Freedom of Choice as Control over Outcomes *

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Abstract

As interactions between individuals are introduced into the freedom of choice literature by the mean of game forms, new issues appear. In particular, in this paper it is argued that individuals face uncertainty with respect to outcomes as they lose the control they implicitely exert over options in the opportunity set framework. A criterion is proposed as to compare alternative game forms in terms of the control they offer to individuals. The CardMin criterion suggests that any game form should be judged on the basis of the strategy offering the lowest number of pairwise different outcomes. An axiomatic characterization is provided in the case of two individuals.

1 Introduction

In this paper what is under scrutiny is the problem of measuring the *control* that individuals exert over the different outcomes they are facing in game forms, as part of the total freedom of choice they enjoy. Freedom of choice has been the subject of a large and stimulating debate and has appeared since fifteen years to be a central concern to social choice theorists who believe that economic policies should no longer be judged in terms of the consequences they have on individuals' well being only, as they are in the traditional welfarist approach.

Rather, the amount of freedom of choice available to the individuals should also be of importance in the evaluation of the policies (see e.g. Rawls (1971), Sen (1985, 1988, 1991) for more detailed discussions). In a concern for formalising the concept of freedom

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of choice, the literature considers that individuals are facing opportunity sets, defined as the sets of all options available to them (see e.g. Arrow (1995), Barbera, Bossert and Pattanaik (2005), Foster (2005), Gravel, Laslier and Trannoy (1998), Pattanaik and Xu (1990), Puppe (1996), Sugden (1998), Suppes (1996), Van Hees (1997) for representative contributions and surveys).

In Bervoets (2007), it has been pointed out that the traditional literature on freedom suffers from its neglect of the interactions that take place between agents forming the society. Indeed, the usual framework assumes that individuals face opportunity sets from which they can choose any option, regardless of any interactive constraints. Individuals are considered alone and this is rather problematic for an attempt to define freedom of choice, especially when one has in mind the famous maxim according to which "the freedom of one begins where the freedom of others ends". In order to account for the interactions, Bervoets (2007) considered game forms rather than opportunity sets as the proper object to appraise individual freedom of choice. In game forms, as opposed to opportunity sets, individuals are not able to determine the final outcome of the social situation by their own choice. Rather, each individual can choose a strategy which, along with the choice of others, will determine the social outcome. In the interactive framework, options (i.e. strategies[1]) and outcomes differ, contrary to the traditional framework of opportunity sets in which both coincide. In the present paper the game form framework is adopted.

The appraisal of freedom of choice in a social context raises some subtle issues, one of which being the *uncertainty* about the final outcome to which a particular individual decision may lead. Uncertainty and control are regarded as two opposite issues herein, as less control for individuals over outcomes of the game form implies more uncertainty and vice-versa. Eventhough one agrees with the views defended throughout the freedom of choice literature that freedom should be measured on the basis of the quantity and the quality of choices available, it seems clear that the availability of a large number of possible outcomes that have very low probability of occurrence does not contribute much to the freedom enjoyed by the individual. Hence, among other important features, the control exerted by individuals over outcomes should be one issue raised when trying to evaluate freedom of choice.

Uncertainty has been the subject of large attention in decision theory, where it is assumed either in a "partial" form - individuals are assumed to know the probabilities of occurrence of the different outcomes[2] (see e.g. Kreps (1988) or Machina (1987) for accessible surveys) - or in a complete form - individuals know the set of outcomes associated to every option but have no information about probabilities or likelihood ranking for these outcomes (see e.g. Bossert (1997), Bossert, Pattanaik and Xu (2000), Nitzan and Pattanaik (1984), Pattanaik and Peleg (1984) for representative pieces). The present contribution is in line with this second stream of literature because complete

[1]Throughout the terms strategy and option will be used indifferently.

[2]According to the distinction suggested by Knight (1921), this litterature refers to *risk* rather than uncertainty.

uncertainty is assumed but it differs because game forms are compared instead of sets.

Another difference between this and the traditional approach on uncertainty is that individuals are not equipped with preferences. This is because the concern of this paper is about a *pure* notion of control. What is under scrutiny is the meaning of "individual i has total (or high, low, partial ...) control over outcomes", independently from the preferences this individual has [3]. One may argue that it is valuable for players to have a high level of control over outcomes they desire and it is also valuable to have control over outcomes they highly dislike, as to prevent those outcomes to be realised. This cannot be done without introducing preferences. As one can see, it is conceivable to connect the notion of control with that of preferences, but such a connection is not provided in this paper. Rather, what is regarded is a notion of predictability of the outcomes by individuals.

Another literature worth mentioning followed Sen's Liberal Paretian Paradox (Sen (1970)) concerning individual rights. After three decades of intensive debate on this issue, a disputed (see e.g. Sen (1983)) majority seems to have reached a consensus that rights should be modelled using game forms rather than the traditional social choice framework (see e.g. Deb (1994,2004), Deb, Pattanaik and Razzolini (1997), Gaertner, Pattanaik and Suzumura (1992), Gärdenfors (1981), Peleg (1984, 1998)). Within the game form framework, effectivity functions have been under scrutiny as functions assigning rights to individuals or to coalitions, giving them the power of restricting the possible outcomes of the game form into a special subset (see e.g. Deb (1994), Moulin (1983), Peleg (1984, 1998) on that particular issue). Effectivity functions and control are related as the more effective an individual, the more control he exerts over outcomes. However, this literature is mainly devoted to the analysis of different notions of effectivity and their influence on the modelling of rights, and it has not produced yet, to the best of my knowledge, methods for comparing alternative game forms on the basis of the "effectivity" individuals enjoy.

Focusing on the case of two individuals forming the society, an axiomatic characterisation is obtained of a criterion called the CardMin, that compares two game forms in terms of the control they offer. A CardMin strategy for player i is a strategy leading for individual i to the lowest number of pairwise different outcomes in the game form. The CardMin criterion then compares two game forms on the basis of the number of outcomes associated to the CardMin strategy in each game form. While this method for comparing game forms may seem naive, among other reasons because it does not consider the justifications for which individuals may value control, it has the merit of being, as far as I am aware of, a first attempt to introduce that issue into the freedom of choice literature in the game form framework.

The rest of the paper is organised as follows. In the next section will be introduced the formal framework, section 3 then presents the axioms used in the characterisation

[3]This relates to the way in which Pattanaik and Xu (1990) measure a *pure* notion of freedom of choice

result given in section 4. Independance of the axioms is shown in the same section, and reasons why the extension to n individuals reveals problematic are discussed through an example. Section 5 concludes.

2 Notations

Let $N = \{1, 2\}$ be the set of players. A *game form* is defined with the following elements (besides the set of individuals):
- $\mathcal{A}_i$ a set of all conceivable strategies individual i can face. Subsets of $\mathcal{A}_i$ will be denoted by A_i, B_i... and the cartesian product $A_1 \times A_2$ will be denoted as A.
- X a set of all social outcomes generated by the strategies in $\mathcal{A}_1$ and $\mathcal{A}_2$.
- An outcome function $g : \mathcal{A}_1 \times \mathcal{A}_2 \longrightarrow X$. This function associates a unique social outcome to any combination of individual strategies. An outcome is thus given by $g(a_1, a_2) = x$ where $a_1 \in A_1$, $a_2 \in A_2$ and $x \in X$.

A game form is given by the pair (A, g). Let $\mathcal{G}$ be the set of all game forms and let $\mathcal{G}(g) \subset \mathcal{G}$ be the set of all game forms generated by the outcome function g. The analysis will be led throughout this paper with $\mathcal{G}(g)$ rather than $\mathcal{G}$, but this restriction of domain is made for notational convenience only. Considering any game forms in $\mathcal{G}$ would yield unnecessary notational complexity in what follows; however the characterisation result of section 4 would still hold without this restriction, given some slight changes in the axioms used. Hence, and so long as there is no ambiguity, game forms in $\mathcal{G}(g)$ are designated uniquely by the set A, droping the outcome function g from the notation.

Let $\succeq_i$ be defined for all $i \in N$ as a transitive binary relation over $\mathcal{G}(g)$. For all $A, B \in \mathcal{G}(g)$, $A \succeq_i B$ means that "the control over outcomes offered by the game form A is at least as great for individual i as the control over outcomes offered by game form B". $\succ_i$ and $\sim_i$ are the antisymmetric and symmetric parts of $\succeq_i$. The cardinality of any set D will be denoted $\#D$.

From now on the analysis will be made from individual 1's standpoint ($i = 1$). This is of course without any loss of generality.

Definition 1 *A strategy $a_2 \in \mathcal{A}_2$ is $(\{a_1\} \times A_2)$-neutral for individual 1 if $\exists\, s_2' \in A_2$, $a_2 \neq s_2'$, such that $g(a_1, a_2) = g(a_1, s_2')$*

This definition says that a strategy available to individual 2 is neutral for individual 1, with respect to strategy a_1, if the outcome generated by that neutral strategy could be generated using another strategy in A_2.

Remark 1 *If $a_2 \in \mathcal{A}_2$ is $(\{a_1\} \times \{s_2'\})$-neutral for individual 1 then of course s_2' is $(\{a_1\} \times \{a_2\})$-neutral for individual 1*

3 Axioms

Some axioms about the way specific sets should be compared in terms of control are discussed in this section.

Axiom 1 $\forall a_1, b_1 \in \mathcal{A}_1, \forall a_2, b_2 \in \mathcal{A}_2,\ \{a_1\} \times \{a_2\} \sim_1 \{b_1\} \times \{b_2\}$

This axiom simply says that all game forms consisting of one strategy only should be considered indifferent in terms of control. In other words, in these game forms player 1 has trivially full control over the unique possible outcome so a very natural requirement is that both games offer the same control. If control was to be apprehended in a larger framework considering individual preferences, axiom 1 should take them into account in some way. In its form, this axiom suggests that what is under scrutiny is a “pure” notion of control.

Axiom 2 *If $a_2 \in \mathcal{A}_2$ is not $(\{a_1\} \times \{s'_2\})$-neutral for individual 1 then*

$$\{a_1\} \times \{s'_2\} \succ_1 \{a_1\} \times \{a_2, s'_2\}$$

This axiom is inspired by the second axiom used in Pattanaik and Xu (1990), although it is used reversely, stating that any set with two elements offers strictly less control than the singletons it contains. It is again very natural that individual 1 should strictly have a better evaluation of control over one unique possible outcome than over a set of two possible outcomes, including that of the first game form. Indeed, in the second game form, the choice of the final outcome fully relies on the other player’s decision, hence there is a “total” uncertainty about the final outcome whereas there is none in the first game form. The predictability of the outcome is much higher in the first case than in the second, justifying the ranking suggested by axiom 2.

Axiom 3 *If $a_2 \in \mathcal{A}_2$ is $(\{a_1\} \times A_2)$-neutral then $\{a_1\} \times A_2 \sim_1 \{a_1\} \times A_2 \cup \{a_2\}$.*

Whenever individual 1’s set is a singleton, adding a new strategy to individual 2 that generates no new outcome, leaves individual 1 indifferent. This axiom, although seemingly natural and intuitive, is an important conceptual one. Indeed, options in game forms can be described on the basis of the *vectors* of contingent outcomes that they generate. Alternatively options can be described on the basis of the *sets* of possible outcomes. This second approach leads to some loss of information as what only matters is whether outcomes exist as the result of at least one combination of options, independently of how many times they appear in the game form. However in terms of control this loss of information should not influence our judgment as no probabilities are attached to outcomes. Hence the fact that one outcome appears more frequently in the game form than another does not tell us that it will appear more frequently than the other once the game is played. If one could easily admit that the generation of a new outcome by addition of a strategy to individual 2's set would strictly reduce individual 1's control over the final outcome and therefore that the set $\{x, y, z\}$ would offer less control than the set $\{x, y\}$, it seems natural to state that both vectors (x, y) and (x, y, y), yielding each of them the same set $\{x, y\}$, offer the same amount of control to individual 1. Axiom 3 says that the information lost with the choice of the set-based model does not intervene in the ranking of sets. (For a more detailed discussion about set-based or vector-based models, see Pattanaik and Peleg (1984)).

Axiom 4 *If $a_2 \in \mathcal{A}_2$ is not $(\{a_1\} \times A_2)$-neutral and $b_2 \in \mathcal{A}_2$ is not $(\{b_1\} \times B_2)$-neutral for individual 1 then*

$$\{a_1\} \times A_2 \succeq_1 \{b_1\} \times B_2 \Longleftrightarrow \{a_1\} \times A_2 \cup \{a_2\} \succeq_1 \{b_1\} \times B_2 \cup \{b_2\}$$

This axiom concerns only game forms for which individual 1 has a singleton as a strategy set. In those cases it is required that adding (or removing) exactly one possible outcome to two different game forms cannot reverse the ranking that preexisted. Notice that the strategies added on both sides are not necessarily the same, making this axiom a strong one. However, note once again that it is valid only when individual 1 is facing a singleton, keeping silent on the cases where individual 1 has a larger set of strategies. This implies that the addition of one option to individual 2's set of options generates only one additional outcome to the considered game form. As axiom 1 states that both options added are judged to be indifferent one with the other, axiom 4 associated to axiom 1 states that the addition of indifferent outcomes on both sides cannot reverse the ranking.

Axiom 5 *If $A_1 \times A_2 \succeq_1 B_1 \times A_2$ then $A_1 \times A_2 \sim_1 A_1 \cup B_1 \times A_2$*

Note that this axiom, contrary to the four previous ones, considers what happens for some changes affecting individual 1's set of strategies, assuming individual 2's set is fixed.

To make interpretation clearer, axiom 5 should be divided in two parts. On the one side, adding a set of strategies offering less control than the available strategies do, cannot increase the overall control in the game form. This is a very common requirement in social choice theory (known as the Gardenförs principle), stating that adding to a set an element that is worse than the elements of the set cannot increase its overall appraisal.

On the other hand, offering more strategies to an individual, whether better or worse, cannot decrease his evaluation of the control she enjoys over the outcomes. Actually, if the individual is not interested in the new strategies, she is free not to choose them, making it difficult to accept that she could be penalised by the addition of strategies. Her situation is thus not worse. If it is required that the addition does not strictly reduce the overall control nor does it strictly increase it, then the overall control in both game forms is equivalent.

Eventhough this axiom seems not unreasonnable, it fails to be satisfactory in the case in which individual 2's set of strategy is a singleton. In that special case, individual 1 has total control over every outcome of the game form as he decides on his own which one will come out. There is no uncertainty. It is then more valuable to have a full choice over a larger set of outcomes, nevertheless axiom 5 says that whatever the size of individual 1's set the control is the same. This is due to the fact that what is measured here is the predicability of the outcome rather than the freedom of choice the individual enjoys. Predictability is full for individual 1 whether he can choose from a singleton or from a huge set.

4 Characterization result

As to illustrate the definitions that follow, consider the game form where $A_1 = \{s, t, u\}$ and $A_2 = \{a, b, c, d\}$:

$$(A_{ex}, g) = \begin{array}{ccccc} & a & b & c & d \\ s & \mathbf{x} & \mathbf{y} & \mathbf{z} & \mathbf{z} \\ t & \mathbf{y} & \mathbf{w} & \mathbf{x} & \mathbf{z} \\ u & \mathbf{x} & \mathbf{y} & \mathbf{x} & \mathbf{x} \end{array}$$

Definition 2 *Let* $D(a_i, A) = \{g(a_i, a_2); \forall a_2 \in A_2\}$

$D(a_i, A)$ is the set of outcomes associated to strategy a_i played by individual 1. Cardinality of $D(a_i, A)$ may be smaller than that of A_2. For instance, $D(t, A_{ex}) = \{y, w, x, z\}$ and $D(u, A_{ex}) = \{x, y\}$.

Definition 3 *Let* $Z(A) = \{D(a_i, A); \forall a_i \in A_1\}$.

Hence, $Z(A_{ex}) = \{\{x, y, z\}, \{y, w, x, z\}, \{x, y\}\}$. To draw a parallel with the framework in which no interactions are taken into account, the set $Z(A)$ could be viewed as the opportunity set of individual 1. Indeed, individual 1 can choose freely any of the sets $D(a_i, A) \in Z(A)$, like she would choose any option in her opportunity set. Still, the main difference between both frameworks is that options are a complete description of the world whereas here they are not. They are sets of uncertain prospects.

Definition 4 *Let a CardMin strategy for individual 1 be defined as* $a_* \in A_1$ *such that* $\#D(a_*, A) \leq \#D(a_i, A)\ \forall D(a_i, A) \in Z(A)$. *Furthermore,* $CM(A) = \#D(a_*, A)$.

The CardMin strategy is not defined uniquely, it is one of individual 1's strategies that generates the least possible outcomes in the game. For instance, in A_{ex}, $\#D(s, A) = 3$, $\#D(t, A) = 4$ and $\#D(u, A) = 2$. Therefore $u \in A_1$ is a CardMin strategy, it is the only one and $CM(A_{ex}) = 2$.

Let us finally define the CardMin criterion, $\succeq_1^{CM}$.

Definition 5 $\succeq_1^{CM}$ *is a transitive binary relation comparing game forms defined as:*

$$A \succeq_1^{CM} B \Longleftrightarrow CM(A) \leq CM(B)$$

The CardMin criterion compares game forms in terms of the smallest set of outcomes that individual 1 can be facing when choosing his strategy. In terms of control over outcomes, it seems reasonable to state that the vector (x, y) offers exactly the same amount of control as the vector (x, y, x, y, x). Indeed, the state of the world in both cases will lie, once the game has been played, in exactly the same set of alternatives, $\{x, y\}$. Moreover, it can be stated that the set $\{x, y\}$ offers more control than the set $\{w, z, a, b, c\}$. This is because the state of the world, once the game is played, will lie within a smaller set of alternatives. Individual 1 has less uncertainty about his future in the first case or said differently, the final outcome is more predictable in the first than in the second case.
If it is a pure notion of control we are interested in, abstracting from any other consideration such as preferences, then it seems reasonnable that individual 1 will choose in his set of strategies the one offering the less possible different outcomes. In the game A_{ex}, player 1, willing to reduce as much as possible the uncertainty about the outcome, or willing to get the most control he can over those outcomes, will choose strategy u. This is what suggests the CardMin criterion. Again, this criterion would be more difficult to accept if preferences were considered.

In order to illustrate, consider the alternative game form

$$A'_{ex} = \begin{array}{cccc} & a' & b' & c' \\ s' & \mathbf{l} & \mathbf{m} & \mathbf{n} \\ t' & \mathbf{m} & \mathbf{m} & \mathbf{m} \\ u' & \mathbf{p} & \mathbf{q} & \mathbf{r} \end{array}$$

In A_{ex}, the CardMin strategy is u and it yields $D(u, A_{ex}) = \{x, y\}$ while in the second the CardMin strategy is t', yielding $D(t', A'_{ex}) = \{m\}$. According to the CardMin criterion, $A'_{ex} \succ_1^{CM} A_{ex}$ as in the second case individual 1 can be entirely sure about what will be the outcome, unlike in the first case.

Remark 2 *If the outcome function g is such that $g(a_1, a_2) \neq g(b_1, b_2)$ for all $a_1, b_1 \in \mathcal{A}_1, b_1, b_2 \in \mathcal{A}_2$, there are no neutral strategies whatever the sets A_1 and A_2. The CardMin criterion $\succeq_1^{CM}$ is then equivalent to the criterion ranking game forms according to the size of the set A_2: $A_1 \times A_2 \succeq_1^{CM} B_1 \times B_2 \Longleftrightarrow \#B_2 > \#A_2$.*

Let us now turn to the characterization result.

Theorem 1 *A transitive binary relation $\succeq_1$ satisfies Axioms 1 to 5 if and only if $\succeq_1 = \succeq_1^{CM}$*

Proof of Theorem 1:
The CardMin criterion $\succeq_1^{CM}$ is transitive and satisfies axioms 1 to 5.

In order to prove the other implication, the following definition will be necessary. For any option $a_1 \in A_1$, let $\bar{A}_2(a_1)$ and $N_2(a_1)$ be defined as subsets of A_2 such that:
(i) $A_2 = \bar{A}_2(a_1) \cup N_2(a_1)$ and $\bar{A}_2(a_1) \cap N_2(a_1) = \emptyset$
(ii) $\forall a_2 \in N_2(a_1), \exists\, s_2' \in \bar{A}_2(a_1)$ such that a_2 is $(\{a_1\} \times \{s_2'\})$-neutral
(iii) $\forall a_2 \in \bar{A}_2(a_1), s_2' \in \bar{A}_2(a_1)$, $g(a_1, a_2) \neq g(a_1, s_2')$.

In words, $\bar{A}_2(a_1)$ and $N_2(a_1)$ are a partition of A_2 where $\bar{A}_2(a_1)$ is a set such that there is no pair of strategies that are neutral one with the other with respect to a_1 and $N_2(a_1)$ is a set such that every strategy is neutral for at least one strategy in $\bar{A}_2(a_1)$ with respect to a_1. Of course, $N_2(a_1)$ can be the empty set, and the sets $\bar{A}_2(s)$ and $N_2(s)$ are not defined uniquely. However, throughout the proof when these sets are needed it is implicitely assumed that one particular set $\bar{A}_2(a_1)$ has been chosen and that it is the same that is used along. Notice finally that eventhough the set $\bar{A}_2(a_1)$ can sometimes be constructed in several ways, all possible sets $\bar{A}_2(a_1)$ have the same cardinality.

Consider two game forms $A = A_1 \times A_2$ and $B = B_1 \times B_2$ and assume a_* is one CardMin strategy for individual 1 in the first game form while b_* is one CardMin strategy in the second game form. First the following implications are shown :

$$\#D(a_*, A) = \#D(b_*, B) \Longrightarrow \{a_*\} \times \bar{A}_2(a_*) \sim_1 \{b_*\} \times \bar{B}_2(b_*)$$

$$\#D(a_*, A) > \#D(b_*, B) \Longrightarrow \{a_*\} \times \bar{A}_2(a_*) \prec_1 \{b_*\} \times \bar{B}_2(b_*)$$

Next it is shown that any game form $\{a_1\} \times A_2$ such that individual 1's set of strategies is a singleton, is indifferent for individual 1 to the game form $\{a_1\} \times \bar{A}_2(a_1)$. Finally, the third step proves that any game form $A_1 \times A_2$ is indifferent for individual 1 to the game form $\{a_*\} \times A_2$ where a_* is one Cardmin strategy for individual 1. Putting the three steps together in the last part will complete the proof.

The proof of step 1 is very much inspired by the proof found in Pattanaik and Xu (1990) for the Cardinal criterion, although it requires carefull attention for its adaptation. Consider the game forms $A_1 \times A_2$ and $B_1 \times B_2$. Consider then the game form $\{a_1\} \times \bar{A}_2(a_1)$ where a_1 is any strategy in A_1. In that game form there is only one single row of outcomes, those associated to strategy a_1, all of them being pairwise different. In the same way, consider the game form $\{b_1\} \times \bar{B}_2(b_1)$ where b_1 is any strategy in B_1. We thus have

$$Z(\{a_1\} \times \bar{A}_2(a_1)) = D(a_1, \{a_1\} \times \bar{A}_2(a_1))$$

$$Z(\{b_1\} \times \bar{B}_2(b_1)) = D(b_1, \{b_1\} \times \bar{B}_2(b_1))$$

and

$$\#D(a_1, \{a_1\} \times \bar{A}_2(a_1)) = \#\bar{A}_2(a_1)$$

$$\#D(b_1, \{b_1\} \times \bar{B}_2(b_1)) = \#\bar{B}_2(b_1)$$

The argument proceeds by induction on the cardinality of the sets $\bar{A}_2(a_1)$ and $\bar{B}_2(b_1)$. Assume $\#\bar{A}_2(a_1) = \#\bar{B}_2(b_1) = 1$. According to axiom 1, $\{a_1\} \times \bar{A}_2(a_1) \sim_1 \{b_1\} \times \bar{B}_2(b_1)$. Now assume that for any game forms $\{a_1\} \times \bar{A}'_2(a_1)$ and $\{b_1\} \times \bar{B}'_2(b_1)$ such that $\#\bar{A}'_2(a_1) = \#\bar{B}'_2(b_1) = n-1$ we have $\{a_1\} \times \bar{A}'_2(a_1) \sim_1 \{b_1\} \times \bar{B}'_2(b_1)$ and assume that $\bar{A}_2(a_1) = \{s_1, \ldots, s_n\}$ and $\bar{B}_2(b_1) = \{t_1, \ldots, t_n\}$. Then in particular, $\{a_1\} \times \{s_1, \ldots, s_{n-1}\} \sim_1 \{b_1\} \times \{t_1, \ldots, t_{n-1}\}$. By construction, s_n is not $(\{a_1\} \times \{s_1, \ldots, s_{n-1}\})$-neutral and t_n is not $(\{b_1\} \times \{t_1, \ldots, t_{n-1}\})$-neutral, so axiom 4 can be applied. It yields $\{a_1\} \times \bar{A}_2(a_1) \sim_1 \{b_1\} \times \bar{B}_2(b_1)$.

It has thus been shown that $\#D(a_1, A) = \#D(b_1, B) \Longrightarrow \{a_1\} \times \bar{A}_2(a_1) \sim_1 \{b_1\} \times \bar{B}_2(b_1)$ and this is true $\forall a_1, b_1$ so this is true in particular for a_* and b_*.

Assume now that $\#\bar{A}_2(a_1) = n > m = \#\bar{B}_2(b_1)$ and $n = m + k$. Let us write $\bar{A}_2(a_1) = \{s_1, \ldots, s_m, s_{m+1}, \ldots, s_n\}$ and denote $\bar{S}_2 = \{s_1, \ldots, s_m\}$.

According to axiom 2, s_{m+1} being not $(\{a_1\} \times \{s_1\})$-neutral, $\{a_1\} \times \{s_1\} \succ_1 \{a_1\} \times \{s_1, s_{m+1}\}$. Hence applying axiom 4 yields to $\{a_1\} \times \{s_1, s_2\} \succ_1 \{a_1\} \times \{s_1, s_2, s_{m+1}\}$. Applying again axiom 4 a few times, one reaches $\{a_1\} \times \bar{S}_2 \succ_1 \{a_1\} \times (\bar{S}_2 \cup \{s_{m+1}\})$. One more step gives us $\{a_1\} \times (\bar{S}_2 \cup \{s_{m+2}\}) \succ_1 \{a_1\} \times (\bar{S}_2 \cup \{s_{m+1}, s_{m+2}\})$. But again, $\{a_1\} \times (\bar{S}_2 \cup \{s_{m+1}\}) \sim_1 \{a_1\} \times (\bar{S}_2 \cup \{s_{m+2}\})$ because the cardinality of both sets is the same, so transitivity gives $\{a_1\} \times \bar{S}_2 \succ_1 \{a_1\} \times (\bar{S}_2 \cup \{s_{m+1}, s_{m+2}\})$.

Carrying on the same way, the addition of $s_{m+3}, \ldots, s_n$ respectively yields $\{a_1\} \times \bar{S}_2 \succ_1 \{a_1\} \times \bar{A}_2(a_1)$. But by what was shown previously, $\{a_1\} \times \bar{S}_2 \sim_1 \{b_1\} \times \bar{B}_2(b_1)$ as both sets $\bar{S}_2$ and $\bar{B}_2(b_1)$ have the same cardinality. Finally by transitivity $\{b_1\} \times \bar{B}_2(b_1) \succ_1 \{a_1\} \times \bar{A}_2(a_1)$.

It has thus been shown that $\#D(a_1, A) < \#D(b_1, B) \Longrightarrow \{a_1\} \times \bar{A}_2(a_1) \succ_1 \{b_1\} \times \bar{B}_2(b_1)$ and this is true $\forall a_1, b_1$ so this is true in particular for a_* and b_*. The first step is now completed.

Now, consider any game form $\{a_1\} \times A_2$ and the associated game form $\{a_1\} \times \bar{A}_2(a_1)$. Let $A_2 \setminus \bar{A}_2(a_1) = N_2(a_1) = \{c_1, \ldots, c_k\}$.[4] According to axiom 3, c_1 beeing $(\{a_1\} \times \bar{A}_2(a_1))$-neutral, we have $\{a_1\} \times \bar{A}_2(a_1) \sim_1 \{a_1\} \times (\bar{A}_2(a_1) \cup \{c_1\})$. Using the same procedure, adding on one by one every strategy in $A_2 \setminus \bar{A}_2(a_1)$, one will reach $\{a_1\} \times \bar{A}_2(a_1) \sim_1 \{a_1\} \times A_2$. This is true for all a_1 so in particular, $\{a_*\} \times \bar{A}_2(a_*) \sim_1 \{a_*\} \times A_2$, and this concludes the second step.

For the third step, consider again a_* as one CardMin strategy for individual 1. By definition, $\#D(a_*, A) \leq \#D(a_i, A)\ \forall D(a_i, A) \in Z(A)$. According to the first part of the proof, this yields $\{a_*\} \times \bar{A}_2(a_*) \succeq_1 \{a_i\} \times \bar{A}_2(a_i)$. But as just seen $\{a_*\} \times \bar{A}_2(a_*) \sim_1 \{a_*\} \times A_2$ and $\{a_i\} \times \bar{A}_2(a_i) \sim_1 \{a_i\} \times A_2$ so transitivity gives $\{a_*\} \times A_2 \succeq_1 \{a_i\} \times A_2$. By axiom 5, this implies that $\{a_*\} \times A_2 \sim_1 \{a_*, a_i\} \times A_2$.

[4]Of course it could the case that $N_2(a_1)$ is empty. In that case the second step of the proof is immediate.

Again, a_* is the CardMin strategy so $\{a_*\} \times A_2 \succeq_1 \{a_j\} \times A_2$ is true for any other $a_j \in A_1$. By transitivity, $\{a_*, a_i\} \times A_2 \succeq_1 \{a_j\} \times A_2$ so using again axiom 5, we get $\{a_*, a_i\} \times A_2 \sim_1 \{a_*, a_i, a_j\} \times A_2$ and by transitivity, $\{a_*\} \times A_2 \sim_1 \{a_*, a_i, a_j\} \times A_2$. Going on the same way, one can finally reach $\{a_*\} \times A_2 \sim_1 A_1 \times A_2$.

Hence, putting the three steps in the reverse order, if a_* is one CardMin strategy, then $A_1 \times A_2 \sim_1 \{a_*\} \times A_2$. Furthermore, $\{a_*\} \times A_2 \sim_1 \{a_*\} \times \bar{A}_2(a_*)$. This implies that comparing two game forms $A_1 \times A_2$ and $B_1 \times B_2$ is equivalent to comparing $\{a_*\} \times \bar{A}_2(a_*)$ and $\{b_*\} \times \bar{B}_2(b_*)$. But these two game forms can be compared on the basis of the cardinality of the sets $D(a_*, A)$ and $D(b_*, B)$. $\blacksquare$

This criterion gives a discriminating power to one particular set of strategies in the game form, that is the set of all CardMin strategies. One weakness is thus the fact that here any contribution to the overall control of the other strategies available is ignored.

Consider the game forms A'_{ex} and $A''_{ex} = \begin{array}{ccc} & a' & d' \\ s' & \mathbf{l} & \mathbf{l} \\ t' & \mathbf{m} & \mathbf{m} \\ u' & \mathbf{p} & \mathbf{r} \end{array}$

The CardMin criterion ranks A'_{ex} and A''_{ex} as indifferent though obviously the second game form should be ranked best, given that individual 1 has total control over outcome m in A'_{ex} while he has total control over l and m in A''_{ex} which is of course better. A''_{ex} offers a full amount of control over a larger set of outcomes than A'_{ex} does. One way of avoiding this would be to consider a lexicographic extension of the CardMin criterion. In case of a tie with $\succeq_1^{CM}$, the lexicographic extension would then compare game forms by looking at the second smaller set of outcomes and so on. This is left for further research.

Next the independence of the axioms used in the characterisation in Theorem 1 is proved.

Proposition 1 *Axioms 1, 2, 3, 4 and 5 are independent*

Proof : In order to show that the axioms are independent, let us define some criteria such that in turn each axiom is violated although all the others are satisfied.
- Let $\succeq^{CM\alpha}$ be defined exactly as $\succeq^{CM}$, except for the comparison of singletons. Singletons such as considered in axiom 1 will then be compared on the basis of any predefined ranking (alphabetical order, any exogenous ranking...), allowing for strict comparisons. Then $\succeq^{CM\alpha}$ satisfies axioms 2 to 5, but can eventually violate axiom 1.
- Let $\succeq^{CMax}$ be defined as the reverse of $\succeq^{CM}$, that is $A \succeq^{CMax} B \iff CM(A) \geq CM(B)$. Then axioms 1, 3, 4 and 5 are satisfied, however, axiom 2 is violated.
- Let $\succeq^{Card2}$ be defined as $A_1 \times A_2 \succeq^{Card2} B_1 \times B_2 \iff \#A_2 \leq \#B_2$. $\succeq^{Card2}$ violates axiom 3 but satisfies all others.
- Let $\succeq^{Par}$ be defined as follows: if the parity of $\#A_2$ is the same as the parity of $\#B_2$ (i.e. if they are both odd or both even), then $A_1 \times A_2 \succeq^{Par} B_1 \times B_2 \iff$

$A_1 \times A_2 \succeq^{CM} B_1 \times B_2$. If $\#A_2$ is odd and $\#B_2$ is even, then $A_1 \times A_2 \succ^{Par} B_1 \times B_2$. $\succeq^{Par}$ satisfies all axioms except for axiom 4.
- Let $\succeq^{Cardlex}$ be defined as follows: $A_1 \times A_2 \succ^{Cardlex} B_1 \times B_2 \iff \#A_1 > \#B_1$ or ($\#A_1 = \#B_1$ and $CM(A_1 \times A_2) < CM(B_1 \times B_2)$); $A_1 \times A_2 \sim^{Cardlex} B_1 \times B_2 \iff \#A_1 = \#B_1$ and $CM(A_1 \times A_2) = CM(B_1 \times B_2)$. $\succeq^{Cardlex}$ violates axiom 5 but satisfies all others. ■

The generalisation to the case of n individuals should be the next step but issues appear that are far from trivial. Although it does not seem at first to be critical, the generalisation turns out to be complex essentially because of the definition of the criteria itself. As seen earlier, in order to implement $\succeq_1^{CM}$, one has first to compute $CM(A)$ by determining the set $\bar{A}_2(a_i)$ for every $a_i \in A_1$ [5]. In the case of two individuals, there is no difficulty in "filtering" the set A_2 as to obtain $\bar{A}_2$ by taking out all options leading to replicated outcomes. In the case of n individuals, no such operation is possible. To see this, consider the following simple example with three individuals.

$A_1 = \{a_1\}, A_2 = \{b_1, b_2\}, A_3 = \{c_1, c_2\}$ and let the outcome function g be such that $g(a_1, b_1, c_1) = x$, $g(a_1, b_1, c_2) = y$, $g(a_1, b_2, c_1) = z$, $g(a_1, b_2, c_2) = x$. Hence, $g(a_1, b_1, c_1) = g(a_1, b_2, c_2)$, so in order to compute $CM(A)$ one option in individual 2 or 3's set must be taken out in order to end up with only one strategy leading to outcome x. However, when reducing A_2 to $\{b_1\}$ by erasing option b_2, the outcome z is artificially thrown away at the same time as outcome x, which would be a mistake. Erasing b_1 instead of b_2 takes away outcome y together with outcome x, so no option can be withdrawn form the set A_2. In the same way, there is no option one can take away from A_3 in order to suppress the outcome x without suppressing at the same time another outcome mistakefully. Hence, there is no way of "filtering out" the outcomes when there are more than two individuals.

5 Conclusion

The purpose of this paper was to introduce the notion of control over outcomes in a social context into the literature about freedom of choice. Albeit a characterisation result has been reached, the CardMin criterion carries some weaknesses. There are at least two interesting lines of research that should be explored in order to push the analysis a bit further. First, by using the same framework, some more subtle criteria, allowing for every strategy to contribute to the overall control, would deserve attention for characterization. A criterion that ranks game forms according to the mean value of the cardinality of the set of outcomes associated to every strategy of individual 1 could be an example of such a criterion. Second, it seems important in order to connect the notion of control with that of freedom of choice, to explore the reasons why individuals might desire to have control over outcomes, thereby bringing preferences

[5] see definitions (i) to (iii) in the proof on Theorem 1

into the framework. This second line of research should focus on the questions raised by the conflicting concepts of freedom of choice as quality, quantity and/or diversity of choices (see Bervoets and Gravel (2007) on the issue of diversity) as well as with the complexity that interactions bring into the field.

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