Smith and Rawls Share a Room[*]

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Abstract

We consider one-to-one matching (roommate) problems in which agents (students) can either be matched as pairs or remain single. The aim of this paper is twofold. First, we review a key result for roommate problems (the "lonely wolf" theorem) for which we provide a concise and elementary proof. Second, and related to the title of this paper, we show how the often incompatible concepts of stability (represented by the political economist Adam Smith) and fairness (represented by the political philosopher John Rawls) can be reconciled for roommate problems.

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1 Roommate Markets

We consider one-to-one matching problems in which agents can either be matched as pairs or remain single. Gale and Shapley (1962, Example 3) introduced these so-called roommate problems as follows: "An even number of boys wish to divide up into pairs of roommates." A very common extension of this problem is to allow also for odd numbers of agents and to consider the formation of pairs and singletons (rooms can be occupied either by one or by two agents). The class of roommate problems also include as special cases the well-known marriage problems (Gale and Shapley, 1962).[1]

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[1]There is a large literature on the marriage problem; see, for instance, Roth and Sotomayor (1990) and the two-sided matching bibliography on Al Roth's game theory, experimental economics, and market design page. In comparison, relatively few papers and books deal with roommate problems; some of the key references concerning roommate problems are Chung (2000); Diamantoudi et al. (2004); Gusfield and Irving (1989); Tan (1991).

The aim of this paper is twofold. First, we review a key result for roommate problems for which we provide a concise and elementary proof. Second, and related to the title of this paper, we show how the often incompatible concepts of equilibrium or stability (represented by Adam Smith[2]) and fairness (represented by John Rawls[3]) can be reconciled for roommate problems.

Formally, a *roommate problem* (Gale and Shapley, 1962) is a pair $(N, (\succeq_i)_{i \in N})$ where N is a finite set of agents and, for each $i \in N$, $\succeq_i$ is a total order over N.[4] For each $i \in N$, we interpret $\succeq_i$ as agent i's preferences over sharing a room with any of the agents in $N \backslash \{i\}$ and having a room for himself (or consuming an outside option such as living off-campus). Preferences are strict, i.e., $k \succeq_i j$ and $j \succeq_i k$ if and only if $j = k$. The strict preference relation associated with $\succeq_i$ is denoted by $\succ_i$. A solution to a roommate problem, a *matching* μ, is a partition of N in pairs and singletons. Alternatively, we describe a matching by a function $\mu : N \rightarrow N$ of order two, i.e., for all $i \in N$, $\mu(\mu(i)) = i$. Agent $\mu(i)$ is agent i's match, i.e., the agent with whom he is matched to share a room (possibly himself). If $\mu(i) = i$ then we call i a single.

A *marriage problem* (Gale and Shapley, 1962) is a roommate problem $(N, (\succeq_i)_{i \in N})$ such that N is the union of two disjoint sets M and W (men and women), and each agent in M (respectively W) prefers being alone to being matched with any other agent in M (respectively W).

A matching μ is *blocked by a pair* $\{i, j\} \subseteq N$ (possibly $i = j$) if $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$. If $\{i, j\}$ blocks μ, then $\{i, j\}$ is called a blocking pair for μ. A matching is *individually rational* if there is no blocking pair $\{i, j\}$ with $i = j$. A matching is *stable* if there is no blocking pair. A roommate problem is *solvable* if the set of stable matchings is non-empty. Gale and Shapley (1962) showed that all marriage problems are solvable and provided an unsolvable roommate problem (Gale and Shapley, 1962, Example 3).

The following is a simplified version of Gale and Shapley's example with three agents: $2 \succ_1 3 \succ_1 1$, $3 \succ_2 1 \succ_2 2$, and $1 \succ_3 2 \succ_3 3$. Clearly, the matching where all agents are singles is not stable (any two agents can block). So, assume two agents share a room. Then, the single agent is the best roommate for one of these two agents and hence a blocking pair can be formed. Tan (1991) provided a necessary and sufficient condition for the existence of a stable matching.

Note that for marriage problems, an individually rational matching never matches two men or two women, i.e., the partition consists of man-woman pairs and singletons.

2 Lonely Wolves, Medians, and Compromise

Our starting point is a solvable roommate problem. Typically there are multiple stable matchings and with choice comes the opportunity to select a particularly appealing stable

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[2] Adam Smith (1723–1790), a political economist, propagated the view that individuals even though interested only in their own gains will still advance public interest (Smith, 1796).

[3] John Rawls (1921–2002), a political philosopher, discussed important aspects of fairness and justice particularly suited for economic applications (Rawls, 1971).

[4] A total order is a binary relation that satisfies reflexivity, antisymmetry, transitivity, and totality or comparability.

matching, for example a stable matching that maximizes the number of matched pairs. It turns out that no such selection is possible because an agent who is single at one stable matching is also single at all other stable matchings (Gusfield and Irving, 1989, Theorem 4.5.2). In fact, Gale and Sotomayor (1985, Proposition 1) already proved this "lonely wolf" theorem for marriage problems. Here we provide an elementary proof.

Theorem 1 *The lonely wolf theorem*
Let μ and μ' be stable matchings. Then, μ and μ' have the same set of single agents, i.e., $\mu(i) = i \Leftrightarrow \mu'(i) = i$.

Proof   Suppose $\mu(i_1) = i_1 \neq i_2 = \mu'(i_1)$ for some $i_1, i_2 \in N$. Since preferences are strict and μ' is individually rational, $i_2 \succ_{i_1} i_1$.

Define $i_3 := \mu(i_2)$. Then, $i_3 \succ_{i_2} i_1$ (otherwise $\{i_1, i_2\}$ is a blocking pair for μ) and $i_3 \neq i_2$ (otherwise $\{i_2\}$ is a blocking pair for μ'). Define $i_4 := \mu'(i_3)$. Then, $i_4 \succ_{i_3} i_2$ (otherwise $\{i_2, i_3\}$ is a blocking pair for μ') and $i_4 \neq i_3$ (otherwise $\{i_3\}$ is a blocking pair for μ). Next, define agents $i_5, i_6, \ldots$ by

$$i_{p+1} := \begin{cases} \mu(i_p) & \text{if } p \text{ is even;} \\ \mu'(i_p) & \text{if } p \text{ is odd.} \end{cases}$$

Similarly as before it follows that for all $p \geq 2$, $i_{p+1} \succ_{i_p} i_{p-1}$ and $i_{p+1} \neq i_p$. Since N is finite, there is a $p' \geq 1$ with

$$i_{p'+1} = i_l \text{ for some } l \leq p'. \tag{1}$$

Let $\bar{p}$ be the smallest $p' \geq 1$ that satisfies (1). By showing that this leads to a contradiction, we prove that $\mu(i_1) = i_1$ implies $\mu'(i_1) = i_1$.

Note that $i_3 \notin \{i_1, i_2\}$ and therefore $\bar{p} \geq 3$. In general, for all $p \geq 2$, $i_{p+1} \notin \{i_p, i_{p-1}\}$. Hence, $\bar{p} \geq 3$ and $\bar{p} > l + 1$ for all $l \leq \bar{p}$ with $i_{\bar{p}+1} = i_l$.

Let $l \leq \bar{p}$ with $i_{\bar{p}+1} = i_l$. Suppose $\bar{p}$ is even, i.e., $i_{\bar{p}+1} = \mu(i_{\bar{p}})$. Then, $\mu(i_{\bar{p}}) = i_l$. If $l = 1$, then from the definition of i_1, $i_{\bar{p}} = i_1$, contradicting that $\bar{p}$ is the smallest $p' \geq 1$ that satisfies (1). If l is odd and $l \neq 1$, then $i_{l-1} = \mu(i_l) = i_{\bar{p}}$, contradicting the definition of $\bar{p}$. If l is even, then $i_{l+1} = \mu(i_l) = i_{\bar{p}}$, contradicting the definition of $\bar{p}$.

Suppose $\bar{p}$ is odd, i.e., $i_{\bar{p}+1} = \mu'(i_{\bar{p}})$. Then, $\mu'(i_{\bar{p}}) = i_l$. If $l = 1$, then from the definition of i_2, $i_{\bar{p}} = i_2$, contradicting the definition of $\bar{p}$. If l is odd and $l \neq 1$, then $i_{l+1} = \mu'(i_l) = i_{\bar{p}}$, contradicting the definition of $\bar{p}$. If l is even, then $i_{l-1} = \mu'(i_l) = i_{\bar{p}}$, contradicting the definition of $\bar{p}$. ∎

Since no selection can be based on the set of matched agent, we next try to find a stable matching that will be perceived as fair by the agents. Imagine that we ask each agent to rank all stable matchings according to his preferences. For two (stable) matchings μ and μ', $\mu \succeq_i \mu' \Leftrightarrow \mu(i) \succeq_i \mu'(i)$. Note that since an agent might be matched to the same agent in several stable matchings, this ranking is not strict. Clearly, we cannot always give the best match to every agent, but can we implement fairness by finding a matching that matches each agent with his k-th ranked match? We explain this idea in our next example and show that this idea of fairness or compromise (at least if there is an even number of stable matchings) is not always feasible.

Example 1 *An even number of stable matchings and no compromise*
Consider the following roommate problem with 8 agents and preferences as listed in the table below. The first column, for instance, means that $8 \succ_1 2 \succ_1 7 \succ_1 4 \succ_1 1 \succ_1 5 \succ_1 3 \succ_1 6$.

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$	$\succ_8$
8	3	4	2	6	7	8	5
2	5	6	1	3	5	6	7
7	1	2	3	8	4	1	1
4	4	5	5	2	2	2	2
1	6	1	6	1	6	3	8
5	8	7	8	5	1	4	6
3	7	3	7	7	3	5	4
6	2	8	4	4	8	7	3

There are 4 stable matchings:

$$\mu_1 = \{\{1,2\}, \{3,4\}, \{5,6\}, \{7,8\}\}$$
$$\mu_2 = \{\{1,2\}, \{3,4\}, \{5,8\}, \{6,7\}\}$$
$$\mu_3 = \{\{1,4\}, \{2,3\}, \{5,6\}, \{7,8\}\}$$
$$\mu_4 = \{\{1,4\}, \{2,3\}, \{5,8\}, \{6,7\}\}$$

The following (weak) rankings on matchings are induced ($\sim$ denotes indifferences):

$$\mu_1 \sim_1 \mu_2 \succ_1 \mu_3 \sim_1 \mu_4$$
$$\mu_3 \sim_2 \mu_4 \succ_2 \mu_1 \sim_2 \mu_2$$
$$\mu_1 \sim_3 \mu_2 \succ_3 \mu_3 \sim_3 \mu_4$$
$$\mu_3 \sim_4 \mu_4 \succ_4 \mu_1 \sim_4 \mu_2$$
$$\mu_1 \sim_5 \mu_3 \succ_5 \mu_2 \sim_5 \mu_4$$
$$\mu_2 \sim_6 \mu_4 \succ_6 \mu_1 \sim_6 \mu_3$$
$$\mu_1 \sim_7 \mu_3 \succ_7 \mu_2 \sim_7 \mu_4$$
$$\mu_2 \sim_8 \mu_4 \succ_8 \mu_1 \sim_8 \mu_3$$

Agents 1, 2, and 4 order their matches at the 4 stable matchings as $2, 2, 4, 4$; $3, 3, 1, 1$; and $1, 1, 3, 3$ respectively. If agent 1 is given his first or second choice, then he is matched with agent 2 who then receives his third or fourth choice. If agent 1 is given his third or fourth choice, then he is matched with agent 4 who then receives his first or second choice. It follows that matching all agents with a k-th ranked match is not possible. $\diamond$

The impossibility result exhibited in the example is due to the fact that there is an even number of (distinct) stable matchings. Next, we show that for roommate problems with an odd number of stable matchings a compromise matching where each agent is matched to a match of the same rank is possible. In fact, we prove that for any odd number of stable matchings, a stable matching at which each agent is matched to his "median" match always exists. Thus, for roommate problems with an odd number of stable matchings Adam Smith (who stands for stability) and John Rawls (who stands for fairness) represent compatible criteria and hence "can share a room."[5]

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[5]Example 1 is an example where the equilibrium and the fairness principles represented by Adam Smith and John Rawls are incompatible: there they will never be able to share a room.

The next lemma, which appeared in Gusfield and Irving (1989, Lemma 4.3.9), facilitates the proof of our main result.

Lemma 1 *Let μ and μ' be stable matchings. Let $i \in N$. Suppose $\mu'(i) \neq \mu(i) = j$ for some $j \in N$. Then, $j, \mu'(i) \neq i$. Moreover, $\mu(i) \succ_i \mu'(i)$ implies $\mu'(j) \succ_j \mu(j)$.*

Proof   By Theorem 1, $j, \mu'(i) \neq i$. Suppose $\mu(i) \succ_i \mu'(i)$ and $\mu(j) \succeq_j \mu'(j)$. Since preferences are strict and $\mu(j) = i \neq \mu'(j)$, $\mu(j) \succ_j \mu'(j)$. Hence, $\{i, j\}$ is a blocking pair for μ', contradicting stability. ∎

Our main result, Theorem 2, extends Theorem 4.3.5 in Gusfield and Irving (1989) from three to any odd number of stable matchings.

Let $\mu_1, \ldots, \mu_{2k+1}$ be an odd number of (possibly non-distinct) stable matchings. Let each agent rank these matchings according to his preferences as explained before (Example 1). We denote agent i's $(k+1)$-st ranked match by $med\{\mu_1(i), \ldots, \mu_{2k+1}(i)\}$.

Theorem 2 *Smith and Rawls share a room*
Let $\mu_1, \ldots, \mu_{2k+1}$ be an odd number of (possibly non-distinct) stable matchings. Then, $\mu^* : N \to N$ defined by

$$\mu^*(i) := med\{\mu_1(i), \ldots, \mu_{2k+1}(i)\} \text{ for all } i \in N$$

is a well-defined stable matching. We call μ^* the median matching *of $\mu_1, \ldots, \mu_{2k+1}$.*

In the context of the linear programming approach to so-called bistable matching problems, Sethuraman and Teo (2001, Theorem 3.2) mentioned this result (without proof) as an interesting structural property of stable roommate matchings. Here we provide an elementary proof.

Proof   First, we show that μ^* is a well-defined matching, i.e., μ^* is of order 2. Let $i \in N$ with $\mu^*(i) \neq i$. We have to prove that $\mu^*(\mu^*(i)) = i$. Let $j := \mu^*(i)$. Without loss of generality, $\mu_1(i) \succeq_i \mu_2(i) \succeq_i \cdots \succeq_i \mu_{2k}(i) \succeq_i \mu_{2k+1}(i)$. Then, $\mu^*(i) = \mu_{k+1}(i) = j$ and $\mu_{k+1}(j) = i$. Let $S = \{\mu_1, \ldots, \mu_{2k+1}\}$. Define

$$\begin{aligned} \overline{B}_i &:= \{\mu \in S \mid \mu \succeq_i \mu_{k+1}\}, \\ W_i &:= \{\mu \in S \mid \mu \prec_i \mu_{k+1}\}, \\ B_j &:= \{\mu \in S \mid \mu \succ_j \mu_{k+1}\}, \text{ and} \\ \overline{W}_j &:= \{\mu \in S \mid \mu \preceq_j \mu_{k+1}\}. \end{aligned}$$

Clearly, $\{\overline{B}_i, W_i\}$ and $\{B_j, \overline{W}_j\}$ are partitions of S, where possibly $W_i = \emptyset$ or $B_j = \emptyset$.

By Lemma 1, $W_i \subseteq B_j$. Suppose there exists $\mu \in B_j \setminus W_i$. Since $\overline{B}_i \cup W_i = S$, $\mu \in \overline{B}_i$. Thus, $\mu \in \overline{B}_i \cap B_j$. Since $\mu \in B_j$, $\mu \succ_j \mu_{k+1}$. In particular, since $\mu_{k+1}(j) = i$, $\mu(j) \neq i$. Then, since $\mu \in \overline{B}_i$, $\mu \succ_i \mu_{k+1}$. Hence, $\{i, j\}$ is a blocking pair for μ_{k+1}, contradicting stability. So, $W_i = B_j$. Since $\{\overline{B}_i, W_i\}$ and $\{B_j, \overline{W}_j\}$ are partitions of S, $\overline{B}_i = \overline{W}_j$. Hence,

$$\overbrace{\mu_1 \succeq_i \mu_2 \succeq_i \cdots \succeq_i \mu_{k+1} \sim_i \cdots}^{\overline{B}_i} \succ_i W_i$$

$$W_i \succ_j \underbrace{\cdots \sim_j \mu_{k+1} \succeq_j \{\mu_1, \ldots, \mu_k\}}_{\overline{B}_i}$$

5

In particular, it follows that $\mu^*(j) = \mu_{k+1}(j)$. Hence, $\mu^*(\mu^*(i)) = \mu^*(j) = \mu_{k+1}(j) = i$.

We now prove that μ^* is stable. By definition, μ^* is individually rational. Suppose there is a blocking pair $\{i, j\}$ with $i \neq j$, i.e., $j \succ_i \mu^*(i)$ and $i \succ_j \mu^*(j)$. Then, i prefers j to at least $k + 1$ stable matchings in $\{\mu_1, \dots, \mu_{2k+1}\}$. Similarly, j prefers i to at least $k + 1$ stable matchings in $\{\mu_1, \dots, \mu_{2k+1}\}$. Hence, for at least one stable matching $\mu \in \{\mu_1, \dots, \mu_{2k+1}\}$, $j \succ_i \mu(i)$ and $i \succ_j \mu(j)$, contradicting stability. ∎

We can easily extend the result of Theorem 2 to an even number of stable matchings (Sethuraman and Teo, 2001, Theorem 3.3).

Corollary 1 *Smith and Rawls (almost) share a room*
Let $\mu_1, \dots, \mu_{2k}$ be an even number of (possibly non-distinct) stable matchings. Then, there exists a stable matching at which each agent is assigned a match of rank k or $k + 1$.

Proof By Theorem 2, the median of the first $2k - 1$ stable matchings, i.e., $\mu^* = med\{\mu_1, \dots, \mu_{2k-1}\}$, is a well-defined and stable matching. Then, for any $i \in N$,

agent i is matched at μ^* with his $\begin{cases} k\text{-th ranked match} & \text{if } \mu^*(i) \succeq_i \mu_{2k}(i); \\ (k+1)\text{-st ranked match} & \text{if } \mu^*(i) \preceq_i \mu_{2k}(i). \end{cases}$ ∎

In the next example we calculate the median matching for three stable matchings and demonstrate that the "median operator" is not closed, i.e., that the resulting median matching is not always one of the stable matchings that were used to calculate it.

Example 2 *An odd number of stable matchings and median matchings*
Consider the following roommate problem with 8 agents and preferences as listed in the table below.

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$	$\succ_6$	$\succ_7$	$\succ_8$
4	7	2	6	2	1	4	7
2	4	8	2	6	2	6	2
5	6	5	5	4	8	1	5
3	8	4	8	7	3	3	1
8	1	6	3	8	4	5	4
7	5	1	1	3	5	2	6
6	3	7	7	1	7	8	3
1	2	3	4	5	6	7	8

There are 5 stable matchings:

$$\mu_1 = \{\{1, 3\}, \{2, 4\}, \{5, 7\}, \{6, 8\}\}$$
$$\mu_2 = \{\{1, 3\}, \{2, 7\}, \{4, 5\}, \{6, 8\}\}$$
$$\mu_3 = \{\{1, 7\}, \{2, 4\}, \{3, 6\}, \{5, 8\}\}$$
$$\mu_4 = \{\{1, 8\}, \{2, 4\}, \{3, 6\}, \{5, 7\}\}$$
$$\mu_5 = \{\{1, 8\}, \{2, 7\}, \{3, 6\}, \{4, 5\}\}$$

μ_4 is the median matching of $\{\mu_1, \mu_2, \mu_3, \mu_4, \mu_5\}$ and of $\{\mu_1, \mu_3, \mu_5\}$. ◇

For marriage problems we even have a stronger result. Let $\mu_1, \ldots, \mu_k$ be k (possibly non-distinct) stable matchings for a marriage problem and assume that each agent ranks these matchings according to his/her preferences. Using linear programming tools, Teo and Sethuraman (1998, Theorem 2) showed that the map that assigns to each man his l-th (weakly) best match and to each woman her $(k-l+1)$-st (weakly) best match determines a well-defined and stable matching. We explain and discuss (generalized) medians as compromise solutions for two-sided matching problems in Klaus and Klijn (2006). Below we provide an elementary proof of this "generalized median result" for marriage problems.

Consider a marriage problem $(M \cup W, (\succeq_i)_{i \in N})$. Let $\mu_1, \ldots, \mu_k$ be k (possibly non-distinct) stable matchings. Let each agent rank these matchings according to his/her preferences as explained before. Formally, for each $i \in M \cup W$ there is a sequence of matchings $(\mu_1^i, \ldots, \mu_k^i)$ such that $\{\mu_1^i, \ldots, \mu_k^i\} = \{\mu_1, \ldots, \mu_k\}$ and for any $l \in \{1, \ldots, k-1\}$, $\mu_l^i(i) \succeq_i \mu_{l+1}^i(i)$. Thus, for any $l \in \{1, \ldots, k\}$, at μ_l^i agent i is assigned to his/her l-th (weakly) best match (among the k stable matchings).

For any $l \in \{1, \ldots, k\}$, we define the generalized median stable matching α_l as the function $\alpha_l : M \cup W \to M \cup W$ defined by

$$\alpha_l(i) := \begin{cases} \mu_l^i(i) & \text{if } i \in M; \\ \mu_{(k-l+1)}^i(i) & \text{if } i \in W. \end{cases}$$

Theorem 3 *Marriage and compromise (generalized medians)*
Let $\mu_1, \ldots, \mu_k$ be k (possibly non-distinct) stable matchings for a marriage problem. Then, for any $l \in \{1, \ldots, k\}$, α_l is a well-defined stable matching.

Proof Let $l \in \{1, \ldots, k\}$. First, we show that α_l is a well-defined matching, i.e., α_l is of order 2. Let $m \in M$ with $\alpha_l(m) = w \in W$. We have to prove that $\alpha_l(\alpha_l(m)) = m$. Without loss of generality, $\mu_1(m) \succeq_m \mu_2(m) \succeq_m \cdots \succeq_m \mu_l(m) \succeq_m \cdots \succeq_m \mu_{k-1}(m) \succeq_m \mu_k(m)$ and $\alpha_l(m) = \mu_l(m)$. Then, $\mu_l(m) = \alpha_l(m) = w$ and $\mu_l(w) = m$. By arguments similar to those in the first part of the proof of Theorem 2,[6]

$$\mu_1 \succeq_m \cdots \succeq_m \mu_{l-1} \succeq_m \quad \mu_l \quad \succeq_m \mu_{l+1} \succeq_m \cdots \succeq_m \mu_k$$
$$\{\mu_k, \ldots, \mu_{l+1}\} \succeq_w \quad \mu_l \quad \succeq_w \{\mu_{l-1}, \ldots, \mu_1\}$$

In particular, $\mu_l(w)$ is the $(k-l+1)$-st ranked match for woman w and therefore $\alpha_l(w) = \mu_l(w)$. Hence, $\alpha_l(\alpha_l(m)) = \alpha_l(w) = \mu_l(w) = m$.

We now prove that α_l is stable. By definition, α_l is individually rational. Suppose there is a blocking pair $\{m, w\}$ with $m \in M$ and $w \in W$, i.e., $w \succ_m \alpha_l(m)$ and $m \succ_w \alpha_l(w)$. Then, m prefers w to at least $k-l+1$ stable matchings in $\{\mu_1, \ldots, \mu_k\}$. Similarly, w prefers m to at least l stable matchings in $\{\mu_1, \ldots, \mu_k\}$. Hence, for at least one matching $\mu \in \{\mu_1, \ldots, \mu_k\}$, $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$, contradicting stability. ∎

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[6]Recall that in the first part of the proof of Theorem 2 only the relative rankings of agent i's mates with respect to μ_{k+1} mattered – which exact rank μ_{k+1} had did not play any role in that part of the proof.

Fleiner (2002, Theorem 5.5), Klaus and Klijn (2006, Theorem 3.2), and Sethuraman et al. (2006, Theorem 9) generalized Theorem 3 to college admissions problems (Gale and Shapley, 1962) with responsive preferences,[7] in which students have to be matched to colleges based on the students' and the colleges' preferences over the other side of the market and colleges' capacity constraints. It is well-known that for this class of two-sided matching problems the set of stable matchings is still non-empty and has the "lonely wolf" property, i.e., the set of single students does not vary from one stable matching to another (see, for instance, Roth and Sotomayor, 1990, Lemma 5.6 and Theorem 5.12). Martínez et al. (2000) introduced the domain of q-separable and substitutable preferences for colleges that contains the domain of responsive preferences. They prove that even for this larger domain the set of stable matchings is non-empty and that the "lonely wolf" property holds. Surprisingly, we cannot extend Theorem 3 to the domain of q-separable and substitutable preferences as the following example shows.

Example 3 *No compromise for q-separable and substitutable preferences*
Consider Martínez et al.'s (2000, Example 2) college admissions problem with 4 students s_1, s_2, s_3, s_4, 2 colleges C_1 and C_2 with 2 seats each, and preferences as listed in the table below. The colleges' preferences are q-separable and substitutable (for details see Martínez et al., 2000).

$\succ_{C_1}$	$\succ_{C_2}$	$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
$\{s_1, s_2\}$	$\{s_3, s_4\}$	C_2	C_2	C_1	C_1
$\{s_1, s_3\}$	$\{s_2, s_4\}$	C_1	C_1	C_2	C_2
$\{s_2, s_4\}$	$\{s_1, s_3\}$				
$\{s_3, s_4\}$	$\{s_1, s_2\}$				
$\{s_1, s_4\}$	$\{s_1, s_4\}$				
$\{s_2, s_3\}$	$\{s_2, s_3\}$				
$\{s_1\}$	$\{s_1\}$				
$\{s_2\}$	$\{s_2\}$				
$\{s_3\}$	$\{s_3\}$				
$\{s_4\}$	$\{s_4\}$				

There are 4 stable matchings:

$$\mu_1 = \{\{C_1, s_1, s_2\}, \{C_2, s_3, s_4\}\}$$
$$\mu_2 = \{\{C_1, s_1, s_3\}, \{C_2, s_2, s_4\}\}$$
$$\mu_3 = \{\{C_1, s_2, s_4\}, \{C_2, s_1, s_3\}\}$$
$$\mu_4 = \{\{C_1, s_3, s_4\}, \{C_2, s_1, s_2\}\}$$

Considering the first three matchings, one straightforwardly checks that matching each agent with its median match is not a matching: C_1 would be matched with $\{s_1, s_3\}$ but at the same time s_3 would be matched with C_2.   ◇

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[7]By responsiveness (Roth, 1985), a college's preference relation over sets of students is related to its ranking of single students in the following way: the college always prefers to add an acceptable student to any set of students (provided this does not violate the capacity constraint) and it prefers to replace any student by a better student.

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