Stability and Manipulation in Representative Democracies *

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Abstract

This paper is devoted to the analysis of all constitutions equipped with electoral systems involving two step procedures. First, one candidate is elected in every jurisdiction by the electors in that jurisdiction, according to some aggregation procedure. Second, another aggregation procedure collects the names of the jurisdictional winners in order to designate the final winner. It appears that whenever individuals are allowed to change jurisdiction when casting their ballot, they are able to manipulate the result of the election except in very few cases. When imposing a paretian condition on every jurisdiction's voting rule, it is shown that, in the case of any finite number of candidates, any two steps voting rule that is not manipulable by movement of the electors necessarily gives to every voter the power of overruling the unanimity on its own. A characterization of the set of these rules is next provided in the case of two candidates.

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1 Introduction

In this paper we are interested in all constitutions with electoral systems such that the winner of the election is designated in a two step procedure, like for instance Great-Britain, Canada or the US electoral systems. In these constitutions, the country is

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divided into jurisdictions in which elections are held at the first step as to choose the jurisdictional winners. In the second step, the government is appointed by an aggregation procedure over all the jurisdictional winners. The question that is under scrutiny is that of the *stability* and the *manipulability by movement* of the voters of constitutions with such two step electoral procedures.

The issue of manipulability by movement of such systems has currently been at stake, as suggest the following recent examples. During the 2005 campaign in Great-Britain for the election of the Members of Parliament (MPs), various websites and forums have been created invoking *tactical vote* (votedorset.net, helpbeathoward.org.uk, tacticalvoter.net, ditchdavis.com...). On votedorset.net, we could find the following advertisement before the election

> "Four Lib Dems are seeking suitable Labour pairs, and our target is Oliver Letwin's seat (a Conservative MP). We would like to swap all four of our tactical votes for Labour in Burton (East Staffordshire) with four equivalent tactical votes for the Lib Dems in Dorset West."

For electoral purposes Britain is divided into constituencies, each of which returns one MP to the House of Commons. The British electoral system is based on the relative majority method - usually referred to as the plurality rule, sometimes also called the "first past the post" principle - which means the candidate with more votes than any other is elected. The leader of the political party which wins most seats (although not necessarily most votes) at a general election, or who has the support of a majority of members in the House of Commons, is by convention invited by the Sovereign to form the new government. The purpose of the websites about tactical vote is to change what would be the result of the election if everyone voted in his constituency. They thus offer to match people located in different constituencies and willing to exchange their votes in order to beat their commonly less desired candidate, the Conservative in this case. However, these voters give up their first choice, hoping that their pair will do the same in their constituency. Consequently, although the overall number of votes granted to every party is the same, the result of the final election might be reversed by such a manipulation.

Such an idea was earlier implemented by the ecologist party's supporters backing Ralph Nader, a candidate running in the 2000 election in the US. There again, voters connected to the net (see, among others, www.voteswap.com) could decide to vote for Gore instead of Nader in states where the election was tight between Gore and Bush, while all paired voters would vote for Nader in states where Gore had a large victory planned. Although Gore won the popular vote by more than 540.000 votes, Bush was elected after his victory in Florida, which happened to be the swing state, by 537 votes only. There is no doubt that such a manipulation could have been very efficiently implemented, in order to reverse the result of that presidential election. For more on vote trading implementing through internet sites, see the recent paper by Hartvigsen (2006).

From a logical standpoint, such a manipulation is in fact logically equivalent to allowing voters to change the constituency they are voting in. Indeed, two voters swapping votes amounts to voters swapping residency. We thus suggest that electoral procedures involving two steps are in general subject to such manipulability by movement of the electors.

In what follows, stability refers to the fact that, in such constitutions, the result of the two step election should be the same, whatever the location of the individuals when they cast their ballot. Conversely, an unstable constitution would be such that for two different partitions of the set of individuals into jurisdictions, the election could yield two different outcomes. Stability, such as defined here, does not refer to any change in the electors' vote profile, rather it is only concerned with their location. In turn, a constitution such that some electors can not only change the result of the election by changing jurisdiction but can change it *in their favour*, will be said to be manipulable. Manipulability by movement, such as considered in this paper, is different from the standard notion of manipulation à la Gibbard-Satterthwaite (1973, 1975). Indeed, in our analysis we assume that individuals are allowed to change jurisdiction, but when doing so, they do not change their vote. Contrary to the Gibbard-Satterthwaite result, we are not concerned by the fact electors vote for their best alternative or not, nevertheless we assume they behave identically, whatever the jurisdiction they vote in.

Issues about such manipulations, that do not involve vote swapping, are raised within the European community. Considering the lot of *nomadic populations*, whose civic rights are rarely guaranteed, two European deputies, Alima Boumedienne-Thiery and Olivier Duhamel advocate the creation of a "European right to vote" for nomadic people, allowing them to vote in whichever country they are in at the moment of the European elections. Actually, this idea of an European right to vote is to be extended to all European citizens, as Alima Boumedienne-Thiery and Olivier Duhamel (2000) recommend :

> "in a context of extreme mobility, it is fundamental that the individual be a citizen wherever he is, as more and more persons will spend a considerable amount of their time abroad"

If any voter has the right to vote in any country for European elections, it seems important to be equipped with electoral rules that are non manipulable by the movement of the voters. As a last illustrating example, let us recall that French citizens living abroad are allowed, when voting, to choose the constituency in which they cast their ballot for some of the French elections.

One solution to this problem of could be simply resolved by prohibiting electors from changing jurisdiction to cast their ballot. However, eventhough electors are not allowed to change jurisdiction, the designers of the election may try to artificialy make them move. This is known under the name of gerrymandering. Indeed, the design of boundaries is sometimes subject to serious contentious in the United States, suggesting that politicians aim at altering the results of the election by doing so. Gerrymandering is, from a logical point of view, equivalent to manipulation by movements of the electors. Instead of the voters themselves, the designers of the jurisdictions' boundaries are the manipulators.

Since Arrow's impossibility theorem (see Arrow (1963)), the axiomatic approach of social choice theory has provided a floodgate of negative or positive results for the analysis of voting rules. Surprisingly enough, little attention has been devoted to the study of two step procedures, though a few contributions have already proposed normative conditions to judge the quality of a federal system (as two step procedures), the oldest being due to Penrose (1946) and Banzhaf (1968): both suggested that in federal unions, every voter should be given the same power, that is the same probability to influence the outcome. More recently, Felsenthal and Machover (1999) as well as Barberà and Jackson (2004) have come with the following suggestion: a federal state should maximize on average the total utility of the citizens. Feix, Lepelley, Merlin and Rouet (2004) have suggested that the way to attribute mandates to a state should minimize the probability of electing a candidate who does not obtain the majority of the popular vote over the whole union. However, all these papers assume the same structure: there is a two party competition in every state and the party which obtains the majority of the votes gets all the mandates for this state. Then some kind of quota rule is used at the federal level[1]. Furthermore, there is a long tradition of contributions on the rounding off problem, that is, the search of rules to allocate the mandates proportionally to the populations of the different members (see the classical book of Balinski and Young (1982)). This tradition assumes from the start that the voting rule is the proportional rule.

It is important to notice that from a conceptual point of view, the framework considered here is different from that of Arrow's theorem. First, given the two step procedure and allowing for voters to change jurisdiction, it appears that the global electoral rule must be defined by the specific partition of the electors in addition to the vote profile. Contrary to the traditional arrovian framework, more than just the preferences are required here in order to define the federal constitution voting procedure. Hence, the fact that the rule depends on the partition of the electors makes it possible for one given vote profile to yield to different winners, according to the partition. For that reason, the combination of jurisdictional rules that are social choice functions, with a federal rule that is also a social choice function, cannot be summarized into one unique social choice function, hence departing from the arrovian framework[2]. Next, the imposition of a sovereignty condition, as will be done throughout, makes it impossible to define

[1]The second level of aggregation will be called throughout the federal level, for convenience only. It is in fact an abuse of language as there are federal systems in which elections are direct (Brasil for instance) and systems that are not federal but in which elections are indirect (United Kingdom for instance).

[2]In its original form, the Arrow theorem considers social welfare functions. The remark just made, pointing out the difference with our framework lies elsewhere than in this distinction between social welfare functions and social choice functions. It relies on the fact that we take into account more information than the preferences.

a dictatorship in this framework, as will be shown further. This also implies that the framework considered differs from that of the Gibbard-Satterthwaite theorem about manipulation.

In this paper, we study two step voting procedures in a general framework that does not restrict the particular voting method used at every step of the procedure. So far, the only requirement made on these jurisdictional voting rules is that of Jurisdiction Sovereignty, stating that the result of the election in every jurisdiction should only depend on the votes casted by the individuals within that jurisdiction. Thus the result should be independent of what happens in the neighbor jurisdictions. We then assume that individuals submit one vote for one candidate in their jurisdiction, the votes being aggregated by the jurisdictional voting rules. Next, the candidates winning at first step are collected and aggregated at the second step, with a federal voting rule.

The question examined in this paper is that of the existence of constitutions involving two step procedures whose results are independent from the location of voters and are, therefore, stable and non manipulable in the sense given to these words above. In the case of an arbitrary finite number of candidates, the answer to that question is positive, although the set of rules allowing for "movement-proofness" is very small. When imposing a jurisdictional paretian condition, according to which a jurisdiction should elect a given candidate when he receives the unanimity of votes in that jurisdiction, the rules we are left with are rather undemocratic[3]. The first set of rules satisfying stability is the Constant Constitution, stating that whoever the winners are at the first step (the jurisdiction level), the winner at the second step is always the same. The federal voting rule is thus constant with respect to the vote profile. The second set of rules satisfying stability all share an undesirable feature. Hence, as to satisfy stability, if the rule is not the Constant Constitution, it is necessary that every individual in the society is given the power of overruling the unanimity. Individuals are then said to be pivotal.

In the more specific case of two candidates, the same analysis leads to a complete characterization result, for which the set of rules satisfying stability as well as the jurisdictional paretian condition is defined. This set consists of the Constant Constitution and the so called Unanimity Rules, that requires unanimity for one candidate for him to be elected, while any other situation elects his rival. Next, the paretian condition is relaxed into a condition called Local Faithfulness, following Young (1974)'s terminology, stating that when a jurisdiction is formed by a single individual, the winner in that jurisdiction should be the one chosen by that individual. We then observe that the set of rules satisfying stability is still formed of the Constant Constitution and the Unanimity Rules, but a third class of rules appear. It is the rather strange class of Parity Rules, that has not yet, at the best of our knowledge, been studied. A Parity Rule designates in every jurisdiction as well as at the aggregate level, a candidate ac-

[3]The term "democratic" does not refer to any precise definition, rather it is the general feeling individuals share about what is democratic or not. As a more rigorous statement, the set of rules satisfying stability do not treat the alternatives equally (May's neutrality condition is violated), which seems to be an consensual feature of democracy.

cording to the parity of the votes the candidate receives. Thus, it appears thus that no constitution involving a two step electoral procedure can be democratic and stable at the same time. The positive answer that is provided to the question examined herein is thus to be taken as an impossibility result.

At this stage, it is useful to mention some previous works on representative democracies, due to Murakami (1966), Fishburn (1971) and Fine (1972) concentrating on the specific case where the collective decisions rest on the majority rule between two alternatives. These studies completed May (1952)'s characterization of the majority rule by relaxing the anonymity assumption. Murakami's main result states that a representative system using the majority rule at each stage is characterized by neutrality, monotonicity, and non-dictatorship. The analysis presented herein departs from these works as it considers any finite number of candidates and it does not assume majority rule from the start. It also is restrained to the case of two steps procedures. In this regard, this contribution is closer to the recent ones of Laffond and Lainé (1999, 2000), Chambers (2006) and Perote-Peña (2005), which also consider more than two alternatives and two steps procedures. Differences and similarities between these various contributions will be discussed at the end of section 5.

The rest of the paper is organized as follows. In the next section, we describe the formal framework used for modelling two step voting procedures and we present several normative properties that may be imposed on them. In particular, we discuss the different stability properties that prevent the outcome of these voting procedures from being manipulable by a move of the electors. Section 3 provides the main theorem of the paper for the general case of any finite number of candidates. Section 4 provides a more complete characterization result in the case of two candidates and explores the consequences of relaxing the paretian condition into the milder Local Faithfulness condition. Section 5 gives some additional comments and concludes.

2 The general framework

2.1 Notations and Definitions

Let the finite and fixed set of candidates be defined as $A = \{a_1, a_2, \ldots, a_p\}$, the fixed set of voters is $N = \{1, \ldots, i, \ldots, n\}$, with $n \geq 3$, and $J = \{J_1, \ldots, J_j, \ldots, J_m\}$ is the set of jurisdictions, with $m \geq 2$. We also assume that $n > m$.

Let σ be a partition function from N to $\{1, 2, \ldots, m\}$. Formally, $\forall i \in N$, $\sigma(i) = j \Leftrightarrow i \in J_j$, with $J_1 \cup J_2 \cup \ldots \cup J_m = N$ and $J_j \cap J_k = \emptyset$. In what follows, we will consider partition functions in Σ, defined as the set of all partitions such that $\forall\, j \in \{1, \ldots, m\}$, $\sigma^{-1}(j) \neq \emptyset$. Hence for any partition σ in Σ, there is no empty jurisdiction and as $n > m$, there is at least one jurisdiction with at least two individuals.

Individuals are equipped with a linear ordering $\succ_i$ on A, a linear ordering being a binary relation that is reflexive, transitive, complete and antisymmetric. Individuals vote for one unique candidate in A in the jurisdiction they reside in. It is important at this stage to note that individuals can vote according to their linear ordering, i.e. for their favorite candidate[4], but they might vote for any other candidate in A. Indeed, in what follows, we analyse what happens when electors change the jurisdiction they vote in. However we always assume that when they move, the electors do not change the candidate they vote for. Thus what we are looking at is manipulation by movement of voters, it is not manipulation à la Gibbard-Satterthwaite.

$\pi \in A^n$ denotes a *vote profile.* Typically, a vote profile is identified with a vectors of a's, b's, ... where the i^{th} coordinate indicates individual i's vote. $\pi|_i$ denotes individual i's vote and for any subset S of N, we denote by $\pi|_S$ the restriction of π to S.

For an election, the j-th jurisdiction J_j will provide a jurisdictional winner designated by the social choice function f_j:

$$\begin{array}{rcl} f_j : \Sigma \times A^n & \rightarrow & A \\ (\sigma, \pi) & \rightarrow & z \in A \end{array}$$

We impose the following very mild condition on the functions $\{f_j\}_{j=1,m}$:

Definition 1 Jurisdiction Sovereignty

$$\textit{If } [\pi|_{J_j} = \pi'|_{J_j}] \textit{ and } [\sigma(i) = j \Leftrightarrow \sigma'(i) = j] \textit{ then } f_j(\sigma, \pi) = f_j(\sigma', \pi') \, \forall j$$

In words, the result of an election in jurisdiction J_j is independent from what happens in any other jurisdiction $J_{j'}$. This property is an independence condition that makes a federation consistent. Indeed, the indirect characteristic of the electoral system would vanish without it. The set of all social choice functions satisfying Jurisdiction Sovereignty is denoted by $\mathcal{F}$.

The m jurisdictional winners yield a jurisdictional vote profile $\Pi \in A^m$, called a *federal profile.* The federation then appoints a federal winner using the federal social choice function g defined as follows:

$$\begin{array}{rrcl} g : & A^m & \rightarrow & A \\ & \Pi = (z_1, \ldots, z_m) & \rightarrow & z \in A \end{array}$$

A *federal constitution* is given by a $(m+1)$-tuplet $C = (g, f_1, \ldots, f_m)$, with $f_j \in \mathcal{F}$, $\forall j$.

For a given constitution, the federal winner of the election will be denoted by:

$$g(f_1(\sigma, \pi), \ldots, f_m(\sigma, \pi)) = g(f(\sigma, \pi))$$

Remark 1 *Although functions f_j and g are social choice functions, the combination of these in a two steps procedure does not generate a social choice function. Strictly speaking, a social choice function has for only argument the preferences of the voters, whereas, in order to define the voting procedure of the federal constitution, it is necessary to take as an argument the partition of the individuals in addition to the vote profile.*

[4]Since A is finite and $\succ_i$ is antisymmetric, there is a unique best element in A for every individual.

The following example illustrates the previous remark, by showing that it is possible for one vote profile π to yield different results.

Assume there are 2 candidates, $A = \{a, b\}$, and there are 5 electors in the society, partitioned into 3 jurisdictions J_1, J_2, J_3. Assume furthermore that the jurisdictional rules as well as the federal voting rule are all given by the majority rule. In case of a tie in a jurisdiction (resp. at the federal level), a will arbitrarily be designated as the winner in that jurisdiction (resp. in the federation). Consider now the voting profile $\pi = (a, a, a, b, b)$ where individuals 1 to 3 vote for a while individuals 4 and 5 vote for b. Partition σ is such that $\sigma(1) = \sigma(2) = \sigma(3) = 1$, $\sigma(4) = 2$ and $\sigma(5) = 3$. Thus the winner in J_1 is a, while the winner in J_2 and J_3 is b, so that $\Pi = (a, b, b)$. Therefore, $g(f(\sigma, \pi)) = g(\Pi) = b$. Consider now another partition σ' such that $\sigma'(1) = 1$, $\sigma'(2) = \sigma'(3) = \sigma'(4) = 2$ and $\sigma'(5) = 3$. In that case, the winner in J_1 and in J_2 is a, while the winner in J_3 is b. Therefore, $\Pi' = (a, a, b)$, so that $g(f(\sigma', \pi)) = a$.

2.2 Properties

We define some properties we might impose on the different social choice functions. The first two axioms are imposed on the jurisdiction functions f_j:

Axiom 1 Local Faithfulness

$$\textit{If } J_j = \{i\} \textit{ then } f_j(\sigma, \pi) = \pi|_i$$

Faithfulness was first introduced by Young (1974) for his characterization of the Borda Choice Correspondence. It requires that when society consists of a single individual, it should choose its most preferred alternative. As in our setting we restrain this condition to jurisdictional rules only, we call it local.

Axiom 2 Local Pareto

$$\textit{If } \pi|_i = \{z\} \ \forall i \in J_j \textit{ then } f_j(\sigma, \pi) = z$$

When every individual in a jurisdiction votes for the same candidate, that one should be elected in his jurisdiction. Local Pareto implies Local Faithfulness.

We now turn to the conditions imposed on the federal constitution C as to convey the different ideas about manipulation by movement.

Axiom 3 Individual Stability

$$\forall \pi \in A^n, \forall i \in N, g(f(\sigma, \pi)) = g(f(\sigma', \pi)),$$

$$\textit{for all } \sigma, \sigma' \in \Sigma \text{ , } \sigma(h) = \sigma'(h) \ \forall h \neq i \textit{ and } \sigma(i) \neq \sigma'(i)$$

This property requires that no single voter can affect the outcome alone by changing jurisdiction (without changing his vote).

Axiom 4 Stability

$$\forall \pi \in A^n,\ g(f(\sigma,\pi)) = g(f(\sigma',\pi))\ \ \forall \sigma,\, \sigma' \in \Sigma$$

Stability is a generalized version of Individual Stability. It states that no coalition of voters can change the result of a federal election by simply changing jurisdictions. Therefore, the result of the election should be independent from the partition of the individuals. Recall that the framework considered in order to define federal constitutions departs from the traditional arrovian framework, by the fact that the voting procedure depends on the partition of the individuals. Thus Stability is a condition such that this non arrovian framework coincides with the arrovian one.

As a logical negation, one can define the corresponding *Individual Instability* axiom, according to which there exists at least one vote profile such that one individual can change the outcome of the election on its own by changing jurisdiction. Similarly, we can define the corresponding *Instability* axiom by saying that a federal constitution C is unstable if and only if it is not stable, i.e. there is at least one vote profile such that a group of voters can change the result by changing their voting jurisdictions.

In line with the instability axioms, we define the two *manipulability* axioms:

Axiom 5 Individual Manipulability

$$\exists\ \pi \in A^n, \exists\ i \in N, \exists\ \sigma, \sigma' \in \Sigma \ \textit{satisfying}\ \sigma(h) = \sigma'(h)\ \forall h \neq i\ \textit{and}\ \sigma(i) \neq \sigma'(i):$$

$$g(f(\sigma',\pi)) \succ_i g(f(\sigma,\pi))$$

With Individual Manipulability, not only can a single voter change the outcome of an election by changing jurisdiction, he can also do that in his favor. Said differently, $g(f(\sigma,\pi))$ is not a Nash Equilibrium in the game where strategies are given by the choice of a jurisdiction.

Axiom 6 Manipulability

$$\exists \pi \in A^n, \exists G \subset N, \exists \sigma,\, \sigma' \in \Sigma\ \textit{satisfying}\ \sigma(h) = \sigma'(h)\ \forall h \notin G\ \textit{and}\ \sigma(i) \neq \sigma'(i)\ \forall i \in G:$$

$$g(f(\sigma',\pi)) \succ_i g(f(\sigma,\pi))\, \forall i \in G$$

When a constitution is manipulable, coalitions of voters can change the result of a federal election by changing jurisdictions and they can do it in the favor of all the members of the coalition. $g(f(\sigma,\pi))$ is not a Nash Equilibrium with respect to coalition movements.

The following theorem shows how the concepts of stability and manipulability relate one with the other.

Theorem 1 *For any set of candidates A, a constitution C is unstable if and only if it is individually manipulable.*

Proof of Theorem 1 : 1- A constitution C is individually unstable if and only if it is individually manipulable : if C is individually unstable, then $\exists \pi \in A^n$ $\exists i \in N$, $\exists \sigma, \sigma' \in \Sigma$, satisfying $\sigma(i) \neq \sigma'(i)$ and $\sigma(h) = \sigma'(h)$ $\forall h \neq i$ and $g(f(\sigma, \pi)) \neq g(f(\sigma', \pi))$. As preferences are linear, either $g(f(\sigma, \pi)) \succ_i g(f(\sigma', \pi))$, or the reverse. So individual i can manipulate g by moving from σ' to σ or from σ to σ'.
2- A constitution C is individually stable if and only if it is stable: Assume C is individually stable, but fails to satisfy Stability. Thus, there exists $\pi \in A^n$ and σ, $\sigma' \in \Sigma$ such that $g(f(\sigma, \pi)) \neq g(f(\sigma', \pi))$. As $n > m$, there exists a sequence of partitions, $\sigma = \sigma_0$, σ_1, $\ldots$, σ_k, $\ldots$, $\sigma_l = \sigma'$ such that only one voter moves in between σ_k and σ_{k+1}. At some point in the sequence, we get $g(\sigma_k, \pi) \neq g(\sigma_{k+1}, \pi)$, a contradiction. ■

It can seem undesirable that a coalition of voters who do not change their votes but who change jurisdictions could invert the result of an election. However, if one does not feel uncomfortable with that, he might feel uncomfortable with the fact that the result of an election depends only on the location of one single voter. Theorem 1 states that these two facts are identical.

2.3 Pivotal Individuals

In every constitution, it seems reasonable to ask for the decision not to be concentrated in the hand of one unique voter. We express part of this idea with the following definitions, stating that one individual alone cannot alter the outcome while all the other individuals vote unanimously for the same candidate.

Before proceeding, we define the *Unanimous vote profile* π_z as the vote profile such that $\pi|_i = z$ for all i in N, for any alternative z in A. It follows that π_a is the unanimous vote profile for candidate a, π_b is the unanimous vote profile for candidate b etc...

Definition 2 Pivotal individual in $\boldsymbol{\pi_z}$
Voter i is called pivotal in π_z with $z \in A$ if $\exists$ $y \in A$, $z \neq y$ and π such that $\pi|_i = y$ and $\pi|_h = z$ $\forall h \in N \setminus \{i\}$ and $g(f(\sigma, \pi_z)) \neq g(f(\sigma, \pi))$ $\forall \sigma$.

Definition 3 Pivotal individual
Voter i is called pivotal if $\exists$ $z \in A$ such that i is pivotal in π_z.

A pivotal individual is thus pivotal for at least one unanimous vote profile. Note that a pivotal individual i does not have the power to impose his best choice to society, he can only change the winner on his own by voting for y at some unanimous profile π_z (but his best choice could be a third candidate). Furthermore, a pivotal individual is defined only with respect to cases for which all other individual are unanimous on candidate z. Hence a pivot in π_z might not be able to change the outcome of the election if at least one other voter votes for another candidate than z. Being pivotal is thus not equivalent to having a veto power.

3 A Result on Stable Constitutions

In this section the main theorem of the paper is presented. It says that any constitution satisfying Stability and Local Pareto necessarily gives the power to every elector of being pivotal, unless it is the Constant Constitution, which definition is provided below.

Definition 4 Constant Constitution
A constitution $C = (g, f_1, \dots, f_m)$ is called constant if there exists $z \in A$ such that $\forall \sigma \in \Sigma,\ \forall \pi \in A^n,\ \ g(f(\sigma, \pi)) = z$

A Constant Constitution is a voting rule such that, whatever the results of the jurisdiction elections, the federal winner is always the same. These rules are totally independent from the electors. A Constant Constitution is a large class of rules, as it is defined irrespective of the properties satisfied by the jurisdictional rules f_j as well as of the properties of the federal rule g. Among others we define the following subclass:

Definition 5 LP Constant Constitution
A federal constitution C is LP constant if it is a Constant Constitution for which every jurisdiction rule f_j satisfies Local Pareto.

We now turn to some preliminary propositions which will be useful to state and prove the main theorem.

Proposition 1 *Let A contain any finite number p of alternatives. Assume a federal constitution $C = (g, f_1, \dots, f_m)$ satisfies Local Pareto and Stability. If one individual is pivotal in π_{a_j} then every individual is pivotal in π_{a_j}.*

Proof of Proposition 1 : Assume that individual i is pivotal in π_{a_j} by voting for candidate a_k. Consider the vote profile π where every individual votes for a_j except individual i who votes for a_k, and the partition σ such that $J_1 = \{i\}$. Then $g(f(\sigma, \pi_{a_j})) \neq g(f(\sigma, \pi))$ where, by Local Pareto, $g(f(\sigma, \pi)) = g(a_k, a_j, \dots, a_j)$. Consider then the vote profile π' where every individual votes for a_j except individual h who votes for a_k, and the partition σ' such that $J_1 = \{h\}$. Then $g(f(\sigma, \pi)) = g(a_k, a_j, \dots, a_j) = g(f(\sigma', \pi'))$ and hence $g(f(\sigma', \pi')) \neq g(f(\sigma, \pi_{a_j}))$. But Stability implies that $g(f(\sigma', \pi')) = g(f(\sigma, \pi'))$. Thus individual h is also pivotal in π_{a_j}. ■

Proposition 2 *Let A contain any finite number p of alternatives. Assume a federal constitution $C = (g, f_1, \dots, f_m)$ satisfies Local Pareto and Stability. For any two federal profiles Π and $\Pi' \in A^m$ such that Π and Π' both contain the same s different alternatives, with $s < m$, then $g(\Pi) = g(\Pi')$.*

Proof of Proposition 2 :
The case $m = 2$ is trivial as necesarily $s = 1$.

Consider $m > 2$ and assume, without loss of generality, that the federal profile Π contains the s alternatives a_1 to a_s. Because $s < m$, the federal profile Π has at least one alternative that is present more than once. Consider any federal profile Π' such that $\Pi'|_j = \Pi|_j$ for all $j \neq k$ and $\Pi'|_k \neq \Pi|_k$, and such that Π' contains also the same s different alternatives a_1 to a_s. Hence only one coordinate has changed from Π to Π', and both profiles have exactly the same number of different alternatives. For instance, assume $\Pi_{ex} = a_1, a_2, \dots, a_{s-1}, a_s, a_s, \dots a_s$.
Then $\Pi'_{ex} = a_1, a_2, \dots, a_{s-1}, a_s, a_s, \dots a_s, a_1$ is such a federal profile, while $\Pi'_{ex2} = a_s, a_2, \dots, a_{s-1}, a_s, a_s, \dots a_s, a_s$ is not (there are only $s-1$ different alternatives in Π'_{ex2}).

We first show that for two such profiles, $g(\Pi) = g(\Pi')$. In order to construct Π', it is necessary that the alternative that has been changed in Π is present more than once in Π. Assume, without loss of generality, that $\Pi = \Pi_{ex}$ and $\Pi' = \Pi'_{ex}$, thus in J_m, the alternative a_s has been replaced by alternative a_1. Consider the individuals vote profile π in which two individuals vote for a_1, every alternative a_2 to a_{s-1} receives one vote and the $n - s$ remaining individuals vote for a_s. This vote profile, with two adequate partitions σ and σ', can generate both the profiles Π and Π'. Indeed, consider σ such that the two individuals voting for a_1 are together in J_1, the jurisdictions J_2 to J_{s-1} are singletons formed respectively of the individual voting for a_2 to that voting for a_{s-1}, and jurisdictions J_s to J_m are filled with the individuals voting for a_s. Using Local Pareto, this partition generates the federal profile Π_{ex}. Consider then the partition σ' such that J_1 is a singleton formed of one of the a_1 voters, jurisdictions J_2 to J_{m-1} do not change, and J_m is a singleton formed of the second a_1 voter. Using Local Pareto again, this partition generates the federal profile Π'_{ex}.

In both cases, the individuals vote profile is the same, as only the partition has changed. Therefore, using Stability, we have

$$g(a_1, a_2, \dots, a_{s-1}, a_s, a_s, \dots a_s) = g(a_1, a_2, \dots, a_{s-1}, a_s, a_s, \dots a_1)$$

As a general result, any federal profiles Π and Π' such that $\Pi'|_j = \Pi|_j$ for all $j \neq k$ and $\Pi'|_k \neq \Pi|_k$, and such that Π' has the same number s of different alternatives, both elect the same winner.

In order to conclude, we use the fact that any federal profile Π', containing exactly s different alternatives, can be obtained starting from Π, another federal profile containing exactly the same s different alternatives, by a sequence of replacements of one alternative by another, each of these replacements leaving the new federal profile with exactly the s same different alternatives. ■

Before stating and proving the main theorem of this section, one more intermediate proposition is given. It says that when the constitution C satisfies Local Pareto and Stability, then the federal social choice function g is anonymous. Let us first define anonymity.

Axiom 7 Federal Anonymity
A federal social choice function $g : A^m \longrightarrow A$ satisfies anonymity if and only if $g(\Pi) =$

$g(\mu(\Pi))$ where Π is a federal vote profile $(f_1(\sigma,\pi),\ldots,f_m(\sigma,\pi))$ in A^m and μ is a permutation on $\{1,\ldots,m\}$

Federal Anonymity is an axiom stating that any change in the name of the jurisdictions leaves the result unchanged. No jurisdiction has more power than another.

Proposition 3 *Let A contain any finite number p of alternatives. Consider a constitution $C = (g, f_1, \ldots, f_m)$ which satisfies Stability and Local Pareto. Then g satisfies Federal Anonymity.*

Proof of Proposition 3 :
Given Local Pareto, it is possible to generate any federal profile Π by forming homogenous partitions, for which any individual in J_j votes for $\Pi|_j$.

Any permutation μ on the name of the jurisdictions generates a federal profile $\mu(\Pi)$ that can be obtained as the result of the corresponding permutation on the set of individuals, i.e. the partition σ' such that $\sigma(i) = j$ and $\mu(j) = j' \Longrightarrow \sigma'(i) = j'$. Since C satisfies Stability, $g(f(\sigma,\pi)) = g(f(\sigma',\pi))$. Consequently, $g(\Pi) = g(\mu(\Pi))$, so g is anonymous. ∎

The theorem can now be stated.

Theorem 2 *Let A contain any finite number p of alternatives. If a constitution $C = (g, f_1, \ldots, f_m)$ satisfies Stability and Local Pareto then C is either a LP Constant Constitution or in C every individual is pivotal.*

Proof of Theorem 2:
The sketch goes as follows. We assume that in C, no individual is pivotal, although C satisfies Local Pareto and Stability. This will imply that C can only be a LP Constant Constitution. This implication, said differently, also states that if a federal constitution C satisfies Stability, Local Pareto and is not a LP Constant Constitution, then in C there is at least one pivotal individual. By proposition 1, the theorem will then be proved.

The proof is led by induction over natural numbers. First we show that all the possible federal profiles containing only one or two alternatives give the same winner, which we call z. Next we assume that any federal profile containing s or less different alternatives, $s < m$, also gives z as the winner and we show that this implies that any federal profile containing $s+1$ different alternatives gives z as the winner of the election. A last argument will then be given to show that it is unnecessary to go beyond $s \geq m$.

Assume thus that C satisfies Local Pareto, Stability, and no individual is pivotal. Consider any alternative a_j. The vote profile π_{a_j} is such that every individual votes for candidate a_j, leading by Local Pareto to $g(f(\sigma,\pi_{a_j})) = g(a_j,\ldots,a_j)$. Call this winner z and consider the vote profile such that every elector votes for a_j, except one who votes for any other alternative a_k. Consider the partition σ' such that the a_k

voter is alone in the last jurisdiction J_m. By Local Pareto, this vote profile leads to $g(a_j, \ldots, a_j, a_k)$ and as it is assumed that no individual is pivotal, this implies that $g(a_j, \ldots, a_j, a_k) = g(a_j, \ldots, a_j) = z$.

Given proposition 2, this implies that any federal profile containing alternatives a_j and a_k yields z as the winner. Thus,

$$g(a_j, \ldots, a_j) = \ldots g(a_j, \ldots, a_j, a_k, \ldots, a_k) = \ldots = g(a_j, a_k, \ldots, a_k)$$

Furthermore, this last term of the above equality can be generated by the voting profile in which every individual votes for a_k except individual 1 who votes for a_j, if individual 1 is put alone in jurisdiction J_1. But again, no individual is pivotal, so that when individual 1 changes his vote to a_k, the result of the election is still z. Hence, $g(a_k, \ldots, a_k) = z$.

We have shown that for any candidate a_j, if the winner associated to the unanimous vote profile π_{a_j} is z, then any unanimous vote profile π_{a_k} should elect z, as well as any federal profile containing exactly two different alternatives.

Assume now that the result of the election is z whenever there are s or less different alternatives in the federal profile, with $s < m$. Consider the vote profile such that one individual votes for a_1, one individual votes for a_2, one individual votes for a_3, ..., one individual votes for a_{s-2}, $n - s$ individuals vote for alternative a_{s-1} and finally the two last individuals vote for any alternative a_i, with $a_i \in \{a_1, \ldots a_{s-1}\}$. Forming homogeneous jurisdictions and using Local Pareto, this particular vote profile generates the federal profile $g(a_1, a_2, \ldots, a_{s-2}, \underbrace{a_{s-1}, \ldots, a_{s-1}}_{m-s+1}, a_i)$. This federal profile contains $s-1$ different alternatives, so according to the hypothesis, the winner is z. Hence

$$g(a_1, a_2, \ldots, a_{s-2}, a_{s-1}, \ldots, a_{s-1}, a_i) = z \text{ for any } a_i \in \{a_1, \ldots a_{s-1}\}$$

Next, when the last two individuals vote for the same alternative a_j with $a_j \in \{a_s, \ldots a_p\}$, the federal profile generated is $g(a_1, a_2, \ldots, a_{s-2}, a_{s-1}, \ldots, a_{s-1}, a_i)$, which contains s different alternatives, so by hypothesis, the winner is still z. Hence

$$g(a_1, a_2, \ldots, a_{s-2}, a_{s-1}, \ldots, a_{s-1}, a_j) = z \text{ for any } a_j \in \{a_s, \ldots a_p\}$$

Summarizing,

$$g(a_1, a_2, \ldots, a_{s-2}, a_{s-1}, \ldots, a_{s-1}, f_m) = z \text{ whatever the value of } f_m$$

Consider now the same partition of the individuals, and assume the last two individuals vote respectively for a_s and a_{s+1}. We have no information about the outcome of the last jurisdiction, as Local Pareto does not apply. However, this vote profile will generate a federal profile of the form $g(a_1, a_2, \ldots, a_{s-2}, a_{s-1}, \ldots, a_{s-1}, f_m)$ which will give z as the winner, as just stated above. Hence there is one individual vote profile containing $s + 1$ different candidates for which the winner is still z.

This specific vote profile, which contains $s+1$ alternatives, can now be used with a different partition: the first $s-2$ jurisdictions are the same, i.e. they are singletons. The jurisdictions J_{s-1} to J_{m-2} are filled with the $n-s$ electors voting for a_{s-1}, jurisdiction J_{m-1} is a singleton formed by the a_s voter and last, J_m is a singleton formed by the a_{s+1} voter. Using Local Pareto, this vote profile and this partition generate the federal profile

$$g(a_1, a_2, \dots, a_{s-2}, \underbrace{a_{s-1}, \dots, a_{s-1}}_{m-s}, a_s, a_{s+1})$$

Notice that the vote profile has not changed, only the partition has changed. Therefore, according to Stability, the winner is still z. Thus there is a federal profile containing $s+1$ alternatives, and this profile gives z as the winner.

There are two possibilities here: $s+1 < m$ or $s+1 = m$. In the first case, proposition 2 helps us conclude, as we will then have z as the winner whenever there are $s+1$ different alternatives at the federal level. In the second case, there are as much alternatives as jurisdictions, so that every jurisdiction elects a different candidate. Proposition 3 states that g is anonymous, implying that all the possible federal profiles give the same winner.

What remains to prove is the case where $s \geq m$. Notice, however, that any vote profile containing p alternatives, with $p > m$, will generate a federal profile containing no more than m alternatives. Proposition 3 thus helps us concluding.

As a consequence, we have shown that for any set of candidates $A = \{a_1, \dots, a_p\}$, if C satisfies Local Pareto, Stability, and has no pivotal individual, then all the possible federal profiles give the same winner. Therefore, the federal constitution is a LP Constant Constitution. ■

Although this result is not complete, as there is no exact characterization given of stable voting rules, theorem 2 is sufficient in order to give a negative answer to the question raised in this paper, that of the existence of non-manipulable constitutions. No matter what exact set of rules is defined, they all share a crucial weakness, that of letting every individual be pivotal.

As mentioned, the theorem does not give an entire characterization of the set of rules that do allow for Stability along with Local Pareto. This characterization is, we believe, unnecessary to stress out that manipulation is a possible byproduct of two stage elections, as pivotal individuals are undesirable enough. However, it seems interesting to provide some examples of rules other than the Constant Constitution which are stable. For that purpose, the specific case of two candidates is examined in the following.

4 The case of two candidates

In this section we restrain our analysis to the case of two candidates, $A = \{a, b\}$.

In what remains we use the following notation : $\pi_0 = \pi_a$ is the unanimous profile for a, $\pi_k^{l_1 \dots l_k}$ denotes the vote profile such that individuals $l_1, \dots, l_k$ have switched their vote to b, while individuals $l_{k+1}, \ldots, l_n$ still vote for a. Ultimately we have $\pi_n^{l_1 \ldots l_n} = \pi_n = \pi_b$, the profile such that every voter votes for b.

4.1 Local Pareto

In this section we give a characterization of the set of rules which satisfy both Stability and Local Pareto. To do this, we first define what we call the class of Unanimity Rules.

As to be more formal, we use the following notation: when $z \in A$ is either one of the two candidates, we denote by $\bar{z}$ his opponent. For instance, if $z = b$, then $\bar{z}$ denotes a.

Definition 6 Unanimity Rule
Let $(x, z) \in A^2$. A federal constitution is a Unanimity Rule (x, z) if :

- $\forall\ J_j$, $[\forall\, i \in J_j\ \pi|_i = x \Longrightarrow f_j(\sigma, \pi) = x]$ *and* $[\exists\, h \in J_j,\ \pi|_h = \bar{x} \Longrightarrow f_j(\sigma, \pi) = \bar{x}]$

- $[\forall J_j \in J,\ f_j(\sigma, \pi) = x \Longrightarrow g(f(\sigma, \pi)) = z]$ *and* $[\exists J_{j'} \in J,\ f_{j'}(\sigma, \pi) = \bar{x} \Longrightarrow g(f(\sigma, \pi)) = \bar{z}]$.

In words, a unanimity rule is such that at both the jurisdictional and the federal levels, unanimity is required for one candidate to be elected while his opponent wins in all other cases. As $A = \{a, b\}$, Unanimity Rule is a class of four rules only: Unanimity Rule (a, a); Unanimity Rule (a, b); Unanimity Rule (b, a) and Unanimity Rule (b, b). For instance, Unanimity Rule (a, a) is such that in every jurisdiction, if a receives unanimity then a is elected at the jurisdiction level, otherwise b is. At the federal level, if a is unanimous then a is the winner, otherwise b is. Therefore, when every individual votes for a, the federal winner is a. In any other case, the federal winner is b.
In turn, Unanimity Rule (b, a) is such that a is elected as the final winner if and only if everyone votes for b. In any other case, b is elected.

We now turn to the characterization theorem.

Theorem 3 *When $A = \{a, b\}$, a federal constitution $C = (g, f_1, \ldots, f_m)$ satisfies Local Pareto and Stability if and only if C is a LP Constant Constitution or is a Unanimity Rules.*

Proof of Theorem 3 :

- Implication $\Longleftarrow$

By definition, the LP Constant Constitution is stable and satisfies Local Pareto.
Unanimity Rule is a class of rules that satisfy Local Pareto. Moreover they satisfy Stability. Indeed, if unanimity is required for x and voter i is a x voter, then when he changes jurisdiction, it does neither affect the jurisdiction he left nor the jurisdiction he

is arriving in. If voter i is a $\bar{x}$ voter, then the jurisdiction he left may become unanimous about x, but the one he arrives in will not be. In any case the winner of the election will remain the same.

- Implication $\Longrightarrow$

Assume C satisfies Local Pareto and Stability. Then, according to theorem 2, we know that C is either the LP Constant Constitution or in C, every individual is pivotal.

Notice that any vote profile containing both options a and b can yield, using Local Pareto and building homogenous partitions, to federal profiles containing both options. We can thus use Proposition 2 to state that

$$g(f(\sigma, \pi_1^{l_1})) = \ldots = g(f(\sigma, \pi_{n-1}^{l_1 \ldots l_{n-1}})) = z$$

Hence not only every federal profile containing both options give the same winner, it is also true for every vote profile. The only remaining vote profiles for which nothing can be said are π_0 and $\pi_n^{l_1 \ldots l_n}$. Recalling that whenever z is either of the two candidates, $\bar{z}$ denotes his opponent, there are four possible cases:
(1) $g(f(\sigma, \pi_0)) = z$ and $g(f(\sigma, \pi_n^{l_1 \ldots l_n})) = z$;
(2) $g(f(\sigma, \pi_0)) = z$ and $g(f(\sigma, \pi_n^{l_1 \ldots l_n})) = \bar{z}$;
(3) $g(f(\sigma, \pi_0)) = \bar{z}$ and $g(f(\sigma, \pi_n^{l_1 \ldots l_n})) = z$;
(4) $g(f(\sigma, \pi_0)) = \bar{z}$ and $g(f(\sigma, \pi_n^{l_1 \ldots l_n})) = \bar{z}$.

Note that along any two sequences of individual changes $l_1, ..., l_n$ and $l'_1, ..., l'_n$, it is not possible that we are in two different cases. This is the consequence of proposition 1 according to which the existence of one pivotal individual implies that every individual is pivotal.

We show here that case (4) cannot hold. Assume there exists a sequence $l_1 \ldots l_n$ such that $g(f(\sigma, \pi_0)) = \bar{z}$, $g(f(\sigma, \pi_1^{l_1})) = z$, $\ldots$, $g(f(\sigma, \pi_{n-1}^{l_1 \ldots l_{n-1}})) = z$ and $g(f(\sigma, \pi_n^{l_1 \ldots l_n})) = \bar{z}$. Consider any partition σ such that $J_m = \{l_1, l_n\}$. Using Local Pareto, $g(f(\sigma, \pi_0)) = g(a, \ldots, a, a) = \bar{z}$. In $\pi_1^{l_1}$, now, we have $g(f(\sigma, \pi_1^{l_1})) = g(a, \ldots, a, f_m) = z$. Hence, necessarily, $f_m = b$.
Now let all the other individuals switch their votes. We will achieve $g(f(\sigma, \pi_{n-1}^{l_1 \ldots l_{n-1}})) = g(b, \ldots, b, f_m) = g(b, \ldots, b) = z$ and this is a contradiction given that $g(f(\sigma, \pi_n^{l_1 \ldots l_n})) = g(b, \ldots, b) = \bar{z}$. Hence there is no sequence such as case (4).

We have shown that for any sequence $l_1, \ldots, l_n$ we necessarily have case (1), (2) or (3). In case (1) the outcome remains constant throughout the sequence while in cases (2) and (3) all individuals are pivotal. Case (1) is a Constant Constitution and Local Pareto is satisfied so it is a LP Constant Constitution. It remains proving that cases (2) and (3) generate the four possible Unanimity Rules.

Consider that every sequence is in case (3). Given Local Pareto, we know that in π_0 we get $g(f(\sigma, \pi_0)) = g(a, \ldots, a) = \bar{z}\ \forall \sigma$. Consider σ such that the $m - 1$ first jurisdictions are singletons and the last one is filled with all the remaining voters. Then, in $\pi_1^{l_1}$ we get $g(f(\sigma, \pi_1^{l_1})) = g(b, a, \ldots, a) = z$. According to proposition 3, g is anonymous, so that whenever the federal profile has one jurisdiction in which b is elected, while all the others elect a, then the federal winner is z.

We also get $g(f(\sigma, \pi_2^{l_1 l_2})) = g(b, b, a, \ldots, a) = z$. Again, Federal Anonymity says that whenever the federal profile is formed of two b's and $m-2$ a's, the winner is z. Going on until π_{m-1} we reach $g(b, \ldots, b, a) = z$. By Federal Anonymity, whenever the profile is formed of $m-1$ b's and one a, the winner is z

Finally, in π_n we have $g(b, \ldots, b, b) = z$. Hence the only possibility for $\bar{z}$ to win the election is for profile π_0. That means that $\bar{z}$ can win only if every jurisdiction votes for a and if every voter of the jurisdiction votes for a. That rule is thus either Unanimity Rule (a, a) or Unanimity Rule (a, b), according to the value of z.

In the same way, case (2) will lead to Unanimity Rule (b, a) and Unanimity Rule (b, b). ■

Remark 2 *Although it was not a specific requirement, the rules characterized in Theorem 3, if not in the set of the LP Constant Constitution, all have in common that the rules at the jurisdiction level are identical, i.e.* $f_j = f_{j'}$ *for all* j, j'.

4.2 Local Faithfulness

In this subsection, the consequences of weakening the Local Pareto condition into the Local Faithfulness one are explored. Trivially, Local Faithfulness is satisfied whenever Local Pareto is assumed, but the reverse is not true. Indeed, Local Faithfulness says that in the case a jurisdiction is a singleton, the choice of the jurisdiction should be the choice of the individual composing it. Yet Local Faithfulness keeps silent on what happens in jurisdictions with at least two individuals.

We now define two other classes of rules in order to state the theorem.

Definition 7 LF Constant Constitution
A federal constitution C is LF constant if it is a Constant Constitution for which every jurisdiction rule f_j *satisfies Local Faithfulness.*

Next we define what we call the class of Parity Rules. In words, a parity rule is such that at both the jurisdictional and the federal levels, the winner is designated uniquely by the parity of the votes he or his opponent receives.

Definition 8 The Parity Rule
Let $(x, y, z) \in A^3$. *A federal constitution is a Parity Rule* (x, y, z) *if:*

- in every jurisdiction, if the number of voters for candidate x *is odd, then* x *is the jurisdiction winner. If that number is even then* $\bar{x}$ *wins.*

- *at the federal level, if the number of jurisdictions that have chosen y is odd, then z is the winner. If that number is even, then $\bar{z}$ is the winner.*

This class of rules may look rather peculiar and has, to the best of our knowledge, never been studied. It bases the election on the parity of the votes. As $A = \{a, b\}$, Parity Rule is a class of eight rules: Parity Rule (a,a,a); Parity Rule (a,a,b); Parity Rule (a,b,a); Parity Rule (b,a,a); Parity Rule (a,b,b); Parity Rule (b,a,b); Parity Rule (b,b,a) and Parity Rule (b,b,b).

For instance, Parity Rule (a,b,a) is the following: in every jurisdiction, if the number of votes for a is odd then a is the winner in the jurisdiction, otherwise b is. At the federal level, if the number of jurisdictions that voted for b is odd, then a is the winner, otherwise b is. As a matter of illustration of this rule, assume $N = \{1, 2, 3, 4\}$, $m = 3$ and the partition of society is given by σ such that $J_1 = \{1\}, J_2 = \{2\}, J_3 = \{3, 4\}$. For the unanimous vote profile π_a, we obtain $f_1(\sigma, \pi_a) = a$, $f_2(\sigma, \pi_a) = a$, $f_3(\sigma, \pi_a) = b$ and thus $g(f(\sigma, \pi_a)) = a$, as the number of jurisdictions voting for b is odd. In π_1^3 (i.e. the profile such that everyone votes for a except individual 3 who votes for b), we obtain $f_1(\sigma, \pi_1^3) = a$, $f_2(\sigma, \pi_1^3) = a$, $f_3(\sigma, \pi_1^3) = a$ and thus $g(f(\sigma, \pi_1^3)) = b$. In $\pi_2^{3,4}$ we obtain $g(f(\sigma, \pi_2^{3,4})) = a$, then $g(f(\sigma, \pi_3^{2,3,4})) = b$ and $g(f(\sigma, \pi_b)) = a$

One interesting feature of this class of rules is, as we will see further in more details, that every time an elector changes his vote, he changes the outcome of the election. Thus if another individual changes his vote at the same time they both neutralise themselves.

A few preliminary lemmas and propositions are needed, all of them being based on the following assumption:

Assumption A_k **:** *Assume that the constitution C is such that there exists a k, $2 \leq k < n$, such that for any k', $k' \leq k$, we have $g(f(\sigma, \pi_{k'}^{l_1 \ldots l_{k'}})) = z$ for any sequence of individuals $l_1, \ldots, l_{k'}$ that have switched their vote from alternative a to alternative b. Assume furthermore that there exists an individual l_{k+1} such that $g(f(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}})) = \bar{z}$.*

We are thus assuming that the constitution always gives the same winner, until $k+1$ individuals have changed their vote. We then have the following:

Lemma 1 *Given assumption A_k, if $\sigma(l_{k+1}) = j$, $f_j(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}}) \neq f_j(\sigma, \pi_k^{l_1 \ldots l_k})$*

When he changes his vote from a to b, voter l_{k+1} changes the result of any jurisdiction he belongs to.

Proof of Lemma 1: Assume on the contrary that there exists a σ such that $f_j(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}}) = f_j(\sigma, \pi_k^{l_1 \ldots l_k})$. Thus:

$$\begin{aligned} g(f(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}})) &= g(f_j(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}}), f_{-j}(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}})) \\ &= g(f_j(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}}), f_{-j}(\sigma, \pi_k^{l_1 \ldots l_k})) \text{ by Jurisdiction Sovereignty} \\ &= g(f_j(\sigma, \pi_k^{l_1 \ldots l_k}), f_{-j}(\sigma, \pi_k^{l_1 \ldots l_k})) \\ &= g(f(\sigma, \pi_k^{l_1 \ldots l_k})) = z, \text{ a contradiction with } A_k. \end{aligned}$$

$\blacksquare$

Lemma 2 *Given assumption A_k, if $\sigma(l_{k+1}) = \sigma(l_k) = j$, $f_j(\sigma, \pi_k^{l_1 \ldots l_k}) = f_j(\sigma, \pi_{k-1}^{l_1, \ldots l_{k-1}})$.*

When voting in the same jurisdiction as l_{k+1}, voter l_k could not change the result of his jurisdiction J_j.

Proof of Lemma 2: Assume the contrary. $f_j(\sigma, \pi_k^{l_1 \ldots l_k}) \neq f_j(\sigma, \pi_{k-1}^{l_1 \ldots l_{k-1}})$. Thus, by Lemma 1, $f_j(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}}) = f_j(\sigma, \pi_{k-1}^{l_1 \ldots l_{k-1}})$ and:

$$\begin{aligned}
g(f(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}})) &= g(f_j(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}}), f_{-j}(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}})) \\
&= g(f_j(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}}), f_{-j}(\sigma, \pi_{k-1}^{l_1 \ldots l_{k-1}})) \text{ by Jurisdiction Sovereignty} \\
&= g(f_j(\sigma, \pi_{k-1}^{l_1 \ldots l_{k-1}}), f_{-j}(\sigma, \pi_{k-1}^{l_1 \ldots l_{k-1}})) \\
&= g(f(\sigma, \pi_{k-1}^{l_1 \ldots l_{k-1}})) = z, \text{ a contradiction with } A_k.
\end{aligned}$$

$\blacksquare$

Lemma 3 *Given assumption A_k, if $j = \sigma(l_{k+1}) \neq \sigma(l_k) = j'$, $f_{j'}(\sigma, \pi_k^{l_1 \ldots l_k}) \neq f_{j'}(\sigma, \pi_{k-1}^{l_1 \ldots l_{k-1}})$.*

When voting in a different jurisdiction than l_{k+1}, voter l_k changes the result of his jurisdiction by changing his vote from a to b.

Proof of Lemma 3: Assume the contrary, $f_{j'}(\sigma, \pi_k^{l_1 \ldots l_k}) = f_{j'}(\sigma, \pi_{k-1}^{l_1 \ldots l_{k-1}})$.

$$\begin{aligned}
g(f(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}})) &= g(f_{j'}(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}}), f_{-j'}(\sigma, \pi_{k+1}^{l_1 \ldots l_{k+1}})) \\
&= g(f_{j'}(\sigma, \pi_k^{l_1 \ldots l_k}), f_{-j'}(\sigma, \pi_k^{l_1 \ldots l_{k-1} l_{k+1}}) \text{ by Jurisdiction Sovereignty} \\
&= g(f_{j'}(\sigma, \pi_{k-1}^{l_1 \ldots l_{k-1}}), f_{-j'}(\sigma, \pi_k^{l_1 \ldots l_{k-1} l_{k+1}})) \\
&= g(f_{j'}(\sigma, \pi_k^{l_1 \ldots l_{k-1} l_{k+1}}), f_{-j'}(\sigma, \pi_k^{l_1 \ldots l_{k-1} l_{k+1}})) \text{ by Jurisdiction Sovereignty} \\
&= g(f(\sigma, \pi_k^{l_1 \ldots l_{k-1} l_{k+1}}))
\end{aligned}$$

But $l_1 \ldots l_{k-1}, l_{k+1}$ forms a sequence of k individuals who have switched their preference. So by assumption A_k, $g(f(\sigma, \pi_k^{l_1 \ldots l_{k-1} l_{k+1}})) = z$, a contradiction. $\blacksquare$

Proposition 4 *Let $A = \{a, b\}$. If a federal constitution $C = (g, f_1, \ldots, f_m)$ satisfies Local Faithfulness and Stability, for any $k, 1 \leq k \leq n-2$, assumption A_k cannot hold.*

Proof of Proposition 4

We divide this proof into two steps. First we show that the proposition holds for any $k, 1 \leq k \leq m-1$. Recall that $n \geq m+1$ by assumption. Therefore, in the case $n = m+1$, the proof will be finished at this stage. However, in the case where $n > m+1$, we will need to show in a second step that the proposition holds for any $k, m \leq k \leq n-2$

<u>**First step:**</u> Let us assume that $k, 1 \leq k \leq m-1$ and assumption A_k holds. The three lemmas can then be applied.

Consider the sequence $l_1 \dots l_{k+1}$ such that $g(f(\sigma, \pi_{k'}^{l_1 \dots l_{k'}})) = g(f(\sigma, \pi_k^{l_1 \dots l_k})) = z$ for any $k' < k$ and $g(f(\sigma, \pi_{k+1}^{l_1 \dots l_{k+1}})) = \bar{z}$. Let σ be the partition such that $J_1 = \{l_1\}, \dots, J_{k-1} = \{l_{k-1}\}$, the jurisdictions $J_k, \dots, J_{m-1}$ are singletons of a voters, and J_m is filled with l_k, l_{k+1} and all the remaining a voters. (In the case where $k = 1$ then $J_1, \dots, J_{m-1}$ are singletons of a voters, and J_m is filled with l_1, l_2 and all the remaining a voters.)
In σ, voters l_k and l_{k+1} are together in the same jurisdiction. We can thus state, using Local Faithfulness and the three lemmas :

$$\begin{array}{lrcl}
\text{At } \pi_{k-1}^{l_1 \dots l_{k-1}} : & g(\underbrace{b, \dots, b}_{k-1}, \underbrace{a, \dots, a}_{m-k}, f_m) & = & z \\
\text{At } \pi_{k}^{l_1 \dots l_{k}} : & g(\underbrace{b, \dots, b}_{k-1}, \underbrace{a, \dots, a}_{m-k}, f_m) & = & z \text{ by Lemma 2} \\
\text{At } \pi_{k+1}^{l_1 \dots l_{k+1}} : & g(\underbrace{b, \dots, b}_{k-1}, \underbrace{a, \dots, a}_{m-k} \bar{f}_m) & = & \bar{z} \text{ by Lemma 1}
\end{array}$$

As a consequence,

$$g(\underbrace{b, \dots, b}_{k-1}, \underbrace{a, \dots, a}_{m-k+1}) \neq g(\underbrace{b, \dots, b}_{k-1}, \underbrace{a, \dots, a}_{m-k}, b)$$

Consider now the partition σ' such that $J_1 = \{l_1\}, \dots, J_{k-1} = \{l_{k-1}\}$, $J_k = \{l_{k+1}\}$, the jurisdictions $J_{k+1}, \dots, J_{m-1}$ are singletons of a voters, and J_m is filled with l_k and all the remaining a voters. (In the case where $k = 1$ then $J_1 = \{l_2\}$, $J_2, \dots, J_{m-1}$ are singletons of a voters, and J_m is filled with l_1 and all the remaining a voters.)
In σ', voters l_k and l_{k+1} are separated in different jurisdictions. We can thus state, using Local Faithfulness and the three lemmas :

$$\begin{array}{lrcl}
\text{At } \pi_{k-1}^{l_1 \dots l_{k-1}} : & g(\underbrace{b, \dots, b}_{k-1}, \underbrace{a, \dots, a}_{m-k}, f'_m) & = & z \\
\text{At } \pi_{k}^{l_1 \dots l_{k}} : & g(\underbrace{b, \dots, b}_{k-1}, \underbrace{a, \dots, a}_{m-k}, \bar{f}'_m) & = & z \text{ by Lemma 3}
\end{array}$$

As a consequence,

$$g(\underbrace{b, \dots, b}_{k-1}, \underbrace{a, \dots, a}_{m-k+1}) = g(\underbrace{b, \dots, b}_{k}, \underbrace{a, \dots, a}_{m-k}, b)$$

which contradicts with the conclusion obtained for σ.

However, for $k \geq m$ the partition σ' cannot be constructed as there are only m jurisdictions in the federation. We need thus proceed differently.

Second step: Let us assume that $k, m \leq k \leq n-2$ and assumption A_k holds. The proof follows the same lines as that of the first step using different partitions.

Using partition σ such that $J_1 = \{l_1\}, \ldots, J_{m-2} = \{l_{m-2}\}$, $J_{m-1} = \{l_n\}$ and J_m is filled with $l_{m-1}, \ldots, l_k, l_{k+1}$ and all the remaining a voters, one can apply lemmas 1 and 2 as to show that

$$g(\underbrace{b, \ldots, b}_{m-2}, a, a) \neq g(\underbrace{b, \ldots, b}_{m-2}, a, b)$$

Considering next the partition σ' such that $J_1 = \{l_1\}, \ldots, J_{m-2} = \{l_{m-2}\}$, $J_{m-1} = \{l_{k+1}\}$, and J_m is filled with $l_{m-1}, l_m, \ldots, l_k$ and all the remaining a voters, one can apply lemma 3 as to obtain

$$g(\underbrace{b, \ldots, b}_{m-2}, a, a) = g(\underbrace{b, \ldots, b}_{m-2}, a, b)$$

$\blacksquare$

Before turning to the theorem, we need this last proposition:

Proposition 5 *Let $A = \{a, b\}$. If C satisfies Stability and Local Faithfulness and has at least one pivotal individual, then every individual is pivotal.*

Proof of Proposition 5 : Assume individual l_1 is pivotal in π_a. Thus $g(f(\sigma, \pi_0)) = z$ and $g(f(\sigma, \pi_1^{l_1})) = \bar{z}$. Pick any individual, say l_i, and assume he is not pivotal in π_a. Let σ be the partition such that $J_1 = \{l_1\}$, jurisdictions J_2 to J_{m-1} are filled with a single voter (different from l_i) and J_m is filled with individual l_i together with all the remaining voters. Using Local Faithfulness, we get $g(f(\sigma, \pi_0)) = g(a, a, \ldots, a, f_m) = z$ and $g(f(\sigma, \pi_1^{l_1})) = g(b, a, \ldots, a, f_m) = \bar{z}$.

Now consider partition σ' such that individuals l_1 and l_i are inverted : $J_1 = \{l_i\}$ and J_m is filled with individual l_1 and the remaining voters. Using Stability, $g(f(\sigma', \pi_0)) = z$ and by Local Faithfulness, $g(f(\sigma', \pi_0)) = g(a, a, \ldots, a, f'_m) = z$. Individual l_1 being pivotal, we still have $g(f(\sigma, \pi_1^{l_1})) = \bar{z}$, so necessarily, $g(a, a, \ldots, a, \bar{f'_m}) = \bar{z}$. To avoid a contradiction, it is necessary that $f_m \neq \bar{f'_m}$, so we can deduce that $f'_m = f_m$. Hence $g(f(\sigma', \pi_0) = z$.

Finally, consider the profile $\pi_1^{l_i}$ where individual l_i only has changed his vote. For the partition σ' we get $g(f(\sigma', \pi_1^{l_i})) = g(b, a, \ldots, a, f_m)$, and this, as we just have seen, is equal to $\bar{z}$. Therefore individual l_i is also pivotal in π_a. And this is true $\forall l_i$. $\blacksquare$

We can now state and prove the theorem 4.

Theorem 4 *When $A = \{a, b\}$, a federal constitution $C = (g, f_1, \ldots, f_m)$ satisfies Local Faithfulness and Stability if and only if C is either an LF Constant Constitution, a Unanimity Rule or a Parity Rule.*

Proof of Theorem 4:

- Implication $\Longleftarrow$

By definition, the LF Constant Constitution class is stable and satisfies Local Faithfulness. Unanimity Rule is a class of rules that all satisfy Local Faithfulness (as they satisfy Local Pareto). Moreover they also satisfy Stability.

Parity Rule is a class of rules that satisfies Local Faithfulness. Indeed, when a voter is alone, his parity is odd so his candidate wins in his jurisdiction. The class also satisfies Stability. To see this, simply notice that when a voter changes jurisdiction, his movement has two effects: it changes the parity of the jurisdiction he was in and it changes the parity of the jurisdiction he arrives in. Thus two jurisdictions change votes and therefore the parity at the federal level does not change. The rule is thus stable.

- Implication $\Longrightarrow$

Assume C, a federal constitution, satisfies Local Faithfulness and Stability. Proposition 4 can then be applied, and hence assumption A_k cannot hold for any $k, 1 \leq k \leq n-2$. Let us analyse what the negation of Assumption A_k implies.

- Either $g(f(\sigma, \pi_0)) = z$ and $\exists\, l_1$ such that $g(f(\sigma, \pi_1^{l_1})) = \bar{z}$. Then individual l_1 is pivotal in π_a.
- Or $g(f(\sigma, \pi_0)) = g(f(\sigma, \pi_1^{l_1})) = z$ for any l_1. Then necessarily $g(f(\sigma, \pi_2^{l_1 l_2})) = z$ for any sequence l_1, l_2,..., $g(f(\sigma, \pi_{n-1}^{l_1 \dots l_{n-1}})) = z$ for any sequence $l_1, \dots l_{n-1}$. There are then two subcases. Either $g(f(\sigma, \pi_n)) = \bar{z}$ and then necessarily individual l_n is pivotal in π_b. Or $g(f(\sigma, \pi_n)) = z$ and then C is a Constant Constitution satisfying Local Faithfulness.

Thus either C is an LF Constant Constitution or in C, there is at least one pivotal individual, which in turn implies, according to proposition 5, that every individual is pivotal.

Notice that the reasoning we just held can also be made when starting from the profile π_n where all individuals vote for b, and going towards π_0. We thus have the following: if for any sequence $l_1 \dots l_{n-1}$, $g(f(\sigma, \pi_n)) = g(f(\sigma, \pi_{n-1}^{l_1 \dots l_{n-1}})) = z$, then $g(f(\sigma, \pi_{n-2}^{l_1, \dots l_{n-2}})) = \dots = g(f(\sigma, \pi_1^{l_1})) = z$.

We next proceed to the determination of the set of all the rules such that every individual is pivotal. Before doing so, however, it is usefull to give the lines of what follows. We know that if no individual is pivotal, we are left with the Constant Constitution. Indeed, the difficult case arises when all individuals are pivotal. We are thus confronted with two possible cases when considering all possible vote profiles from π_0 to π_n: either the result of the election changes only once in that sequence, or it changes more than once. In the first case we will be left with the Unanimity Rules, while in the second, only the Parity Rules will remain. All cases are considered below, however, before analysing what possible rules are generated, we point out the following property:

If there exists a pair of individuals l_i, l_h such that $g(f(\sigma, \pi_0)) = z$, $g(f(\sigma, \pi_1^{l_i})) = \bar{z}$ and $g(f(\sigma, \pi_2^{l_i l_h})) = \bar{z}$, then we have $g(f(\sigma, \pi_2^{l_k l_l})) = \bar{z}\ \forall l_k, l_l \in N$.
To see this, assume $g(f(\sigma, \pi_0)) = z$ and $g(f(\sigma, \pi_1^{l_i})) = \bar{z}\ \forall l_i \in N$. Assume furthermore that there exists two individuals l_i and l_h such that $g(f(\sigma, \pi_2^{l_i l_h})) = \bar{z}$. Consider then the partition σ such that $J_1 = \{l_i\}$ and individuals l_h and l_k are together in jurisdiction J_2. We get:

$$g(f(\sigma, \pi_0)) = g(a, f_2, f_3, \ldots, f_m) = z$$

$$g(f(\sigma, \pi_1^{l_i})) = g(b, f_2, f_3, \ldots, f_m) = \bar{z}$$

As individual l_i is pivotal, then l_h and l_k are as well, according to proposition 5. This implies that l_h and l_k change the outcome of their jurisdiction when they switch votes.

$$g(f(\sigma, \pi_1^{l_h})) = g(a, \bar{f}_2, f_3, \ldots, f_m) = \bar{z}$$

$$g(f(\sigma, \pi_1^{l_k})) = g(a, \bar{f}_2, f_3, \ldots, f_m) = \bar{z}$$

From that we can derive that $g(f(\sigma, \pi_2^{l_i l_h})) = g(b, \bar{f}_2, f_3, \ldots, f_m) = \bar{z}$.
But we also have $g(f(\sigma, \pi_2^{l_i l_k})) = g(b, \bar{f}_2, f_3, \ldots, f_m)$ and thus $g(f(\sigma, \pi_2^{l_i l_k})) = \bar{z}$.

Summarizing, if $g(f(\sigma, \pi_2^{l_i l_h})) = \bar{z}$ then $g(f(\sigma, \pi_2^{l_i l_k})) = \bar{z}$ for any l_k. Hence if there is one common individual in both sequences, as it is the case here with l_i, then the outcome in π_2 should be $\bar{z}$. Therefore, we can derive that if $g(f(\sigma, \pi_2^{l_i l_k})) = \bar{z}$, then $g(f(\sigma, \pi_2^{l_k l_l})) = \bar{z}$, as l_k is now a common individual in both sequences. Hence, if there is one pair of individuals such that $g(f(\sigma, \pi_2^{l_i l_h})) = \bar{z}$, then $g(f(\sigma, \pi_2^{l_k l_l})) = \bar{z}$ for any pair of individuals l_k, l_l. This fact will help us in what follows, as we now analyse the possible rules that are generated by the fact that every individual is pivotal.

Case 1: assume $g(f(\sigma, \pi_0)) = g(f(\sigma, \pi_1^{l_i})) = z\ \forall l_i \in N$. Then according to the negation of assumption A_k we have $g(f(\sigma, \pi_k^{l_1 \ldots l_k})) = z$, for any sequence of k individuals, $1 \leq k \leq n-1$.

- Case 1a: If $g(f(\sigma, \pi_n)) = z$ then C is an LF Constant Constitution.
- Case 1b: If $g(f(\sigma, \pi_n)) = \bar{z}$ then the outcome of the election is z for any vote profile except in π_n where the outcome is $\bar{z}$.

Case 2: assume $g(f(\sigma, \pi_0)) = z$ and $g(f(\sigma, \pi_1^{l_i})) = \bar{z}\ \forall l_i \in N$.

- Case 2a: If $\exists\, l_h$ such that $g(f(\sigma, \pi_2^{l_i l_h})) = \bar{z}$ then according to what has been said just before, we know that $g(f(\sigma, \pi_2^{l_k l_l})) = \bar{z}\ \forall\, l_k, l_l \in N$. We can then apply the negation of A_k to derive $g(f(\sigma, \pi_k^{l_1 \ldots l_k})) = \bar{z}$, $\ldots$, $g(f(\sigma, \pi_{n-1}^{l_1 \ldots l_{n-1}})) = \bar{z}$.
  - Case 2aa: If $g(f(\sigma, \pi_n)) = \bar{z}$ then the outcome of the election is $\bar{z}$ for any vote profile except in π_0 where the outcome is z. This case is symmetric to case 1b.

– Case 2ab: assume $g(f(\sigma, \pi_n)) = z$. Then the outcome is z for profiles π_0 and π_n and it is $\bar{z}$ for any other profile. We show here that this case cannot occur as it leads to a contradiction. Consider the partition σ such that $J_1 = \{l_1\}$, $J_2 = \{l_2\}$, $\ldots$, $J_{m-2} = \{l_{m-2}\}$, $J_{m-1} = \{l_n\}$ and jurisdiction J_m is filled with all the $n - m + 1$ remaining individuals $l_{m-1}, \ldots, l_{n-1}$. Using Local Faithfulness, we have: $g(f(\sigma, \pi_0)) = g(a, \ldots, a, f_m) = z$. As every individual is pivotal, we have $g(f(\sigma, \pi_1^{l_{m-1}})) = \bar{z}$ so necessarily, $g(f(\sigma, \pi_1^{l_{m-1}})) = g(a, \ldots, a, \bar{f}_m)$. As $g(f(\sigma, \pi_k^{l_1 \ldots l_k})) = \bar{z}$ for any $k, 1 \leq k \leq n - 1$, in particular we have $g(f(\sigma, \pi_{n-m+1}^{l_{m-1} \ldots l_{n-1}})) = \bar{z}$ so necessarily $g(f(\sigma, \pi_{n-m+1}^{l_{m-1} \ldots l_{n-1}})) = g(a, \ldots, a, \bar{f}_m)$. Hence, when all individuals in jurisdiction J_m change their vote to b, the jurisdiction votes for $\bar{f}_m$. This implies that for π_n we get $g(f(\sigma, \pi_n)) = g(b, \ldots, b, \bar{f}_m)$. Let us step back to the profile $\pi_m^{l_1, \ldots l_{m-2} l_n l_{m-1}}$ where every individual that is alone in his jurisdiction has changed his vote and where only one individual in jurisdiction J_m has changed his. For that profile, the outcome of the election is by hypothesis $\bar{z}$. We then have, using Local Faithfulness, $g(f(\sigma, \pi_m^{l_1, \ldots l_{m-2} l_n l_{m-1}})) = g(b, \ldots, b, \bar{f}_m) = \bar{z}$, so $g(f(\sigma, \pi_n)) = g(b, \ldots, b, \bar{f}_m) = g(f(\sigma, \pi_m^{l_1, \ldots l_{m-2} l_n l_{m-1}})) = \bar{z}$ and this is a contradiction. Hence there is no such case as 2ab.

- Case 2b: For every individual l_h we have $g(f(\sigma, \pi_2^{l_i l_h})) = z$

We thus have four cases: case 1a generates the LF Constant Constitutions. We next show that cases 1b and 2aa generate all the Unanimity Rules while case 2b generates all Parity Rules.

Cases 1b and 2aa generate all Unanimity Rules

The two cases are symmetric. We will thus show that case 2aa generates Unanimity Rules (a, a) and (a, b). The same reasoning can then be led to show that case 1b generates Unanimity Rules (b, a) and (b, b).

Let the partition σ be such that the $m - 1$ first jurisdictions are singletons and the last one is filled with all the remaining voters. Then in π_0, using Local Faithfulness, we get $g(f(\sigma, \pi_0)) = g(a, \ldots, a, f_m) = z$. We must show that necessarily $f_m = a$. Then it will follow that the only configuration such that z wins the election is given by the federal profile $\Pi = a, \ldots, a$ that can only be obtained by the vote profile π_0. All the other vote profiles make $\bar{z}$ the winner and therefore, according to the value of z, we will get either Unanimity Rule (a, a) or Unanimity Rule (a, b).

Assume $f_m = b$. Then $g(f(\sigma, \pi_0)) = g(a, \ldots, a, b) = z$ and $g(f(\sigma, \pi_k^{l_1 \ldots l_k})) = \bar{z}$ for any k, $1 \leq k \leq n$. Therefore, we have $g(f(\sigma, \pi_1^{l_m})) = \bar{z} = g(a, \ldots, a, \bar{f}_m) = g(a, \ldots, a, a)$. Let all the voters that are alone change one after the other their vote. This leads us to $g(b, a, \ldots, a, a) = g(b, b, a, \ldots, a, a) = \ldots = g(b, \ldots, b, a) = \bar{z}$. Hence, any combination such that jurisdiction J_m votes for a gives $\bar{z}$ as the winner.

Consider now another partition σ' such that jurisdiction J_m is a singleton. Then in π_0, necessarily jurisdiction J_m chooses a. But we have just seen that all the combinations such that J_m chooses a give $\bar{z}$ as the winner, so this implies $g(f(\sigma', \pi_0)) = \bar{z}$. This is thus a contradiction with Stability; the cause of this contradiction is the assumption that $f_m = b$. Hence $f_m = a$.

Case 2b generates all Parity Rules

Assume without loss of generality that n, the number of electors is odd and we have $g(f(\sigma, \pi_0)) = z$, $g(f(\sigma, \pi_1^{l_i})) = \bar{z}\ \forall i \in N$ and $g(f(\sigma, \pi_2^{l_i l_h})) = z\ \forall i, h \in N$. Consider the partition σ such that the $m-1$ first jurisdictions are singletons and the m-th jurisdiction is filled with the $n-m+1$ remaining voters. Using Local Faithfulness we get $g(f(\sigma, \pi_0)) = g(a, \ldots, a, f_m) = z$. In π_1, we can change any individual's vote as to reach

$$g(f(\sigma, \pi_1^{l_i})) = g(b, a, ..a, f_m) = g(a, ..a, b, a, ..a, f_m) = g(a, ..a, b, f_m) = g(a, ..a, \bar{f_m}) = \bar{z}$$

Again, in π_2 the outcome is z for any pair l_i, l_h so

$$g(f(\sigma, \pi_2^{l_i l_h})) = g(b, b, a, \ldots, a, f_m) = g(a, ..a, b, a, ..a, b, a, ..a, f_m) = g(a, \ldots, a, b, b, f_m) = z$$

Necessarily, if two individuals in jurisdiction m change their vote, then by Stability, the result in J_m will be f_m, so

$$g(f(\sigma, \pi_2^{l_i l_h})) = g(a, \ldots, a, f_m) = z$$

Consider then the profile $\pi_3^{l_i, l_h, l_k}$. As $g(f(\sigma, \pi_2^{l_i l_h})) = g(a, \ldots, a, f_m)$, if individual l_k is in one of the $m-1$ first jurisdictions, then we get

$$g(f(\sigma, \pi_3^{l_i, l_h, l_k})) = g(b, a, \ldots, a, f_m) = \bar{z}$$

Hence in π_3 the outcome is $\bar{z}$, like we had in π_1. Changing the vote of every individual in turn we get

$$g(f(\sigma, \pi_3^{l_i, l_h, l_k})) = g(b, b, \ldots, a, \bar{f_m}) = g(b, b, b, a, \ldots, a, f_m)$$

and all these give $\bar{z}$ as the winner, by Stability.

Next we reach π_4 for which the outcome is again changed to z etc...

As to completely characterize the rule we have just described, it remains determining the value of f_m. Assume $f_m = a$. We notice that when the number of b's present at the aggregate level is even, the outcome is z, whereas an odd number of b's yields $\bar{z}$ as the winner[5]. Therefore the rule at the aggregate level is the Parity Rule. In the same way, we notice that in jurisdiction J_m the outcome depends on the parity of b's

[5]We focus on the parity of b's as a matter of illustration. Of course the same remark can be made with respect to the parity of a's at the aggregate level

(resp. of a's) in the vote profile of that jurisdiction. Therefore the voting rule in J_m is the Parity Rule. Finally, it is possible to lead the same reasoning when inverting the jurisdictions so that the rule in every jurisdiction is the Parity Rule. The same can be said if $f_m = b$ and all the cases we face generate the eight Parity Rules. ■

Theorem 4 says roughly the same as theorem 3. Its contribution, apart from yielding a strange set of rules, is to show that the non-democratic feature of theorem 1 is due to the stability condition rather than to the Local Pareto one, as relaxing it into Local Faithfulness does not help achieving more acceptable rules.

Remark 3 *As for theorem 3 we notice that, if C is not in the class of the Constant Constitutions, all the rules characterized imply that all the jurisdiction rules are identical, $f_j = f_{j'}$.*

5 Comments and Conclusion

5.1 Dictatorship

As a surprising feature, one can notice that dictatorship is not in the set of stable rules for constitutions satisfying either Local Pareto or the weaker Local Faithfulness condition. This is all the most surprising as dictatorship is usually the rule one obtains in the more classical impossibility results, such as Arrow's and Gibbard and Satterthwaite's theorems.

The reason is the following: in a federal constitution, the description of the voting procedure has to include a definition of the jurisdictional rules as well as the federal voting rule. Hence, if individual i is a dictator, it can be detected in the jurisdiction in which i votes. However, at the federal level, there is no information remaining about the identity of the voters, as the rule is based on the results in all jurisdictions.

The most natural way to define dictatorship would be to state that the jurisdiction in which i votes will elect according to i's choice, while all the other jurisdictions elect the same candidate as the one elected in i's jurisdiction. However, this implies that the result of the other jurisdictions changes whenever individual i changes his choice, a violation of Jurisdiction Sovereignty.

There is, however, a second way to define dictatorship in a federal constitution satisfying Jurisdiction Sovereignty, which is rather complicated. It is the following: all jurisdictions elect the same candidate a_j, thus independently from individuals' choices, except that one containing individual i, which elects the candidate chosen by individual i. At the federal level, if all jurisdictions have elected candidate a_j, then a_j is the winner. If one jurisdiction has voted for candidate $a_k, a_k \neq a_j$, then the winner is a_k. This definition gives to individual i the power of a dictator.

However, this definition violates Local Faithfulness (and thus Local Pareto). It then appears that dictatorship is excluded from our framework, either by the imposition of

Jurisdiction Sovereignty, or by that of Local Faithfulness.

5.2 A comprehensive comparison of recent results

For a long time, the works of Murakami (1966) Fishburn (1971) and Fine (1972) were the only contributions on the axiomatic analysis of two step voting rules. Clearly, the 2000 US presidential elections, as well as the debates about the European constitutions that sprung out after the Nice treaty have revived the analysis of two step voting rules in social choice theory. In this section we put our results into perspective by mentioning recent contributions on the same issue.

Laffond and Lainé (1999, 2000) started to analyze this issue in a series of published or unpublished papers since 1999, by focusing on its relationship with the Ostrogorski's Paradox (see Nurmi (1999)) and the use of tournament solutions. Their model assumes that voters have strict preference over a set X of alternatives, of cardinality greater than 3. The number of voters in the society is odd, as well as the number of jurisdictions. Moreover, each jurisdiction has the same odd number of voters. Tournament solutions can then be used in a two step procedure: the preferences of the voters in a jurisdiction can be aggregated into a tournament via the majority rule, and the m tournaments obtained in this way are aggregated into a federal tournament, which solution will select a federal winner. The question is then to know whether a tournament solution exists such that its implementation to the preference profile of n voters coincides with the two step procedure described above. Lainé and Laffond clearly show that direct and representative democracy may lead to mutually inconsistent decisions when the society uses the majority rule and tournament solutions. The strength of their results comes from their ability to get an impossibility result with equal size constituencies, while we need single voter jurisdictions to build our proofs.

Chambers (2003) reaches some impossibility results similar to ours although the framework considered therein is quite different[6]. Two main assumptions, departing from our study, are made in that contribution. First, the jurisdictional voting rules are all identical, i.e. $f_j = f \ \forall j$, and the federal voting rule is also the same, i.e. $g = f$. This assumption therefore restricts the set of rules one can look for. As another consequence of this assumption, it becomes impossible to impose different conditions on the rules f_j and g. For instance, whereas Local Pareto only applies to the jurisdictional rules in our case, the same Pareto condition (which Chambers names Unanimity) applies on both levels. Similarly, Anonymity is imposed on f from the start. Second, Chambers assumes that the rule f should be defined for any finite number of voters. Thus the voting rule used is defined for a variable population size, hence for any number of jurisdictions. The results might not hold in the case of a fixed number n of individuals or for a

[6]We first heard about Chambers' work in March 2005, although they existed since 2003, while we reached our main results in April 2004. This explains why, although the results are similar, both approach model similar issues in completely different manners.

fixed number m of jurisdictions. In this framework, the non manipulability condition imposed is "Representative Consistency", stating that the indirect voting procedure should yield the same result as the direct voting rule, i.e. $f(f(\pi), \ldots, f(\pi)) = f(\pi)$. As the number of jurisdictions is not fixed, the outcome should not change whether the population is split in 2, 10 or any finite number of jurisdictions. Chambers reaches characterization results for any finite number of alternatives, with a general theorem stating that Anonymity, Representative Consistency and Unanimity together define the class of Priority Rules, which coincides with the union of our Unanimity Rules and Constant Constitutions in the case $A = \{a, b\}$.

The work by Perote-Peña uses a similar framework: the f_j's are identical to g, the number of jurisdiction may vary (though the total number of voters is fixed), the anonymity condition is needed together with a gerrymander-proof assumption (here called independence of institution formation, IIF) and the Pareto-Unanimity condition. The main contribution is that Perote-Peña assumes that the voters can express their whole preferences (represented by a weak ordering or a linear ordering) during the voting process. He proves that IFF, Pareto and Neutrality are incompatible with Anonymity. However, when droping Anonymity, he shows the existence of some rules, the characterization of which still awaits.

5.3 Concluding remarks

This paper was devoted to the study of manipulation by the movement of voters in federal constitutions. It appears that the only voting procedure that is non manipulable when imposing a local paretian condition and when requiring in addition that individuals should not have the power of overruling unanimous profiles, is the class of Constant Constitutions. This class of rules is such that the winner of the election is totally independent from the votes of the electors. This result is thus to be taken as an impossibility one.

When restricting the analysis to the case of two candidates, and when assuming that the result of the election should depend on the individuals' votes, the set of Unanimity Rules is characterized by the stability property when the local paretian condition is imposed. When relaxing this mild condition into a milder condition of Local Faithfulness, according to which any jurisdiction formed of a singleton should elect the candidate chosen by the lonely voter, another class of rules appears, that of Parity Rules. These rules designate the winner of the election according to the parity of the number of votes this candidate or his opponent receives.

All these negative results, together with ours, show that the existence of gerrymandering by the election designers or of manipulating by the electors, in two step electoral systems, is a robust fact. Whatever the framework, all conclusions go in the same direction. As a consequence, it seems that the noble principle defended by the two European deputies, according to which every European citizen should have the right to vote in any country he is at the time of the European election, would generate undesirable manipulations. Unless the voting procedure is direct.

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