consult the beginning of Section 5 to review the context of the definition of S. In view of (64) we have, for each $f \in L^2(\mathbb{R}^n)$,

$$\begin{aligned}\|S\,\chi_B * f\|_2 &\le C\,\|T^1 f\|_2 + \|b * Tf\|_2 \\ &\le C\,(\|Tf\|_2 + \|\widehat{b}\|_\infty \|Tf\|_2) \\ &= C\,\|Tf\|_2 \,.\end{aligned}$$

By Plancherel, the above L^2 inequality translates into a pointwise inequality between the Fourier multipliers, namely,

$$|\widehat{S\,\chi_B}(\xi)| \le C\,|\widehat{p.v.K}(\xi)| = C\,\frac{|Q(\xi)|}{|\xi|^{2N+1}}\,. \tag{65}$$

Notice that Q has plenty of zeros because it has zero integral on the sphere. Our next aim is to use (65) to show that P_{2j+1} vanishes where Q does. For each function f on $\mathbb{R}^n$ set $Z(f) = \{x \in \mathbb{R}^n : f(x) = 0\}$.

Lemma (Zero Sets Lemma)**.**

$$Z(Q) \subset Z(P_{2j+1})\,, \quad 0 \le j \le N\,.$$

Proof. We know that S has an expression of the form (see (42))

$$S(x) = \sum_{L=N+1}^{2N} \sum_{j=0}^{L-N-1} c_{L,j}\; P_{2j+1}(x)\; |x|^{2(L-N-j-1)}\,.$$

Since $\widehat{\chi_B} = G_m(2\pi)^m$, $m = n/2$, Lemma 3 yields

(66)

$$\begin{aligned}&\widehat{S\,\chi_B}(\xi) = \\ &= S(\imath\,\partial)\,\widehat{\chi_B}(\xi) \\ &= \imath(2\pi)^{n/2} \sum_{L=N+1}^{2N} \sum_{j=0}^{L-N-1} c_{L,j}\,(-1)^{L-N}\; P_{2j+1}(\partial)\; \triangle^{L-N-j-1}\, G_{\frac{n}{2}}(\xi) \\ &= (2\pi)^{n/2} \sum_{L=N+1}^{2N} \sum_{j=0}^{L-N-1} \sum_{k=0}^{L-N-j-1} c_{\,L,\,j,\,k}\; P_{2j+1}(\xi)\; |\xi|^{2(L-N-j-1-k)} G_{\frac{n}{2}+2(L-N)-1-k}(\xi)\,.\end{aligned}$$

The function $G_p(\xi)$ is, for each $p \ge 0$, a radial function which is the restriction to the real positive axis of an entire function [GrCF, B.6]. Set $\xi = r\,\xi_0$, $|\xi_0| = 1$, $r \ge 0$. Then

$$(2\pi)^{-n/2}\widehat{S\chi_B}(r\xi_0) = \sum_{p=0}^{\infty} a_{2p+1}(\xi_0)\, r^{2p+1}\,, \tag{67}$$

and the power series has infinite radius of convergence for each ξ_0. Assume now that $Q(\xi_0) = 0$. Then, by (65), $\widehat{S\chi_B}(r\xi_0) = 0$ for each $r \ge 0$, and hence

$a_{2p+1}(\xi_0) = 0$, for each $p \geq 0$. For $p = 0$ one has $a_1(\xi_0) = P_1(\xi_0)\, C_1$, where

$$C_1 = \sum_{L=N+1}^{2N} c_{L,0,L-N-1}\, G_{\frac{n}{2}+L-N}(0)\,.$$

It will be shown later that $C_1 \neq 0$, and then we get $P_1(\xi_0) = 0$. Let us make the inductive hypothesis that $P_1(\xi_0) = \cdots = P_{2j-1}(\xi_0) = 0$. Then we obtain, if $j \leq N-1$, $a_{2j+1}(\xi_0) = P_{2j+1}(\xi_0)\, C_{2j+1}$, where

$$C_{2j+1} = \sum_{L=N+1+j}^{2N} c_{L,j,L-N-j-1}\, G_{\frac{n}{2}+L-N+j}(0)\,. \tag{68}$$

Since we will show that $C_{2j+1} \neq 0$, $P_{2j+1}(\xi_0) = 0$, $0 \leq j \leq N-1$. We have

$$0 = Q(\xi_0) = \sum_{j=0}^{N} \gamma_{2j+1}\, P_{2j+1}(\xi_0)\,,$$

and so we also get $P_{2N+1}(\xi_0) = 0$. Therefore the zero sets Lemma is completely proved provided we have at our disposition the following formula, which in particular shows that $C_{2j+1} \neq 0$, $0 \leq j \leq N-1$.

Lemma 6.

$$C_{2j+1} = \frac{1}{2^{\frac{n}{2}}}\, \frac{(-1)^j}{4^j(2j+1)\Gamma(\frac{n}{2}+2j+1)}, \quad 0 \leq j \leq N-1\,.$$

The proof of Lemma 6 is lengthy and rather complicated from the computational point of view, and so we postpone it to Section 8. □

Notice that, although the constants C_{2j+1} are non-zero, they become rapidly small as the index j increases and they oscillate around 0.

The reason why Lemma 6 is involved is that one has to trace back the exact values of the constants C_{2j+1} from the very beginning of our proof of (64). This forces us to take into account the exact values of various constants. For instance, those which appear in the expression of the fundamental solution of $(-\Delta)^{1/2}\triangle^N$ and the constants $A_0, A_1, \ldots, A_{2N}$ in formula (30). Finally, we need to prove some new identities involving a triple sum of combinatorial numbers, in the spirit of those that can be found in the book of R. Graham D. Knuth and O. Patashnik [GKP].

The remaining of the proof of the necessary condition is basically a plain translation of what was done in the even case. One first completes the proof of the polynomial case by an appropriate division process. Then the general case must be faced. We reduce to the polynomial case by truncating the spherical harmonics expansion of Ω. Denoting by S_N the analogue of S at the truncated level we set $\xi = r\,\xi_0$, with $|\xi_0| = 1$ and $r > 0$. Rewrite (67) with S replaced by S_N and

a_{2p+1} by a^N_{2p+1} :

$$(2\pi)^{-n/2}\widehat{S_N\chi_B}(r\xi_0)=\sum_{p=0}^{\infty}a^N_{2p+1}(\xi_0)\,r^{2p+1}\,.$$

As in the even case, it is a remarkable key fact that for a fixed p the sequence of the a^N_{2p+1} stabilizes for N large. This fact depends on a laborious computation of various constants and will be proved in Section 8 in the following form.

Lemma 7. *If $p+1\le N$, then $a^N_{2p+1}=a^{p+1}_{2p+1}$.*

If $p\ge 0$ and $p+1\le N$ we set $a_{2p+1}=a^N_{2p+1}$. We need an estimate for the a^N_{2p+1} , which will be proved as well in Section 8.

Lemma 8. *We have, for a constant C depending only on n,*

$$|a_{2p+1}|\le\frac{C}{p!\,4^p}\sum_{j=0}^{p}\|P_{2j+1}\|_\infty\,,\quad 0\le p\le N-1\,, \tag{69}$$

and

$$|a^N_{2p+1}|\le\frac{C}{4^p}\,\frac{\binom{N+\frac{n}{2}-\frac12}{N}}{\binom{N-\frac12}{N}}\sum_{j=0}^{N-1}\|P_{2j+1}\|_\infty\,,\quad 1<N\le p\ . \tag{70}$$

8. Proof of the combinatorial Lemmata

This section will be devoted to prove lemmas 6, 7 and 8 stated and used in the preceding sections. The arguments are parallel to those of the even case, but many different computations have to be performed. Owing to the intricate combinatorics involved we prefer to write carefully down all calculations.

For the proof of Lemma 6 (see Section 7) we need to have explicit expressions for the constants C_{2j+1} and for this we need to carefully trace back the path that led us to them. To begin with we need a formula for the coefficients A_L in (30) and for that it is essential to have the expression for a fundamental solution $E_N=E^n_N$ of $(-\Delta)^{1/2}\triangle^N$. Recall that $\triangle^N(E_N)(x)=c_n|x|^{1-n}$ in $\mathbb{R}^n$, where the normalization constant c_n is chosen so that $\widehat{c_n/|x|^{n-1}}(\xi)=1/|\xi|$. One has

$$E_N(x)=c_n|x|^{2N+1-n}(\alpha(n,N)+\beta(n,N)\log|x|^2)\ ,$$

where α and β are constants that depend on n and N .To write in close form α and β we consider different cases. Write $m=\dfrac{n-1}{2}$

Case 1: n is even. Then

$$\alpha(n,N) = \left(\prod_{j=0}^{N-1} (2N+1-n-2j)\,(2N+1-2(j+1))\right)^{-1}$$
$$= \left(\binom{N-m}{N}(2N)!\right)^{-1}$$

and

$$\beta(n;N)=0.$$

Case 2: n is odd and $2N+1-n<0$. Then

$$\alpha(n,N) = \left(\prod_{j=0}^{N-1} (2N+1-n-2j)\,(2N+1-2(j+1))\right)^{-1}$$
$$= \left(\frac{(-2)^N(m-1)!}{(m-N-1)!}\right)^{-1}\frac{2^{N-1}(N-1)!}{(2N-1)!} = \frac{(-1)^N(m-N-1)!(N-1)!}{2(m-1)!(2N-1)!}$$

and

$$\beta(n;N)=0.$$

Case 3: n is odd and $2N+1-n \geq 0$. Then

$$\beta(n,N) = \left((-1)^{m+1}2(m-1)!\,(N-m)!\,\frac{(2N-1)!}{(N-1)!}\right)^{-1}$$

and $\alpha(n,N)$ is a constant which we don't need to precise.

Recall that the constants the constants $A_0, A_1, ..., A_{2N}$ are chosen so that the function (see (30))

$$\varphi(x) = E(x)\,\chi_{\mathbb{R}^n\setminus\overline{B}}(x) + (A_0 + A_1\,|x|^2 + ... + A_{2N}\,|x|^{4N})\,\chi_B(x)\,,$$

and all its partial derivatives of order not greater than $2N$ extend continuously up to ∂B.

Lemma 9. *For $L = N+1,\dots,2N$ we have*

$$A_L = c_n\,\frac{(-1)^{L+N}\binom{L+m-N-1}{L-N}\binom{N+m}{2N-L}}{(2N)!\binom{L}{N}}\,.$$

Proof. Let $m=(n-1)/2$ and set $t=|x|^2$, so that

$$E_N^n(x) \equiv E(t) = t^{N-m}(\alpha + \beta\log(t))\,. \tag{71}$$

Let $P(t)$ be the polynomial $\sum_{L=0}^{2N} A_L t^L$. By Corollary 2 in Section 2 we need that

$$P^{k)}(1) = E^{k)}(1)\,,\quad 0\leq k\leq 2N\,.$$

By Taylor's expansion we have that $P(t) = \sum_{i=0}^{2N} \frac{E^{i)}(1)}{i!}(t-1)^i$, and hence, by the binomial formula applied to $(t-1)^i$,

$$A_L = \sum_{i=L}^{2N} \frac{E^{i)}(1)}{i!}(-1)^{i-L}\binom{i}{L}\,, \quad 0 \le L \le 2N\,.$$

Now we want to compute $E^{i)}(1)$. Clearly

$$\left(\frac{d}{dt}\right)^i (t^{N-m}) = (N-m)\cdots(N-m-i+1)t^{N-m-i}$$

and it is zero when m is integer and $i > N-m$. Notice that the logarithmic term in (71) only appears when the dimension n is odd (then m is integer) and $N \ge m$. In this case, for each $i \ge N+1$

$$\left(\frac{d}{dt}\right)^i (t^{N-m}\log t) = (N-m)!(-1)^{i-N+m-1}(i-N+m-1)!\, t^{-i+N-m}\,.$$

Hence, for $i \ge N+1$, we obtain

$$\begin{aligned}\frac{E^{i)}(1)}{c_n} &= \alpha(n,N)(N-m)\cdots(n-m-i+1)\\ &\qquad + \beta(n,N)(N-m)!(-1)^{i-N+m-1}(i-N+m-1)!\,.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{A_L}{c_n} &= (-1)^L\alpha(n,N)\sum_{i=L}^{2N}(N-m)\cdots(N-m-i+1)\frac{(-1)^i}{i!}\binom{i}{L}\\ &\quad + (-1)^{L-N+m-1}\beta(n,N)(N-m)!\sum_{i=L}^{2N}(i-N+m-1)!\frac{\binom{i}{L}}{i!}.\end{aligned} \tag{72}$$

Let's remark that for the case n odd and $N \ge m$ the first term in (72) is zero, while for the cases n even or n odd and $N < m$ the second term is zero because $\beta(n,N) = 0$. This explains why we compute below the two terms separately.

For the first term we show that

$$\sum_{i=L}^{2N}(N-m)\cdots(N-m-i+1)\frac{(-1)^i}{i!}\binom{i}{L} = (-1)^L\binom{N-m}{L}\binom{m+N}{2N-L}. \tag{73}$$

Indeed, the left hand side of (73) is, setting $k = i - L$,

$$\frac{1}{L!}\sum_{k=0}^{2N-L}(N-m)\cdots(N-m-L-k+1)\frac{(-1)^{L+k}}{k!} =$$

$$= (-1)^L\binom{N-m}{L}\sum_{k=0}^{2N-L}\binom{m+L-N+k-1}{k}$$

$$= (-1)^L\binom{N-m}{L}\binom{N+m}{2N-L},$$

where the last identity comes from ([GKP, (5.9), p. 159]) .

To compute the second term we first show that

$$\sum_{i=L}^{2N}(i-N+m-1)!\frac{1}{i!}\binom{i}{L} = \frac{(L-N+m-1)!}{L!}\binom{N+m}{2N-L}. \tag{74}$$

As before, setting $k = i - L$ and applying [GKP, (5.9), p. 159], we see that the left hand side of (74) is

$$\frac{1}{L!}\sum_{k=0}^{2N-L}(L+k-N+m-1)!\frac{1}{k!} =$$

$$= (L-N+m-1)!\sum_{k=0}^{2N-L}\binom{m+L-N+k-1}{k}$$

$$= \frac{(L-N+m-1)!}{L!}\binom{N+m}{2N-L}.$$

We are now ready to complete the proof of the lemma distinguishing 3 cases.

Case 1: n even.

Since $\beta(n, N) = 0$, replacing in (72) $\alpha(n, N)$ by its value and using (73) we get, by elementary arithmetics,

$$\frac{A_L}{c_n} = (-1)^L\frac{(-1)^L\binom{N-m}{L}\binom{N+m}{2N-L}}{2N\binom{N-m}{N}(2N-1)!} = (-1)^{L+N}\frac{\binom{L+m-N-1}{L-N}\binom{N+m}{2N-L}}{(2N)!\binom{L}{N}}.$$

Case 2: n is odd and $2N + 1 - n < 0$.

As in case 1 $\beta(n, N) = 0$, and we proceed similarly using (73) to obtain

$$\frac{A_L}{c_n} = \binom{N-m}{L}\binom{N+m}{2N-L}\frac{(-1)^N(m-N-1)!(N-1)!}{2(m-1)!(2N-1)!}$$

$$= (-1)^{L+N}\frac{\binom{L+m-N-1}{L-N}\binom{N+m}{2N-L}}{(2N)!\binom{L}{N}}.$$

Case 3: n is odd and $2N+1-n \geq 0$.

Replacing in (72) $\alpha(n,N)$ and $\beta(n,N)$ by their values and using (74) we get, by elementary arithmetics,

$$\begin{aligned}\frac{A_L}{c_n} =&(-1)^{L-N+m-1}\beta(n,N)(N-m)!\frac{(L-N+m-1)!}{L!}\binom{N+m}{2N-L}\\ =&\frac{(-1)^{N+L}(N-1)!(L-N+m-1)!}{2(m-1)!(2N-1)!L!}\binom{N+m}{2N-L}\\ =&(-1)^{L+N}\frac{\binom{L+m-N-1}{L-N}\binom{N+m}{2N-L}}{(2N)!\binom{L}{N}}\,.\end{aligned}$$

□

Proof of Lemma 6. Recall that (see (68))

$$C_{2j}=\sum_{L=N+1+j}^{2N} c_{L,\,j,\,L-N-j-1}\, G_{\frac{n}{2}+L-N+j}(0)\,.$$

Thus, we have to compute the constants $c_{L,j,k}$ appearing in the expression (66) for $\widehat{S\,\chi_B}(\xi)$. For that we need the constants $c_{L,j}$ appearing in the formula (42) for $S(x)$. We start by computing $P_{2j+1}(\partial)\Delta^{N-j}(|x|^{2L})$. Using (40) and Lemma 4 one gets

$$P_{2j+1}(\partial)\Delta^{N-j}(|x|^{2L})=\frac{2^{2N+1}L!(N-j)!}{(L-N-j-1)!}\binom{L-1+\frac{n}{2}}{N-j}P_{2j+1}(x)|x|^{2(L-N-j-1)}$$

if $L-N-j-1\geq 0$ (and $=0$ if $L-N-j-1<0$).

As in (45) (Section 3), we express $P_{2j+1}(\partial)\Delta^{L-N-j-1}G_{\frac{n}{2}}(\xi)$ using Lemma 3 applied to $f(r)=G_{\frac{n}{2}}(r)$ and the homogeneous polynomial $L(x)=P_{2j+1}(x)\,|x|^{2(L-N-j-1)}$. We obtain

$$\begin{aligned}&P_{2j+1}(\partial)\Delta^{L-N-j-1}G_{\frac{n}{2}}(\xi)=\\ &=\sum_{k\geq 0}\frac{1}{2^k k!}\Delta^k(P_{2j+1}(x)|x|^{2(L-N-j-1)})\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{2(L-N)-1-k}G_{\frac{n}{2}}(\xi)\\ &=\sum_{k\geq 0}\frac{(-1)^{k+1}}{2^k k!}\Delta^k(P_{2j+1}(x)|x|^{2(L-N-j-1)})\,G_{\frac{n}{2}+2(L-N)-1-k}(\xi)\\ &=\sum_{k=0}^{L-N-j-1}\frac{(-1)^{k+1}}{2^k k!}4^k\frac{(L-N-j-1)!}{(L-N-j-1-k)!}k!\binom{\frac{n}{2}+j+L-N-1}{k}\\ &\qquad\times P_{2j+1}(\xi)|\xi|^{2(L-N-j-1-k)}\,G_{\frac{n}{2}+2(L-N)-1-k}(\xi)\,.\end{aligned}$$

In view of the definitions of $Q(x)$ and $S(x)$,

$$
\begin{aligned}
&S(x) = \\
&= -Q(\partial)\left(\sum_{L=0}^{2N} A_L |x|^{2L}\right) = -\sum_{L=0}^{2N} A_L \sum_{j=0}^{N} \gamma_{2j+1} P_{2j+1}(\partial) \Delta^{N-j}(|x|^{2L}) \\
&= -\sum_{L=N+1}^{2N} \sum_{j=0}^{L-N-1} A_L \gamma_{2j+1} \frac{2^{2N+1} L!(N-j)!}{(L-N-j-1)!} \binom{L-1+\frac{n}{2}}{N-j} P_{2j+1}(x)|x|^{2(L-N-j-1)} \\
&= \sum_{L=N+1}^{2N} \sum_{j=0}^{L-N-1} c_{L,j} P_{2j+1}(x)|x|^{2(L-N-j-1)} \,,
\end{aligned}
$$

where the last identity defines the $c_{L,j}$. In Section 5 (66) we set

$$
\begin{aligned}
&\widehat{S\,\chi_B}(\xi) = \\
&= S(\imath\,\partial)\,\widehat{\chi_B}(\xi) \\
&= \imath(2\pi)^{n/2} \sum_{L=N+1}^{2N} \sum_{j=0}^{L-N-1} c_{L,j}\,(-1)^{L-N}\; P_{2j+1}(\partial)\; \triangle^{L-N-j-1}\, G_{\frac{n}{2}}(\xi) \\
&= (2\pi)^{n/2} \sum_{L=N+1}^{2N} \sum_{j=0}^{L-N-1} \sum_{k=0}^{L-N-j-1} c_{L,\,j,\,k}\, P_{2j+1}(\xi)\,|\xi|^{2(L-N-j-1-k)} G_{\frac{n}{2}+2(L-N)-1-k}(\xi)\,.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
c_{L,j,k} &= \imath c_{L,j}(-1)^{L-N} \frac{(-1)^{k+1}}{2^k} 4^k \frac{(L-N-j-1)!}{(L-N-j-1-k)!} \binom{\frac{n}{2}+j+L-N-1}{k} \\
&= \imath(-1)^{L+k+N} A_L \gamma_{2j+1} \frac{2^{2N+1} L!(N-j)!}{(L-N-j-1-k)!} \binom{L-1+\frac{n}{2}}{N-j} \\
&\qquad\qquad 2^k \binom{\frac{n}{2}+j+L-N-1}{k}\,.
\end{aligned}
$$

Replacing A_L by the formula given in lemma 9 and performing some easy arithmetics we get

(75)

$$
\begin{aligned}
c_{L,j,k} &= \imath(-1)^k c_n \gamma_{2j+1} \frac{2^{2N+1} L!(N-j)!\binom{L-1+\frac{n}{2}}{N-j} 2^k \binom{\frac{n}{2}+j+L-N-1}{k}}{(L-N-j-1-k)!} \frac{\binom{L+m-N-1}{L-N}\binom{N+m}{2N-L}}{(2N)!\binom{L}{N}} \\
&= \imath(-1)^k c_n \gamma_{2j+1} \frac{2^k (N-j)!(n-1)\binom{L-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+j+L-N-1}{k}\binom{N+\frac{n}{2}-\frac{1}{2}}{N}}{(2N-L)!(L-N+\frac{n}{2}-\frac{1}{2})(L-N-j-1-k)!\binom{N-\frac{1}{2}}{N}}\,.
\end{aligned}
$$

The final computation of the C_{2j+1} is as follows.

$$
\begin{aligned}
C_{2j+1} &= \\
&= \sum_{L=N+1+j}^{2N} c_{L,\,j,\,L-N-j-1}\, G_{\frac{n}{2}+L-N+j}(0) \\
&= \qquad [\text{by the explicit value of } G_p(0) \text{ given in (77) below}] \\
&= \sum_{L=N+1+j}^{2N} c_{L,\,j,\,L-N-j-1} \frac{1}{2^{\frac{n}{2}+L-N+j}\Gamma(\frac{n}{2}+L-N+j+1)} \\
&= \qquad [\text{by (75)}] \\
&= -\imath c_n \sum_{L=N+1+j}^{2N} \frac{(-1)^{L-N-j}\gamma_{2j+1}2^{L-N-j-1}(N-j)!(n-1)}{(L-N+\frac{n}{2}-\frac{1}{2})(2N-L)!\binom{N-1/2}{N}} \\
&\qquad\qquad \frac{\binom{L-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+j+L-N-1}{L-N-j-1}\binom{N+\frac{n}{2}-1/2}{N}}{2^{\frac{n}{2}+L-N+j}\Gamma(\frac{n}{2}+L-N+j+1)} \\
&= -\frac{\imath c_n\gamma_{2j+1}(N-j)!(n-1)\binom{N+\frac{n}{2}-1/2}{N}}{2^{\frac{n}{2}+2j+1}\binom{N-1/2}{N}} \\
&\qquad \sum_{L=N+1+j}^{2N} \frac{(-1)^{L+N+j}\binom{L-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+j+L-N-1}{L-N-j-1}}{(L-N+\frac{n}{2}-\frac{1}{2})(2N-L)!\Gamma(\frac{n}{2}+L-N+j+1)} \\
&= \qquad [\text{setting} \quad L=i+N+j+1] \\
&= \frac{\imath\, c_n\gamma_{2j+1}(N-j)!(n-1)\binom{N+\frac{n}{2}-1/2}{N}}{2^{\frac{n}{2}+2j+1}\binom{N-1/2}{N}} \\
&\qquad \sum_{i=0}^{N-j-1} \frac{(-1)^i\binom{N+i+j+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+2j+i}{i}}{(i+j+\frac{n}{2}+\frac{1}{2})(N-j-i-1)!\Gamma(\frac{n}{2}+i+2j+2)} \\
&= \qquad [\text{because } \Gamma(\frac{n}{2}+i+2j+2)=\Gamma(\frac{n}{2}+2j)\prod_{k=0}^{i+1}(\frac{n}{2}+2j+k)] \\
&= \frac{\imath\, c_n\gamma_{2j+1}(N-j)!(n-1)\binom{N+\frac{n}{2}-1/2}{N}}{2^{\frac{n}{2}+2j+1}\binom{N-1/2}{N}\Gamma(\frac{n}{2}+2j)} \\
&\qquad \sum_{i=0}^{N-j-1} \frac{(-1)^i\binom{N+i+j+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+2j+i}{i}}{(i+j+\frac{n}{2}+\frac{1}{2})(N-j-i-1)!\prod_{k=0}^{i+1}(\frac{n}{2}+2j+k)}
\end{aligned}
$$

$$= \qquad \text{[using Lemma 10 below]}$$

$$= \frac{\imath\, c_n \gamma_{2j+1} (N-j)!(n-1)\binom{N+\frac{n}{2}-1/2}{N}}{2^{\frac{n}{2}+2j+1}\binom{N-1/2}{N}\Gamma(\frac{n}{2}+2j)} \frac{2\binom{N+1/2}{N-j}}{(2N+1)(2j+\frac{n}{2})} \frac{\Gamma(\frac{n}{2}+j+1/2)}{\Gamma(\frac{n}{2}+N+1/2)}$$

$$= \qquad \text{[substituting the value given in (15) in } \gamma_{2j+1}\text{]}$$

$$= c_n \left(\frac{\pi}{2}\right)^{\frac{n}{2}} \frac{(n-1)\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}+\frac{1}{2})} \frac{(-1)^j}{4^j(2j+1)\Gamma(\frac{n}{2}+2j+1)}$$

$$= \qquad \text{[recalling the exact value of } c_n\text{]}$$

$$= \frac{1}{2^{\frac{n}{2}}} \frac{(-1)^j}{4^j(2j+1)\Gamma(\frac{n}{2}+2j+1)}. \qquad \square$$

Lemma 10. *For each $j = 0, \dots, N-1$*

$$\sum_{i=0}^{N-j-1} \frac{(-1)^i \binom{N+i+j+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+2j+i}{i}}{(i+j+\frac{n}{2}+\frac{1}{2})(N-j-i-1)!\prod_{k=0}^{i+1}(\frac{n}{2}+2j+k)} = \frac{2\binom{N+1/2}{N-j}}{(2N+1)(2j+\frac{n}{2})} \frac{\Gamma(\frac{n}{2}+j+\frac{1}{2})}{\Gamma(\frac{n}{2}+N+\frac{1}{2})}.$$

Proof. Denote the left hand side by S. Using the identity $\Gamma(A) = \Gamma(A-k)\binom{A-1}{k}k!$, for any non-negative integer k, and elementary arithmetics one gets

$$\frac{\Gamma(\frac{n}{2}+N+\frac{1}{2})\binom{N+i+j+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+2j+i}{i}}{\Gamma(\frac{n}{2}+j+\frac{1}{2})(i+j+\frac{n}{2}+\frac{1}{2})\prod_{k=0}^{i+1}(\frac{n}{2}+2j+k)}$$
$$= \binom{N+i+j+\frac{n}{2}}{N-j-1}\binom{N+\frac{n}{2}-1}{N-i-j-1}\binom{\frac{n}{2}+i+j-1}{i}\frac{(N-i-j-1)!}{(2j+\frac{n}{2})(N-j)},$$

and so

$$S = \frac{\Gamma(\frac{n}{2}+j+\frac{1}{2})}{\Gamma(\frac{n}{2}+N+\frac{1}{2})(2j+\frac{n}{2})(N-j)}$$
$$\sum_{i=0}^{N-1-j}(-1)^i\binom{N+i+j+\frac{n}{2}}{N-j-1}\binom{N+\frac{n}{2}-1}{N-i-j-1}\binom{\frac{n}{2}+i+j-\frac{1}{2}}{i}$$

$$= \qquad \left[\text{because } \binom{a+i}{i} = (-1)^i \binom{-a-1}{i}\right]$$

$$= \frac{\Gamma(\frac{n}{2}+j+\frac{1}{2})}{\Gamma(\frac{n}{2}+N+\frac{1}{2})(2j+\frac{n}{2})(N-j)}$$

$$\sum_{i=0}^{N-1-j} (-1)^i \binom{N+i+j+\frac{n}{2}}{N-j-1} \binom{N+\frac{n}{2}-1}{N-i-j-1} \binom{-\frac{n}{2}-j-\frac{1}{2}}{i}$$

= [by the triple-binomial identity (5.28) of ([GKP], p. 171), see (76) below]

$$= \frac{\Gamma(\frac{n}{2}+j+\frac{1}{2})}{\Gamma(\frac{n}{2}+N+\frac{1}{2})(2j+\frac{n}{2})(N-j)} \binom{N+\frac{n}{2}+j}{0} \binom{N-\frac{1}{2}}{N-j-1}$$

$$= \frac{\Gamma(\frac{n}{2}+j+\frac{1}{2})}{\Gamma(\frac{n}{2}+N+\frac{1}{2})(2j+\frac{n}{2})} \binom{N+\frac{1}{2}}{N-j} \frac{2}{2N+1}\,.$$

For the reader's convenience and later reference we state the triple-binomial identity (5.28) of [GKP] :

(76)

$$\sum_{k=0}^{n} \binom{m-r+s}{k} \binom{n+r-s}{n-k} \binom{r+k}{m+n} = \binom{r}{m} \binom{s}{n} \qquad m, n \geq 0 \text{ integers.} \qquad \square$$

Our next task is to prove Lemma 7 and Lemma 8. Setting $\xi = r\,\xi_0$ in (66) we obtain

$$\frac{\widehat{S_N\,\chi_B}(r\xi_0)}{(2\pi)^{n/2}} =$$

$$= \sum_{L=N+1}^{2N} \sum_{j=0}^{L-N-1} \sum_{k=0}^{L-N-j-1} c_{L,\,j,\,k}\; P_{2j+1}(r\xi_0)\; |r\xi_0|^{2(L-N-j-1-k)} G_{\frac{n}{2}+2(L-N)-1-k}(r\xi_0)$$

[make the change of indexes $L = N + s$ and $|\xi_0| = 1$]

$$= \sum_{s=1}^{N} \sum_{j=0}^{s-1} \sum_{k=0}^{s-j-1} c_{N+s,\,j,\,k}\; P_{2j+1}(\xi_0)\; r^{2(s-k)-1} G_{\frac{n}{2}+2s-1-k}(r)$$

$$= \sum_{j=0}^{N-1} \sum_{s=j+1}^{N} \sum_{k=0}^{s-j-1} c_{N+s,\,j,\,k}\; P_{2j+1}(\xi_0)\; r^{2(s-k)-1} G_{\frac{n}{2}+2s-1-k}(r)$$

$$:= \sum_{p=0}^{\infty} a^N_{2p+1}(\xi_0) r^{2p+1}.$$

In order to compute the coefficients $a^N_{2p+1}(\xi_0)$ we substitute the power series expansion of $G_q(r)$ [GrCF, B.2], namely,

$$G_q(r) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,\Gamma(q+i+1)} \frac{r^{2i}}{2^{2i+q}}\,, \tag{77}$$

in the last triple sum above.

Proof of Lemma 7. We are assuming that $0 \le p \le N-1$. It is crucial to remark that, for this range of p, after introducing (77) in the triple sum above, only the values of the index j satisfying $0 \le j \le p$ are involved in the expression for a^N_{2p+1}. Once (77) has been introduced in the triple sum one should sum, in principle, on the four indexes i, j, s and k. But since we are looking at the coefficient of r^{2p+1} we have the relation $2(s-k)-1+2i = 2p+1$, which actually leaves us with three indexes. The range of each of these indexes is easy to determine and one gets

$$a^N_{2p+1} = \sum_{j=0}^{p} P_{2j+1}(\xi_0) \sum_{i=0}^{p-j} \sum_{s=p-i+1}^{N} c_{N+s,\,j,\,s-(p-i)-1}$$
$$\times \text{ coefficient of } r^{2i} \text{ from } G_{\frac{n}{2}+s+p-i}(r)\,.$$

In view of (77)

$$\begin{aligned}
a^N_{2p+1} &= \\
&= \sum_{j=0}^{p} P_{2j+1}(\xi_0) \sum_{i=0}^{p-j} \sum_{s=p-i+1}^{N} c_{N+s,\,j,\,s-(p-i)-1} \frac{(-1)^i}{i! 2^{i+\frac{n}{2}+s+p}\Gamma(\frac{n}{2}+s+p+1)} \\
&= \qquad [\text{by the expression (75) for } c_{L,j,k}\,] \\
&= \imath\, c_n \sum_{j=0}^{p} P_{2j+1}(\xi_0) \sum_{i=0}^{p-j} \sum_{s=p-i+1}^{N} \frac{(-1)^i(-1)^{s-(p-i)-1}\gamma_{2j+1}}{i! 2^{i+\frac{n}{2}+s+p}\Gamma(\frac{n}{2}+s+p+1)} \\
&\qquad \frac{2^{s-(p-i)-1}(N-j)!(n-1)\binom{N+s-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+j+s-1}{s-(p-i)-1}\binom{N+\frac{n}{2}-\frac{1}{2}}{N}}{(s+\frac{n}{2}-\frac{1}{2})(N-s)!(p-i-j)!\binom{N-\frac{1}{2}}{N}}
\end{aligned}$$

$$= (-1)^{p+1} c_n \frac{\binom{N+\frac{n}{2}-\frac{1}{2}}{N}}{\binom{N-\frac{1}{2}}{N}} \sum_{j=0}^{p} P_{2j+1}(\xi_0) \frac{(-1)^j \pi^{\frac{n}{2}} \Gamma(j+\frac{1}{2})(N-j)!}{\Gamma(\frac{n}{2}+j+\frac{1}{2}) 2^{\frac{n}{2}+2p+1}} \sum_{i=0}^{p-j} \frac{1}{i!(p-i-j)!}$$

$$\sum_{s=p-i+1}^{N} \frac{(-1)^s \binom{N+s-1+\frac{n}{2}}{N-j} \binom{\frac{n}{2}+j+s-1}{s-(p-i)-1}}{(s+\frac{n}{2}-\frac{1}{2})(N-s)!\Gamma(\frac{n}{2}+s+p+1)}$$

In Lemma 11 below we give a useful compact form for the last sum. Using it we obtain

$$\frac{a_{2p+1}^N}{c_n} =$$

$$(-1)^{p+1} \frac{\binom{N+\frac{n}{2}-\frac{1}{2}}{N}}{\binom{N-\frac{1}{2}}{N}} \sum_{j=0}^{p} P_{2j+1}(\xi_0) \frac{(-1)^j \pi^{\frac{n}{2}} \Gamma(j+\frac{1}{2})(N-j)!}{\Gamma(\frac{n}{2}+j+\frac{1}{2}) 2^{\frac{n}{2}+2p+1}} \sum_{i=0}^{p-j} \frac{1}{i!(p-i-j)!}$$

$$\frac{(-1)^{p+1-i}(N-p-1)!(p+1-j)!\Gamma(p+\frac{n}{2}-i+\frac{1}{2})}{(N-j)!\Gamma(N+\frac{n}{2}+\frac{1}{2})\Gamma(\frac{n}{2}+2p+2-i)}$$

$$\binom{N-\frac{1}{2}}{N-p-1} \binom{\frac{n}{2}+2p-i+1}{p+1-j}.$$

Easy arithmetics with binomial coefficients gives

$$\frac{\binom{N+\frac{n}{2}-\frac{1}{2}}{N}(N-p-1)!\binom{N-\frac{1}{2}}{N-p-1}}{\binom{N-\frac{1}{2}}{N}\Gamma(N+\frac{n}{2}+\frac{1}{2})} = \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(p+\frac{3}{2})}$$

We finally get the extremely surprising identity

$$\frac{a_{2p+1}^N}{c_n} = \frac{\Gamma(\frac{1}{2})(\frac{\pi}{2})^{\frac{n}{2}}}{2^{2p+1}\Gamma(\frac{n}{2}+\frac{1}{2})\Gamma(p+\frac{3}{2})} \sum_{j=0}^{p} \frac{(-1)^j \Gamma(j+\frac{1}{2}) P_{2j+1}(\xi_0)}{\Gamma(\frac{n}{2}+j+\frac{1}{2})} \sum_{i=0}^{p-j} \frac{(-1)^i \Gamma(\frac{n}{2}+p-i+\frac{1}{2})}{i!(p-i-j)!\Gamma(\frac{n}{2}+p-i+j+1)}, \tag{78}$$

in which N has miraculously disappeared. Thus Lemma 7 is proved. □

Proof of Lemma 8. We start by proving the inequality (69), so that $0 \leq p \leq N-1$. We roughly estimate $a_{2p+1} = a_{2p+1}^N$ by putting the absolute value inside the sums in (78). The absolute value of each term in the innermost sum in (78) is obviously not greater than 1 and there are at most $p+1$ terms. The factor in front of $P_{2j+1}(\xi_0)$ is again not greater than 1 in absolute value. Denoting by C the terms that depend only on n we obtain the desired inequality (69).

We turn now to the proof of inequality (70). Recall that

$$\begin{aligned}\frac{\widehat{S_N\,\chi_B}(r\xi_0)}{(2\pi)^{n/2}} &= \sum_{p=0}^{\infty} a^N_{2p+1}(\xi_0) r^{2p+1} \\ &= \sum_{j=0}^{N-1}\sum_{s=j+1}^{N}\sum_{k=0}^{s-j-1} c_{N+s,\,j,\,k}\; P_{2j+1}(\xi_0)\; r^{2(s-k)-1} G_{\frac{n}{2}+2s-1-k}(r)\,.\end{aligned}$$

Replacing $G_{\frac{n}{2}+2s-1-k}(r)$ by the expression given by (77) we obtain, as before, a sum with four indexes. Now we eliminate the index i of (77) using $s-k+i = p+1$. Hence

$$\begin{aligned} a^N_{2p+1} &= \\ &= \sum_{j=0}^{N-1} P_{2j+1}(\xi_0) \sum_{s=j+1}^{N}\sum_{k=0}^{s-j-1} c_{N+s,\,j,\,k} \times \text{coefficient of } r^{2(p+1-s+k)} \text{ from } G_{\frac{n}{2}+2s-1-k}(r) \\ &= \sum_{j=0}^{N-1} P_{2j+1}(\xi_0) \sum_{s=j+1}^{N}\sum_{k=0}^{s-j-1} c_{N+s,\,j,\,k} \frac{(-1)^{p+1-s+k}}{(p+1-s+k)!\Gamma(\frac{n}{2}+p+s+1)2^{2p+1+\frac{n}{2}+k}} \\ &= \sum_{j=0}^{N-1} P_{2j+1}(\xi_0) \sum_{k=0}^{N-1-j}\sum_{s=j+k+1}^{N} c_{N+s,\,j,\,k} \frac{(-1)^{p+1-s+k}}{(p+1-s+k)!\Gamma(\frac{n}{2}+p+s+1)2^{2p+1+\frac{n}{2}+k}} \end{aligned}$$

$$= c_n(-1)^{p+1}\frac{\binom{N+\frac{n}{2}-\frac{1}{2}}{N}\pi^{\frac{n}{2}}(n-1)}{4^p\binom{N-\frac{1}{2}}{N}\,2^{\frac{n}{2}+1}}\sum_{j=0}^{N-1}\frac{(-1)^j\Gamma(j+\frac{1}{2})}{\Gamma(\frac{n}{2}+j+\frac{1}{2})}P_{2j+1}(\xi_0)\sum_{k=0}^{N-1-j}\sum_{s=j+k+1}^{N}$$
$$\frac{(-1)^s(N-j)!\binom{N+s-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+j+s-1}{k}}{(p+1-s+k)!\Gamma(\frac{n}{2}+p+s+1)(s+\frac{n}{2}-\frac{1}{2})(N-s)!(s-j-1-k)!}\,.$$

The second identity is just (77). The third is a change of the order of summation and the latest follows from the formula (75) for the constants $c_{l,j,k}$ and some simplifications.

In view of the elementary fact that

$$(N-j)!\binom{N+s-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+j+s-1}{k} = \frac{\Gamma(s+\frac{n}{2}+N)}{k!\Gamma(s+\frac{n}{2}+j-k)}$$

we get

$$\left|\sum_{k=0}^{N-1-j}\sum_{s=j+k+1}^{N}\frac{(-1)^s(N-j)!\binom{N+s-1+\frac{n}{2}}{N-j}\binom{\frac{n}{2}+j+s-1}{k}}{(p+1-s+k)!\,\Gamma(\frac{n}{2}+p+s+1)(s+\frac{n}{2}-\frac{1}{2})(N-s)!(s-j-1-k)!}\right|\le$$

$$\le\sum_{k=0}^{N-1-j}\frac{1}{k!}\sum_{s=j+k+1}^{N}\frac{1}{\Gamma(s+\frac{n}{2}+j-k)(p+1-s+k)!}$$

$$\frac{1}{(\frac{n}{2}+p+s)(s+\frac{n}{2}-\frac{1}{2})(N-s)!(s-j-1-k)!}$$

$$\le\sum_{k=0}^{N-1-j}\frac{1}{k!}\sum_{s=j+k+1}^{N}\frac{1}{(s-j-1-k)!}\le e^2\,,$$

where in the first inequality we used that, since $N\le p$,

$$\frac{\Gamma(s+\frac{n}{2}+N)}{\Gamma(\frac{n}{2}+p+s+1)}\le\frac{1}{\frac{n}{2}+p+s}\,.$$

The proof of (70) is complete. □

Lemma 11. *Let $N-1\ge p\ge j+i\ge 0$ be non-negative integers and set $m=p+1-i$. Then*

$$\sum_{s=0}^{N-m}\frac{(-1)^s\binom{\frac{n}{2}+N+m+s-1}{N-j}\binom{\frac{n}{2}+j+m+s-1}{s}}{(m+s+\frac{n}{2}-\frac{1}{2})(N-m-s)!\Gamma(\frac{n}{2}+2m+i+s)}=$$

$$=\frac{(N-m-i)!\,(m+i-j)!\,\Gamma(m+\frac{n}{2}-\frac{1}{2})}{(N-j)!\,\Gamma(\frac{n}{2}+2m+i)\,\Gamma(N+\frac{n}{2}+\frac{1}{2})}\binom{N-\frac{1}{2}}{N-m-i}\binom{\frac{n}{2}+2m+i-1}{m+i-j}.$$

Proof. Denote the left hand side by S. Using repeatedly the identity $\Gamma(x+1)=x\,\Gamma(x)$ and arithmetics with binomial coefficients we have

$$\frac{\binom{\frac{n}{2}+N+m+s-1}{N-j}\binom{\frac{n}{2}+j+m+s-1}{s}}{\Gamma(\frac{n}{2}+2m+i+s)}=$$

$$=\frac{(N-m-i)!\,(m+i-j)!}{(N-j)!\,s!\,\Gamma(\frac{n}{2}+2m+i)}\binom{\frac{n}{2}+N+m+s-1}{N-m-i}\binom{\frac{n}{2}+2m+i-1}{m+i-j}.$$

Then

$$
\begin{aligned}
S = {} & \frac{(N-m-i)!\,(m+i-j)!}{(N-j)!\,\Gamma(\frac{n}{2}+2m+i)} \binom{\frac{n}{2}+2m+i-1}{m+i-j} \\
& \qquad\qquad \sum_{s=0}^{N-m} \frac{(-1)^s \binom{\frac{n}{2}+N+m+s-1}{N-m-i}}{s!(N-m-s)!(m+s+\frac{n}{2}-\frac{1}{2})} \\
= {} & \frac{(N-m-i)!\,(m+i-j)!}{(N-j)!\,\Gamma(\frac{n}{2}+2m+i)} \binom{\frac{n}{2}+2m+i-1}{m+i-j} D(m,i)\ ,
\end{aligned}
$$

where the last identity defines $D(m,i)$. The only task left is the computation of the sum $D(m,i)$. The identity

$$
\frac{1}{m+s+\frac{n}{2}-\frac{1}{2}} = \frac{1}{m+\frac{n}{2}-\frac{1}{2}} \left(1 - \frac{s}{m+s+\frac{n}{2}-\frac{1}{2}}\right),
$$

yields the expression

$$
\begin{aligned}
D(m,i) = {} & \frac{1}{(m+\frac{n}{2}-\frac{1}{2})(N-m)!} \sum_{s=0}^{N-m} (-1)^s \binom{N-m}{s} \binom{\frac{n}{2}+N+m+s-1}{N-m-i} \\
& - \sum_{s=1}^{N-m} \frac{(-1)^s \binom{\frac{n}{2}+N+m+s-1}{N-m-i}}{s!(N-m-s)!} \frac{s}{(m+\frac{n}{2}-\frac{1}{2})(m+s+\frac{n}{2}-\frac{1}{2})}\ .
\end{aligned}
$$

The first sum in the above expression for $D(m,i)$ turns out to vanish for $i \geq 1$. This is because

$$
\begin{aligned}
& \sum_{s=0}^{N-m} (-1)^s \binom{N-m}{s} \binom{\frac{n}{2}+N+m+s-1}{N-m-i} \\
& \qquad\qquad = (-1)^{N-m} \binom{N+m+\frac{n}{2}-1}{-i} = 0\ ,
\end{aligned}
$$

where the first identity follows from [GKP, (5.24), p.169] and the second from the fact that $\binom{m}{n} = 0$ if n is a negative integer. Hence, setting $r = s-1$,

$$D(m,i) = -\sum_{s=1}^{N-m} \frac{(-1)^s \binom{\frac{n}{2}+N+m+s-1}{N-m-i}}{s!(N-m-s)!} \frac{s}{(m+\frac{n}{2}-\frac{1}{2})(m+s+\frac{n}{2}-\frac{1}{2})}$$

$$= \frac{1}{(m+\frac{n}{2}-\frac{1}{2})} \sum_{r=0}^{N-(m+1)} \frac{(-1)^r \binom{\frac{n}{2}+N+(m+1)+r-1}{N-(m+1)-(i-1)}}{r!(N-(m+1)-r)!((m+1)+r+\frac{n}{2}-\frac{1}{2})}$$

$$= \frac{1}{(m+\frac{n}{2}-\frac{1}{2})} D(m+1,i-1)\ .$$

Repeating the above argument i times we obtain that

$$D(m,i) = \frac{1}{(m+\frac{n}{2}-\frac{1}{2})(m+\frac{n}{2}+\frac{1}{2})\cdots(m+\frac{n}{2}+i-\frac{3}{2})} D(m+i,0).$$

To compute $D(m+i,0)$ or $D(p+1,0)$ we use the elementary identity

$$\frac{\Gamma(N+\frac{n}{2}+\frac{1}{2})}{\Gamma(p+\frac{n}{2}+\frac{1}{2})s!(N-p-1-s)!(p+s+\frac{n}{2}+\frac{1}{2})} =$$

$$= \binom{N+\frac{n}{2}-\frac{1}{2}}{N-p-1-s}\binom{p+s+\frac{n}{2}-\frac{1}{2}}{s},$$

from which we get

$$D(p+1,0) =$$

$$= \sum_{s=0}^{N-p-1} \frac{(-1)^s \binom{\frac{n}{2}+N+p+s}{N-p-1}}{s!(N-p-1-s)!(p+s+\frac{n}{2}+\frac{1}{2})}$$

$$= \frac{\Gamma(p+\frac{n}{2}+\frac{1}{2})}{\Gamma(N+\frac{n}{2}+\frac{1}{2})} \sum_{s=0}^{N-p-1} (-1)^s \binom{\frac{n}{2}+N+p+s}{N-p-1}\binom{N+\frac{n}{2}-\frac{1}{2}}{N-p-1-s}\binom{p+s+\frac{n}{2}-\frac{1}{2}}{s}$$

$$= \frac{\Gamma(p+\frac{n}{2}+\frac{1}{2})}{\Gamma(N+\frac{n}{2}+\frac{1}{2})} \sum_{s=0}^{N-p-1} \binom{\frac{n}{2}+N+p+s}{N-p-1}\binom{N+\frac{n}{2}-\frac{1}{2}}{N-p-1-s}\binom{-p-\frac{n}{2}-\frac{1}{2}}{s}$$

$$= \frac{\Gamma(p+\frac{n}{2}+\frac{1}{2})}{\Gamma(N+\frac{n}{2}+\frac{1}{2})}\binom{N-\frac{1}{2}}{N-p-1},$$

where in the third identity we applied [GKP, (5.14), p. 164] and the latest equality is consequence of the triple-binomial identity (76) [GKP, (5.28), p.171] (for $k=s$, $n=N-p-1$, $m=0$, $r=N+p+\frac{n}{2}$ and $s=N-\frac{1}{2}$) . Consequently,

$$\begin{aligned} D(m,i) &= \frac{\Gamma(m+\frac{n}{2}+i-\frac{1}{2})}{\Gamma(N+\frac{n}{2}+\frac{1}{2})}\binom{N-\frac{1}{2}}{N-m-i}\frac{1}{(m+\frac{n}{2}-\frac{1}{2})\cdots(m+\frac{n}{2}+i-\frac{3}{2})} \\ &= \frac{\Gamma(m+\frac{n}{2}-\frac{1}{2})}{\Gamma(N+\frac{n}{2}+\frac{1}{2})}\binom{N-\frac{1}{2}}{N-m-i}, \end{aligned}$$

which completes the proof of the lemma. □

9. Failure of the pointwise estimate (3)

In this section we give a proof that (3) is false, more direct than the one in [MV], and we show the connection with the algebra of operators already mentioned.

Theorem 7. *The following pointwise inequality is false for functions in $L^2(\mathbb{R})$:*

$$H^*f(x) \le C\,M(Hf)(x)\,, \quad x\in\mathbb{R}\,. \tag{79}$$

Remark. Notice that the Theorem implies that there is no good-lambda inequality between $H^*(f)$ and $H(f)$.

Replacing f by $H(f)$ in (79) and recalling that $H(Hf)=-f$, $f\in L^2(\mathbb{R})$, we see that (79) is equivalent to saying that

$$H^*(H(f))(x)\le C\,M(f)(x)\,,\quad x\in\mathbb{R}\,,$$

for any $f\in L^2(\mathbb{R})$.

Lemma 12. *The operator $f\to H^*(Hf)$ fails to be of weak type $(1,1)$.*

Proof. To prove the Lemma it is enough to show that if $f=\chi_{(0,1)}$, then there are positive constants m and C such that whenever $x>m$,

$$H^*(Hf)(x)\ge C\,\frac{\log x}{x} \tag{80}$$

Indeed, choosing $m>e$ if necessary, we have

$$\begin{aligned} \sup_{\lambda>0}\lambda\,|\{x\in\mathbb{R}: H^*(Hf)(x)>\lambda\}| &\ge \sup_{\lambda>0}\lambda\left|\{x>m:\frac{\log x}{x}>C^{-1}\lambda\}\right| \\ &= C\sup_{\lambda>0}\lambda\left|\{x>m:\frac{\log x}{x}>\lambda\}\right| \ge C\sup_{\lambda>0}\lambda\,(\varphi^{-1}(\lambda)-e), \end{aligned}$$

where φ is the decreasing function $\varphi:(e,\infty)\to(0,e^{-1})$, given by $\varphi(x)=\frac{\log x}{x}$. To conclude observe that the right hand side of the estimate is unbounded as $\lambda\to 0$:

$$\lim_{\lambda\to0}\lambda\varphi^{-1}(\lambda)=\lim_{\lambda\to\infty}\varphi(\lambda)\lambda=\infty.$$

To prove (80) we recall that for $f=\chi_{(0,1)}$

$$Hf(y)=\log\frac{|y|}{|y-1|}.$$

Let $m > 1$ big enough that will be chosen soon. Take $x > m$. Hence, by definition,

$$H^*(Hf)(x) \geq \left| \int_{|y-x|>m+x} \frac{1}{y-x} \log \frac{|y|}{|y-1|} \, dy \right|$$

and splitting the integral in the obvious way

$$\int_{-\infty}^{-m} \frac{1}{y-x} \log \frac{-y}{-y+1} \, dy + \int_{2x+m}^{\infty} \frac{1}{y-x} \log \frac{y}{y-1} \, dy =$$

$$= \int_{m}^{\infty} \frac{1}{x+y} \log \frac{y+1}{y} \, dy + \int_{2x+m}^{\infty} \frac{1}{y-x} \log \frac{y}{y-1} \, dy = A(x) + B(x),$$

where both $A(x), B(x)$ are positive. Hence

$$H^*(Hf)(x) \geq A(x).$$

Since

$$\log(1 + \frac{1}{y}) \approx \frac{1}{y}$$

as $y \to \infty$, there is a constant $m > 1$ such that whenever $y > m$

$$\frac{1}{2} < \frac{\log(1 + \frac{1}{y})}{\frac{1}{y}} < \frac{3}{2}.$$

Hence, with this constant m we have

$$A(x) = \int_{m}^{\infty} \frac{1}{x+y} \log \left(1 + \frac{1}{y}\right) dy \approx \int_{m}^{\infty} \frac{1}{x+y} \frac{dy}{y} = \frac{1}{x} \log \frac{y}{x+y} \Big|_{m}^{\infty} \approx \frac{\log x}{x},$$

which proves (80).

Notice that B is better behaved :

$$B(x) \leq \int_{2x+m}^{\infty} \frac{1}{y-x} \log \frac{y}{y-1} \, dy \leq \int_{2x+m}^{\infty} \frac{2}{y} \frac{dy}{y} \leq \frac{1}{x}. \qquad \square$$

10. Composition of operators : positive results

We first discuss a proof of (26) in Lemma 2 using standard arguments except for a point that will be supplied. We mention that in [Le1] there is a different argument.

Let $x \in \mathbb{R}^n$ and let $Q = Q(x, r)$ be an arbitrary cube centered at x and sidelength r. It is enough to show that there exists $C > 0$ such that for some constant $c = c_Q$

$$\frac{1}{|Q|} \int_Q |Tf(y) - c| \, dy \leq C \, Mf(x). \tag{81}$$

Let $f = f_1 + f_2$, where $f_1 = f\,\chi_{2Q}$. If we pick $c = (T(f_2))_Q$, we can estimate the left hand side of (81) by a multiple of

$$\frac{1}{|Q|}\int_Q |T(f_1)(y)|\,dy + \frac{1}{|Q|}\int_Q |T(f_2) - (T(f_2))_Q|\,dy = I + II.$$

To take care of II we use the regularity of the kernel in a standard way as in [GrMF, p. 153]. We omit the details. Hence we have

$$II \le C\,Mf(x).$$

To control I we use (25). Hence, since the support of f_1 is contained in $2Q$ we have

$$\begin{aligned} I &\le \frac{C}{|Q|}\int_{4Q} |T(f_1)(y)|\,dy \\ &\le \frac{C}{|Q|}\int_{4Q} M(f_1)(y)\,dy \le C\,\frac{C}{|4Q|}\int_{4Q} M(f)(y)\,dy \le C\,M^2(f)(x). \end{aligned}$$

Proof of Theorem 2. To prove a) we use part a) of Coifman-Fefferman's Theorem 6 and part a) Fefferman-Stein's Theorem 4:

$$\begin{aligned} \int_{\mathbb{R}^n} \Big(T_1^* \circ T_2(f)(x)\Big)^p w &\le \int_{\mathbb{R}^n} \Big(M \circ T_2 f(x)\Big)^p w(x)dx \\ &\le C\int_{\mathbb{R}^n} \Big(M^{\#} \circ T_2 f(x)\Big)^p w(x)dx \le C\int_{\mathbb{R}^n} (M^2 f)^p\, w \end{aligned}$$

where in the last estimate we have used (26) in Lemma 2. This yields (16) and concludes the proof of the first part of the theorem.

To prove (17) we use similar arguments except that we use part b) of both Theorems 6 and 4:

$$\begin{aligned} \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : |T_1^* \circ T_2 f| > t\}) &\le \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : M(T_2 f)(y) > t\}) \\ &\le \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n \colon M^{\#}(T_2 f)(y) > t\}) \\ &\le \sup_{t>0} \frac{1}{\Phi(\frac{1}{t})} w(\{y \in \mathbb{R}^n : M^2(f)(y) > t\}). \end{aligned}$$

To prove b) in Theorem 2 we use a similar argument. The main difference is that we use first Cotlar's improved estimate from Theorem 5. Indeed, this is used after an application of Theorem 6 of Coifman and Fefferman:

$$\begin{aligned}\int_{\mathbb{R}^n} \Big(T_1^* \circ T_2^*(f)(x)\Big)^p w &\le \int_{\mathbb{R}^n} \Big(M \circ T_2^* f(x)\Big)^p w(x)dx \\ &\le \int_{\mathbb{R}^n} \Big(M \circ M_\delta \circ T_2 f(x)\Big)^p w(x)dx + \int_{\mathbb{R}^n} M^2 f(x)^p w(x)dx \\ &= I + II.\end{aligned}$$

We just need to control I. For this we remark that

$$M \circ M_\delta f \le c_\delta M f(x). \tag{82}$$

Hence by Fefferman-Stein's theorem 4

$$\begin{aligned} I &\le C_\delta \int_{\mathbb{R}^n} \Big(M \circ T_2 f(x)\Big)^p w(x)dx \\ &\le C_\delta \int_{\mathbb{R}^n} \Big(M^\# \circ T_2 f(x)\Big)^p w(x)dx \le C \int_{\mathbb{R}^n} \Big(M^2 f(x)\Big)^p w(x)dx \end{aligned}$$

where in the last estimate we have used (26) from Lemma 2.

We are left with the proof of (82). Let $x \in \mathbb{R}^n$ and let $Q = Q(x,r)$ be an arbitrary cube centered at x with sidelength r. We have to show that

$$\frac{1}{|Q|} \int_Q M_\delta f(y)\, dy \le C\, Mf(x).$$

Let $f = f_1 + f_2$, where $f_1 = f\, \chi_{2Q}$. We can estimate the left hand side by a multiple of

$$\frac{1}{|Q|} \int_Q M_\delta f_1(y)\, dy + \frac{1}{|Q|} \int_Q M_\delta f_2(y)\, dy = I + II.$$

To take care of II we use that it is roughly constant on Q by [GrMF, p. 299]. Hence we have

$$II \le C\, M_\delta f(x) \le C\, Mf(x).$$

To control I we use that $\delta < 1$ and that the maximal operator is bounded on $L^{1/\delta}(\mathbb{R}^n)$. We obtain

$$I \le \frac{C_\delta}{|Q|} \int_{2Q} |f(y)|\, dy \le C\, M(f)(x).$$

This concludes the proof of the first part of b) of Theorem 2. The proof of the second part is similar to the proof of part a), except for the fact that one uses the method we have just described. We leave the details to the interested reader. □

Proof of Corollary 1. By homogeneity it is enough to assume $t = 1$ and hence we just need to prove

$$w(\{y \in \mathbb{R}^n : |T_1^* \circ T_2 f(y)| > 1\}) \le C \int_{\mathbb{R}^n} \Phi(|f(y)|) w(y)dy.$$

Now, $\Phi = t(\log(e+t)) \approx t(1+\log^+ t)$ is submultiplicative, that is, $\Phi(ab) \leq \Phi(a)\,\Phi(b)$, $a,b \geq 0$. In particular, Φ is doubling. We have by Theorem 2 and (8)

$$\begin{aligned}
w(\{y\in\mathbb{R}^n : |T_1^*\circ T_2 f(y)|>1\}) &\leq C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}w(\{y\in\mathbb{R}^n : |T_1^*\circ T_2 f(y)|>t\})\\
&\leq C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}w(\{y\in\mathbb{R}^n : M^2 f(y)>t\})\\
&\leq C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\int_{\mathbb{R}^n}\Phi(\frac{|f(y)|}{t})w(y)dy\\
&\leq C\sup_{t>0}\frac{1}{\Phi(\frac{1}{t})}\int_{\mathbb{R}^n}\Phi(|f(y)|)\Phi(\frac{1}{t})w(y)dy\\
&= C\int_{\mathbb{R}^n}\Phi(|f(y)|)w(y)dy\,,
\end{aligned}$$

which completes the proof. □

Proof of inequality (19). It is enough by translation invariance to consider $z=0$ in (19), that is,

$$B^*(B(f))(0) \leq C\,(B^2)^*(f)(0) + C\,M(f)(0).$$

Recall that

$$B^* f(z) = \sup_{\epsilon>0}|B_\epsilon f(z)|, \quad z\in\mathbb{C}\,,$$

with

$$B_\epsilon f(z) = \int_{|w-z|>\epsilon} f(z-w)\frac{1}{w^2}\,dw\,.$$

To prove the inequality at 0 we use that (see [MV]) for any h

$$B^*(h) = \widetilde{M}(Bh)\,,$$

where

$$\widetilde{M}(g)(z) = \sup_{\epsilon>0}\Big|\frac{1}{\pi\epsilon^2}\int_{D(z,\epsilon)} g(w)\,dA(w)\Big|.$$

Hence, it is enough to show that

$$\widetilde{M}(B^2 f)(0) \leq C\,(B^2)^*(f)(0) + C\,M(f)(0).$$

By dilation invariance is enough to estimate the integral of $B^2 f$ on the unit disc D. Clearly

$$\int_D B^2 f(w)\,dA(w) = \int f(w)\,B^2(\chi_D)(w)\,dA(w)\,,$$

and so we need to compute $B^2(\chi_D)$. For this we use the basic property of B, namely

$$\frac{\partial}{\partial z}\varphi = -\frac{1}{\pi}B(\frac{\partial \varphi}{\partial \overline{z}}),$$

which holds for appropriate classes of functions φ.

Integrating the function $\chi_D(z)$ in $\overline{z}$ one gets the function

$$\varphi(z) = \overline{z}\chi_D(z) + \frac{1}{z}\chi_{D^c}(z),$$

and so

$$-\frac{1}{\pi}B(\chi_D)(z) = \frac{\partial \varphi}{\partial z} = -\frac{1}{z^2}\chi_{D^c}(z).$$

Following the same strategy for the function $-\frac{1}{z^2}\chi_{D^c}(z)$ we get

$$-\frac{1}{\pi}B^2(\chi_D)(z) = (-2\frac{\overline{z}}{z^3} + \frac{3}{z^4})\chi_{D^c}$$

and so

$$-\frac{1}{\pi}\int f(w)\, B^2(\chi_D)(w)\, dA(w) = -2\int_{D^c} f(w)\,\frac{\overline{w}}{w^3}\, dA(w) + 3\int_{D^c} f(w)\,\frac{1}{w^4}\, dA(w).$$

Last term is bounded by a multiple of $Mf(0)$ since, after putting the absolute value inside the integral, one is convolving with a non-negative decreasing integrable kernel. Alternatively one may just integrate in dyadic annuli centered at 0. For the first term we simply observe that the function $-2\frac{\overline{w}}{w^3}$ is the kernel of B^2 and hence

$$-2\int_{D^c} f(w)\,\frac{\overline{w}}{w^3}\, dA(w)$$

is the truncation at level 1 of $B^2 f(0)$ (see the definition just after (10)). $\square$

Proof of Theorem 3. By dilating and translating it is enough to prove that

$$|R^1(T(f))(0)| \le C(|S^2 f(0)| + Mf(0)),$$

where R^1 and S^2 are the truncations of R and S at levels 1 and 2 respectively (see the definition just after (10)).

Denote by K_0, K and K_1 the kernels of R, T and S respectively. Let B be the unit ball of $\mathbb{R}^n$. It was shown in [MOV] that, because R is an even higher order Riesz transform, its kernel off the unit ball is in the range of R. More precisely, there exists a polynomial b such that

$$K_0(y)\chi_{B^c}(y) = R(b\chi_B)(y), \quad y \in \mathbb{R}^n.$$

Thus, since $R \circ T = S + c\,I$,

$$\begin{aligned} R^1(Tf(0)) &= \int_{|y|\geq 1} K_0(y) Tf(y)dy \\ &= \int R(b\chi_B)(y) Tf(y)dy \\ &= \int b(y)\chi_B(y) Sf(y)dy + c\int b(y)\chi_B(y)f(y)dy \\ &= I + II. \end{aligned}$$

Clearly, II is bounded by $C\|b\|_{L^\infty(B)}\,(Mf)(0)$. On the other hand,

$$\begin{aligned} I &= \int S(b\chi_B)(y)f(y)dy \\ &= \int_{2B} S(b\chi_B)(y)f(y)dy + \int_{(2B)^c} S(b\chi_B)(y)f(y)dy \\ &= III + IV. \end{aligned}$$

Using Lemma 5 in [MOV] we get that III is bounded by

$$CMf(0)(\|b\|_{L^\infty(B)} + \|b\|_{\mathrm{Lip}(1,B)})\,,$$

where $\|b\|_{\mathrm{Lip}(1,B)}$ is the Lipschitz semi-norm of b on B. Since the kernel K_1 of S is smooth off the origin we have

$$S(b\chi_B)(y) = K_1(y)\int b\chi_B + \|b\|_{L^\infty(B)}\,O(\frac{1}{|y|^{n+1}})\,, \quad |y| > 2\,.$$

Thus,

$$\begin{aligned} |IV| &\leq C\|b\|_\infty |\int_{(2B)^c} K_1(y)f(y)dy| + C\|b\|_\infty \int_{(2B)^c} \frac{|f(y)|}{|y|^{n+1}}dy \\ &\leq C\|b\|_\infty(|S^2f(0)| + Mf(0))\,, \end{aligned}$$

which completes the proof. □

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