Improving the Risk Concept: A Revision of Arrow-Pratt Theory in the Context of Controlled Dynamic Stochastic Environments

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Abstract

In the literature on risk, one generally assume that uncertainty is uniformly distributed over the entire working horizon, when the absolute risk-aversion index is negative and constant. From this perspective, the risk is totally exogenous, and thus independent of endogenous risks. The classic procedure is "myopic" with regard to potential changes in the future behavior of the agent due to inherent random fluctuations of the system. The agent's attitude to risk is rigid. Although often criticized, the most widely used hypothesis for the analysis of economic behavior is risk-neutrality. This borderline case must be envisaged with prudence in a dynamic stochastic context. The traditional measures of risk-aversion are generally too weak for making comparisons between risky situations, given the dynamic complexity of the environment. This can be highlighted in concrete problems in finance and insurance, context for which the Arrow-Pratt measures (in the small) give ambiguous results (see, Ross, 1981). We improve the Arrow-Pratt approach (1964, 1971a, 1971b), which takes into account only attitudes towards small exogenous risks, by integrating in the analysis potentially high endogenous risks which are under the control of the agent. This point of view has strong implications on the agent's adaptive behavior towards risk in a changing environment. It can be seen as a step further in the refinement of the risk-aversion concept. It is necessary to have a very good understanding of the way the evolution of the environment affects the risk perception of rational decision-makers. Based on multiple theoretical and empirical arguments, this new approach offers an elegant study of the close relationship between behavior, attitude and perceived risk.

Keywords: Controlled dynamic stochastic system, optimal trajectory, closed-loop strategy, feedback-and-forward information, rational decision-maker, dynamic learning, endogenous risk-aversion, adaptive risk management, optimal risk-aversion threshold, excessive risk-averse behavior, risk perception, changing risk behavior.

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1. Introduction

The most common attitude of economic agents in decision-making problems is one generated by risk-aversion. Such a behavior is characteristic for large gains as well as large losses. Risk-aversion and rationality are generally considered together. Rational agents are goal oriented, they are values and reference points and base their decisions on uncertain future. They are confronted with multiple risks, which are generally different in different contexts. The decision adopted is not made independently but jointly with other decisions, which places the agents in risky situations. Decisions taken to avoid, even partially, a source of risk can be affected by the presence of others.

It must distinguish between quantifiable risks (when the objective probabilities are supposed to be known) and inherently unmeasurable uncertainties (when the objective probabilities are not given in advance). In other words, it must distinguish between decisions under risk and decisions under uncertainty.

Traditionally, risk-aversion is equivalent to the concavity of the utility function (viewed as the measure upon which the agent bases his decisions). Formally speaking, that means that for any arbitrary risk, the agent will prefer the sure amount equal to the expected value of the risk rather than the risk itself. However, the condition of concavity is just a way of expressing risk-averse preferences.

In the literature on risk, two polar cases can be distinguished according to the degree of risk-aversion exhibited by the economic agent, namely, the risk-neutral case and respective the infinitely risk-averse case. Both cases are based on strong and non-realistic assumptions.

The objective of this paper is to analyze how the traditional results are modified when the risk attitude of economic agents depends on the system evolution. It offers an elegant solution to the inconvenients that arise when modelling the behavior to risk as in ARROW-PRATT traditional approach. Several realistic scenarios are analyzed in the context of a finite discrete-time dynamic environment.

The proposed analysis investigates on the decision-makers' psychology to risk, providing new perspectives of research for theorists and empirical analysts.

2. Problem Statement

Consider a rational decision-maker characterized by a consistent and efficient outcome oriented behavior (SEE, DREZE, J. H., 1990 AND V., WALSH, 1996) who faces uncertainty vis-à-vis of the future evolution of a stochastic environment. He disposes of an optimal set of control instruments employed in order to drive the system as close as possible to a desired reference trajectory which ensures its equilibrium and stability. These instruments are optimally selected, on the basis of a non-decreasing endogenous information set.

We consider the context of a closed-loop control strategy, the information being utilized in real time. In this case, the agent's policy does not require some large periods of engagement. The control rules are thus sensitive to the choice of the working horizon. This may be the case when the relevant information acquisition cost is high, most likely due to permanent random shocks in the system or because of the slow inertia of the environment. The decision-maker tries to reduce the uncertainty related to his actions by acquiring information from the beginning of the control to the moment of decision. He learns from errors and makes a self-assessment of his risky actions. We call this a recurrent behavior.

The feedback control responds not only to the effects of random inputs, but also to the measurement errors as well. Thus, it is not necessary to be able to identify and measure the sources of disturbance. The feedback learning process is progressive, allowing to adjust the agent's ex-ante anticipations about the system trend. In this way, the agent will minimize the difference between actual and assumed system characteristics by monitoring its random fluctuations.

The degree of information embedded in the observation of the state variable generally depends on the selected values for the control variables, so that the extent of learning about the latent parameters can be directly influenced by the decision-maker. It is the context of a rational active learning (specific to a dynamic modelling) which allows for the decision-maker to experiment. In a multiperiod setting, he can learn about the consequences of his actions through experimentation. It helps to better anticipate the environment trend and thus to avoid undesirable scenarios in the future. The environment can incite to active learning. The agent bases his actions on his state of knowledge at the point where the actions are taken. He has some influence over the rate at which information arrives. His behavior may generate information. The active learning makes the agent more experienced over time. However, despite of potential benefits from active learning in stochastic optimization problems (SEE, AMONG OTHERS, EASLEY & KIEFER, 1988 AND KIEFER & NYARKO, 1989), the potential for learning is very limited if the model is very noisy.

The paper is organized as follows. Section 3 discusses the general model. Section 4 deals with the choice of the criterion which ensures an unique solution for the problem of optimization and control. Section 5 summarizes the traditional approach of ARROW-PRATT (1964, 1971A, 1971B). Section 6 introduces the concept of endogenous risk-aversion. Four distinct cases are discussed: a) when agent's risk-aversion is based on a truncated history of the process and rational anticipations of the system behavior in the future; b) when agent's risk-aversion is based only on a truncated history of the process; c) when agent's risk-aversion is based on past overall performances (a progressive history); d) when agent's risk-aversion is based only on rational anticipations of the system behavior in the future. We also analyze here the case of high potential shifts and fluctuating systems. A comparative empirical analysis based on multiple simulation results is given in this direction. Several qualitative results on adaptive risk management / perception are given in Section 7. Three distinct cases are discussed: a) the context of small system deviations; b) the context of small and large system deviations; c) the context of large system deviations. Section 8 introduces the concept of excessive risk-averse decision-maker. This is defined with respect to an optimal risk-aversion threshold chosen by the decision-maker before starting the control, which must not be exceeded during the control period. We analyze, in this context, two distinct types of economic agents: a) more risk-averse by nature; and b) less risk-averse by nature. A differentiation of agents with common / distinct types becomes thus possible. Section 9 refines the analysis by taking into account potential sensitive periods which can influence the choice of the optimal risk-aversion threshold. Section 10 discusses the general case where the agent is characterized by a changing risk behavior, that is, when he becomes (depending on the system evolution / perception) risk-averse, (almost) risk-neutral or risk-lover. A characterization of the agent's type in function of his individual preferences is provided. Section 11 draws several important conclusions and makes suggestions for further research.

3. The Model

The type of model we analyze corresponds to a Data Generating Process (DGP) which is dynamic, non-linear and managed by a system of discrete simultaneous equations.

If the action of the agent is purely quantitative, we say that he disposes of an instrument (or control variable) with the help of which he tries to control the environment in an optimal way.

Let $x_t \in \mathbf{R}^q$ be the value of the control variable at time t. Note that x_t is not strictly exogenous, in general the actions being dependent variables on the history and current state of the system. Different contexts of decision making generally call for different actions.

Let $y_t \in \mathbf{R}^p$ be the observable (not controlled) target variable and let $z_t \in \mathbf{R}^r$ be an exogenous variable not subjected to the control and hence observed outside the system considered. This may be forecasted but cannot be influenced by the decision-maker.

Whether or not the variable z_t is exogenous depends upon whether or not that variable can be taken as "given" without losing information for the purpose at hand. Specifically, the exogeneity of the variable z_t depends on the parameters of interest of the decision-maker as well as on the purpose of the model (statistical inference, forecasting, or policy analysis). Variations in the process z_t over time will generate variations in the process x_t.

Denote by $\mathbf{X}_t \overset{not.}{=} \{..., x_{-1}, x_0, x_1, ..., x_t\}$ the history of the process x until the date t and similar for $\mathbf{Y}_t$ and $\mathbf{Z}_t$. Thus, we allow for the current state variable to depend not only on the agent's current decision but also on an arbitrarily complex history $\mathbf{X}_t$. We make the following basic assumptions:

Assumption 1. The evolution of the environment is modelled by a nonlinear extended-memory process (i.e., with strong dependence between observations, SEE CLEMHOUT & WAN JR, 1985) endogenously generated according to the structural state equation:

$$y_t = F(\mathbf{Y}_{t-1}, \mathbf{X}_t, \mathbf{Z}_t, \boldsymbol{\beta}_t, t) + u_t, \quad t \in \mathbf{Z}$$

where $u_t \in \mathbf{R}^p$ (exogenous environmental "white noise") is the specific "risk" modelled like a normal random variable, $u_t \sim i\mathcal{N}(0, \boldsymbol{\Psi})$. However, nothing constrains the DGP to be stable. The parameter of interest $\boldsymbol{\beta}_t$ (which specifies the structure of the model) varies according to the information available at date t.

The adjustment function F is assumed twice continuously differentiable with respect to the parameter-vector $\boldsymbol{\beta}_t \in \mathcal{B} \subset \mathbf{R}^k$. We make no particular assumption on the functional form of F. In practice, the true distribution of the data is never known with precision. Several types of errors can occur and bias the choice of the function F. The shape of F and the value of the parameters of interest are generally determined from the behavior of the decision-maker.

In general, y_t is not a GAUSSIAN process. Non-linearity between y_t and $\mathbf{Y}_{t-1}$ implies non-normality (dynamic asymmetry) but non-normality does not necessarily imply non-linearity.

We point out that the variable t plays the role of a synthesis variable in the econometric model.

Assumption 2. The agent's objective is to constrain the system to follow a feasible optimal path (aspiration level) $\eta = \{y_1^g, y_2^g, ..., y_T^g\}$ by selecting the control variable x_t in a suitable way. This is a pre-specified condition which cannot be changed during the control period. We say that η characterizes the decision-maker's (discrete) preferences on the environment.

The targets and the dynamics are modelled simultaneously. The targets must be compatible with the state of the system.

Taking into account foreseeable movements in y as well as possible economic constraints, the agent will fix some optimal bounds l_t $(t = 1, ..., T)$ such that $0 < y_t^g \leq l_t < 1$ (the constraints may change over time). It is important to note that the optimal bounds l_t are generally correlated with the agent's type.

Since a real time control process is necessarily discrete, we cannot hope to converge with precision to any target value, but only to a neighborhood of it. When the process of control is finished, the decision-maker will obtain a stochastic neighbouring-optimal trajectory which is expected to be close to the optimal path η. More is non-linear the model, more it will be difficult to track the targets.

When there is no cost on the control, the decision-maker should not have for objective to follow a fixed optimal trajectory. It is the case of a "myopic" (pseudo optimal) behavior.

Assumption 3. The timing of the control is as follows: At each period t, the agent implements a risky action x_t (after the exogenous variable z_t was observed) that will be an external stimulus for the system. This is purported to contribute to the equilibrium and stability of the system. A shock u_t is carried out and the agent observes the output y_t (the impulse response) which allows for extracting a dynamic signal about the environment trend. The question is: how this signal will influence the agent's risk behavior?

The information revealed by the output signal can increase the precision of the next control instrument and decrease the agent's risk-aversion in the future. This output together with the corresponding action provides information on the data generating process. The agent emploies this signal for strategic learning (specific to a closed-loop strategy) in order to reduce the uncertainty on the system behavior and thus, to drive the system as close as possible to the desired reference path η.

The uncertainty is ex-post reduced only, after the informative output-signal has been received. The shock u_t will have a persistent effect on y_t that will disappear gradually over time.

Assumption 4. The optimality of the instrument is considered with respect to a global criterion (preference function) which measures the system deviations, $\Delta y_t \overset{not.}{=} y_t - y_t^g$, $t = 1, ..., T$. Let $W(y_1, y_2, ..., y_T)$ be this criterion, supposed twice continuously differentiable and strictly convex, at least in the feasible area of the model.

Assumption 5. The decision problem is to find the optimal values of the control instruments which minimize the agent's preference function by taking into account the constraint relationships that exist between the controlled, partially controlled and uncontrolled variables.

The agent estimates his optimal policy $\widehat{x}(.) \overset{not}{=} (\widehat{x}_1, \widehat{x}_2, ..., \widehat{x}_T)$ before to know the value of y_0. He obtains a random policy conditional to y_0:

$$\widehat{x}\,(.) = \underset{x(\cdot)}{\arg\min} E_x[W(y_1, y_2, ..., y_T) \mid y_0]$$

where E_x is the expactation with respect to the controlled stochastic process induced by the rule of decision x.

In practice, the initial state y_0 can be fixed or random. It is crucial to treat the initial value correctly and to measure its impact. Small variations of the initial conditions can have large effects on the long-run outcomes.

It is very likely that difference between ex-ante decisions and ex-post results (i.e., between ex-ante and ex-post optimality) exists.

This is the classic context, where the hypothesis of risk-neutrality for the decision-maker is adopted for the entire period of control.

4. Choice of the Criterion

In order to avoid several local minima and thus to have a unique solution for the optimal control problem, a necessary condition is to use a strictly convex criterion.

Using the parsimony criterion, we seek for the simplest strictly convex loss function. It is the quadratic approximation which satisfies this condition. Interpretation is simple: a quadratic objective function may be considered as a good local approximation of the true preferences, exactly as a model approaches the behavior of the environment around the observed variables.

This is reasonable because it induces a high penalty for large deviations of the state from the target but a relatively weak penalty for small deviations. If we start with a much more complicated loss function, we can always specify a quadratic approximation which will solve the problem or achieve the goal.

Nothing prevents to suppose that the loss function is additively recursive, on the one hand in order to simplify the determination of the formula for the optimal instrument and, on the other hand, because it makes possible to apply the BELLMAN's (1961) optimality principle.

A limited expansion of second order of $W_{[1,T]}(y_1, ..., y_T)$ around a given feasible point $Y^g = (y_1^g, ..., y_T^g)$ gives us:

$$W_{[1,T]}(y_1, ..., y_T) = \Delta Y' K \Delta Y + 2\Delta Y' d + c$$

where

$$K_{(pT\times pT)} \overset{not}{=} \left[\frac{\partial^2 W}{\partial Y \partial Y'}\right]_{Y^g}, \quad d \overset{not}{=} \left[\frac{\partial W}{\partial Y'}\right]_{Y^g}$$

The function W being strictly convex and twice continuously differentiable, the matrix K is symmetrical and positive semidefinite.

The decision criterion is supposed to be a function that puts weight (or measure) on the possible outcomes indicating their desirability or undesirability. We have:

$$W_{[1,T]}(y_1, ..., y_T) \overset{def}{=} \sum_{t=1}^{T} W_t(y_t)$$

where W_t is a local quadratic loss function, strictly convex and twice differentiable.

By eliminating the constant (which leaves the minimization invariant), we can use the following deviation-dependent asymmetrical specification for W_t:

$$W_t(y_t) = (y_t - y_t^g)' K_t (y_t - y_t^g) + 2(y_t - y_t^g)' d_t$$

where a prime denotes transpose.

Asymmetry derives from the difference in penalty costs which the decision-maker may attach to errors, depending on whether these are errors of shortfall or errors of overshooting about the targets.

Generally, the decision to choose certain parameters K_t and d_t reflects the decision-maker's priorities and also depend on the available quantity of information concerning the future development of the system parameters. However, it is far from probable that the decision-maker will be able to assign values to the weights which represent his preferences correctly.

If the future evolution of the system is unpredictable, then the best weighting matrix K_t which can be selected is the identity matrix, while the best value for d_t is the unity vector. If K_t is not diagonal, then the penalties also attach to the covariances of the state variable deviations from its desired level.

The weights employed are anything but objective, since the deviation of the target variable may be not of the same importance at different moments in time. The idea is to choose the parameters which yield a smoother (i.e., less fluctuating) control and so a more stable system.

At each period t, the parameters K_t and d_t are updated and new optimal values are chosen in order to fulfill the requirements of the decision-maker. These requirements are based on policy values presented at each period, and do not require any direct information about the actual weighting that the decision-maker may have in mind.

5. Sensitive Criterion to Risk: Static Approach

In the static approach of expected utility, the measure of risk-aversion is given by the ARROW-PRATT index, which requires the existence of a VON NEUMANN-MORGENSTERN utility function.

Let $W_{[1,T]}$ be the usual quadratic non-symmetrical criterion and let U be the global utility of the control given by:

$$U(W_{[1,T]}) = \frac{2}{\varphi}\left\{\exp(-\frac{\varphi}{2}\cdot W_{[1,T]}) - 1\right\}$$

which verifies:

$$-\frac{U''(W_{[1,T]})}{U'(W_{[1,T]})} = \frac{\varphi}{2}, \ \forall \ W_{[1,T]}$$

where a prime denotes the partial derivative with respect to $W_{[1,T]}$.

Therefore, $\frac{\varphi(W_{[1,T]})}{2}$ (the absolute index of ARROW-PRATT) can be interpreted as a measure of local risk-aversion at a particular $W_{[1,T]}$, U being a global CARA utility.

This is the case of a totally exogenous risk, when the agent's attitude to risk does not change during the entire working horizon. This way to capture the risk is non-realistic.

The optimal strategy will be φ-dependent:

$$s_{\varphi}^{g}(.) = \underset{x_1,...,x_T}{\arg\max} E_0[U(W_{[1,T]})]$$

$$s_{\varphi}^{g}(t) \overset{not}{=} \widehat{x}_t(I_{t-1}) \mid y_0, \ \ \forall \ t = 1,...,T,$$

where I_t is the information set acquired until the date t and updated each time as new observation becomes available.

We define the sensitive criterion to risk by:

$$\gamma_0(\varphi) \overset{def}{=} E_0(W_{[1,T]})$$

If $\varphi \cdot V_0(W_{[1,T]}(y_1,...,y_T))$ is small, then a limited expansion of second order justifies the following approximation:

$$\gamma_0(\varphi) \approx -E_0(W_{[1,T]}(y_1,...,y_T)) + \frac{\varphi}{4}\cdot V_0(W_{[1,T]}(y_1,...,y_T))$$

We distinguish three distinct cases:

a) If $\varphi < 0$, then the function $U(W_{[1,T]})$ is negative, strictly concave and decreasing. Note that the sign of U has no particular importance. When one maximizes $\gamma_0(\varphi)$, the above approximation shows that the variability is penalized (the agent is afraid of large accidental values of $W_{[1,T]}$).

There is a strong correlation between pessimism / prudence and risk-aversion. Indeed, if φ decreases, the agent is convinced that some large values of $W_{[1,T]}$ appear more and more frequently and thus he will have some pessimistic expectations. The agent will be characterized by a significative loss aversion.

b) If $\varphi > 0$, then the function $U(W_{[1,T]})$ is negative, strictly convex and decreasing. The line $W = -2/\varphi$ is an horizontal asymptote towards infinity of the curve $U(W_{[1,T]})$ and $(0,0) \in Graph(U)$.

The situation is opposed to the previous case. There is rather an interest for moderate values of $W_{[1,T]}$ than for extreme values. We say that the agent is optimistic (or risk-lover).

c) If $\varphi = 0$, then we obtain (using the HOSPITAL's rule):

$$\lim_{\varphi \to 0} \gamma_0(\varphi) = -E_0[W_{[1,T]}(y_1, ..., y_T)]$$

Thus:

$$\min E_0[W_{[1,T]}(y_1, ..., y_T)] \Leftrightarrow \max \gamma_0(\varphi)$$

the term in the left-hand being the specific usual criterion from the risk-neutral case. We can speak about a dual control problem. We have that $U(W_{[1,T]}) \underset{\varphi \to 0}{\to} -W_{[1,T]}$ (the quadratic loss function is obtained as a special case).

If $\varphi_1 \leq 0 \leq \varphi_2$, then we can write the following inequalities:

$$U_{\varphi_1}(W^*_{[1,T]}) \leq U_0(W^*_{[1,T]}) \leq U_{\varphi_2}(W^*_{[1,T]})$$

We give below a graphical representation of the three above cases.

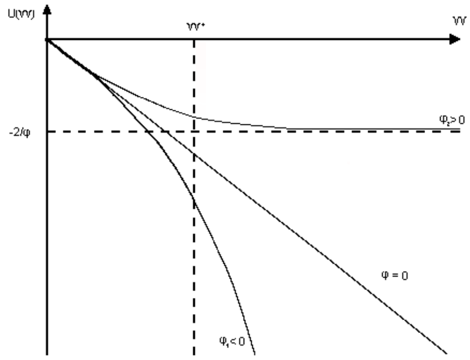

Figure 1: Risk-Aversion / Risk-Neutrality / Risk-Taking

Despite the fact that risk-averse agents hate uncertainty whereas the risk-neutral are indifferent (their behavior remains unchanged), they often place a lower value on forecasting than risk-neutral decision-makers do.

The need for a new approach requires a consistent study concerning the agent's attitude to risk in complex stochastic environments. It is important to have a very good understanding of the way the evolution of the environment affects the risk perception of rational decision-makers.

6. Adaptive Endogenous Risk-Aversion

6.1. General Framework

Although there is a large amount of literature on risk (SEE, AMONG OTHERS, FRIEDMAN, M. AND L. J. SAVAGE, 1948, BERNOULLI, D., 1954/1738, PRATT, J. W., 1964, ARROW, K. J., 1951, 1971, D. KAHNEMAN & A. TVERSKY, 1979, ROSS, S. A., 1981, YAARI, M., 1987, KIMBALL, M. S., 1993, RABIN, M. AND R. H., THALER, 2001), there is no theoretical and empirical work for comparing the degree of risk-aversion of rational decision-makers in the context of controlled dynamic stochastic environments.

Our purpose is to extend the ARROW-PRATT traditional approach, which takes into account only attitudes towards small exogenous risks, to the context of potentially high endogenous risks. To do this, we focus our analysis in the general context of a varying-risk environment.

The history of the process as well as the agent's anticipations on the system behavior in the future are closely linked to his attitude to risk over time. We formalize this point of view for a general class of models and we detail the positive effects that it implies in the context of a dynamic stochastic system which evolves over a finite and discrete horizon.

Because the behavior of the environment changes continuously, the agent's sensitivity to risk will also change, particularly when these changes are significant. We can speak about a systematic risk assumed by the agent at each period of control.

There will be a better risk management due to a better learning (even if it is not perfect). Following a permanent risk adjustment process, the agent's objective is to diminish his degree of risk-aversion over time. He moves from risk avoidance to risk elimination.

Consider a family of exponential anticipative local utility functions (performance criteria) U_t generated by a dynamic absolute risk-aversion index φ_t and an evolutive loss $W_{[1,t]}$:

$$U_t(W_{[1,t]},\varphi_t) \overset{def}{=} \frac{2}{\varphi_t}\left\{\exp(-\frac{\varphi_t}{2}\cdot W_{[1,t]})-1\right\},\quad t=1,...,T$$

with

$$W_{[1,t]} \overset{def}{=} \sum_{s=1}^{t} W_s(y_s)$$

It follows that:

$$-\frac{U''(W_{[1,t]},\varphi_t)}{U'(W_{[1,t]},\varphi_t)}=\frac{\varphi_t}{2}$$

where a prime denotes the partial derivative with respect to $W_{[1,t]}$.

Therefore, $\frac{\varphi_t(W_{[1,t]})}{2}$ measures locally (at the point $W_{[1,t]}$) the decision-maker's risk-aversion. We define the local sensitive criterion to dynamic risk (at time t) by:

$$\gamma_0(\varphi_t) \overset{def}{=} E_0 U_t(W_{[1,t]},\varphi_t)$$

We have:

$$\min E_0[W_{[1,t]}(y_1,...y_t)] \Leftrightarrow \max \gamma_0(\varphi_t)$$

To illustrate why the proposed risk-aversion index is informative about how attitudes towards risk of the decision-maker change along the working horizon, we prove several theoretical results in this direction.

Proposition 1. A more (less) risk-averse decision-maker is characterized by a smaller (higher) local anticipative utility level.

Proof. In the appendix. ■

This result characterizes the agent's utility in function of the risk-aversion index evolution over time. The objective is not to exceed a fixed limit threshold for the index over which the agent becomes excessively risk-averse (being characterized by an over-pessimism). This strategy prevents to exceed some threshold utility level under which the agent's ex-post preferences are suboptimal. Over-pessimism is an induced deviating behavior with respect to perceived states of the system.

At each period of control, the interest of the agent is to choose an optimal action with a higher ex-ante expected utility. We provide below a suggestive graphical representation of the above result.

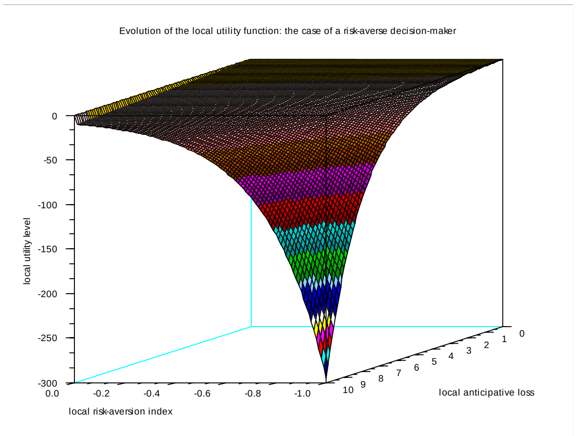

Figure 2:

In an evolving environment, the utility function obviously does not remain constant over time. It changes at each stage of the control, even if not significantly (the case of smooth preferences). It does not exclude the possibility to have the same level of utility for various periods. In this direction, we note a large class of decision rules which is representable by a variation of the utility between each two consecutive periods (SEE, GILBOA, 1989). It comes to consider the agent at different periods as though he were different individuals (SEE, ALLAIS, 1947). It is important to stress that the current decision can affect the agent's utility level in the future. The decision-maker can benefit from the learning of preferences. This idea was developed in first by STIGLER AND BECKER (1977). However, it is far from probable that the decision-maker exactly maximizes his utility (as well-being) at each stage of the control. We rather face a nearly optimization behavior, where the control variable is continuously and optimally adjusted over time to maximize some objective function (SEE, AMONG OTHERS, VAN DE STADT, H. ET AL., 1985, AND VARIAN, 1990). The stochastic disturbance in the system will produce random shocks in the decision-maker's preferences.

Before introducing new theoretical considerations on the concept of risk-aversion, a natural question arises: how become the standard conclusions when modelling the attitude to risk of the decision-maker according to the past and future dynamics of the system?

6.2. Risk-Aversion Based on a Truncated History of the Process and Rational Anticipations of the System Behavior in the Future

The environment evolution can change the agent's attitude to risk over time. In a dynamic context, the agent can fully take advantage from the learning benefits. He can influence the likelihood of the environment states by using a reinforcement of the active learning. We say that the agent is not myopic in the sense of expecting. Future anticipations play an important role in how the agent will decide what strategic actions and optimal risk to take. Suppose that the agent is a strategic decision-maker. He thinks about the future. Depending on the way the agent perceives future outcomes, both risk sensitivity and optimal decisions will be affected during the process of optimization and control. An increased power of prediction is an efficient method in order to reduce the uncertainty about the future system trajectory. The forecast is updated each time as new observation becomes available. The future is regarded as an extended present. A correct evaluation of the past is crucial for making optimal predictions in the future. This is necessary for an optimal assessment of the agent's risk-aversion over time.

We make the following useful notations:

$$S_{t,\ p\_d} \overset{not.}{=} \parallel y_{t-1} - y^g_{t-1} \parallel^2 + ... + \parallel y_{t-k_1} - y^g_{t-k_1} \parallel^2$$
(the sum of squared past deviations at time t)

$$S_{t,\ a\_f\_d} \overset{not.}{=} \parallel y^a_{t|I_t} - y^g_t \parallel^2 + ... + \parallel y^a_{t+k_2|I_{t+k_2}} - y^g_{t+k_2} \parallel^2$$
(the sum of squared anticipated future deviations at time t)

$$S_{t,\ w\_p\_d} \overset{not.}{=} \parallel y_{t-1} - y^g_{t-1} \parallel^2 L_{t-1} + ... + \parallel y_{t-k_1} - y^g_{t-k_1} \parallel^2 L_{t-k_1}$$
(the weighted sum of squared past deviations at time t)

$$S_{t,\ w\_a\_f\_d} \overset{not.}{=} \parallel y^a_{t|I_t} - y^g_t \parallel^2 \overline{L}_t + ... + \parallel y^a_{t+k_2|I_{t+k_2}} - y^g_{t+k_2} \parallel^2 \overline{L}_{t+k_2}$$
(the weighted sum of squared anticipated future deviations at time t)

where y^g_{t+i} $(i = 0, ..., k_2)$ represent fixed targets in the future (taking into account foreseeable movements in y), $y^a_{t+i|I_{t+i}}$ $(i = 0, ..., k_2)$ are expected values of the target variable at time $t + i$ based on non-decreasing endogenous information sets I_{t+i} and L_{t-j_1} $(j_1 = 1, ..., k_1)$, $\overline{L}_{t+j_2}$ $(j_2 = 0, ..., k_2)$ are weighting scalars attached to the system deviations (in the past and future) with respect to the equilibrium path η. Permanent fluctuations in the system target variable generate time-varying risk-aversion for the economic agent.

We are now in a position to give a definition of the agent's risk-aversion index by taking into account past performances of the system (a truncated history) and rational anticipations of the system behavior in the future.

Definition 1. Using t to denote time, the absolute risk-aversion index $\varphi^{r\_a}_{t,\ p\_f}$ evolves according to:

$$\varphi^{r\_a}_{t,\ p\_f} \overset{def.}{=} \frac{S_{t,\ w\_p\_d} + S_{t,\ w\_a\_f\_d}}{\sqrt{(S_{t,\ p\_d} + S_{t,\ a\_f\_d})^2 + l}},\ t = 1, ..., T$$

where $l \geq 1$ is a positive integer which characterizes the agent's type, and:

$$1 \leq k_1 < T,\ k_2 \geq 0,\ 1 \leq k_1 + k_2 \leq T - 1$$
(fixed integers)

$$-1 < L_{t-1} \leq ... \leq L_{t-k_1} \leq 0,\ \ -1 < \overline{L}_t \leq ... \leq \overline{L}_{t+k_2} \leq 0$$

The weights may differ across individuals. They are updated during the control period each time as new observation becomes available. The decision-maker gives a higher importance to the past and future deviations which are closer to the moment of implementation of a new optimal action. Smaller the weight is, higher is the importance given by the agent to the system deviation from his local objective. Given the potential destabilizing role of a long memory, the agent will include in the analysis only a limited history of the process. Distant past observations might increase significantly the biais of the estimators in the econometric model. It generally exists an arbitrary element as regards the choice of the backward lag k_1. The objective is to find the better compromise between fit and complexity. The larger the forward lag k_2 is, the more the prediction error increases. Distant forecasts are difficult to formulate due to unpredictable external disturbances (which generally affect the system performance).

The risk can be interpreted like the agent's degree of confidence in the future. It decreases with uncertainty. It is only by taking into account both, the past and the expected future, that the agent can optimally evaluate the risk in a changing environment. It allows for a better risk allocation at each period of control. A mixing of objectivity and subjectivity will always characterize the agent's degree of risk-aversion. The complexity of this mixing is given by the changing environment design and the agent's typology. Generally, there is an inherent inertia effect of the environment due to its capacity of reaction.

Remark 1. We underline here the local nature of the agent's risk-aversion. This is defined for a neighborhood of the optimal fixed targets $y^g_{t-1}, ..., y^g_{t-k_1}$ and respective $y^g_t, ..., y^g_{t+k_2}$. It will therefore exist some neighborhood effects of the system dynamics on the risk-aversion index. There is a strong relationship between the optimal targets selected by the decision-maker and his attitude to risk during the control period. Smaller values of the targets are correlated with a smaller degree of risk-aversion and hence a higher sensitivity to risk for the decision-maker.

Remark 2. The higher (lower) the degree of risk-aversion at time t, the lower (higher) the absolute risk-aversion index $\varphi^{r\_a}_{t,\ p\_f}$. It is important to distinguish between local risk-aversion (at each period t) and global risk-aversion (over the whole).

Remark 3. There is no loss of generality as regards the upper and lower bounds of the absolute risk-aversion index $\varphi^{r\_a}_{t,\ p\_f}$ (by construction, it belongs to $(-1, 0]$) because we can always find an one-to-one function from $(-1, 0]$ to $(a, 0]$, with $a < -1$ a fixed real number.

Remark 4. Large deviations of the system with respect to the decision-maker's fixed targets leads to the following conditions:

$$\parallel y_{t-j_1} - y^g_{t-j_1} \parallel \ \gg 1,\ j_1 = 1, ..., k_1$$

$$\parallel y^a_{t+j_2|I_{t+j_2}} - y^g_{t+j_2} \parallel \ \gg 1,\ j_2 = 0, ..., k_2$$

Proposition 2. The absolute risk-aversion index function $t \longrightarrow \varphi^{r\_a}_{t,\ p\_f}$ $(t = 1, ..., T)$ is non-monotonous over time. Risk perception changes with the system behavior as well as the way the agent interprets its evolution.

Proof. In the appendix. ■

In real world, the risk is not uniformly distributed over the entire working horizon. During the control period, the decision-maker becomes more or less risk-averse according to the system fluctuating behavior. Uncertainty does not necessarily diminish over time. It verifies for simple systems as well as for complex dynamic environments. We give below a suggestive graphic in this sense.

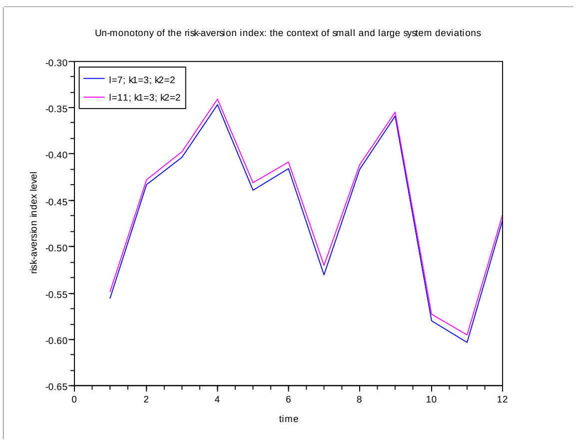

Figure 3:

It is interesting to remark that for smaller (higher) deviations of the system, we obtain a smaller (higher) value of the quadratic loss function but a higher (smaller) value of the risk-aversion index. In other words, $\varphi^{r\_a}_{t,\,p\_f}$ is a decreasing function of $W_{[1,t]}$. The decision-maker more readily accepts the risk when his loss is decreased. His behavior is characterized by a smaller degree of loss aversion. It changes with increasing gains. We call this the Decreasing Absolute Dynamic Risk-Aversion (DADRA) assumption.

Remark 5. In dynamic systems with a smooth evolution (but not only) it may be possible to obtain the same index value at distinct periods of time. Mathematically speaking, this behavior is due to the non-injectivity of the index function. We illustrate below two examples of scenarios when all system deviations are small (high).

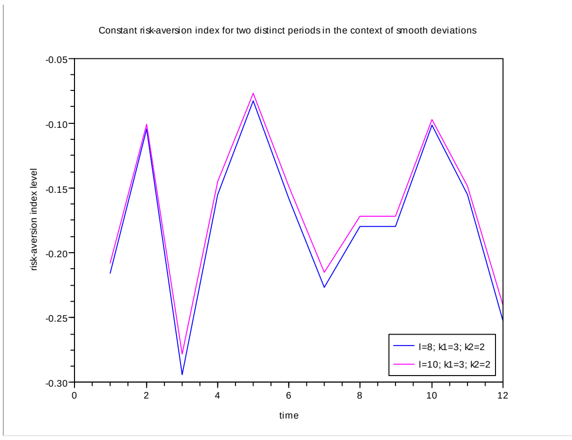

Figure 4:

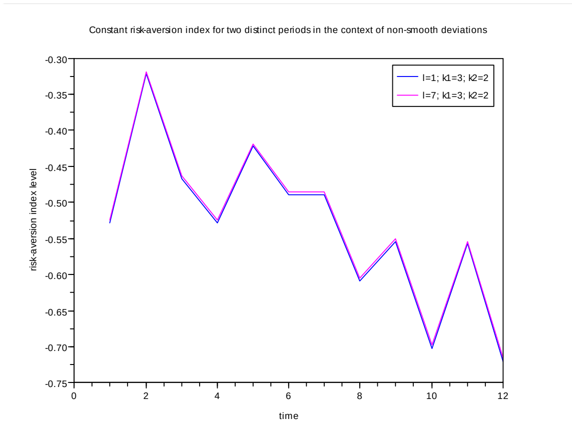

Figure 5:

6.3. Risk-Aversion Based only on Past Performances (a Truncated History of the Process)

Suppose that the decision-maker takes into account only a truncated history of the system in the process of estimation of the index. More exactly, only the most informative information is considered. In this particular context, we propose the following definition for the risk-aversion index:

Definition 2.

$$\varphi^{r\_a}_{t,\,p} \overset{def.}{=} \frac{\| y_{t-1} - y^g_{t-1} \|^2 L_{t-1} + \ldots + \| y_{t-k_1} - y^g_{t-k_1} \|^2 L_{t-k_1}}{\sqrt{(\| y_{t-1} - y^g_{t-1} \|^2 + \ldots + \| y_{t-k_1} - y^g_{t-k_1} \|^2)^2 + l}}, \; t = \overline{1,T}; \; k_1 = \overline{1, T-1}$$

with $l \geq 1$, a fixed integer which characterizes the agent's type, and

$$-1 < L_{t-1} \leq \ldots \leq L_{t-k_1} \leq 0$$

some weighting scalars attached to the system deviations with respect to the optimal path η. For further details, see PROTOPOPESCU, D., 2003A.

Two system deviations with respect to the agent's fixed targets are said to be comparable if and only if their ratio is very close to 1 in magnitude.

Proposition 3. Suppose that all system deviations are comparable. In this context, the agent is characterized by a higher degree of risk-aversion at the same stage of the control if a smaller value for the backward lag k_1 is considered in the estimation of the local index $\varphi^{r\_a}_{t,\,p}$.

Proof. In the appendix. ■

This result proves the role played by the system history in estimating the agent's degree of risk-aversion. It is important to optimally use all the available information from the system. It improves the risk assessment during the control period. When all system deviations are small and comparable, we say that it is characterized by a smooth evolution (i.e., without significative time-shifts). By contrast, a model with relevant time-shifts is characterized by large (comparable) deviations.

It is interesting to underline that the shift amplitude of the system target variable is computed with respect to the (fixed) optimal reference path η. We present below a graphic illustration of this theoretical result.

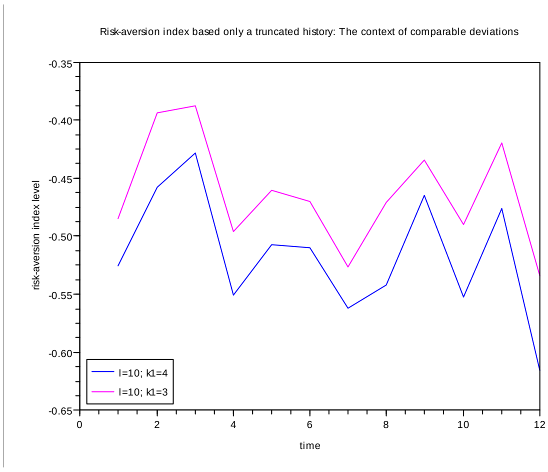

Figure 6:

6.3.1. High Potential Shifts

Shift happens. The agent learns from failures. Every high deviation (seen as a failure) is analyzed in order to avoid unexpected fluctuations of the system in the future. The moment when a large deviation from the target is perceived is significant. We call a high positive shift a context where the system is characterized by consecutive small levels of performance followed by a high level one. In the opposite case, we call this a high negative shift.

The agent's objective during the period of control is to obtain smooth forward shifts with respect to the fixed optimal targets. It contributes to the dynamic equilibrium and stability of the system. The risky shift phenomenon is one of the key issues in economics, and in the study of controlled dynamic systems in particular.

The concept of risk-aversion is appropriate to dynamic stochastic environments whose behavior change significantly over time. It is the case of high fluctuating systems.

Proposition 4. Consider the following two opposite scenarios: **i**) the transition of the system is from consecutive small deviations to a large deviation; and **ii**) the transition of the system is from consecutive large deviations to a small deviation. For this type of scenarios, the agent's degree of risk-aversion is highly dependent on the weights attached to the large deviations of the system.

Proof. In the appendix. ■

It is an astonishing result. Sudden significative changes in the system behavior can affect differently the agent's risk perception. Large shifts are correlated with large deviations of the system with respect to the fixed targets. The shift amplitude is generally different in different contexts, depending on one hand, on the type of transition, and on the other hand, on the size of the transition shock.

We give below three distinct attitudes to risk as regards the agent's individual perception about large fluctuations of the system (in both above contexts).

a) the large deviation of the system at time $t-1$ (in the context of the first scenario) is much more important for the decision-maker than all other (k_1-1) large deviations obtained in the context of the second scenario.

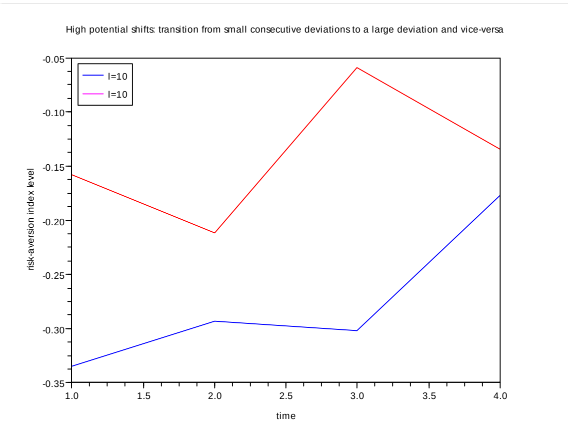

Figure 7:

b) the large deviation of the system at time $t-1$ (in the context of the first scenario) is much less important for the decision-maker than all other (k_1-1) large deviations obtained in the context of the second scenario.

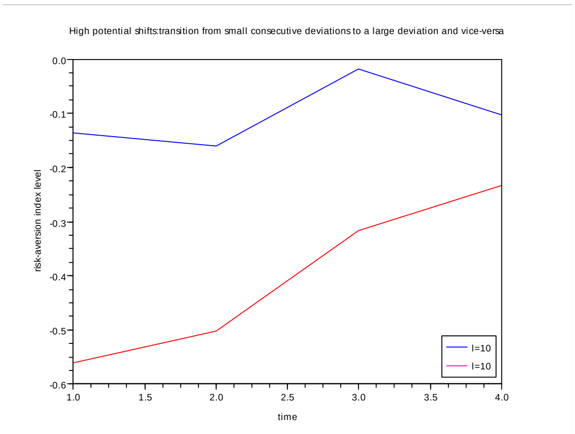

Figure 8:

c) the large deviation of the system at time $t-1$ (in the context of the first scenario) is either much more or much less important for the decision-maker than all other (k_1-1) large deviations obtained in the context of the second scenario.

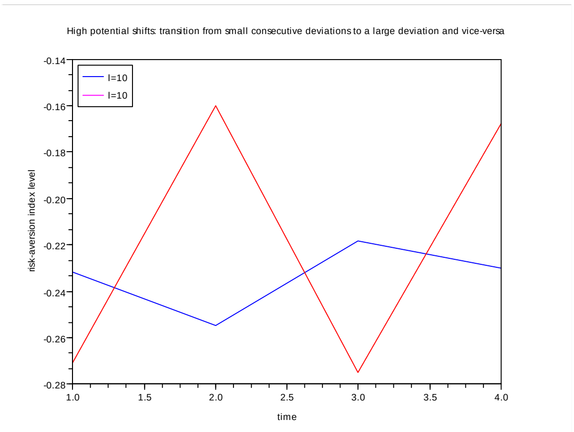

Figure 9:

We underline that for the three above graphics, the risk-aversion index value at time $t = 1, ..., 4$ corresponds, respectively, to the following distinct scenarios: **1**) system transition from four large deviations to one small deviation; **2**) system transition from three large deviations to one small deviation; **3**) system transition from two large deviations to one small deviation; **4**) system transition from one large deviation to one small deviation.

6.3.2. Fluctuating System

In what follows, we analyze the impact of the system fluctuations on the agent's attitude to risk during the period of control. Suppose that all high deviations are comparable in magnitude. We can distinguish two cases, according to the parity of k_1.

Case 1. k_1 is an even number $(k_1 = 2k')$. We can imagine two distinct scenarios:

Scenario 1. System deviations fluctuate successively from one period to another such that: $\| y_{t-2k'} - y^g_{t-2k'} \|$ is high, $\| y_{t-(2k'-1)} - y^g_{t-(2k'-1)} \|$ is small,..., $\| y_{t-2} - y^g_{t-2} \|$ is high, $\| y_{t-1} - y^g_{t-1} \|$ is small.

Scenario 2. System deviations fluctuate successively from one period to another such that: $\parallel y_{t-2k'} - y^g_{t-2k'} \parallel$ is small, $\parallel y_{t-(2k'-1)} - y^g_{t-(2k'-1)} \parallel$ is high,..., $\parallel y_{t-2} - y^g_{t-2} \parallel$ is small, $\parallel y_{t-1} - y^g_{t-1} \parallel$ is high.

For this type of scenarios, the system deviation at time $t-1$ is the most significative for the decision-maker. The value of the parameter L_{t-1} is considerably higher than all others parameters L_{t-i} $(i = 2, ..., k_1)$.

Proposition 5. For an even backward parameter k_1, the agent is characterized by a smaller degree of risk-aversion at time t for the first scenario, even if the negative and positive shifts are in equal number.

Proof. In the appendix. ■

This result proves that the sense of transition is perceived differently by a risk-averse decision-maker. The initial state of transition plays a crucial role in estimating his degree of risk-aversion at a given period of time. His objective is to anticipate significant fluctuations in the target variable. It allows for a more refined perception of large system deviations and thus for an improved assessment of the risk-aversion index over time.

Case 2. k_1 is an odd number $(k_1 = 2k' + 1)$

Proposition 6. For an odd backward parameter k_1, the agent is characterized by a smaller degree of risk-aversion (at the same control period) for the second scenario compared with the first one. We have the following inequality:

$$\varphi^{r\_a\_first.sc.}_{t,\ p,\ k_1=2k'+1} < \varphi^{r\_a\_\sec ond.sc.}_{t,\ p,\ k_1=2k'+1},\ \forall\ t = 1, ..., T$$

Proof. In the appendix. ■

A natural question arises: in real world, does one more often envisage to pass from a high variation of the system to a small one or the opposite? The answer is not obvious. When the system inertia is high, the first scenario is not easy to be carried out. Also, an effective control will prevent the realization of the second scenario. The decision-maker must avoid to much deviate from the fixed targets. A robust control strategy (i.e., with a low sensitivity to changes in the input data) is necessary in order to reach this objective. We give below two suggestive graphics which illustrate the agent's behavior to risk in the context of a fluctuating system. The first (second) graph corresponds to a high (weak) risk-averse decision-maker who implements his optimal policy in the context of a dynamic environment characterized by a large inertia.

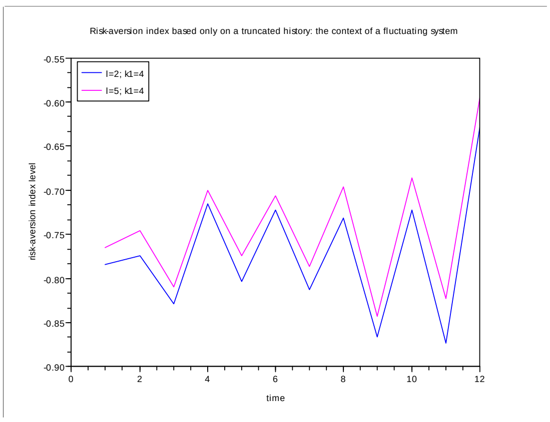

Figure 10:

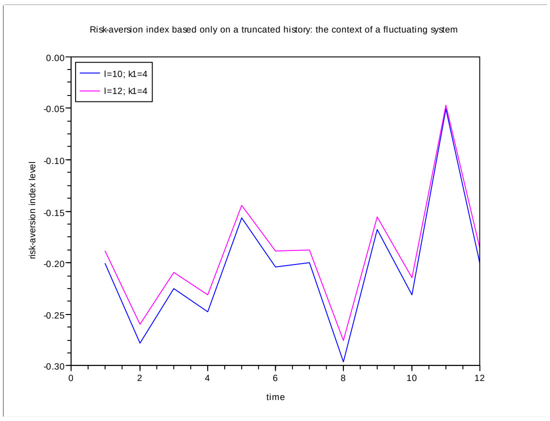

Figure 11:

Effective management of the system inertia will have a positive impact on the agent's attitude to risk. It is interesting to analyze the correlation between the agent's behavior to risk and his degree of inertia. This is an exciting fruitful area for future research, with strong implications in terms of policy-making.

6.4. Risk-Aversion Based on Past Overall Performances (a Progressive History of the Process)

Suppose that the decision-maker has the interest to use a progressive history of the process in the valuation of the risk-aversion index. This may be the case where the horizon length is short and the agent needs more information useful in improving the process of risk-assessment. The index value will be updated each time as new observation becomes available. In this particular context, we define:

Definition 3.

$$\varphi_{t,\ w}^{r\_a} \stackrel{def}{=} \frac{\| y_{t-1} - y_{t-1}^g \|^2 L_{t-1} + ... + \ \ \| y_0 - y_0^g \|^2 L_0}{\sqrt{(\| y_{t-1} - y_{t-1}^g \|^2 + ... + \ \ \| y_0 - y_0^g \|^2)^2 + l}}, \ \ t = 1, ..., T$$

with $l \geq 1$, a fixed integer which characterizes the decision-maker's type, and

$$-1 < L_{t-1} \leq L_{t-2} \leq ... \leq L_0 \leq 0$$

some weighting scalars attached to the system deviations with respect to the optimal reference path η.

We denote by y_0^g the target variable at time $t = 0$ (the initial state of the system) employed by the decision-maker in the previous optimization scheme. It is supposed to be small.

The initial state of the process will influence the system trajectory and implicitly the decision-maker's attitude to risk during the period of control. For a dynamic system characterized by a sensitive dependency of initial conditions, the variations of the index over time will be significative. Agent's sensitivity to risk (generally non-uniform over time) will be more relevant. This behavior may be captured by the shape of the agent's index curve. An empirical analysis in the case of a free-floating initial state can illustrate this risk-sensitive behavior.

We can imagine two distinct scenarios concerning the agent's attitude to risk before to start the control:

a) y_0 is small; in this case, the deviaiton $\parallel y_0 - y_0^g \parallel$ is small and thus the absolute risk-aversion index $\varphi_{1,\,w}^{r-a}$ will have a high value. It is a strategic attitude for the agent to start the control with a smaller degree of risk-aversion.

b) y_0 is high; in this case, the deviation $\parallel y_0 - y_0^g \parallel$ is high and therefore the absolute risk-aversion index $\varphi_{1,\,w}^{r-a}$ will have a small value. This will determine the agent to start the control process.

The observability of the system is dependant only on the system states and the system output. In real world, the relationship between the agent's reaction to the perceived states of nature and his attitude to risk is complex. This is a consequence of the importance the agent places on the system states.

Generally, risk-aversion makes the reaction stronger than risk-neutrality. However, when both players are risk-averse but at very different degrees, the less cautious of the two can have a weaker reaction than in the risk-neutral case. This is an astonishing result that naturally follows from the analysis of the relationship between high deviations and perceived states of nature.

We give below two suggestive graphics which illustrate the variation of the risk-aversion index $\varphi_{1,\,w}^{r-a}$ in function of the system initial state y_0.

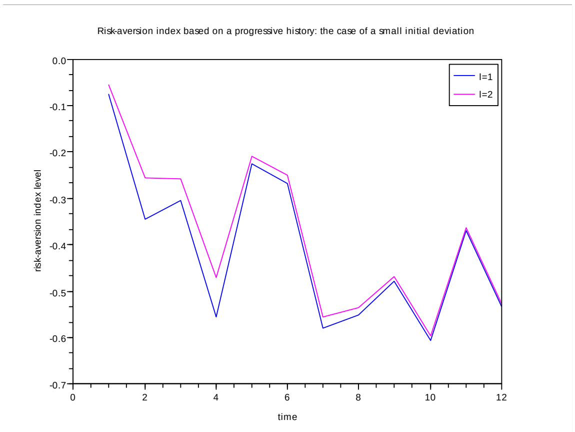

Figure 12:

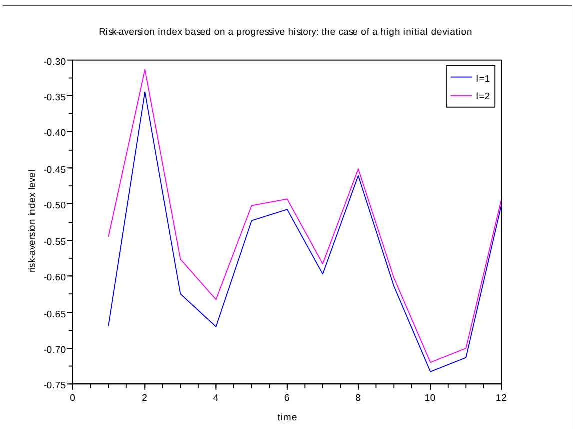

Figure 13:

Proposition 7. A decision-maker who manages more and more hardly the evolution of the system will become more and more risk-averse over time.

Proof. In the appendix. ■

As the number of consecutive failures increases, the economic agent will become less and less confident and thus his degree of risk-aversion will be more and more raised. It is the context when the agent becomes excessively risk-averse during the control period.

This type of scenario is possible when large deviations are correlated with a high inertia of the system. The potential for learning is limited in this particular context.

In a noisy environment, the actions implemented by the agent have a weak impact on the system irregular trend. The agent must avoid to much deviate from the optimal path η.

Reasoning by analogy, we conclude that in the case where the agent controls better and better the system trajectory, he will become more and more confident over time; his degree of risk-aversion will be less and less raised. It is important to note that the two above scenarios are not symmetrical.

We give below a graphical illustration for these two distinct scenarios in the case where the amplitude of the system deviation at time $t = 1$ is relatively small or very large.

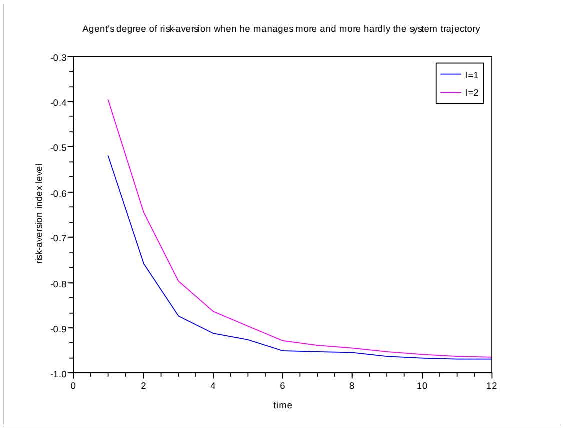

Figure 14:

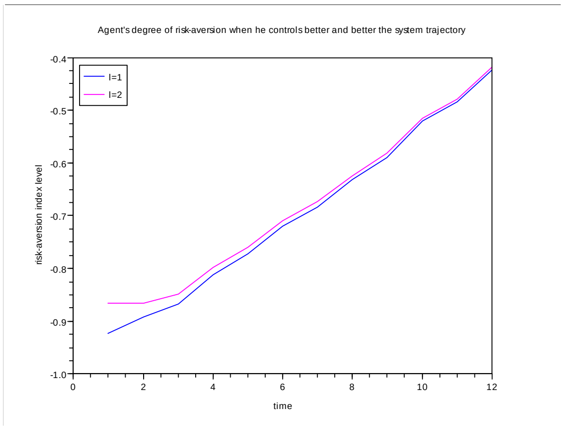

Figure 15:

It is interesting to remark that in the case where the agent more and more hardly controls the system trajectory, the index value decreases more quickly, while in the opposite case, this increases slowly. In other words, it is more easy to loss the control of the system than to improve its trajectory. For the second scenario, the shape of the index curve is almost linear, while for the first scenario, this is characterized by an important degree of nonlinearity. This is a very surprising result, far from intuitive.

6.5. Risk-Aversion Based only on Rational Anticipations of the System Behavior in the Future

In this section, we analyze the case where the agent takes into account only rational anticipations about the system behavior when modelling the risk over time. It implies a risk-attitude adjustment during the period of control. This may be the case where the agent look towards the future rather than evaluating the evolution of the system in the past. In other words, deviations from past targets are ignored except to the extent that they affect the future. In this particular context, the absolute risk-aversion index is defined as follows:

Definition 4.

$$\varphi_{t,\ f}^{r\_a} \overset{def.}{=} \frac{S_{t,\ w\_a\_f\_d}}{\sqrt{(S_{t,\ a\_f\_d})^2 + l}},\ t = 1, ..., T.$$

6.6. Comparing the four Dynamic Risk-Aversion Definitions

Proposition 8. In a dynamic environment, the agent's degree of risk-aversion varies according to his adopted strategy to manage exogenous and endogenous risks.

Proof. In the appendix. ■

Future uncertainty affects the agent's attitude to risk. Anticipation of higher (smaller) system deviations with respect to the fixed targets corresponds to a higher (smaller) degree of risk-aversion only if the agent attributes appropriate weights to these deviations.

More information helps for better forecasting the risk but it does not necessarily decrease the agent's uncertainty. System dynamics and its inherent inertia can produce a significative change in the agent's attitude to risk.

Errors from learning occur. They can be improved but not eliminated when analyzing real phenomena. The potential for learning is limited in a noisy environment.

It is useful to underline that the process of learning is generally non-monotonous over time. We also note that for a risk-averse decision-maker, the benefits of active learning are important. The agent dynamically adjusts his actions in order to minimize the distance between actual state of the system and target trajectory.

Large endogenous risks are not easy to avoid due to their high correlation with high fluctuations of the system. Random transitory (or permanent) shocks in observed movements of the system will influence the agent's degree of risk-aversion.

Contrary to what is generally believed or the intuition should suggests, the sensitivity to risk can be lower when the agent does not explicitly integrate knowledge of the past in the index formula. The observed outputs depend upon his ability to anticipate the future, as well as other factors that are outside the agent's control.

A natural question arises: how to define the optimal trade-off between past and expected future when dealing with adaptive risk perception and optimal risk assessment?

We give below four suggestive graphics (corresponding to the four proposed risk-aversion index definitions) which illustrate numerically the above theoretical result.

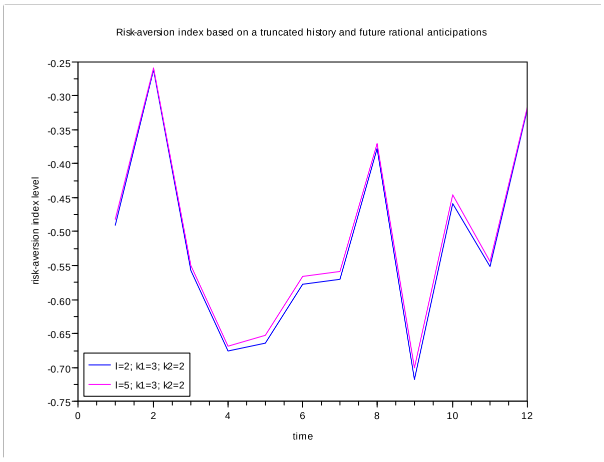

Figure 16:

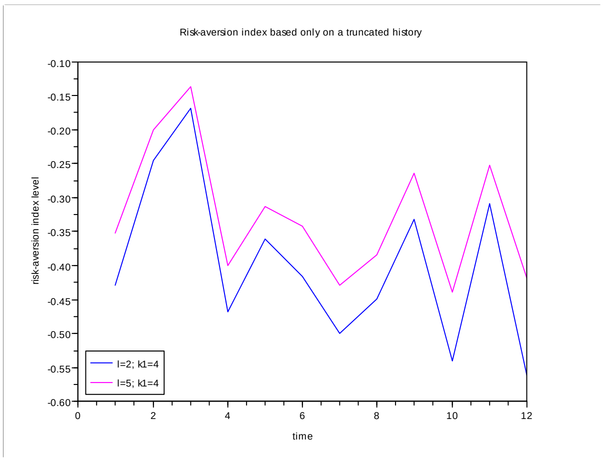

Figure 17:

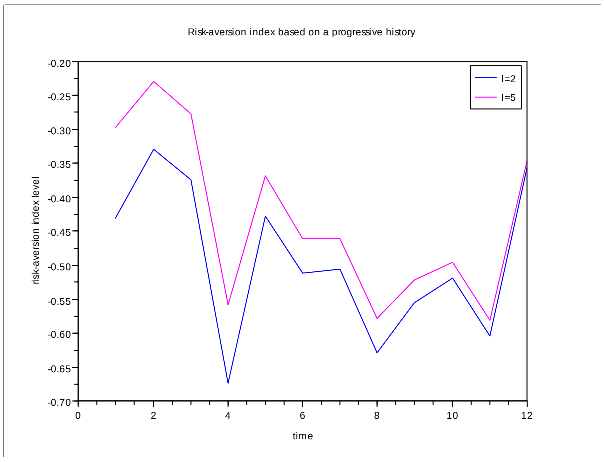

Figure 18:

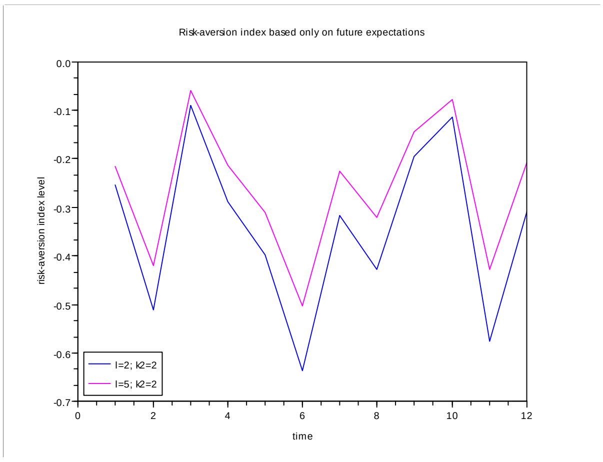

Figure 19:

We have the following index inequalities:

$$\varphi^{r\_a}_{1,\ p\_f} < \varphi^{r\_a}_{1,\ w} < \varphi^{r\_a}_{1,\ p} < \boldsymbol{\varphi}^{r\_a}_{1,\ f},\ \text{while}\ \boldsymbol{\varphi}^{r\_a}_{2,\ f} < \varphi^{r\_a}_{2,\ w} < \varphi^{r\_a}_{2,\ p\_f} < \varphi^{r\_a}_{2,\ p}$$

$$\boldsymbol{\varphi}^{r\_a}_{12,\ p} < \varphi^{r\_a}_{12,\ w} < \varphi^{r\_a}_{12,\ p\_f} < \varphi^{r\_a}_{12,\ f},\ \text{while}\ \boldsymbol{\varphi}^{r\_a}_{6,\ f} < \varphi^{r\_a}_{6,\ p\_f} < \varphi^{r\_a}_{6,\ w} < \boldsymbol{\varphi}^{r\_a}_{6,\ p}$$

$$\boldsymbol{\varphi}^{r\_a}_{3,\ p\_f} < \varphi^{r\_a}_{3,\ w} < \varphi^{r\_a}_{3,\ p} < \varphi^{r\_a}_{3,\ f},\ \text{while}\ \varphi^{r\_a}_{8,\ w} < \varphi^{r\_a}_{8,\ p} < \varphi^{r\_a}_{8,\ f} < \boldsymbol{\varphi}^{r\_a}_{8,\ p\_f}$$

Distinct agents, characterized by distinct risky preferences, have in general distinct perceptions of endogenous risks in a given environment. It generally depends on how they succeed to manage the available information from the system as well as future information from outside the system. It is important to make distinction between risk-aversion before and after starting the control process. They are not generally based on the same information set. Before to start the control, the agent's degree of risk-aversion is measured on the basis of an exogenous information set; it defines the agent's type at a given period of time. After to start the control, the agent's degree of risk-aversion is estimated on the basis of a non-decreasing endogenous information set. In a dynamic evolving context, the agent's type can change. In a static context (one aggregated period of time), it is not justified to fix an arbitrary value for the risk-aversion index. All the more, in a multiple-horizon dynamic environmemt, this procedure does not make sense.

7. Qualitative Consequences on Risk Management

The object of this section is to explore the implications of the proposed risk-aversion concept on the decision-maker's adaptive behavior by developing realistic scenarios for the dynamic environment (for further details, see PROTOPOPESCU, D., 2003B).

7.1. Small System Deviations

Proposition 9. Suppose that all system deviations in the past and future are small. For this type of scenario, the decision-maker is characterized by a small risk-aversion during the control period. When all system deviations are almost null (a borderline case), the decision-maker becomes almost risk-neutral.

Proof. In the appendix. ■

This result reveals the correlation that exists between risk-aversion and highly fluctuating systems. It is interesting to note, in this case, that for small symmetrical deviations with respect to the fixed targets, the decision-maker will adopt the same attitude to risk at time t.

Zero risk can exist if the agent does not attribute any importance to the system deviations. This may be the case of a dynamic system characterized by small deviations. However, when the deviations are (relatively) large, this scenario is not compatible with a rational behavior (SEE, AMONG OTHERS, DOUARD, J., 1996). Because at least one of the system deviations (in the past or future) is strictly positive, the index value is non null. This can be explained by the presence of significative random disturbances, the inherent inertia of the system and the inevitable forecast errors in predicting the system trajectory. The hypothesis of risk-neutrality for the entire working horizon, very often employed in the literature, is very restrictive. In real world, this work-hypothesis is non-realistic, taking into account the dynamic complexity of the environment. Its use in many theoretical and empirical studies is satisfactory only for parsimony purposes. The apparent need for parsimony is derived by the facility brought in the construction of the models. Parsimony may seems desirable, but is in fact not because it generally introduces non-negligible errors in the model. It limits an adaptive endogenous behavior to an exogenous pseudo-optimal one. We give below two suggestive graphics that illustrate the adaptive behavior towards risk of the decision-maker in the context of small (almost null) system deviations.

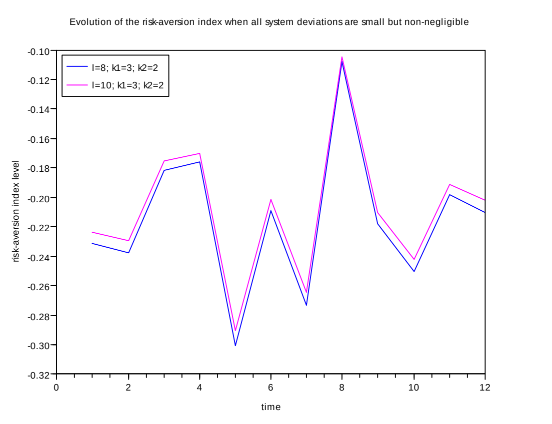

Figure 20:

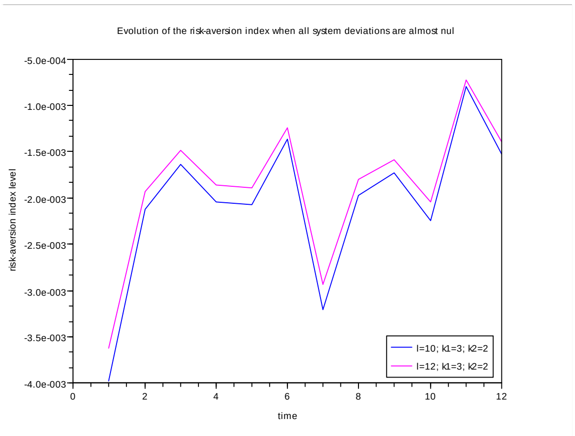

Figure 21:

We observe here the non-monotonous character of the risk-aversion index as well as the intimate correlation between the choice of the parameter l and the agent's risk perception during the control period.

7.2. Small and Large System Deviations

Proposition 10. In the context of a dynamic system characterized by small and large deviations, the agent will not be necessarily more risk-averse for the same control period compared to the case where all system deviations are small. It depends on two distinct factors: **i**) the moment of time when these deviations arrive in the past or are expected to be realized in the future; and **ii**) the weighting scalars the agent will attach to the system deviations with respect to the equilibrium path η.

Proof. In the appendix. ■

The theoretical intuition of this result is based on two evidences: **i**) the effect of high deviations in the past will diminish in time; and **ii**) large expected deviations in distant future will have a non significative effect on the agent's risk sensitivity. Individuals are more risk tolerant on distant horizons. We give below a realistic scenario for the system evolution which confirms this intuition.

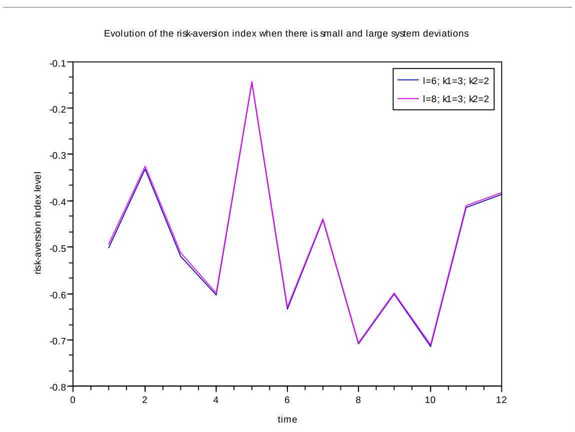

Figure 22:

As is easily seen, at time $t = 5$, we ascertain a higher index value compared to that one obtained when all system deviations are small.

7.3. Large System Deviations

Assumption 6: The parameter l verifies the additional conditions:

$$\textbf{i)}\ l / \parallel y_{t-j_1} - y^g_{t-j_1} \parallel^4 \cong 0, \forall\ j_1 = 1, ..., k_1$$

$$\textbf{ii)}\ l / \parallel y^a_{t+j_2|I_{t+j_2}} - y^g_{t+j_2} \parallel^4 \cong 0, \forall\ j_2 = 0, ..., k_2$$

Proposition 11. When all system deviations are large and comparable, the agent's degree of risk-aversion is highly dependent on the weighting scalars attached to the system deviations.

Proof. In the appendix. ■

There is a relationship between risk-neutrality and perverted perception of the environment. This is the case where the agent does not attribute any importance to the system deviations. When the importance attached is negligible, the agent is characterized by an almost risk-neutral behavior. By contrast, for non-negligible weighting scalars attached to the system deviations, the agent's degree of risk-aversion is significative. Risk perception and environment impact are highly correlated.

It is interesting to note that for this type of scenario, the boundary condition imposed on the fixed targets does not allow for a symmetrical evolution of the dynamic system.

In order to illustrate the above theoretical result, we give below three suggestive graphics which show the strong correlation between the agent's risk perception and the size of the weighting scalars attached to the system deviations.

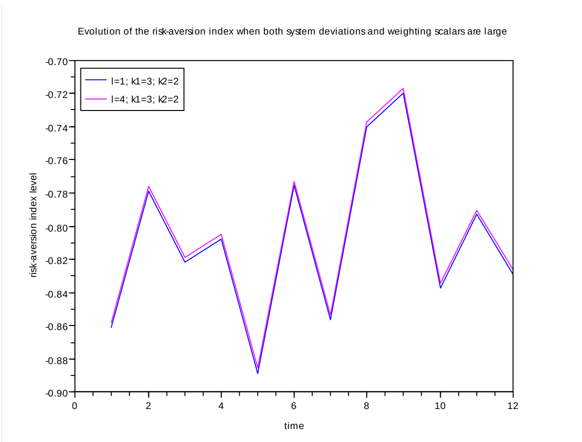

Figure 23:

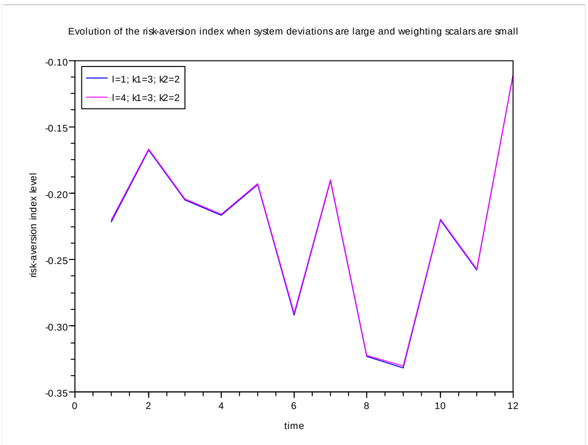

Figure 24:

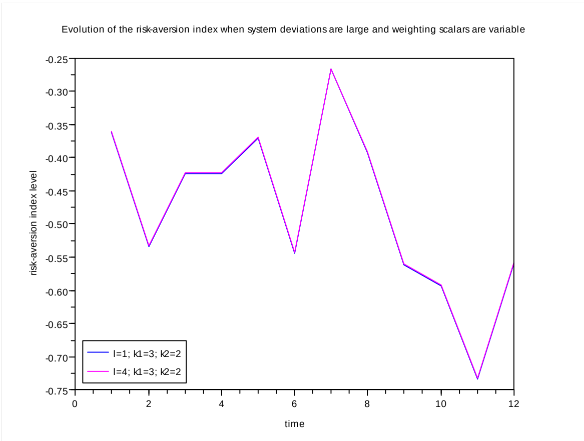

Figure 25:

Proposition 12. Suppose that all system deviations are comparable. A higher value of the sum of k_1(feedback period) and k_2 (forward period) does not necessarily diminish the agent's degree of risk-aversion at the same stage of the control.

Proof. In the appendix. ■

This result proves the trade-off between past and future evolution of the system. Neither the past nor the future have a dominant effect on the agent's risk behavior. In general, risk perception is influenced by the aggregate cumulative effect of both temporal dimensions.

Often the past is a backwards indicator of the future. The weight that the agent places on the future may be correlated with exogenous signals from the past. Depending on the context, the past can provide more (or less) informative signals about the system future trend. They can change the agent's behavior to risk during the period of control. We give below two suggestive graphics which illustrate this theoretical result.

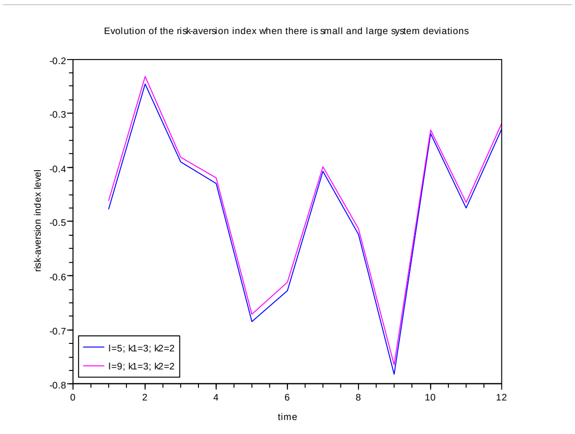

Figure 26:

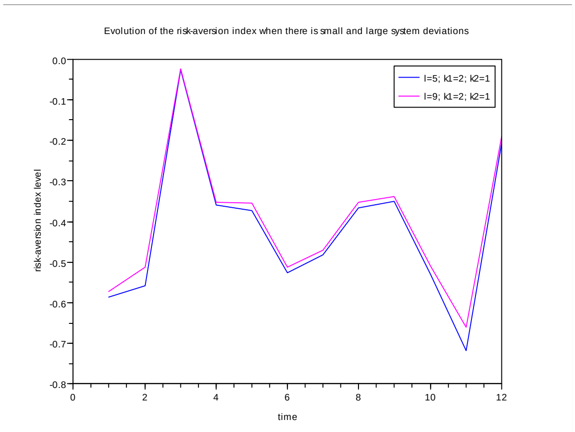

Figure 27:

A smooth (or almost constant) evolution of the system will not significantly change the agent's attitude to risk. In this case, the past and the expected future will have relatively the same impact on the agent's risk behavior.

Generally, the agent's risk sensitivity is highly correlated with a significative change in the dynamic of the system. As a consequence, the more the system is non-linear, the more marked will be the agent's degree of risk-aversion over time.

There is a close relationship between the system inherent inertia and the agent's risk perception. Distinct agents develop individual behaviors with regard to how they respond in similar risky situations. They are most often characterized by different degrees of risk-aversion. However, it may be possible that their attitudes to risk be similar in a given risky situation.

8. Excessive Risk-Averse Decision-Maker

Experimental evidence shows that individuals overweight extreme events. They can modify their individual type behavior. The ability to assess future risks associated with extreme events is increasingly important to decision-makers.

Let $\varphi_{\min}^{p\_f}$ be an optimal risk-aversion threshold fixed by the agent before starting the control and for the entire working horizon. His objective is not to exceed it, if not he becomes excessively risk-averse for the current period of control, being characterized by an extreme pessimism (prudence).

We correlate the exceeding of the threshold $\varphi_{\min}^{p\_f}$ during the control period with at least one high deviation of the system from the agent's fixed targets.

There is a close relationship between the choice of the agent's optimal targets and the possibility to exceed the fixed risk-aversion threshold $\varphi_{\min}^{p\_f}$. Smaller optimal targets will generally imply a smaller absolute risk-aversion index and thus a higher "probability" to exceed the threshold $\varphi_{\min}^{p\_f}$.

An agent with a higher (smaller) risk-aversion before starting the control will choose a smaller (higher) threshold $\varphi_{\min}^{p\_f}$.

If $\varphi_{t,\,p\_f}^{r\_a}$ characterizes the agent's local risk-aversion (at time t), $\varphi_{\min}^{p\_f}$ will characterize his global risk-aversion (over the whole). The optimal threshold $\varphi_{\min}^{p\_f}$ must be selected such that it offers the best characterization of the agent's type. It depends on the particular environmental context.

We must distinguish between $\varphi_{\min}^{p\_f}$ and $\varphi_{1,\,p\_f}^{r\_a}$. It is a strategic attitude for the agent to fix a threshold $\varphi_{\min}^{p\_f}$ inferior to $\varphi_{1,\,p\_f}^{r\_a}$.

The above definition of $\varphi_{\min}^{p\_f}$ give us a good explanation why the absolute risk-aversion index is a non-monotonous function with respect to the time variable.

An increasing index would be in contradiction with the definition of $\varphi_{\min}^{p\_f}$ (it would be never exceeded). In real world, this limit threshold is most often exceeded by risk-averse agents.

In the case of a decreasing index, the dynamic learning would be inefficient. It would constrain the decision-maker to exceed the threshold $\varphi_{\min}^{p\_f}$ over time, whatever his strategy, and hence he would become excessively risk-averse for the remaining period of control. It would be non-realistic.

The decision-maker's global utility function corresponding to the risk-aversion optimal threshold $\varphi_{\min}^{p\_f}$ is given by:

$$U_{[1,T]}(W_{[1,T]},\varphi_{\min}^{p\_f}) \stackrel{def}{=} \frac{2}{\varphi_{\min}^{p\_f}}\{\exp(-\frac{\varphi_{\min}^{p\_f}}{2}\cdot W_{[1,T]})-1\}$$

Proposition 13. Any risk-averse decision-maker is characterized by an optimal risk-aversion threshold $\varphi_{\min}^{p\_f}$ chosen in function of his individual type.

Proof. In the appendix. ■

This result allows for a differentiation of common types as well as a total separation of distinct types. Two common (distinct) types of decision-makers will not generally give the same interpretation to the system evolution. Its endogenous impact is perceived differently by different decision-makers.

With the same control policy instruments and some linear decision rules, they can implement distinct optimal actions for a given period t. However, it may be possible that their actions be the same for the same control period. It is the case of non-linear decision rules. This can be explained by arguments based on the non-injectivity of the control rule, regarded as a function of the index variable.

In this approach, we deal with a continuum of agents' types and heterogenous risk preferences. This type of modelling provides a better description of agents' behavior as decision-makers.

We illustrate below a realistic scenario concerning the choice of the agent's risk-aversion threshold.

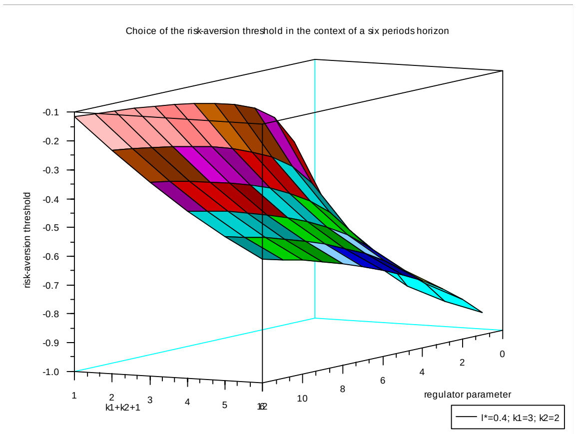

Figure 28:

In the following, we give an interesting theoretical result which characterizes the boundaries of the decision-maker's utlity function during the period of control.

Proposition 14. Any risk-averse decision-maker is characterized by a local utility function whose upper and lower bounds vary according to the risk-aversion index level relative to the optimal threshold $\varphi_{\min}^{p\_f}$.

Proof. In the appendix. ■

An important consequence of the above result is the complete separability of the agent's risk-averse preferences, depending on the index variation with respect to the fixed optimal threshold $\varphi_{\min}^{p\_f}$. It allows to characterize the agent's type in function of his individual preferences.

The inherent disutility of the risk associated with extreme events is also proved from this new perspective. We give below two suggestive graphics in this sense, in the case where $\varphi_{\min}^{p\_f} = -0.5$.

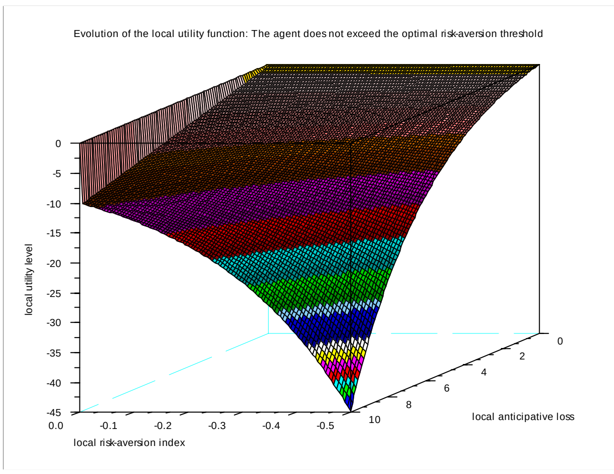

Figure 29:

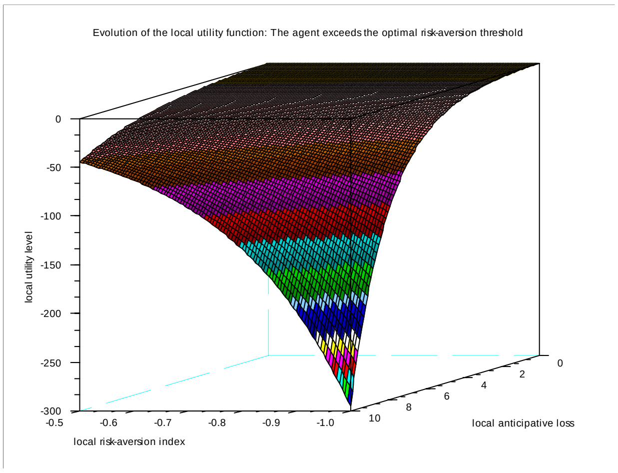

Figure 30:

In order to illustrate the potential correlation between the length of working horizon and the agent's degree of risk-aversion, we give in the following an interesting result in this direction.

Proposition 15. There is a potential effect of the working horizon length on the decision-maker's attitude to risk.

Proof. In the appendix. ■

This approach allows to characterize a closed-loop strategy as regards the attitude to risk of a rational decision-maker in a dynamic environment. It is useful to note that the horizon length may generate variations of risk-aversion over time. We illustrate below a realistic scenario concerning the potential effect of the horizon length on the agent's risk attitude.

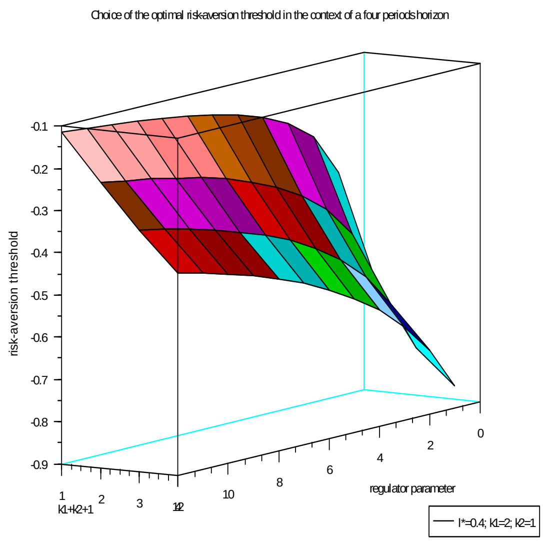

Figure 31:

We remark that in the context of a four periods horizon, the decision-maker can choose an optimal risk-aversion threshold $\varphi_{\min}^{p\_f}$ inferior to -0.45, while for a six periods horizon, $\varphi_{\min}^{p\_f}$ is inferior to -0.57. In other words, the agent's risk-perception (and implicitly his degree of risk-aversion) is dependent on the fixed-horizon length. The above result proves that risk-aversion increases with horizon length.

9. Potential Sensitive Periods

It may exist some significant periods (with an important degree of uncertainty), when the decision-maker has as objective not to exceed a local fixed risk-aversion index level. More exactly, suppose that for each significant period t, the agent chooses, based on his rational expectations, an optimal local value $\varphi_{t,\,p\_f}^{loc.}$ that he wants not to exceed ($\varphi_{t,\,p\_f}^{loc.} \leq \varphi_{t,\,p\_f}^{r\_a}$) and such that $\varphi_{\min}^{p\_f} \leq \varphi_{t,\,p\_f}^{loc.}$. This can be explained by the fact that a global minimum is always smaller or equal than a local one. One should not confuse $\varphi_{t,\,p\_f}^{loc.}$ (which is fixed before starting the control and thus does not depend on the evolution of the system, like $\varphi_{\min}^{p\_f}$; these two thresholds are strictly exogenous, by definition) with $\varphi_{t,\,p\_f}^{r\_a}$ (estimated after the control begins). It is also important to distinguish between $\varphi_{\min}^{p\_f}$ and $\varphi_{t,\,p\_f}^{loc}$. The first one defines the decision-maker's type (for the entire period of control), while the second depends on his objective at time t.

Let t_j ($j = 1, ..., \overline{k}$) be all significant periods on the interval $[1, T]$, where $\{t_1, ..., t_{\overline{k}}\} \subseteq \{1, ..., T\}$. Note that $\varphi_{t_j,\,p\_f}^{loc.}$ are defined only for the periods t_j, while $\varphi_{\min}^{p\_f}$ is chosen by the decision-maker according to his expectations on all $\varphi_{t_j,\,p\_f}^{loc.}$. Consequently, these are $\varphi_{t_j,\,p\_f}^{loc.}$ ($j = 1, ..., \overline{k}$) that influence $\varphi_{\min}^{p\_f}$ and not vice versa. One can say that $\varphi_{t_j,\,p\_f}^{loc.}$ is the equivalent of $\varphi_{\min}^{p\_f}$, but only locally. In this context, we define:

$$\varphi_{\min}^{p\_f} \leq \min\left\{\varphi_{t_1,\,p\_f}^{loc.}, ..., \varphi_{t_{\overline{k}},\,p\_f}^{loc.}\right\}$$

The agent's utility level at time t, defined for a fixed threshold $\varphi_{t,\,p\_f}^{loc.}$, is given by:

$$U_t(W_{[1,t]}, \varphi_{t,\,p\_f}^{loc.}) = \frac{2}{\varphi_{t,\,p\_f}^{loc.}}\{\exp(-\frac{\varphi_{t,\,p\_f}^{loc.}}{2} \cdot W_{[1,t]}) - 1\}$$

We can write:

$$U_t(W_{[1,t]}, \varphi_{\min}^{p\_f}) \leq U_t(W_{[1,t]}, \varphi_{t,\,p\_f}^{loc.}) \leq U_t(W_{[1,t]}, \varphi_{t,\,p\_f}^{r\_a})$$

if and only if

$$\varphi_{\min}^{p\_f} \leq \varphi_{t,\,p\_f}^{loc.} \leq \varphi_{t,\,p\_f}^{r\_a} \;\forall\; t = 1, ..., T.$$

Note that $\varphi_{t_j,\,p\_f}^{loc.}$ can be viewed as a first control-threshold imposed by the agent in order not to exceed $\varphi_{\min}^{p\_f}$. At each stage t_j, it is possible to exceed $\varphi_{t_j,\,p\_f}^{loc.}$ but not $\varphi_{\min}^{p\_f}$. This can be explained by the fact that, in general, the local conditions are more restrictive than the global ones. Since a real time control process is necessarily discrete, the dynamic of the risk-aversion index $\varphi_{t,\,p\_f}^{loc.}$ cannot converge with precision to any fixed optimal value $\varphi_{t_j,\,p\_f}^{loc.}$, but only to a neighborhood of it. When the process of control is finished, the decision-maker will obtain a stochastic neighbouring-optimal trajectory of the risk-aversion index which is expected to be close to the optimal path $\left\{\varphi_{1,\,p\_f}^{loc.}, ..., \varphi_{T,\,p\_f}^{loc.}\right\}$. It is very likely that difference between ex-ante and ex-post behavior towards risk exists.

10. Agent with a Changing Risk Behavior

The way the risk-aversion index is parameterized depends on the problem statement. We refine the analysis by taking into account the case of an economic agent characterized by a changing risk behavior. There may be periods when the agent is risk-averse and others when he becomes (almost) risk-neutral or risk-lover. A mixture of pessimism and optimism can exist (SEE, TOULET, 1982).

We consider the same definition of the index as in **Chapter 6.2**, with the only difference that the weighting scalars attached to the deviations of the system from its reference level lie inside the unit circle. Let us make the following notations:

$$\widetilde{S}_{t,\ w\_p\_d} \overset{not.}{=} \parallel y_{t-1} - y^g_{t-1} \parallel^2 \widetilde{L}_{t-1} + ... + \parallel y_{t-k_1} - y^g_{t-k_1} \parallel^2 \widetilde{L}_{t-k_1}$$

(the weighted sum of squared past deviations at time t)

$$S'_{t,\ w\_a\_f\_d} \overset{not.}{=} \parallel y^a_{t|I_t} - y^g_t \parallel^2 L'_t + ... + \parallel y^a_{t+k_2|I_{t+k_2}} - y^g_{t+k_2} \parallel^2 L'_{t+k_2}$$

(the weighted sum of squared anticipated future deviations at time t)

where

$$-1 < \widetilde{L}_{t-1} \leq ... \leq \widetilde{L}_{t-k_1} < 1, \ -1 < L'_t \leq ... \leq L'_{t+k_2} < 1$$

are weighting scalars attached to observed and expected deviations of the system. Their values are selected according to the importance the agent places on the past and future.

Remark 6. We distinct, in this general context, two completely opposite scenarios:

i) the pure risk-aversion case, defined according to the following boundaries conditions:

$$-1 < \widetilde{L}_{t-1} \leq ... \leq \widetilde{L}_{t-k_1} < 0, \ -1 < L'_t \leq ... \leq L'_{t+k_2} < 0$$

ii) the pure risk-taking case, defined according to the symmetrical boundaries conditions:

$$0 < \widetilde{L}_{t-1} \leq ... \leq \widetilde{L}_{t-k_1} < 1, \ 0 < L'_t \leq ... \leq L'_{t+k_2} < 1$$

A changing risk behavior can be regarded as a mixed case of risk-aversion, risk-neutrality and risk-taking. System evolution will refine the agent's type.

Definition 5. Using t to denote time, the absolute risk index $\varphi^{r\_a\_n\_l}_{t,\ p\_f}$ evolves according to:

$$\varphi^{r\_a\_n\_l}_{t,\ p\_f} \overset{def.}{=} \frac{\widetilde{S}_{t,\ w\_p\_d} + S'_{t,\ w\_a\_f\_d}}{\sqrt{(S_{t,\ p\_d} + S_{t,\ a\_f\_d})^2 + \widetilde{l}}}, \ t = 1, ..., T$$

where $\widetilde{l} \geq 1$ is a positive integer which characterizes the agent's type, and k_i $(i = 1, 2)$ represent the backward (forward) lag parameters verifying the following constraints:

$$1 \leq k_1 < T, \ k_2 \geq 0, \ 1 \leq k_1 + k_2 \leq T - 1$$

In this general context, the boundary condition for the index changes. It is possible to obtain negative values for some periods and positive (or null) values for others. All these values are obtained by a "constant" mixing of objectivity and subjectivity. They depend on several factors including, the available exogenous information from the system, the accuracy of perception and respective the inherent discrepancies between ex-ante anticipations and the corresponding ex-post quantities.

The evolution of the system will not have, in this case, the same impact on the agent's risk perception. His optimal actions will be generally different compared to those selected in a pure risk-aversion (risk-taking) context.

This type of modelling represents an important refinement of the concept of temporal risk-aversion, very often employed in the literature (SEE, AMONG OTHERS, MACHINA, M., 1984 AND VAN DER PLOEG, 1992). A clear useful distinction between temporal and timeless risks is provided by our model. It improves the standard analysis of economic behavior under temporal unceratinty (SEE, AMONG OTHERS, CHAVAS, J-P., AND LARSON, B. A., 1994).

Proposition 16. A more (less) risk-taker decision-maker is characterized by a smaller (higher) local anticipative utility level.

Proof. In the appendix. ■

Agent's preferences level will decrease with the growth of the risk-taker index. His objective is not to exceed a fixed limit threshold for the index over which he becomes excessively risk-taker, and thus he will be characterized by an over-optimism.

This strategical attitude prevents to exceed some threshold utility level over which the agent's ex-post preferences are suboptimal.

In order to reach his objective, the agent must avoid to much deviate from the fixed optimal trajectory η. Over-optimism is an induced deviating behavior with respect to perceived states of the system.

We give below a suggestive graphic which illustrates the evolution of the agent's utility function with respect to the anticipative loss and the risk-taker index.

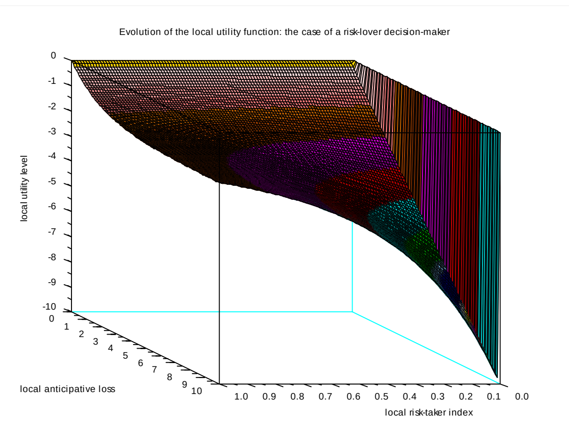

Figure 32:

Proposition 17. Any risk-lover decision-maker is characterized by an optimal risk-taker threshold $\varphi_{\max}^{p\_f}$ chosen in function of his individual type.

Proof. In the appendix. ■

It is useful to note that risk-taking and risk-aversion are non-symmetrical behaviors. They are analyzed in non-symmetrical contexts. We give below a suggestive graphic which illustrates this theoretical result.

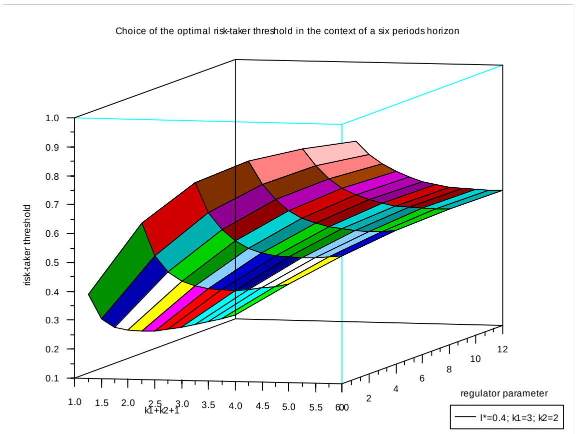

Figure 33:

Proposition 18. Any risk-lover decision-maker is characterized by a local utility function whose upper and lower bounds vary according to the risk-taker index level relative to the optimal threshold $\varphi_{\max}^{p\_f}$.

Proof. In the appendix. ■

A direct consequence of this result is the complete separability of the agent's risk-taking preferences. It depends on the index variation with respect to the fixed optimal threshold $\varphi_{\max}^{p\_f}$. It allows to characterize the agent's type in function of his individual preferences. We give below two suggestive graphics in this sense, in the case where $\varphi_{\min}^{p\_f} = 0.5$.

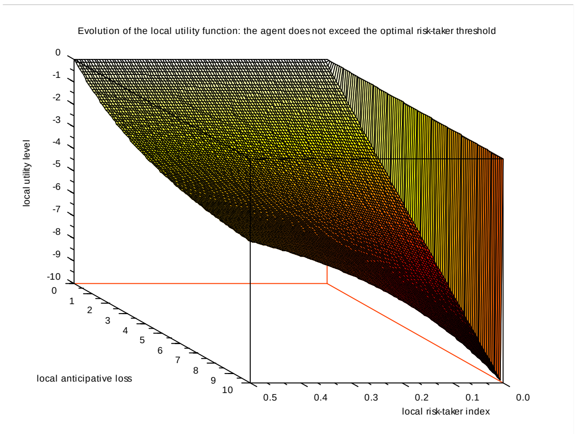

Figure 34:

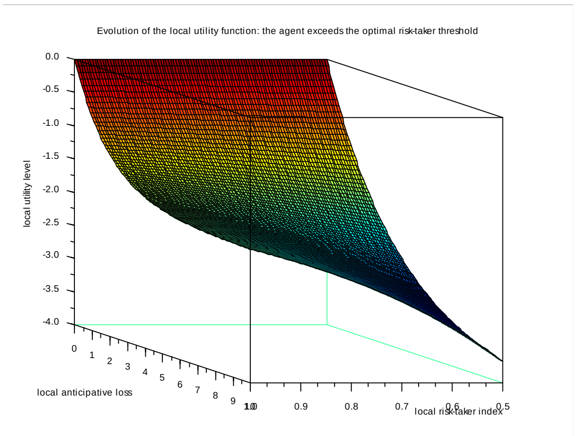

Figure 35:

Proposition 19. There is a potential effect of the working horizon length on the decision-maker's risk-taking bahavior.

Proof. In the appendix. ■

As in the case of risk-aversion, the agent's goal is to reduce the exogenous effect of the horizon length on the optimal policy decisions.

A particular environmental context can change the agent's strategy employed in the process of optimization and control and thus his adaptive behavior to risk. To fix a smaller / higher horizon length generally depends on the agent's strategic objective and his individual type.

This study take a step towards a more refined understanding of the concepts of short and long horizon risk. This is a key topic for future research with implications in a wide range of economic applications intrinsically linked to the study of policy implementation and analysis.

We give below a graphic representation of this theoretical result.

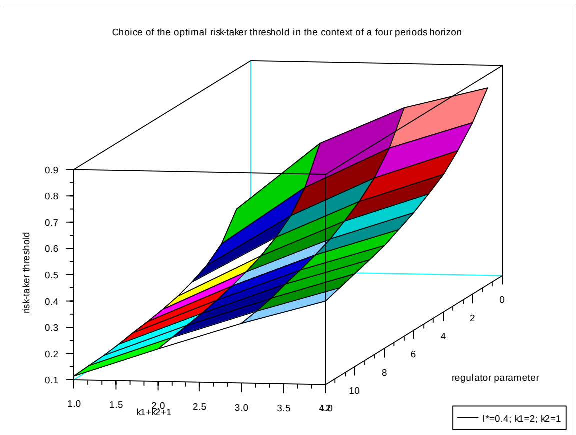

Figure 36:

We remark that in the context of a four periods horizon, the decision-maker can choose an optimal risk-taker threshold $\varphi_{\max}^{p\_f}$ inferior to 0.4, while for a six periods horizon, $\varphi_{\max}^{p\_f}$ is inferior to 0.55.

Proposition 20. Any rational decision-maker with a changing risk behavior is characterized by two optimal thresholds chosen in function of his individual type: a "minimal" risk-aversion threshold $\varphi_{\min}^{p\_f}$ and respective a "maximal" risk-taker threshold $\varphi_{\max}^{p\_f}$.

Proof. In the appendix. ■

The objective of a decision-maker characterized by a changing risk behavior is to maintain the index risk level inside the interval defined by the risk thresholds $\varphi_{\min}^{p\_f}$ and $\varphi_{\max}^{p\_f}$. It can be possible if and only if the decision-maker succeeds to constrain the dynamic system to follow the fixed optimal path η. When at least one of these thresholds are exceeded, the economic agent is characterized by an excessive risk behavior.

Significative changes in the system evolution are necessary such that a transition from an extreme risk behavior to the opposite one be possible. An interesting question is: which of these two extreme attitudes are most probable to arrive in real world? The answer is not obvious.

The risk-neutrality case can be regarded as a transitory state in this dynamic process. A changing risk behavior can be defined as a free floating state with respect to the agent's attitude to risk over time. For a stochastic system characterized by an "uniform" trend towards lower values for the target variable, the agent's behavior to risk is defined in a reasonably neighborhood of the risk-neutrality transitory state.

Generally, risk-taking makes the reaction stronger than risk-neutrality. However, when both players are risk-taker but at very different degrees, the more optimistic of the two can have a weaker reaction than in the risk-neutral case. This is an astonishing result that can be explained by the relationship that exists between high deviations and perceived states of nature.

The environment can encourage risk-taking or induce risk-aversion behavior. Outcomes as large losses can induce risk-aversion, while outcomes as large gains can induce risk-taking.

We give below a realistic scenario which illustrates the strategic attitude to risk of a rational decision-maker characterized by a changing risk behavior during the control period.

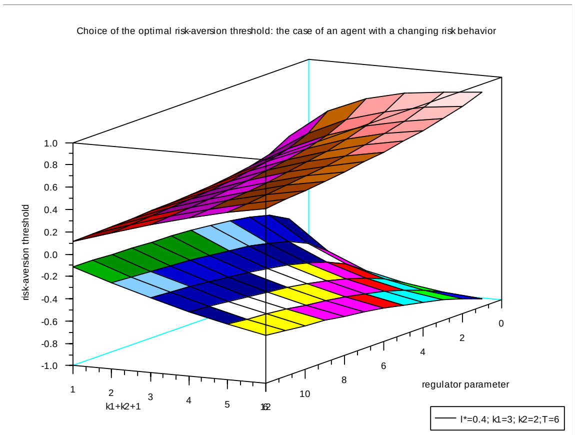

Figure 37:

For this scenario, the agent's objective is not to exceed the optimal fixed thresholds $\varphi_{\min}^{p\_f} = -0.6$ and $\varphi_{\max}^{p\_f} = 0.4$. If $\varphi_{\min}^{p\_f}$ (respective $\varphi_{\max}^{p\_f}$) is exceeded at time t, the agent will implement an excessive risk-averse (respective risk-taking) decision which will be suboptimal. His utility function during the entire control period evolves according to the following formula:

$$U_t(W_{[1,t]}, \varphi_{t,\ p\_f}^{r\_a\_n\_l}) \overset{def}{=} \begin{cases} \frac{2}{\varphi_{t,\ p\_f}^{r\_a\_n\_l}} \left\{ \exp(-\frac{\varphi_{t,\ p\_f}^{r\_a\_n\_l}}{2} \cdot W_{[1,t]}) - 1 \right\} \text{ if } -1 < \varphi_{t,\ p\_f}^{r\_a\_n\_l} < 0 \\ -W_{[1,t]} \text{ if } \varphi_{t,\ p\_f}^{r\_a\_n\_l} = 0 \\ \frac{2}{\varphi_{t,\ p\_f}^{r\_a\_n\_l}} \left\{ \exp(-\frac{\varphi_{t,\ p\_f}^{r\_a\_n\_l}}{2} \cdot W_{[1,t]}) - 1 \right\} \text{ if } 0 < \varphi_{t,\ p\_f}^{r\_a\_n\_l} < 1 \end{cases}$$

We illustrate below its changing evolution as well as the agent's adaptive behavior to risk during the period of control.

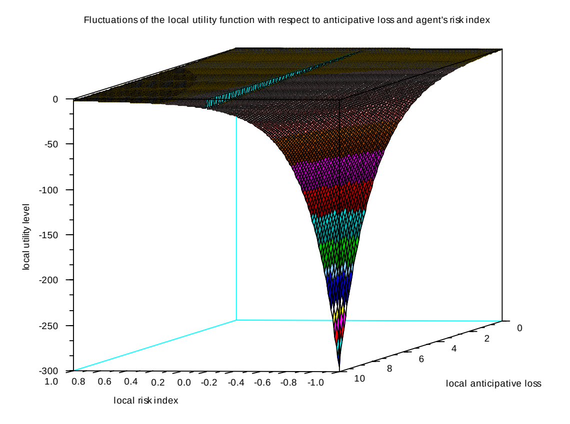

Figure 38:

In order to illustrate the complexity of the agent's behavior to risk in a changing environment, we give below a suggestive plot in the particular case where the objective is not to exceed the optimal thresholds $\varphi_{\min}^{p\_f} = -0.3$ and $\varphi_{\max}^{p\_f} = 0.4$ during the entire period of control.

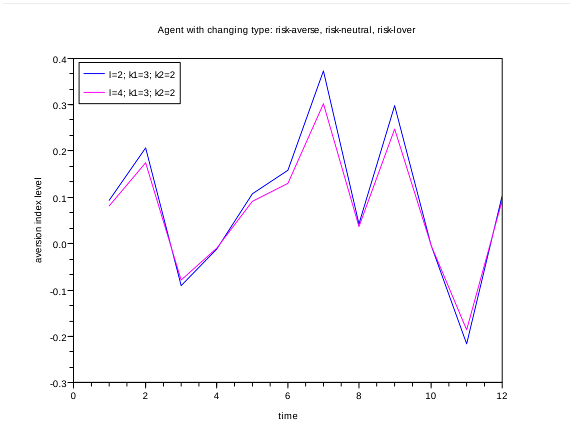

Figure 39:

The action taken at time $t = 1$ will induce a weak risk-taking for the next period; the agent is characterized by a small degree of confidence. A reinforcement of the active learning correlated with an efficient feedback (closed-loop) strategy will progressively decrease the agent's degree of confidence in the two next periods.

At time $t = 4$, the agent becomes (almost) risk-neutral. For the following three periods, we ascertain an increasing tendence in taking higher risks. However, the agent succeeds in not overreaching the fixed optimal threshold $\varphi_{\max}^{p\_f}$.

At time $t = 8$ and $t = 10$, the agent's degree of confidence is almost null. Given the inherent inertia of the system, the agent's risk-aversion will increase (even if not significantly) for the next period. He succeeds in not overreaching the fixed optimal threshold $\varphi_{\min}^{p\_f}$. In other words, the agent's strategic objective is reached.

The agent's optimal policy employed in the process of optimization and control succeeds to constrain the dynamic system to follow the reference trajectory η.

An interesting question arises: from what values, the optimal thresholds $\varphi_{\min}^{p\_f}$ and $\varphi_{\max}^{p\_f}$ chosen by the economic agent, can be regarded as small or large? This generally depends on the particular environmental context as well as on the agent's individual type.

11. Concluding Remarks and Possible Extensions

The present paper contributes to the literature on risk by exploring the crucial role played by the dynamics of the system on the agents' attitudes towards risk. The definitions of the absolute risk-aversion index proposed in this paper offer a variety of advantages. First, they allow to see how endogenous uncertainty is captured over environmental changes and to make better comparisons between risky situations. The agent interprets the system evolution and reveals his adaptive behavior to risk over time. It is only by optimizing the past and present that he will optimally anticipate endogenous risks in the future (regarded as an extended present). Second, they allow for a better risk management by the use of reinforced active learning and closed-loop feedback information. The integration of the risk-aversion index in the agent's optimal policy will improve the control and implicitly the system trajectory. The risk-aversion index provides a good understanding on how the agent's perception to risk evolves over time. Third, they allow to characterize the agent's type according to his degree of risk-aversion and individual preferences. An interesting relationship emerges between behavior to risk and length of working horizon. Although often criticized, the most widely used hypothesis for the analysis of economic behavior is risk-neutrality. It appears as a borderline case in our model. We propose a natural extension of the Arrow-Pratt theory (which takes into account only attitudes towards small exogenous risks) by including in the analysis potentially high endogenous risks (the case of large-amplitude fluctuations). This is necessary in order to overcome the limits imposed by the standard measures of risk-aversion. Furthermore, we refine the analysis by developing the general case of changing risk behavior. This type of modelling has the potential to be a powerful tool for characterizing (based on the system evolution and the importance attached to the past and future) the agent's attitudes towards risk, including risk-aversion, risk-neutrality and risk-taking. This study also contributes to the bounded rationality literature (see, among others, Simon, H. A., 1982a, 1982b, and T. J., Sargent, 1993) in that it allows for a refinement of the agent's individual preferences in discrete time. The proposed analysis can be easily extended to the more general context of stochastic dynamic games (cooperative or non-cooperative), the objective here being to define and characterize the equilibrium of the game according to the optimal risk-sharing between strategic players. We can also study the identification of the emerging endogenous coalitions in interactive strategic environments, in function of the players' types and their risky objectives. Exploring such possibilities appears to be a good topic for further research.

Appendix

Proof of Proposition 1.

Differentiating the expression of $U_t(W_{[1,t]}, \varphi_{t,\, p\_f}^{r\_a})$ with respect to $\varphi_{t,\, p\_f}^{r\_a}$ (for an arbitrary fixed value of $W_{[1,t]}$), we obtain:

$$U_t'(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f}) = \frac{2 \cdot \exp[-\frac{\varphi^{r\_a}_{t,\, p\_f}}{2} \cdot W_{[1,t]}](-\frac{\varphi^{r\_a}_{t,\, p\_f}}{2} \cdot W_{[1,t]} - 1) + 1}{(\varphi^{r\_a}_{t,\, p\_f})^2}$$

Let us define:

$$V_t(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f}) \overset{def.}{=} \exp[-\frac{\varphi^{r\_a}_{t,\, p\_f}}{2} \cdot W_{[1,t]}](-\frac{\varphi^{r\_a}_{t,\, p\_f}}{2} \cdot W_{[1,t]} - 1) + 1$$

as a function of $\varphi^{r\_a}_{t,\, p\_f}$. After simple algebraic computations, we find for the first derivative of V_t:

$$V_t'(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f}) = \varphi^{r\_a}_{t,\, p\_f} \cdot \frac{W^2_{[1,t]}}{4} \exp(-\frac{\varphi^{r\_a}_{t,\, p\_f}}{2} \cdot W_{[1,t]}) < 0,$$

that is, V_t decreases with $\varphi^{r\_a}_{t,\, p\_f}$:

$$\varphi^{r\_a}_{t,\, p\_f} < 0 \Rightarrow V_t(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f}) > V_t(W_{[1,t]}, 0) = 0$$

It follows that $U_t'(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f}) > 0$. The local utility function $U_t(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f})$ is therefore increasing in $\varphi^{r\_a}_{t,\, p\_f}$. We also have that U_t is decreasing in $W_{[1,t]}$. This is in agreement with the real world and strong empirical evidence.

Proof of Proposition 2.

Simple algebraic manipulations show that the sign of the difference:

$$\begin{aligned} \varphi^{r\_a}_{t+1,\, p\_f} - \varphi^{r\_a}_{t,\, p\_f} \quad = \quad & \frac{\xi_0 L_t + ... + \ \xi_{k_1-1} L_{k_1-1} + \overline{\xi}_1 \overline{L}_{t+1} + ... + \ \overline{\xi}_{k_2+1} \overline{L}_{t+1+k_2}}{\sqrt{(\xi_0 + ... + \ \xi_{k_1-1} + \overline{\xi}_1 ... + \ \overline{\xi}_{k_2+1})^2 + l}} \\ & - \frac{\xi_1 L_{t-1} + ... + \ \xi_{k_1} L_{k_1} + \overline{\xi}_0 \overline{L}_t + ... + \ \overline{\xi}_{k_2} \overline{L}_{t+k_2}}{\sqrt{(\xi_1 + ... + \ \xi_{k_1} + \overline{\xi}_0 + ... + \ \overline{\xi}_{k_2})^2 + l}} \end{aligned}$$

is either positive or negative or zero (it is extremely rare), depending on the values taken by the norm-deviations:

$$\underset{\text{(exogenous variables)}}{\xi_{j_1} \overset{not.}{=} \parallel y_{t-j_1} - y^g_{t-j_1} \parallel}, \ j_1 = 0, ..., \ k_1$$

$$\underset{\text{(random variables)}}{\overline{\xi}_{j_2} \overset{not.}{=} \parallel y^a_{t+j_2|I_{t+j_2}} - y^g_{t+j_2} \parallel}, \ j_2 = 0, ..., \ k_2 + 1$$

Proof of Proposition 3.

Suppose that $k_1 < k_1'$. Let us consider that $k_1' = k_1 + k$, with $k \geq 1$ a fixed integer. By hypothesis, we have:

$$L_{t-1} < L_{t-2} < ... < L_{t-k_1} < L_{t-(k_1+1)} < ... < L_{t-(k_1+k)}$$

We can write:

$$\begin{aligned} & \frac{L_{t-1} + ... + \ L_{t-k_1} + ... + \ L_{t-(k_1+k)}}{k_1 + k} - \frac{L_{t-1} + ... + \ L_{t-k_1}}{k_1} \\ = & \frac{-k[L_{t-1} + ... + \ L_{t-k_1}] + k_1[L_{t-(k_1+1)} + ... + \ L_{t-(k_1+k)}]}{k_1(k_1 + k)} \end{aligned}$$

$$= \frac{[-kL_{t-1} + L_{t-(k_1+1)} + ... + L_{t-(k_1+k)}] + ... + [-kL_{t-k_1} + L_{t-(k_1+1)} + ... + L_{t-(k_1+k)}]}{k_1(k_1+k)} \gg 0$$

In other words, we have proved that:

$$k_1 < k_1' \Rightarrow \underset{\text{(the average of } k_1' \text{ weights)}}{\frac{L_{t-1} + ... + L_{t-k_1'}}{k_1'}} \gg \underset{\text{(the average of } k_1 \text{ weights)}}{\frac{L_{t-1} + ... + L_{t-k_1}}{k_1}}$$

Proof of Proposition 4.

Consider first the case where the system transition is from consecutive small deviations to a large deviation. Denote by $\varphi_{t,\,p}^{r\_a,\,s\_h}$ the risk-aversion index at time t for this type of scenario. We have that $\varphi_{t,\,p}^{r\_a,\,s\_h} \rightarrow L_{t-1}$, where L_{t-1} represents the weight attached to the system deviation at time $t-1$. Let us denote by $\varphi_{t,\,p}^{r\_a,\,h\_s}$ the risk-aversion index at time t when the system transition is from consecutive large deviations to a small deviation. In this case, we can write:

$$\varphi_{t,\,p}^{r\_a,\,h\_s} \rightarrow \frac{\widetilde{L}_{t-2} + ... + \widetilde{L}_{t-k_1}}{k_1 - 1}$$

where $\widetilde{L}_{t-2}, ..., \widetilde{L}_{t-k_1}$ are weighting scalars attached to large system deviations (in the context of the second scenario). Depending on the size of the parameters L_{t-1} and $\frac{\widetilde{L}_{t-2}+...+\widetilde{L}_{t-k_1}}{k_1-1}$, we can imagine three distinct situations:

a) the large deviation of the system at time $t-1$ (in the context of the first scenario) is much more important for the decision-maker than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario:

$$\underset{\text{(sufficient condition)}}{L_{t-1} \ll \widetilde{L}_{t-2}, ..., L_{t-1} \ll \widetilde{L}_{t-k_1}} \Rightarrow L_{t-1} \ll \frac{\widetilde{L}_{t-2} + ... + \widetilde{L}_{t-k_1}}{k_1 - 1}$$

In this case, it follows that $\varphi_{t,\,p}^{r\_a,\,s\_h} < \varphi_{t,\,p}^{r\_a,\,h\_s}$, and hence a higher degree of risk-aversion for the same control period in the context of the first scenario compared with the second scenario.

b) the large deviation of the system at time $t-1$ (in the context of the first scenario) is much less important for the decision-maker than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario:

$$\underset{\text{(sufficient condition)}}{L_{t-1} \gg \widetilde{L}_{t-2}, ..., L_{t-1} \gg \widetilde{L}_{t-k_1}} \Rightarrow L_{t-1} \gg \frac{\widetilde{L}_{t-2} + ... + \widetilde{L}_{t-k_1}}{k_1 - 1}$$

In contrast with the previous case, we obtain that $\varphi_{t,\,p}^{r\_a,\,s\_h} > \varphi_{t,\,p}^{r\_a,\,h\_s}$, and thus a smaller degree of risk-aversion at time t in the context of the first scenario compared with the second scenario.

c) the large deviation of the system at time $t-1$ (in the context of the first scenario) is much more or less important for the decision-maker than all other $(k_1 - 1)$ large deviations obtained in the context of the second scenario:

$$L_{t-1} \ll (\text{or } \gg)\ \widetilde{L}_{t-2}, ..., L_{t-1} \ll (\text{or } \gg)\ \widetilde{L}_{t-k_1}$$

In this case, it may be possible to obtain:

either

$$\varphi_{t,\,p}^{r\_a,\,s\_h} < \varphi_{t,\,p}^{r\_a,\,h\_s}$$

or

$$\varphi_{t,\ p}^{r\_a,\ s\_h} > \varphi_{t,\ p}^{r\_a,\ h\_s}$$

This completes the proof.

Proof of Proposition 5.

In the context of the first (second) scenario, we obtain respectively:

$$\varphi_{t,\ p,\ k_1=2k'}^{r\_a\_first.sc.} \to \frac{L_{t-2k'} + L_{t-(2k'-2)} + ... + \ L_{t-2}}{k'}$$

$$\varphi_{t,\ p,\ k_1=2k'}^{r\_a\_\sec ond.sc.} \to \frac{L_{t-(2k'-1)} + L_{t-(2k'-3)} + ... + \ L_{t-1}}{k'}$$

Suppose without loss of generality that:

$$L_{t-2k'} + L_{t-(2k'-2)} + ... + L_{t-2} \gg L_{t-(2k'-1)} + L_{t-(2k'-3)} + ... + \ L_{t-1}$$

It follows that:

$$\varphi_{t,\ p,\ k_1=2k'}^{r\_a\_first.sc.} > \varphi_{t,\ p,\ k_1=2k'}^{r\_a\_\sec ond.sc.}$$

Proof of Proposition 6.

We have respectively:

$$\varphi_{t,\ p,\ k^*=2k'+1}^{r\_a\_first.sc.} \to \frac{L_{t-(2k'+1)} + L_{t-(2k'-1)} + ... + \ L_{t-3} + L_{t-1}}{k'+1}$$

$$\varphi_{t,\ p,\ k^*=2k'+1}^{r\_a\_\sec ond.sc.} \to \frac{L_{t-2k'} + L_{t-(2k'-2)} + ... + \ L_{t-2}}{k'}$$

Suppose without loss of generality that:

$$\frac{L_{t-2k'} + L_{t-(2k'-2)} + ... + L_{t-2}}{k'} \gg \frac{L_{t-(2k'+1)} + L_{t-(2k'-1)} + ... + \ L_{t-3} + L_{t-1}}{k'+1}$$

It follows that:

$$\varphi_{t,\ p,\ k_1=2k'+1}^{r\_a\_first.sc.} < \varphi_{t,\ p,\ k_1=2k'+1}^{r\_a\_\sec ond.sc.}$$

We also have:

$$\varphi_{t,\ p,\ k_1=2k'+1}^{r\_a\_first.sc.} < \varphi_{t,\ p,\ k_1=2k'}^{r\_a\_first.sc.}$$

$$\varphi_{t,\ p,\ k_1=2k'}^{r\_a\_\sec ond.sc.} < \varphi_{t,\ p,\ k_1=2k'+1}^{r\_a\_\sec ond.sc.}$$

This completes the proof.

Proof of Proposition 7.

It results from the following sequence of inequalities:

$$\frac{L_0}{1} \gg \frac{L_0 + L_1}{2} \gg ... \gg \frac{L_0 + ... + \ L_{t-1}}{t}$$

Each above ratio corresponds (in this order) to the estimation of the risk-aversion index at time $\tau = 1, 2, ..., t$. For this type of scenario, it is supposed that $L_1 \ll L_0$. In other words, the impact of the deviations from $\tau = 1$ to $\tau = t$ on the agent's future uncertainty is much more important compared with the impact at $\tau = 0$.

Proof of Proposition 8.

It is easy to see that the sign of the following differences:

$$\varphi_{t,\, p\_f}^{r\_a} - \varphi_{t,\, f}^{r\_a},\ \varphi_{t,\, p\_f}^{r\_a} - \varphi_{t,\, p}^{r\_a},\ \varphi_{t,\, f}^{r\_a} - \varphi_{t,\, p}^{r\_a},\ \varphi_{t,\, f}^{r\_a} - \varphi_{t,\, w}^{r\_a}$$

may be either positive or negative. In other words, the agent's degree of risk-aversion is correlated with his adopted strategy to manage the inevitable risk and uncertainty involved in the process of optimization and control.

Proof of Proposition 9.

Denote by $\xi_{j_1} \overset{not.}{=} \| \ y_{t-j_1} - y_{t-j_1}^g \ \|$, $j_1 = 1, ..., k_1$, and $\overline{\xi}_{j_2} \overset{not.}{=} \| \ y_{t+j_2|I_{t+j_2}}^a - y_{t+j_2}^g \ \|$, $j_2 = 0, ..., k_2$, the system norm-deviations at time $t-1, ..., t-k$, and $t, ..., t+k_2$, respectively. These are supposed to be small but non-negligible. The absolute risk-aversion index $\varphi_{t,\, p\_f}$ evolves according to:

$$\varphi_{t,\, p\_f}^{r\_a} = \frac{\xi_1 L_{t-1} + ... + \ \xi_{k_1} L_{t-k_1} + \overline{\xi}_0 \overline{L}_t + ... + \ \overline{\xi}_{k_2} \overline{L}_{t+k_2}}{\sqrt{(\xi_1 + ... + \ \xi_{k_1} + \overline{\xi}_0 + ... + \ \overline{\xi}_{k_2})^2 + l}},\ t = 1, ..., T.$$

For ease of exposition, assume that $l_t = l^*$ (a small value from $(0, 1)$) for all $t = 1, ..., T$. It follows that:

$$\xi_{j_1} < l^*,\ \forall\ j_1 = 1, ..., k_1 \text{ and } \overline{\xi}_{j_2} < l^*,\ \forall\ j_2 = 0, ..., k_2$$

We have:

$$\begin{aligned} \mid \varphi_{t,\, p\_f}^{r\_a} \mid \ &\leq \frac{\xi_1 + ... + \ \xi_{k_1} + \overline{\xi}_0 + ... + \ \overline{\xi}_{k_2}}{\sqrt{(\xi_1 + ... + \ \xi_{k_1} + \overline{\xi}_0 + ... + \ \overline{\xi}_{k_2})^2 + l}} \\ &< \frac{l^*(k_1 + k_2 + 1)}{\sqrt{(l^*(k_1 + k_2 + 1))^2 + l}} = \frac{1}{\sqrt{1 + l/(l^*(k_1 + k_2 + 1))^2}} \end{aligned}$$

Because the agent anticipates small deviations of the system from the fixed targets, his degree of risk-aversion will be small before starting the control. Consequently, he will choose a high value for the parameter l. As k_1 and k_2 have generally small values, the ratio $\frac{1}{\sqrt{1+l/(l^*(k_1+k_2+1))^2}}$ (and implicitly the risk-aversion index $\varphi_{t,\, p\_f}^{r\_a}$) will be small. We have:

$$-\frac{1}{\sqrt{1 + l/(l^*(k_1 + k_2 + 1))^2}} \leq \varphi_{t,\, p\_f}^{r\_a} < 0$$

In the particular case where $\| \ y_{t-j_1} - y_{t-j_1}^g \ \| \to 0$, $\forall\ j_1 = 1, ..., k_1$ and $\| \ y_{t+j_2|I_{t+j_2}}^a - y_{t+j_2}^g \ \| \to 0$, $\forall\ j_2 = 0, ..., k_2$, it is clear that $\varphi_{t,\, p\_f}^{r\_a} \to 0$ (a borderline case). In real world, this type of scenario is rare to be realized due to continuous endogenous fluctuations of the system. The assumption of risk-neutrality for the entire working horizon is generally non-realistic.

We have the following implication:

$$\varphi_{t,\, p\_f}^{r\_a} \to 0 \Rightarrow U_t(W_{[1,t]}, \varphi_{t,\, p\_f}^{r\_a}) \overset{def}{=} \frac{2}{\varphi_{t,\, p\_f}^{r\_a}}\{\exp(-\frac{\varphi_{t,\, p\_f}^{r-a}}{2} \cdot W_{[1,t]}) - 1\} \to -W_{[1,t]}$$

and thus

$$\max E_0 U_t(W_{[1,t]}, \varphi_{t,p\_f}^{r\_a}) = \min E_0[W_{[1,t]}(y_1, ...y_t)]$$

Proof of Proposition 10.

Suppose there is some large deviations of the system (in the past and future), say $\xi_{\alpha_1}, ..., \xi_{\alpha_{N_1}}$ and $\overline{\xi}_{\beta_1}, ..., \overline{\xi}_{\beta_{N_2}}$, respectively. We have:

$$\alpha_{1,...,}\alpha_{N_1} \in \{1, ..., k_1\},\ N_1 < k_1 \text{ and } \beta_1, ..., \beta_{N_2} \in \{0, ..., k_2\},\ N_2 < k_2.$$

For simplicity, assume that all high deviations of the system are comparable. Denote by $\varphi_{t,\ p\_f}^{r\_a,\ s\_h}$ the risk-aversion index at time t in the case when system deviations are mixed (i.e., small and large). Its value is defined by the following expression:

$$\frac{\xi_1 L_{t-1} + ... + \xi_{\alpha_{N_1}} L_{t-\alpha_{N_1}} + ... + \ \xi_{k_1} L_{t-k_1} + \overline{\xi}_0 \overline{L}_t + ... + ... + \ \overline{\xi}_{\beta_{N_2}} \overline{L}_{t+\beta_{N_2}} + ... + \ \overline{\xi}_{k_2} \overline{L}_{t+k_2}}{\sqrt{(\xi_1 + ... + \ \xi_{\alpha_{N_1}} + ... + \ \xi_{k_1} + \overline{\xi}_0 + ... + \ \overline{\xi}_{\beta_{N_2}} + ... + \ \overline{\xi}_{k_2})^2 + l}}$$

where

$$\xi_{j_1},\ j_1 = 1, ..., k_1,\ j_1 \neq \alpha_{N_1^*},\ N_1^* = 1, ..., N_1$$

and

$$\overline{\xi}_{j_2},\ j_2 = 0, ..., k_2,\ j_2 \neq \alpha_{N_2^*},\ N_2^* = 0, ..., N_2$$

represent the system small deviations.

Denote by $\varphi_{t,p\_f}^{r\_a,\ s}$ the risk-aversion index at time t in the case where there is only small deviations of the system from the fixed targets. We can write:

$$\varphi_{t,\ p\_f}^{r\_a,\ s} = \frac{\xi_1 L_{t-1} + ... + \ \xi_{k_1} L_{t-k_1} + \overline{\xi}_0 \overline{L}_t + ... + \ \overline{\xi}_{k_2} \overline{L}_{t+k_2}}{\sqrt{(\xi_1 + ... + \ \xi_{k_1} + \overline{\xi}_0 + ... + \ \overline{\xi}_{k_2})^2 + l}}$$

Without loss of generality, we also assume that all small deviations of the system are comparable. We have the following inplications:

$$\varphi_{t,\ p\_f}^{r\_a,\ s\_h} \rightarrow \frac{L_{t-\alpha_1} + ... + L_{t-\alpha_{N_1}} + \overline{L}_{t+\beta_1} + ... + \overline{L}_{t+\beta_{N_2}}}{\sqrt{(N_1 + N_2)^2 + l}}$$

and

$$\varphi_{t,\ p\_f}^{r\_a,\ s} \rightarrow \frac{L_{t-1} + ... + L_{t-k_1} + \overline{L}_t + ... + \overline{L}_{t+k_2}}{\sqrt{(k_1 + k_2 + 1)^2 + l}}$$

Depending on the moment when high deviations arrive in the past or are expected to be realized in the future as well as on the weighting scalars attached to the system deviations, we can obtain two distinct scenarios:

$$\frac{L_{t-\alpha_1} + ... + L_{t-\alpha_{N_1}} + \overline{L}_{t+\beta_1} + ... + \overline{L}_{t+\beta_{N_2}}}{\sqrt{(N_1 + N_2)^2 + l}} \ll (\text{or } \gg) \frac{L_{t-1} + ... + L_{t-k_1} + \overline{L}_t + ... + \overline{L}_{t+k_2}}{\sqrt{(k_1 + k_2 + 1)^2 + l}}$$

It follows that:

$$\varphi_{t,\ p\_f}^{r\_a,\ s\_h} < (\text{or } >)\ \varphi_{t,\ p\_f}^{r\_a,\ s}$$

In other words, the decision-maker's risk-aversion at time t is not necessarily higher when system deviations are mixed, compared to the case where all system deviations are small. This surprising result reveals the complexity of the agent's behavior towards risk in a changing environment.

Proof of Proposition 11.

By hypothesis, we have:

$$\| y_{t-j_1'} - y^g_{t-j_1'} \| / \| y_{t-j_1''} - y^g_{t-j_1''} \| \quad \cong 1 \text{ for } j_1' \neq j_1''; \; j_1', \; j_1'' \in \{1, ..., k_1\}$$

$$\| y^a_{t+j_2'|I_{t+j_2'}} - y^g_{t+j_2'} \| / \| y^a_{t+j_2''|I_{t+j_2''}} - y^g_{t+j_2''} \| \quad \cong 1 \text{ for } j_2' \neq j_2''; \; j_2', \; j_2'' \in \{0, ..., k_2\}$$

$$l / \| y_{t-j_1} - y^g_{t-j_1} \|^4 \cong 0, \forall \; j_1 = \overline{1, k_1}; \; l / \| y^a_{t+j_2|I_{t+j_2}} - y^g_{t+j_2} \|^4 \cong 0, \forall \; j_2 = \overline{0, k_2}$$

It follows that:

$$\varphi^{r\_a}_{t,\, p\_f} \rightarrow \underbrace{\frac{L_{t-1} + ... + L_{t-k_1} + \overline{L}_t + ... + \overline{L}_{t+k_2}}{k_1 + k_2 + 1}}_{\text{(the average of all the weights - in the past and future)}} \in (-1, 0)$$

Depending on the weighting scalars values attached to the system deviations, the risk-aversion index level at time t may be more or less close to -1 or 0. Thus, if all weighting scalars approach -1, then the index value will be close enough to -1. In this case, the agent will be characterized by an excessive risk-aversion. Contrary to what is generally believed, it may be possible to have a small risk-aversion when the system deviations are large. It is the case where all weighting scalars approach 0. We say that the agent is almost "risk-neutral" by nature, that is, there is a small variability in his risk sensitivity over time.

Proof of Proposition 12.

Suppose that all system deviations are comparable. For ease of exposition, we consider that $k_1 < k_1'$ and $k_2 < k_2'$. Let us make the following notations before proceeding: $k_1' \overset{not.}{=} k_1 + k^*$ and $k_2' \overset{not.}{=} k_2 + k^{**}$ with k^*, $k^{**} \geq 1$. We have:

$$\frac{L_{t-1} + ... + L_{t-k_1'} + \overline{L}_t + ... + \overline{L}_{t+k_2'}}{k_1' + k_2' + 1} - \frac{L_{t-1} + ... + L_{t-k_1} + \overline{L}_t + ... + \overline{L}_{t+k_2}}{k_1 + k_2 + 1} =$$

$$\frac{(L_{t-1} + ... + L_{t-k_1}) + L_{t-(k_1+1)} + ... + \; L_{t-(k_1+k^*)} + (\overline{L}_t + ... + \overline{L}_{t+k_2}) + \overline{L}_{t+(k_2+1)} + ... + \overline{L}_{t+(k_2+k^{**})}}{k_1 + k_2 + 1 + k^* + k^{**}}$$

$$- \frac{L_{t-1} + ... + \; L_{t-k_1} + \overline{L}_t + ... + \overline{L}_{t+k_2}}{k_1 + k_2 + 1} =$$

$$\frac{(k_1 + k_2 + 1)[L_{t-(k_1+1)} + ... + \; L_{t-(k_1+k^*)} + \overline{L}_{t+(k_2+1)} + ... + \overline{L}_{t+(k_2+k^{**})}]}{(k_1 + k_2 + 1 + k^* + k^{**})(k_1 + k_2 + 1)}$$

$$- \frac{(k^* + k^{**})(L_{t-1} + ... + \; L_{t-k_1} + \overline{L}_t + ... + \overline{L}_{t+k_2})}{(k_1 + k_2 + 1 + k^* + k^{**})(k_1 + k_2 + 1)}$$

It is easy to see that:

$$\overline{k}(L_{t-1} + ... + \; L_{t-k_1}) < k_1[L_{t-(k_1+1)} + ... + \; L_{t-(k_1+k^*)}] \; \forall \; \overline{k} \in \{k^*, k^{**}\}$$

$$\overline{k}(\overline{L}_t + ... + \overline{L}_{t+k_2}) < (k_2 + 1)[\overline{L}_{t+(k_2+1)} + ... + \overline{L}_{t+(k_2+k^{**})}] \; \forall \; \overline{k} \in \{k^*, k^{**}\}$$

It follows that:

$$(k^* + k^{**})(L_{t-1} + ... + \; L_{t-k_1} + \overline{L}_t + ... + \overline{L}_{t+k_2})$$

$$< 2k_1[L_{t-(k_1+1)} + ... + \; L_{t-(k_1+k^*)}] + 2(k_2 + 1)[\overline{L}_{t+(k_2+1)} + ... + \overline{L}_{t+(k_2+k^{**})}]$$

Let us examine the sign of the following difference:

$$k_1[L_{t-(k_1+1)}+...+\ L_{t-(k_1+k^*)}]+(k_2+1)[\overline{L}_{t+(k_2+1)}+...+\overline{L}_{t+(k_2+k^{**})}]$$
$$-\{k_1[\overline{L}_{t+(k_2+1)}+...+\overline{L}_{t+(k_2+k^{**})}]+(k_2+1)[L_{t-(k_1+1)}+...+\ L_{t-(k_1+k^*)}]\}$$
$$=(k_2+1-k_1)\{[\overline{L}_{t+(k_2+1)}+...+\overline{L}_{t+(k_2+k^{**})}]-[L_{t-(k_1+1)}+...+\ L_{t-(k_1+k^*)}]\}$$

Several distinct scenarios can be taken into account in the analysis:

i)

$$k_2<k_1 \text{ and } \overline{L}_{t+(k_2+1)}+...+\overline{L}_{t+(k_2+k^{**})}\gg L_{t-(k_1+1)}+...+\ L_{t-(k_1+k^*)}$$

ii)

$$k_2>k_1 \text{ and } \overline{L}_{t+(k_2+1)}+...+\overline{L}_{t+(k_2+k^{**})}\ll L_{t-(k_1+1)}+...+\ L_{t-(k_1+k^*)}$$

iii)

$$k_2<k_1 \text{ and } \overline{L}_{t+(k_2+1)}+...+\overline{L}_{t+(k_2+k^{**})}\ll L_{t-(k_1+1)}+...+\ L_{t-(k_1+k^*)}$$

iv)

$$k_2>k_1 \text{ and } \overline{L}_{t+(k_2+1)}+...+\overline{L}_{t+(k_2+k^{**})}\gg L_{t-(k_1+1)}+...+\ L_{t-(k_1+k^*)}$$

For the first two scenarios, we obtain:

$$\frac{L_{t-1}+...+L_{t-k_1'}+\overline{L}_t+...+\overline{L}_{t+k_2'}}{k_1'+k_2'+1}\gg\frac{L_{t-1}+...+L_{t-k_1}+\overline{L}_t+...+\overline{L}_{t+k_2}}{k_1+k_2+1}$$

and hence, a smaller degree of risk-aversion for the agent (at the same stage of the control) when he considers a higher value of the sum of k_1(feedback period) and k_2 (forward period). We underline, in this context, the trade-off between complexity and perception.

For the last two scenarios, the agent's degree of risk-aversion does not necessarily diminish (at the same stage of the control) when the sum of k_1 and k_2 is higher. This is the case where the agent attributes more importance to the past or to the future.

Proof of Proposition 13.

We can write the inequality:

$$\mid \varphi_{t,\,p\_f}^{r\_a}\mid\ <\frac{S_{t,\,p\_d}+S_{t,\,a\_f\_d}}{\sqrt{\left(S_{t,\,p\_d}+S_{t,\,a\_f\_d}\right)^2+l}},\ t=1,...,T.$$

The agent's objective is to constrain the system in such a way that:

$$\parallel y_{t-j_1}-y_{t-j_1}^g\parallel\ <l^*,\ j_1=1,...,k_1$$
$$\parallel y_{t+j_2|I_{t+j_2}}-y_{t+j_2}^g\parallel\ <l^*,\ j_2=0,...,k_2$$

where l^* is a small value from $(0,1)$. It follows that:

$$\mid \varphi_{t,\,p\_f}^{r\_a}\mid\ <\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+l}},$$

that is,

$$-1<-\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+l}}<\varphi_{t,\,p\_f}^{r\_a}<0$$

The objective is not to exceed the fixed optimal threshold $\varphi_{\min}^{p\_f}$ during the entire period of control (i.e., $\varphi_{t,\,p\_f}^{r\_a}>\varphi_{\min}^{p\_f}\ \forall\ t=1,...,T$). Two distinct scenarios can be possible:

Either

$$-1 < \varphi^{p\_f}_{\min,\, more} < -\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2 + l_{more}}}$$

or

$$-\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2 + l_{less}}} < \varphi^{p\_f}_{\min,\, less} < 0$$

with $1 \leq l_{more} < l_{less}$ two regulator parameters which characterize the agent's type.

The first (second) scenario corresponds to a more (less) risk-averse agent. The two fixed thresholds, $\varphi^{p\_f}_{\min,\, more}$ and $\varphi^{p\_f}_{\min,\, less}$, are not exceeded during the control period if and only if the agent succeeds in controlling the system fluctuations.

Proof of Proposition 14.

Suppose that $W_{[1,t]} > 0$, $\forall\; t = 1, ..., T$. We have the following inequality:

$$\exp(-\frac{\varphi^{r\_a}_{t,\, p\_f}}{2} \cdot W_{[1,t]}) - 1 > -\frac{\varphi^{r\_a}_{t,\, p\_f}}{2} \cdot W_{[1,t]} > 0$$

and hence

$$U_t(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f}) < -W_{[1,t]} < 0$$

The agent's local utility level varies with the value of the risk-aversion index. If the agent succeeds to manage the risk at time t (i.e., $\varphi^{p\_f}_{\min} < \varphi^{r\_a}_{t,p\_f} < 0$), then one can write:

$$U_t(W_{[1,t]}, \varphi^{p\_f}_{\min}) < U_t(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f}) < -W_{[1,t]} = \lim_{\varphi^{r\_a}_{t,p\_f} \to 0} U_t(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f})$$

In the case where the agent does not succeed to manage the risk at time t (i.e., $-1 < \varphi^{r\_a}_{t,\, p\_f} < \varphi^{p\_f}_{\min}$), we have:

$$\lim_{\varphi^{r\_a}_{t,p\_f} \to -1} U_t(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f}) = U_t(W_{[1,t]}, -1) < U_t(W_{[1,t]}, \varphi^{r\_a}_{t,\, p\_f}) < U_t(W_{[1,t]}, \varphi^{p\_f}_{\min})$$

What is optimal for the agent in the first context becomes unoptimal in the second context. Optimality generally depends on context and adopted criteria.

Proof of Proposition 15.

We can write the sequence of inequalities:

$$\varphi^{p\_f}_{\min,\, more} < -\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2 + l_{more}}},$$

$$-\frac{T}{\sqrt{T^2 + l_{less}}} < -\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2 + l_{less}}} < \varphi^{p\_f}_{\min,\, less}$$

For a higher (smaller) number of periods T, the value of the parameters k_1 and k_2 can be higher (smaller) and thus the ratio $-\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+l_{less}}}$ (respective $-\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+l_{more}}}$) can take a smaller (higher) value. In other words, a less (more) risk-averse decision-maker can choose a smaller (higher) risk-aversion threshold $\varphi^{p\_f}_{\min,\, less}$ (respective $\varphi^{p\_f}_{\min,\, more}$) depending on the length of the working horizon. We must distinguish between "nature" and type. The agent is considered risk-averse by nature, while his individual type is "more or less risk-averse". The evolution of the environment will refine the agent's type.

Proof of Proposition 16.

Following the same reasoning as in **Proposition 1**, we obtain that $U_t^{r\_l}(W_{[1,t]}, \varphi_{t,\,p\_f}^{r\_l})$ is decreasing in $\varphi_{t,\,p\_f}^{r\_l}$ and respective $W_{[1,t]}$, where:

$$U_t^{r\_l}(W_{[1,t]}, \varphi_{t,\,p\_f}^{r\_l}) \overset{def}{=} \frac{2}{\varphi_{t,\,p\_f}^{r\_l}} \left\{ \exp(-\frac{\varphi_{t,\,p\_f}^{r\_a}}{2} \cdot W_{[1,t]}) - 1 \right\}$$

represents the local utility function at time t of a risk-taker decision-maker.

Proof of Proposition 17.

Consider the case of a risk-taker agent. The weighting scalars verify the following conditions:

$$0 < \widetilde{L}_{t-1} < ... < \widetilde{L}_{t-k_1} < 1,\ 0 < L'_t < ... < L'_{t+k_2} < 1$$

Following the same reasoning as in **Proposition 13**, one can write:

$$\varphi_{t,\,p\_f}^{r\_l} < \frac{\widetilde{S}_{t,\,p\_d} + S'_{t,\,a\_f\_d}}{\sqrt{\left(S_{t,\,p\_d} + S_{t,\,a\_f\_d}\right)^2 + \widetilde{l}}},\ t = 1, ..., T.$$

It follows that:

$$\varphi_{t,\,p\_f}^{r\_l} < \frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \widetilde{l}}},$$

where l^* is a small value from $(0, 1)$ such that:

$$\parallel y_{t-j_1} - y_{t-j_1}^g \parallel \ < l^*,\ j_1 = 1, ..., k_1$$

$$\parallel y_{t+j_2|I_{t+j_2}} - y_{t+j_2}^g \parallel \ < l^*,\ j_2 = 0, ..., k_2$$

The agent's objective is not to exceed a fixed optimal risk-taker threshold $\varphi_{\max}^{p\_f}$ during the control period (i.e., $\varphi_{t,\,p\_f}^{r\_l} < \varphi_{\max}^{p\_f}\ \forall\ t = 1, ..., T$). Two distinct scenarios can be possible:

Either

$$0 < \varphi_{\max,\,less}^{p\_f} < \frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \widetilde{l}_{less}}} < 1$$

or

$$0 < \frac{l^*(k_1 + k_2 + 1)}{\sqrt{[l^*(k_1 + k_2 + 1)]^2 + \widetilde{l}_{more}}} < \varphi_{\max,\,more}^{p\_f} < 1$$

with $1 \leq \widetilde{l}_{more} < \widetilde{l}_{less}$ two regulator parameters which characterize the agent's type.

The first (second) scenario corresponds to a less (more) risk-taker agent. The two fixed thresholds, $\varphi_{\max,\,more}^{p\_f}$ and $\varphi_{\max,\,less}^{p\_f}$, are not exceeded during the control period if and only if the agent succeeds in controlling the system fluctuations.

Proof of Proposition 18.

Following the same reasoning as in **Proposition 14**, we obtain the inequality:

$$U_t^{r\_l}(W_{[1,t]}, \varphi_{t,\,p\_f}^{r\_l}) > -W_{[1,t]},\ t = 1, ..., T$$

where by hypothesis, $W_{[1,t]} > 0$. We distinguish two asymmetrical cases:

i) the agent does not exceed the optimal threshold $\varphi_{\max}^{p\_f}$ at time t (i.e., $\varphi_{t,\,p\_f}^{r\_l} < \varphi_{\max}^{p\_f}$).

ii) the agent exceedes the optimal threshold $\varphi_{\max}^{p\_f}$ at time t (i.e., $\varphi_{t,\,p\_f}^{r\_l} > \varphi_{\max}^{p\_f}$).
In the first case, we have:

$$U_t^{r\_l}(W_{[1,t]}, \varphi_{\max}^{p\_f}) < U_t^{r\_l}(W_{[1,t]}, \varphi_{t,\,p\_f}^{r\_l}) < U_t^{r\_l}(W_{[1,t]}, 0)$$

In the second case, we can write:

$$\lim_{\varphi_{t,p\_f}^{r\_l} \to 0} U_t^{r\_l}(W_{[1,t]}, \varphi_{t,\,p\_f}^{r\_l}) = -W_{[1,t]} < U_t^{r\_l}(W_{[1,t]}, \varphi_{t,\,p\_f}^{r\_l}) < U_t^{r\_l}(W_{[1,t]}, \varphi_{\max}^{p\_f})$$

Proof of Proposition 19.
We can write the sequence of inequalities:

$$\varphi_{\max,\,less}^{p\_f} < \frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2 + \widetilde{l}_{less}}} < \frac{T}{\sqrt{T^2 + \widetilde{l}_{less}}},$$

$$\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2 + \widetilde{l}_{more}}} < \varphi_{\max,\,more}^{p\_f}$$

For a higher (smaller) number of periods T, the value of the parameters k_1 and k_2 can be higher (smaller) and thus the ratio $\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+\widetilde{l}_{less}}}$ (respective $\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+\widetilde{l}_{more}}}$) can take a (higher) smaller value. In other words, a less (more) risk-taker decision-maker can choose a higher (smaller) risk-taker threshold $\varphi_{\max,\,less}^{p\_f}$ (respective $\varphi_{\max,\,more}^{p\_f}$) depending on the length of the working horizon.

Proof of Proposition 20.
We have the inequality:

$$\mid \varphi_{t,\,p\_f}^{r\_a\_n\_l} \mid \ < \frac{\widetilde{S}_{t,\,p\_d} + S'_{t,\,a\_f\_d}}{\sqrt{\left(S_{t,\,p\_d} + S_{t,\,a\_f\_d}\right)^2 + \widetilde{l}}},\ t = 1, ..., T.$$

It follows that:

$$\varphi_{t,\,p\_f}^{r\_a\_n\_l} < \frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2 + \widetilde{l}}} \text{ if } \varphi_{t,\,p\_f}^{r\_a\_n\_l} > 0$$

or

$$\varphi_{t,\,p\_f}^{r\_a\_n\_l} > -\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2 + \widetilde{l}}} \text{ if } \varphi_{t,\,p\_f}^{r\_a\_n\_l} < 0$$

where l^* is a small value from $(0,1)$ such that:

$$\parallel y_{t-j_1} - y_{t-j_1}^g \parallel \ < l^*,\ j_1 = 1, ..., k_1$$

$$\parallel y_{t+j_2|I_{t+j_2}} - y_{t+j_2}^g \parallel \ < l^*,\ j_2 = 0, ..., k_2$$

Following the same reasoning as in **Proposition 13** and **Proposition 17**, we obtain the following distinct scenarios:
i) either

$$-1 < \varphi_{\min,\,more}^{p\_f,\,c\_b} < -\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2 + \widetilde{l}_{more}}}$$

or

$$-\frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+\widetilde{l}_{less}}} < \varphi^{p\_f,\, c\_b}_{\min,\, less} < 0$$

ii) either

$$0 < \varphi^{p\_f,\, c\_b}_{\max,\, less} < \frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+\widetilde{l}_{less}}} < 1$$

or

$$0 < \frac{l^*(k_1+k_2+1)}{\sqrt{[l^*(k_1+k_2+1)]^2+\widetilde{l}_{more}}} < \varphi^{p\_f,\, c\_b}_{\max,\, more} < 1$$

with $1 \leq \widetilde{l}_{more} < \widetilde{l}_{less}$, two regulator parameters which characterize the agent's attitude to risk. This completes the proof.

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