Fairness under Uncertainty with Indivisibilities

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Abstract

I analyze an economy with uncertainty in which a set of indivisible objects and a certain amount of money is to be distributed among agents. The set of intertemporally fair social choice functions based on envy-freeness and Pareto efficiency is characterized. I give a necessary and sufficient condition for its non-emptiness and propose a mechanism that implements the set of intertemporally fair allocations in Bayes-Nash equilibrium. Implementation at the ex ante stage is considered, too. I also generalize the existence result obtained with envy-freeness using a broader fairness concept, introducing the aspiration function.

Keywords: aspiration function, envy-free social choice function, fairness, implementation, indivisible goods, uncertainty

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1 Introduction

Envy-free allocations are allocations for which every agent prefers his own bundle to the ones assigned to other agents. In the economies I deal with, a set of indivisible objects is to be distributed among a group of agents such that individuals consume at most one

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object. In general envy-free allocations might not exist, but when a proper amount of a perfectly-divisible good, typically money, is available in the economy, the set of envy-free allocations is not empty and indeed can be quite large. Alkan, Demange and Gale (1991) and Aragonés (1995) study these economies, the existence of envy-free allocations and how the amount of the divisible good affects the existence results. It is shown that for a sufficiently large amount of money the set of envy-free allocations is not empty. When negative distribution of money is not allowed there exists a minimum level of money that guarantees non-emptiness and implies a unique way to combine objects and money such that these bundles give rise to an envy-free allocation. Based on this result refinements can be defined that reduce the size of the set down to a singleton. It is the case of the so-called Rawlsian solutions proposed by Alkan, Demange and Gale (1991) and the one in Aragonés (1995) that in fact coincide. It is well known that in this environment envy-freeness implies Pareto efficiency and therefore envy-free allocations can be considered as fair ones, too. Also, some nice features of the envy-free set are proper to the indivisible case, as for example its lattice structure.

There are many situations in which this type of models can be useful. Nevertheless there are numerous cases that can not be handled due to the presence of uncertainty. The present work deals with the study of the latter. I study economies with any number of objects and individuals participating in the distribution as long as there are at least as many agents as objects. I review the above mentioned results taking into account economies in which information is not complete in some timing stages. A distinction is made among ex ante, interim and ex post stages; and according to that different envy-free, efficiency and fairness notions are defined. The intersection between the sets of ex ante Pareto optimal and ex post envy-free is particularly interesting and will be called ex ante intertemporally fair. Moulin (1997) pointed out that fairness from an ex ante point of view can be seen as a concept of procedural justice. It is a characteristic of the mechanism or game form itself and is independent of the way the game is played by the agents in the future. Ex post fairness can be interpreted as endstate justice that deals with a particular utility or judgement profile and a particular endstate in a given state of the nature. I take into account both judgement concepts, since my model considers the most restrictive definition for fairness that allows both for ex ante and ex post justice.

I keep the assumption that individuals can consume at most one indivisible object in a given state, but do not constrain the amount of money that can appear in consumption bundles. The former is clearly a restriction, nevertheless my model can account for numerous real-life problems and is accepted as a common assumption in the discrete literature. Two of the classical real-life situations covered by my framework appear in the following examples.

Example 1 *A group of students decides to share a flat. There are as many rooms in the flat as students and it is agreed upon that everyone will have a private room and noone leaves or enters the group later. Rooms are of different characteristics (size, quietness, etc.) and the students have their own private valuations over them. These valuations might be unknown at the time they enter in the flat as they might have never lived in a similar situation before. The problem to be solved - before (ex ante), immediately after (interim) or after (ex post) entering the flat - is to assign a room to every students and decide the share of the rent each of them must pay in a fair way.*

Example 2 *A number of villages has to participate in a flood-protection project. There is a certain number of tasks to be executed and some amount of money available for the realization. Tasks are of different characteristics and the villages are supposed to have their own valuations over them. For example, some tasks might be or might look easier to carry out for a village than for the others, etc. Nevertheless, these villages might have never participated in a similar cooperation and therefore could have some uncertainty when evaluating future possibilities, e.g. for example the ones of success or failure in their tasks. The problem to be solved by the central authority is to assign a task to every village and with it a share of the fixed budget in a possibly fair way.*

As is well known, competitive equilibrium theory runs into difficulties when considering indivisibilities. However, there exists a special case that has been studied in the literature that is tractable. Here that framework with several indivisible objects and a perfectly divisible one (usually thought of as money) is adopted. I prove that ex ante intertemporally fair social choice functions exist whenever certain condition on prior beliefs and preferences holds. Beside of the constructive nature of the proof, the importance of that condition is shown in two simple numeric examples and through identifying an intuitive special case of the condition: If agents have the same prior beliefs and preferences show constant marginal utility of money among agents for a given state, then the set of ex ante intertemporally fair state-contingent allocations is not empty. Some fairness results under uncertainty without indivisibilities are discussed in Gajdos and Tallon (2001). They prove existence in the adopted perfectly divisible framework. I obtain similar results to those in Gajdos and Tallon (2001) according to which the existence of intertemporally fair allocations depends on agents' prior beliefs, for a given state they must be the same for every agent. In contrast to Gajdos and Tallon (2001), here utilities are state dependent, and the necessary and sufficient condition for existence is slightly less restrictive.

After considering the problem of existence I proceed to implementation matters. The literature under certainty offers the characterization of fair allocations, and gives methods to find them once the social planner (or some central government) learns the individuals' preferences. A constructive and very elegant way to find them is presented in Su (1999) that is based on a simple combinatorial lemma due to Sperner in 1928. The leading example in Su (1999) is flat sharing as in Example 1. This paper studies the case in which agents behave strategically, the social planner is not informed about the preferences, and players are not completely informed either. Relying on results in Palfrey and Srivastava (1987), for Bayesian implementability the condition of non-exclusive information is introduced and a mechanism that implements the set of non-wasteful ex post envy-free social choice functions in Bayes-Nash equilibrium is defined. A subsection deals with the problem of implementation at the ex ante stage in which information is symmetric. Due to this fact I conclude that simple mechanisms of the "divide-and-permute" or "cake-cutting" type can be used to implement the set of ex ante intertemporally social choice functions. This result is presented beside of the ones in McAfee (1992) according to which the cake-cutting mechanism produces efficient results under symmetric information, but under asymmetric information it is ex post inefficient in an unusual way.

The fair-division literature has already examined the above implementation problem under certainty with indivisibilities and two players. Crawford and Heller (1979), for example, showed that a modified version of the divide-and-choose mechanism performs well in the adopted set-up.

The present paper is organized as follows: Section 2 introduces the formal model and defines the basic concepts of fairness that are studied, while Section 3 deals with the question of existence. A subsection presents a generalization of the Rawlsian refinement proposed by Alkan, Demange and Gale (1991) and Aragonés (1995). On Section 4 I discuss implementation matters.

Considering the first chapters of this work, it is a self-contained study based on the axiomatically accepted notion of intertemporal fairness that embodies envy-freeness. The literature on distributive justice usually follows a similar path and does not deal with the problematic of choosing fairness criteria. However, an extra section (Section 5) is included that considers the aspiration function as an appropriate tool for studying fairness without restricting attention on a particular concept. Corchón and Iturbe-Ormaetxe (2001) offers a detailed study of fairness in a generalized set-up. Section 5 here can be seen as the adaptation of some very few definitions from Corchón and Iturbe-Ormaetxe (2001) to the uncertainty case with indivisibilities. The most important point in that part of the paper is the generalization of the existence result. I find that under the conditions stated for the envy-free case, and under some restrictions on personal aspirations, an intertemporally fair social choice function exists. Intertemporally fair is now a broader concept that allows among others for envy-free and for the egalitarian solutions. The condition is sufficient and necessary here, too. At the end of the paper I study the problem of implementation of the set of the generalized intertemporally fair social choice functions. I conclude with a positive result: a condition (on the fairness concept) that is necessary for Bayesian monotonicity, i.e. for Bayesian implementation is derived.

2 The model

Let N be a finite set of agents, O a finite set of indivisible objects and S a finite set of possible states of nature. The typical elements of the sets are i, o and s respectively. For simplicity I shall denote the cardinality of the sets by the same symbols N, O and S. There is also a perfectly divisible good in the economy called money, the total available amount of which is $M \in \mathbb{R}$. Each agent can consume at most one indivisible object and any amount of money.

For simplicity and presentational considerations assume that the set of agents and the set of objects have the same cardinality. This is a standard assumption in the literature. If there are at least as many agents as objects one can achieve this situation by introducing worthless null objects. Hence, the analysis holds for any economy with more agents than objects, too. The reverse case in which there are more objects than agents is tractable, too. Alkan, Demange and Gale (1991) present an argument in a set-up without uncertainty. It requires fictitious agents that only value money, and also different definitions for efficiency, envy-freeness, etc. To keep things simple this case is not considered here.

In the economy there is uncertainty concerning the state of the nature. As for timing, I distinguish three stages: In the ex ante stage information is symmetric, but agents have to cope with uncertainty as they do not know which state of nature will occur. The interim stage is the one in which a given state of nature has already occurred, but agents cannot observe it perfectly and may own private information. This informational asymmetry is dissolved at the next, ex post stage when agents are completely informed about the state. There is no aggregate risk in my model, as the list of objects and the amount of money if fixed across all states of the world.

In general, information available to agent i is given by a partition Π^i of the set S, where the event in partition Π^i that contains the state s is denoted by $E^i(s)$. From an ex ante point of view a prior probability distribution can be defined over states, that for agent i and state s will be denoted by $q^i(s) > 0$. I shall assume that the set S does not contain any redundant elements, that is $\cap_i E^i(s) = \{s\}$ for all $s \in S$.

Allocations in this economy will be represented by vectors in $A = (O \times \mathbb{R})^N$. Let $A^i = O \times \mathbb{R}$ denote player i's set of allocation. Now the set of allocations can be expressed as a Cartesian product $A = A^1 \times \ldots \times A^N$. For example, an allocation is given by

$$a = \left(a^1, ..., a^N\right) = \left[\left(o^1, m^1\right), ..., \left(o^N, m^N\right)\right]$$

where a^i stands for the bundle that agent i consumes in which she receives object o^i that may be as well the empty set. The amount of money that agent i enjoys in the given allocation is m^i. The set of feasible allocations is defined as

$$A^f = \left\{ a \in A : o^i \neq o^j \text{ for any } i \neq j \text{ with } o^i \neq \varnothing,\, o^j \neq \varnothing, \text{ and } \sum_{i=1}^{N} m^i \leq M \right\}.$$

An allocation will be called non-wasteful if every agent has an object and the money shares sum up to M. Formally,

$$A^{fnw} = \left\{ a \in A^f : \bigcup_{i=1}^{N} \left\{ o^i \right\} = O, \text{ and } \sum_{i=1}^{N} m^i = M \right\}.$$

Non-wasteful allocations are those feasible ones in which every object finds an owner and the money shares sum up to the total available amount, M.

Let $X = \left\{ x : S \longrightarrow A^f \right\}$ be the set of feasible state-contingent allocations, that also will be referred to as social choice functions.[1] A social choice set is a subset $F \subset X$. The sets of non-wasteful social choice functions (X^{fnw}) and social choice sets (F^{fnw}) are defined in an similar way. Agents' preferences are represented by state-dependent utility functions, $u^i\left(x^i\left(s\right), s\right)$. I shall suppose preferences are quasi-linear in money. If $\phi^i\left(s\right)$ represents agent i's marginal utility of money in state s then the utility function can be written as

$$u^i\left(x^i\left(s\right), s\right) = u^i\left[o^{xi}\left(s\right), m^{xi}\left(s\right), s\right] = u^i\left[o^{xi}\left(s\right), s\right] + \phi^i\left(s\right) \cdot m^{xi}\left(s\right)$$

with $\phi^i\left(s\right) > 0$ finite for all i and s, where $o^{xi}\left(s\right)$ denotes the indivisible object and $m^{xi}\left(s\right)$ the money that agent i consumes in state s according to the social choice function x. Assume that no indivisible object is infinitely desirable or undesirable as compared to money. That is $u^i\left(o^{xi}\left(s\right), s\right)$ is bounded for every i and s. An economy is represented by a list

$$\mathcal{E} = \left(N, O, S, M, \left[q^i\left(s\right)\right]_{i\in N}, \left[u^i\left(s\right)\right]_{i\in N}\right).$$

More notation is introduced in the text when needed.

Before continuing with fairness concepts, recall Example 1 and identify the ex ante, interim and ex post stages. Now states of nature can be defined as utility profiles taking into account how students value the different rooms available in the flat and, in comparison with them, money. Before moving into the new flat, i.e. ex ante, students are not supposed to know how they value the rooms, because they have never lived in the flat before nor

[1] Note that this definition differs from the one generally used in the social choice literature, since now the social choice function is defined for a given economy, precisely over the set of possible states in that economy. However our definition is a standard one in the Bayesian implementation literature.

have any information about their characteristics. They deal with uncertainty of the same type in a symmetric way. At the interim stage, when they arrive and can have a first look around they are able to observe the characteristics of the flat, therefore can tell how they personally value the rooms. Nevertheless, they can not identify the state of the nature, since private valuations may not be announced truthfully or observable. Therefore, students in this stage have to cope with uncertainty in an asymmetric way. Uncertainty disappears at the ex post stage in which, after some time of living together, students know how their flat-mates think about the flat and value its rooms.

2.1 Fairness concepts

In order to analyze fairness in this model I introduce some useful concepts by the following definitions. They consider widely used fairness notions from the literature and continue with the distinction among the three timing stages.

Definition 1 *A non-wasteful social choice function x is ex post Pareto optimal if there is no non-wasteful social choice function y such that*

$$u^i\left[y^i(s), s\right] \geq u^i\left[x^i(s), s\right]$$

for all i in N and all s in S, and with strict inequality for at least one i and one s. Let P_p denote the set of ex post Pareto optimal social choice functions.

Definition 2 *A non-wasteful social choice function x is interim Pareto optimal if there is no non-wasteful social choice function y such that*

$$\sum_{s \in E^i(s^*)} q^i\left[s \mid E^i(s^*)\right] \cdot u^i\left[y^i(s), s\right] \geq \sum_{s \in E^i(s^*)} q^i\left[s \mid E^i(s^*)\right] \cdot u^i\left[x^i(s), s\right]$$

for all i in N and s^* in S, with strict inequality for at least one i and one s^*. Let P_i denote the set of interim Pareto optimal social choice functions.[2]

Definition 3 *A non-wasteful social choice function x is ex ante Pareto optimal if there is no non-wasteful social choice function y such that*

$$\sum_{s \in S} q^i(s) \cdot u^i\left[y^i(s), s\right] \geq \sum_{s \in S} q^i(s) \cdot u^i\left[x^i(s), s\right]$$

for all i in N and with strict inequality for at least one i. Let P_a denote the set of ex ante Pareto optimal social choice functions.

[2] $q^i\left[s \mid E^i(s^*)\right]$ is the probability that agent i assigns to state s conditional on the information she owns after that state s^* has occured.

Non-wastefulness has been included in the definitions for Pareto efficiency, but it is not a requirement for envy-freeness.

Definition 4 *A social choice function x is ex post envy-free if*

$$u^i\left[x^i(s), s\right] \geq u^i\left[x^j(s), s\right]$$

for all i, j in N and s in S. Let EF_p denote the set of ex post envy-free social choice functions.

Definition 5 *A social choice function x is interim envy-free if*

$$\sum_{s \in E^i(s^*)} q^i\left[s \mid E^i(s^*)\right] \cdot u^i\left[x^i(s), s\right] \geq \sum_{s \in E^i(s^*)} q^i\left[s \mid E^i(s^*)\right] \cdot u^i\left[x^j(s), s\right]$$

for all i, j in N and s^* in S. Let EF_i denote the set of interim envy-free social choice functions.

Definition 6 *A social choice function x is ex ante envy-free if*

$$\sum_{s \in S} q^i(s) \cdot u^i\left[x^i(s), s\right] \geq \sum_{s \in S} q^i(s) \cdot u^i\left[x^j(s), s\right]$$

for all i and j in N. Let EF_a denote the set of ex ante envy-free social choice functions.

The literature offers some general results on the structure of these sets. They are summarized in the following propositions.

Proposition 1 *If a social choice function is ex ante Pareto efficient, then it is also interim Pareto efficient. If a social choice function is interim Pareto efficient, then it is also ex post Pareto efficient. That is $P_a \subset P_i \subset P_p$.*

Proof. See Laffont (1981), Mas-Colell, Whinston and Greene (1995) and also check Holmstrom and Myerson (1983). ■

Proposition 2 *If a social choice function is ex post envy-free, then it is also interim envy-free. If a social choice function is interim envy-free, then it is also ex ante envy-free. That is $EF_a \supset EF_i \supset EF_p$.*

Proof. See Gajdos and Tallon (2001). ■

The referred papers give proofs for the perfectly divisible case that can be adapted directly to the present framework. For this reason specific proofs are not included here.

Now fairness can be defined. I use the definition adopted in the literature and call ex ante intertemporally fair those social choice functions that are both ex ante Pareto optimal and also ex post envy-free. The main reason for adopting this definition is that allocations in the intersection cannot be criticized from any point of view and/or any timing stage considered in this paper, as this intersection is the most restrictive among all the possible ones formed by the introduced sets. I emphasize the ex ante point of view in this definition in order to make contrast to the similar interim fairness concept that is introduced later.

Definition 7 *A social choice function x is ex ante intertemporally fair if $x \in P_a \cap EF_p$.*

3 Existence

In this section I focus on the existence of ex ante intertemporally fair allocations. As for the case of economies with uncertainty, but without indivisibilities it is shown in Gajdos and Tallon (2001) that an ex ante intertemporally fair social choice function exists whenever agents' prior beliefs coincide. Alkan, Demange and Gale (1991) study an economy with indivisible goods in a framework without uncertainty, but in other features very similar to ours. They study whether there exist allocations that are both efficient and envy-free. They reach the existence result across two steps: first is shown that non-wasteful envy-free allocations are also Pareto optimal in this specific environment. Then the existence of envy-free allocations is proved, that combined with the previous result establishes non-emptiness for the set of fair allocations. The proof is constructive and can be used to give an algorithm for finding a fair allocation in the indivisible case. In this paper I shall follow a similar path. Unfortunately the set of intertemporally fair social choice functions may be empty. Proposition 3 gives a necessary and sufficient condition under which the set of ex ante intertemporally fair social choice functions in not empty. The basic idea of the proof is that under that condition ex post envy-free social choice functions are also Pareto optimal. More discussion of the result is presented after the proof.

Proposition 3 *Consider the economy*

$$\mathcal{E} = \left\{ N, O, S, M, \left[q^i(s) \right]_{i \in N}, \left[u^i(s) \right]_{i \in N} \right\}.$$

An ex ante intertemporally fair social choice function exists if and only if

$$\text{for all } s,\ s' \text{ in } S \text{ there exists } \gamma(s, s') \in \mathbb{R}^+ \text{ such that}$$
$$q^i(s) \cdot \phi^i(s) = \gamma(s, s') \cdot q^i(s') \cdot \phi^i(s') \text{ for all } i \text{ in } N. \qquad \text{(Condition 1)}$$

Proof. With mathematical terms the proposition says that $EF_p \cap P_a \neq \varnothing$ if and only if Condition 1 holds. The proof is presented in two parts according to the two directions of implication.

1. The if part: I know that the set EF_p is not empty (see for example see Alkan, Demange and Gale (1991)). Therefore it is enough to prove that under Condition 1 any non-wasteful ex post envy-free efficient social choice function is ex ante Pareto efficient. That is for non-wasteful social choice functions I have the inclusion $EF_p \subset P_a$.

Let x be a non-wasteful ex post envy free social choice function, $x^i(s) = [o^{xi}(s), m^{xi}(s)]$ and $y\,(\neq x)$, $y^i(s) = [o^{yi}(s), m^{yi}(s)]$ any non-wasteful social choice function. Let us suppose that y ex ante Pareto dominates x. This means that for every i

$$\sum_{s\in S} q^i(s)\cdot\left\{u^i\left[o^{yi}(s), s\right] + \phi^i(s)\cdot m^{yi}(s)\right\} \geq \sum_{s\in S} q^i(s)\cdot\left\{u^i\left[o^{xi}(s), s\right] + \phi^i(s)\cdot m^{xi}(s)\right\} \tag{1}$$

holds and with strict inequality for some i_0. Since x is ex post envy-free for all i I have:

$$\sum_{s\in S} q^i(s)\cdot\left\{u^i\left[o^{xi}(s), s\right] + \phi^i(s)\cdot m^{xi}(s)\right\} \geq \sum_{s\in S} q^i(s)\cdot\left\{u^i\left[o^{yi}(s), s\right] + \phi^i(s)\cdot m^{x\rho(i)}(s)\right\}, \tag{2}$$

where $\rho(i)$ is the agent who receives object $o^{yi}(s)$ under x in state s. I let $\rho(i)$ denote that agent without making reference to state s, since I do not need to specify it for the proof. The bundle with $o^{yi}(s)$ also consisted of $m^{x\rho(i)}(s)$ units of money besides the object. Finding $\rho(i)$ is like permuting the agents among themselves; and therefore summing for $\rho(i)$ is like summing up for i in every state, and vice versa. Now let us multiply Equations 1 and 2 by an appropriate positive number, $\lambda^i \in \mathbb{R}^+$ for every i, where λ^i is defined such that $\lambda^i \cdot q^i(s_1)\cdot\phi^i(s_1) = 1$ for all i. After that sum up Equations 1 and 2 on both sides for all agents, and take the right end and the left end of the resulting expression

$$\sum_{i=1}^{N}\sum_{s\in S}\lambda^i\cdot q^i(s)\cdot u^i\left[o^{yi}(s), s\right] + \phi^i(s)\cdot m^{yi}(s) >$$
$$> \sum_{i=1}^{N}\sum_{s\in S}\lambda^i\cdot q^i(s)\cdot u^i\left[o^{yi}(s), s\right] + \phi^i(s)\cdot m^{x\rho(i)}(s)$$

That is, $\sum\limits_{i=1}^{N}\sum\limits_{s\in S}\lambda^i\cdot q^i(s)\cdot\phi^i(s)\cdot m^{yi}(s) > \sum\limits_{i=1}^{N}\sum\limits_{s\in S}\lambda^i\cdot q^i(s)\cdot\phi^i(s)\cdot m^{x\rho(i)}(s)$.

Note that fixing s_1 by Condition 1 for any i and s' I have that $\lambda^i\cdot\, q^i(s')\cdot\phi^i(s') = \frac{1}{\gamma(s_1,s')}$. Therefore the last inequality can be rewritten in the following form.

$$\sum_{i=1}^{N}\sum_{s\in S}\frac{m^{yi}(s)}{\gamma(s_1,s)} > \sum_{i=1}^{N}\sum_{s\in S}\frac{m^{xi}(s)}{\gamma(s_1,s)}$$

By non-wastefullness of x I have $\sum\limits_{i=1}^{N} m^{yi}(s) = \sum\limits_{i=1}^{N} m^{xj}(s) = M$ for all s. Now what is left over from the previous inequality is $M\cdot\sum_{s\in S}\frac{1}{\gamma(s_1,s)} > M\cdot\sum_{s\in S}\frac{1}{\gamma(s_1,s)}$ that is clearly

impossible. I have reached a contradiction, hence x is ex ante Pareto optimal.
2. The only if part: Also this part of the proof is by contradiction. I shall show that any non-wasteful social choice function that is ex post envy-free cannot be ex ante Pareto efficient if Condition 1 does not hold. As before, take λ^i such that $\lambda^i \cdot q^i(s_1) \cdot \phi^i(s_1) = 1$ for all i with $\lambda^i \in \mathbb{R}^+$. Let $\delta^i(s) = \frac{1}{\lambda^i \cdot q^i(s) \cdot \phi^i(s)}$. Now take s^* such that $\delta^i(s^*) \neq \delta^j(s^*)$ for some agents $i \neq j$. Note that such a state s^* always exists if Condition 1 does not hold. For simplicity suppose that I have that inequality for agents i_1 and i_2, and also that $\delta^{i_1}(s^*) > \delta^{i_2}(s^*)$. Take any non-wasteful ex post envy-free social choice function, x, and consider the following transfers (distortioning x) among agents: if s^* occurs agent i_1 pays one monetary unit to agent i_2; if s_1 occurs agent i_2 pays $\frac{1}{\delta^{i_1}(s^*)}$ monetary units to agent i_2. With this the expected utilities will change in the following manner.
For agent i_1:

$$q^{i_1}(s_1) \cdot \phi^{i_1}(s_1) \cdot \frac{1}{\delta^{i_1}(s^*)} + q^{i_1}(s^*) \cdot \phi^{i_1}(s^*) \cdot (-1) = 0$$

For agent i_2:

$$q^{i_2}(s_1) \cdot \phi^{i_2}(s_1) \cdot \left(-\frac{1}{\delta^{i_1}(s^*)}\right) + q^{i_2}(s^*) \cdot \phi^{i_2}(s^*) =$$
$$= -\frac{\lambda^{i_2} \cdot \beta}{\delta^{i_1}(s^*)} + \frac{\lambda^{i_2} \cdot \beta}{\delta^{i_2}(s^*)} > 0$$

Clearly, this means an ex ante Pareto improvement that concludes the proof.
■

Condition 1 contains as a special case an intuitive restriction on the economy in order to guarantee the existence of ex ante intertemporally fair social choice functions. It is stated in the following corollary.

Corollary 1 *If agents have the same prior beliefs and preferences show constant marginal utility of money among agents for a given state, then there exist social choice functions that are ex ante intertemporally fair, that is,* $EF_p \cap P_a \neq \varnothing$.

Proof. The result is a direct consequence of Proposition 3, since if agents have the same prior beliefs and preferences show constant marginal utility of money among agents for a given state, then Condition 1 is satisfied. ■

The proof of Proposition 3 is based on the idea that ex post envy-freeness can be sacrificed in order to reach some ex ante Pareto improvement whenever Condition 1 does not hold. The following two examples contain numerical illustration for this in a simple economy with two possible states and one valuable object - plus $(N-1)$ null-objects.

Example 3 considers an ex post envy-free social choice function and allow for money transfers between agents with different marginal utility of money. The resulting state-dependent allocation represents an ex ante Pareto improvement, but it is not ex post envy-free anymore.

Example 3 *Take the case in which* $M = 10$*, preferences show different marginal utility for money for some agents, there are only two states of nature,* s_1 *and* s_2*, and only one indivisible good complemented by* $(N-1)$ *null objects. Let us suppose that* $u^{i_0}\left[o^{i_0}(s), m^i(s), s\right] = \begin{Bmatrix} 20o^{i_0} + m^i \text{ if } s = s_1 \\ 10o^{i_0} + 2m^i \text{ if } s = s_2 \end{Bmatrix}$ *and* $u^i\left(o^i(s), m^i(s), s\right) = \begin{Bmatrix} 20o^i + 2m^i \text{ if } s = s_1 \\ 20o^i + m^i \text{ if } s = s_2 \end{Bmatrix}$ *for all* $i \in N \setminus \{i_0\}$*. Consider also that* $q^i(s) = \begin{Bmatrix} 0.4 \text{ if } s = s_1 \\ 0.6 \text{ if } s = s_2 \end{Bmatrix}$ *for all* $i \in N$*. The following social choice function is non-wasteful and ex post envy free: in every state give the object to* i_0 *and also* $\left(\frac{20}{N} - 10\right)$ *units of money, let each of the other agents receive* $\left(\frac{20}{N}\right)$ *units of money. This social choice function is ex ante Pareto dominated by the following one: if* $s = s_1$*, then give the object to* i_0 *and also* $\left(\frac{20}{N} - 11\right)$ *units of money, let* $i_1 \neq i_0$ *get* $\left(\frac{20}{N} + 1\right)$ *and each of the other agents receive* $\left(\frac{20}{N}\right)$ *units of money; if* $s = s_2$*, then give the object to* i_0 *with* $\left(\frac{20}{N} - 9\right)$ *units of money, give* $\left(\frac{20}{N} - 1\right)$ *money to* $i_1 \neq i_0$ *and let each of the other agents receive* $\frac{20}{N}$ *units of money. The latter social choice function is clearly not ex post envy-free.*

Note that in the same way one can ex ante Pareto improve any ex post envy-free state-contingent allocation whenever there is different marginal utility for money for some agents in the same state and the following condition does not hold:

$$\text{For all } s, s' \ \phi^i(s) = \gamma(s, s') \cdot \phi^i(s') \text{ for all } i \text{ with } \gamma(s, s') \in \mathbb{R}^+. \qquad \text{(Condition 2)}$$

Example 4 considers a similar economy to the one in Example 3, but now agents will not share a common prior distribution and this will allow for ex ante Pareto improvements in the case of any ex post envy-free social choice function.

Example 4 *Take the case in which* $M = 10$*, preferences show the same marginal utility of money for every agent, there are only two states of nature,* s_1 *and* s_2*, and only one indivisible good complemented by* $(N-1)$ *null objects. Let us suppose that* $u^{i_0}\left[o^{i_0}(s), s\right] = \begin{Bmatrix} 20o^{i_0} \text{ if } s = s_1 \\ 10o^{i_0} \text{ if } s = s_2 \end{Bmatrix}$ *and* $u^i\left[o^i(s), s\right] = \begin{Bmatrix} 10o^i \text{ if } s = s_1 \\ 20o^i \text{ if } s = s_2 \end{Bmatrix}$ *for all* $i \in N \setminus \{i_0\}$*. Consider also that* $q^{i_0}(s) = \begin{Bmatrix} 0.2 \text{ if } s = s_1 \\ 0.8 \text{ if } s = s_2 \end{Bmatrix}$ *and* $q^i(s) = \begin{Bmatrix} 0.8 \text{ if } s = s_1 \\ 0.2 \text{ if } s = s_2 \end{Bmatrix}$ *for all* $i \in N \setminus \{i_0\}$*. The following social choice function is non-wasteful and ex post envy free: if* $s = s_1$*, then give the object to* i_0 *and also* $\left(\frac{20}{N} - 10\right)$ *units of money, let each of the other agents receive* $\left(\frac{20}{N}\right)$ *units of money; if* $s = s_2$*, then give the object to* $i_1 \neq i_0$ *with* $\left(\frac{30}{N} - 20\right)$

units of money and let each of the other agents receive $\left(\frac{30}{N}\right)$ *units of money. This social choice function is ex ante Pareto dominated by the following one: if* $s = s_1$*, then give the object to* i_0 *and also* $\left(\frac{20}{N} - 11\right)$ *units of money, let* $i_1 \neq i_0$ *get* $\left(\frac{20}{N} + 1\right)$ *and each of the other agents receive* $\left(\frac{20}{N}\right)$ *units of money; if* $s = s_2$*, then give the object to* $i_1 \neq i_0$ *with* $\left(\frac{30}{N} - 21\right)$ *units of money, give* $\left(\frac{30}{N} + 1\right)$ *money to* i_0 *and let each of the other agents receive* $\frac{30}{N}$ *units of money. The latter social choice function is clearly not ex post envy-free.*

Corollary 2 *Under Condition 2 I have that* $EF_a \cap P_a \neq \varnothing$*,* $EF_i \cap P_i \neq \varnothing$*,* $EF_p \cap P_p \neq \varnothing$*,* $EF_p \cap P_a \neq \varnothing$*,* $EF_p \cap P_i \neq \varnothing$*.*

Proof. The results are direct consequences of previous propositions and the inclusion results among the sets in question. ■

I have not put any restriction on the sign of the amount of money contained in the bundles. The possibility of negative distribution of money might be essential for the existence of intertemporally fair social choice functions. As discussed in Alkan, Demange and Gale (1991) and Aragonés (1995) in the certainty case, if distribution of money is restricted to be positive, then for the existence result to hold one must be sure that there is enough money to be distributed in the economy. This finding can be easily presented for the uncertainty case as well. Supposing that Condition 1 holds, the amount of money in the economy M, that is not state-dependent, should be large enough to be able to assure existence in every state of the nature.

Note that until this point I have been following an ex ante approach, because I have been dealing with a symmetric situation, considering the uncertainty of each agent not knowing which state of nature from S will occur. This is the reason for putting the qualification ex ante before intertemporally fair social choice functions. Nevertheless, fairness for the interim stage can defined in a similar way. It is interesting that if I move to the interim stage, that is I consider for example that state s_1 has occurred then there are only degenerated cases in which ex post envy-freeness combined with non-wastefulness implies interim Pareto optimality. For the formal definition of interim efficiency and more details on the statement check Appendix A.[3]

Condition 1 plays a decisive role in the existence of intertemporally fair social choice functions, as it is required to ensure ex ante Pareto efficiency. I have shown in a formal proof and also illustrated with two examples that without it one can always find an ex ante Pareto improvement. If I were to define intertemporal fairness with the help of ex post Pareto efficiency I could do without Condition 1. According to the fairness literature with certainty an (ex post) envy-free and (ex post) Pareto efficient social choice function

[3]We state the following proposition (Proposition 10) in the appendix, because the interim considerations do not constitute the main objective of this paper.

exists. Since this result holds for every state s I have existence with all the possible definitions of intertemporal fairness that deals with ex post efficiency.

3.1 Structure of the fair set

The set of envy free allocations, in a set-up with indivisible goods without uncertainty, has a nice structure as was shown by Alkan, Demange and Gale (1991). My next results generalize this finding for the case of uncertainty supposed that Condition 1 holds. In particular I show that the set of ex post envy-free social choice functions owns the lattice property. In order to do so some pieces of notation have to be introduced.

Let x and y be two social choice functions and let

$$\overline{u}_x(s) = \left[u_x^1(s), \ldots, u_x^N(s)\right]$$

denote the vector of utility levels that players enjoy according to x in state s. The owner of object o in state s will be referred to as i^o. An other vector $\overline{u}_y(s)$ is defined similarly. Now consider the following sets

$$\begin{aligned} N_x^s &= \left\{i \in N : u_x^i(s) > u_y^i(s)\right\}, \; O_x^s = \left\{o \in O : m^{xi^o}(s) > m^{yi^o}(s)\right\}, \\ N_y^s &= \left\{i \in N : u_y^i(s) > u_x^i(s)\right\}, \; O_y^s = \left\{o \in O : m^{yi^o}(s) > m^{xi^o}(s)\right\}, \\ N_0^s &= \left\{i \in N : u_x^i(s) = u_y^i(s)\right\}, \; O_0^s = \left\{o \in O : m^{xi^o}(s) = m^{yi^o}(s)\right\}. \end{aligned}$$

The social choice function x induces a mapping between N and O for every state s. It attaches to every agent in N an object from O, precisely the object that the agent receives according to x in state s. Alkan, Demange and Gale (1991) in a set-up without uncertainty about the state of nature show that for any state s two ex post envy-free social choice functions x and y are indeed bijections between N_x^s and O_x^s, N_y^s and O_y^s, N_0^s and O_0^s. A consequence of this result is the lattice property for which the following operators are defined. Given vectors a and b

$$\begin{aligned} a \vee b &= c, \text{ where } c_i = \min(a_i, b_i), \\ a \wedge b &= c, \text{ where } c_i = \max(a_i, b_i). \end{aligned}$$

Let $z = x \wedge y$ be a social choice function defined as follows

a) for every state s in S, $o^{zi}(s) = \left\{ \begin{array}{c} o^{xi}(s) \text{ if } i \in N_x^s \\ o^{yi}(s) \text{ if } i \in N_y^s \cup N_0^s \end{array} \right\}$;

b) for every state s in S, $m^{zi}(s) = m^{xi}(s) \wedge m^{yi}(s)$.

Proposition 4 (Lattice property) *If x and y are ex post envy-free social choice functions, then the social choice function $x \wedge y$ is ex post envy-free.*

Proof. I omit the proof as, using the above introduced notation, it is very similar to the one presented in Alkan, Demange and Gale (1991). ■

As shown in Alkan, Demange and Gale (1991), similar result holds with the minimum operator. This result is useful when defining refinements on the set of ex post envy-free social choice functions that can be very large in general. For a given social choice function x and state s I write $\overline{u}_x^{\min}(s) = \min_{i \in N} \overline{u}_x(s)$ and $\overline{u}_x^{\max}(s) = \max_{i \in N} \overline{u}_x(s)$.

Definition 8 *A non-wasteful ex post envy-free social choice function x is called Rawlsian if $\overline{u}_x^{\min}(s) \geq \overline{u}_y^{\min}(s)$ for all non-wasteful ex post envy-free social choice function y and state s.*

The set of Rawlsian ex post envy-free social choice functions is well-defined thanks to the lattice property. A Rawlsian social choice function gives a Rawlsian allocation in every possible state of nature. As in the case without uncertainty one can show that the elements of the set of Rawlsian ex post envy-free social choice functions induce the same utility level profile. The result is trivial if one considers that in a given state s utility levels are the same for all Rawlsian allocations in that state.

4 Implementation

Now that non-emptiness of the ex ante intertemporally fair set is assured under Condition 1, I can turn my attention to implementation matters. I shall suppose that Condition 1 holds and will concentrate on the implementation of the set of non-wasteful ex post envy-free social choice functions. First implementation at the interim stage is considered, i.e. after the occurrence of a given state when information is asymmetric. In the interim set-up Bayesian implementation is the adequate tool. ex ante implementation is studied later in a separate subsection. Now I introduce some extra notation that will be used in this section.

A mechanism for an economy is a pair (Σ, g), where $\Sigma = \Sigma^1 \times \ldots \times \Sigma^N$, $g : \Sigma \to A^f$. A strategy for agent i is $\sigma^i : \Pi^i \to \Sigma^i$. A deception for agent i is a mapping $\alpha^i : \Pi^i \to \Pi^i$, $\alpha = \left(\alpha^1, ..., \alpha^N\right)$.

The following definition of implementation comes from Jackson (1991).

Definition 9 *A mechanism (Σ, g) implements in Bayes-Nash equilibrium a social choice set, F if:*
a) for any $x \in F$ there exists a Bayes-Nash equilibrium σ with $g\{\sigma[E^i(s)]\} = x(s)$ for all s, and
b) for any Bayes-Nash equilibrium σ there exists $x \in F$ with $g\{\sigma[E^i(s)]\} = x(s)$ for all s.

As shown in Jackson (1991) Bayesian incentive compatibility is needed for Bayesian implementability.

Definition 10 *A social choice set F satisfies Bayesian Incentive Compatibility is for all $x \in F$, i, s and $E^i \in \Pi^i$,*

$$\sum_{s \in E^i(s)} q^i \left[s \mid E^i(s) \right] \cdot u^i \left[x^i(s), s \right] \geq \sum_{s \in E^i(s)} q^i \left[s \mid E^i(s) \right] \cdot u^i \left[x^i_{E^i}(s), s \right]$$

with

$$x_{E^i}(s) = \left\{ \begin{array}{c} x \left\{ \left[\underset{j \neq i}{\cap} E^j(s) \right] \cap E^i \right\} \text{ if the argument is not empty} \\ 0 \text{ otherwise} \end{array} \right\}.$$

Unfortunately Bayesian Incentive Compatibility is not guaranteed in general in the model. Some restrictions on the structure of information owned by agents have to be introduced. Palfrey and Srivastava (1987) consider Bayesian implementation in a set-up in which information is non-exclusive. They prove that if there are at least three agents, information is non-exclusive and the social choice set $F \neq \varnothing$ to be implemented satisfies Bayesian monotonicity, then F is indeed implementable. It is not difficult to show that the set of non-wasteful ex post envy-free social choice functions satisfy the requirement of Bayesian monotonicity. In order to assure implementability from now on let us suppose that information is non-exclusive, that is

$$E^i(s) \supset \underset{j \neq i}{\cap} E^j(s) \text{ for all } i \text{ in } N \text{ and } s \text{ in } S. \tag{NEI}$$

Note that the former assumption of no redundant states combine with NEI delivers the fact that $(N-1)$ agents can identify without uncertainty the state that has occurred, i.e. $\underset{j \neq i}{\cap} E^j(s) = \{s\}$ for all i and s. Now let me introduce the following notation that will be useful in defining a mechanism:

$$D(\sigma) = \left\{ s^* \in S : \underset{i \in N \setminus \{j\}}{\cap} \sigma^i = \{s^*\} \text{ for some } j \in N \right\}.$$

The punishment that dissuades agents from deviation in some cases is $\overline{F} \in \mathbb{R}$. It can be interpreted as a fine that players must pay when they fail to reach some agreement to be specified later with the mechanism. As for the example of flat-mates it could be seen as monetary equivalent of all the inconveniences that the lack of agreement can cause, for example the cost of looking for a new flat or flat-mates, or the utility loss due to the persistence of envy. $\overline{F}$ can be found with the help of the following inequality:

$$u^i\left[x^i(s), s\right] > u^i\left(o, \frac{M}{N}, s\right) - \overline{F} \text{ for all } i,\ s,\ o \text{ and non-wasteful}$$
$$\text{ex post envy free social choice function } x.$$

$\overline{F}$ is well-defined since no object is infinitely desirable, and the set of states and the one of objects are both finite. Let us define the mechanism as follows.

Definition 11 **($\mathcal{M}$)** *Every agent has to announce (simultaneously) some elements from the partition Π^i denoted by e^i. Players also have to choose a permutation p^i over the set N and a non-negative integer. The message space for agent i is then $\Sigma^i = \Pi^i \times P^i \times \mathbb{Z}_{0+}$ with a typical element in the form of (e^i, z^i). The outcome of the mechanism, g, is defined as follows.*
a) If $\#D = 1$ and there are at least $(N-1)$ zeros among the z^i, then the outcome is $\left(p^1 \circ \ldots \circ p^N\right)[x(s^0)]$ where $\{s^0\} = D$ and x is some non-wasteful ex post envy-free social choice function. In this case, after the first stage the planner offers a non-wasteful ex post envy-free allocation for s^0, i.e. $x(s^0) \in EF_p \cap F^{fnw}$.
b) If $\#D > 1$ and there are at least $(N-1)$ zeros among the z^i, then let assign objects to agents in a random way (for example in such a way that agent i can get any of the objects with the same probability) and allocate money equally, giving $\frac{M}{N}$ to everyone. Agents in this case are forced to pay a fine of the amount $\overline{F}$ each.
c) In any other case let the agent with the smallest index among those who have announced the largest z^i receive the object of her choice and $\max\{0, M\}$ *amount of money. The other players receive a random object from the ones that have been left over and the following amount of money* $\min\left\{0, \frac{M}{N-1}\right\}$.

In what follows I show that this mechanism can be used to implement the set of ex-post envy-free social choice functions. It is point a) in the above definition according to which the Bayes-Nash equilibrium outcomes of the game are determined. Points b) and c) introduce incentives for reporting the state of nature truthfully be optimal for agents. Under b) agents are severely punished by a fine that amounts $\overline{F}$. In point c) the mechanism contains an integer game that gives incentives to participants to send messages that give rise to situations that fall under point a). Proposition 5 states the formal result and is followed by the formal proof.

Proposition 5 *Let x be a non-wasteful ex post envy-free social choice function. If $N \geq 3$ and information is non-exclusive the mechanism $\mathcal{M}$ implements x in Bayes-Nash equilibrium.*

Proof. Suppose that the conditions of the proposition hold. Let us prove first that for any state of the world every equilibrium outcome of $\mathcal{M}$ is a non-wasteful ex post envy-free allocation.
1.1. Note that there can not exist any equilibrium under c) or b). The first one is the case of an integer game in which, given the others' choice, every player i has incentives to announce a larger integer above the level of

$$max \left\{ z^1, \ldots, z^{i-1}, z^{i+1}, \ldots, z^N \right\}.$$

1.2. Under b) agents are severely punished and have incentives to change their announcements. If there are at least $(N-2)$ zeros among the announced z^i by setting $e^i = \emptyset$ and $z^i = 0$, and inducing case a). Or, if there are less then $(N-1)$ zeros among the announced z^i by the others any agent can switch to case c) by announcing a sufficiently large integer.
1.3. Therefore only case a) can support an equilibrium. Its allocation must be ex post envy-free due to the enclosed permutation game.
In the second part I shall show that any non-wasteful envy-free social choice function can be supported as an equilibrium of the mechanism $\mathcal{M}$.
2. In order to do so let us show that $\sigma^{i*} = \left[E^i(s), p^{id}, 0\right]$ for all i constitutes an equilibrium of the mechanism $\mathcal{M}$ where p^{id} stands for the identity permutation. Note that no agent has incentives to switch to case b), since in that one all agents are seriously punished by a fine that makes them worse off than in any possible result under a). Case c) might be a tempting possibility but for a single player it is impossible to switch directly from a) to c) when the others are playing according to σ^*. ■

In first place the mechanism $\mathcal{M}$ is designed to extract true information from agents. Joining that, the planner is able to find out which state of nature has occurred and her task is to find a particular non-wasteful ex post envy-free social choice function. There exist numerous algorithms that can be used to find envy-free allocations for every state, hence for constructing the ex post envy-free social choice function. For examples check Alkan, Demange and Gale (1991), Aragonés (1995) and Su(1999).

Note that the mechanism is not wasteful in equilibrium and the indivisible objects are always allocated according to the rules of the economy, however it contains threats under case b) that are not budget balanced. These are needed, because even if $(N-1)$ players are able to identify the occurred state, the planner can not always identify the deviator and therefore has to punish everyone to avoid deviations.

4.1 Implementation ex ante

Let us study more carefully the ex ante situation when information is symmetric, i.e. agents have prior beliefs about the occurrence of the states and these beliefs are known to everyone. I do not have to deal with social choice functions or set anymore, but with allocations.[4] Since agents do not know which state will occur, they value these bundles according to their expected utility function. For simplicity I shall define a utility function for this case: $v^i(a^i) = \sum_{s\in S} q^i(s) \cdot u^i(a^i, s)$. Note that previous results hold, meaning that an envy-free allocation is also Pareto optimal in this economy. For v^i it is useless to distinguish between ex ante, interim or ex post concepts, but it is worth to point out that for example its envy-freeness is closely related to the ex ante envy-freeness notion that I had before. The most important change is that before I had social choice functions and now I am working with allocations that do not change with the states of nature - they are no longer state-contingent. For this reason I only consider constant social choice functions in this subsection that simply will be called allocations. Now with redefining my envy-freeness and Pareto-optimality concepts with allocations I have the following result.[5]

Proposition 6 *If Condition 1 holds any non-wasteful ex ante envy-free feasible allocation is ex ante Pareto optimal.*

Proof. Just like in Proposition 3. ■

Now let us turn my attention to implementation matters. Taking into account the previous notes and assuming that Condition 1 holds I have that the well-known "divide and permute"[6] mechanism implements (in Nash equilibrium) the set of ex ante envy free allocations that are also ex ante Pareto optimal according to the last proposition.

Definition 12 *The "divide and permute" mechanism. The message space for agent i is* $\Sigma^i = \left\{ \begin{array}{c} A^{fnw} \times p \text{ if } i = 1,2 \\ p \text{ otherwise} \end{array} \right\}$ *where p denotes the set of all possible permutations in N. The outcome function is g with the following definition*[7]*:*

$$g(\sigma) = \left\{ \begin{array}{cc} (p^n \circ \ldots \circ p^1)(a^1) & \text{if } a^{f1} = a^{f2} \\ \left(o^{\text{rand}}, \frac{M}{N} - \overline{F}\right)^N & \text{otherwise} \end{array} \right\}.$$

[4] Nevertheless there might exist problems in which the ex-ante implementation of a social choice function is interesting. A "divide and permute" type mechanism can be used in those cases, too. The only modification required is that the first two players have to announce a non-wasteful social choice function instead of an allocation.

[5] Note that in this context only notions corresponding to the earlier ex-ante concepts have meanings.

[6] For details check Thomson (1995).

[7] The symbol o^{rand} stands for a random object from O.

The proof of this implementation result is not included here, since my ex ante implementation problem is technically identical to the Nash implementation of envy-free allocations problem studied in the literature, for instance in Thomson (1995). The "divide and permute" or cake-cutting mechanisms that was designed for the two-player divisible case, have many variants and generalizations for situations with more participants and indivisibilities.[8] Unfortunately they do not perform well under uncertainties. McAfee (1992) points out that the cake-cutting mechanism produces efficient results under symmetric information, but under asymmetric information it is ex post inefficient in an unusual way.

5 Existence with generalized fairness concept

The literature on distributive justice usually does not deal with the problematic of choosing fairness criteria. Concepts are very often axiomatically justified, and/or their use is made acceptable intuitively. In this section I enlarge my focus and study some generalized fairness concepts. This allows for more judgements on fairness and does not exclusively deal with envy-freeness.

In order to do so, following the idea in Corchón and Iturbe-Ormaetxe (2001) I define for every agent i a function $\psi^i : A^f \times S \longrightarrow A^i$ which I call state-dependent aspiration function, or simply aspiration function. Let $\psi = \left(\psi^1, \ldots, \psi^N\right)$. The expression $\psi^i \left[x\left(s\right), s\right]$ denotes the personal aspiration of agent i in state s when the social choice function is x.[9] It can be interpreted as the list of bundles for agent i that she thinks are fair in each state, when bundles are assigned according to the social choice function x in the population. Note that the personal aspirations may perfectly be unfeasible together. The next definition identifies the feasible aspiration correspondences.

Definition 13 *Given the social choice function x, the aspiration function ψ is feasible if*

$$\psi^i \left[x\left(s\right), s\right] \in A^f \text{ for all } i \text{ and } s.$$

I can generalize the fairness concepts with the help of the aspiration function also in the uncertainty case. Envy-free social choice functions, for example, will be a special case of the satisfactory ones defined below. For this to be true, one should think about personal aspirations, for a given state and social choice function, as the best bundle owned by any agent in the given state and according to the given social choice function.[10]

[8] For examples check Brams and Taylor (1996).

[9] The notation $\psi^i \left(x\left(s\right), s\right)$ is redundant, since $x\left(s\right) \in A^f$ gives the allocation for the whole set of agents in state s. Therefore it is clear that we are dealing with aspirations for state s and we could simply write $\psi^i \left(x\left(s\right)\right)$. However the notation in longer form is more in line with the formal definition and for this reason is kept.

[10] For more explanation, intuition and results under certainty check Corchón, Iturbe-Ormaetxe (2001).

Definition 14 *A feasible social choice function x, given ψ, is ex post satisfactory if*

$$u^i\left[x^i(s), s\right] \geq u^i\left\{\psi^i\left[x(s), s\right], s\right\} \text{ for all } i \text{ and } s.$$

Definition 15 *A feasible social choice function x, given ψ, is ex ante satisfactory if*

$$\sum_{s\in S} q^i(s)\cdot u^i\left[x^i(s), s\right] \geq \sum_{s\in S} q^i(s)\cdot u^i\left\{\psi^i\left[x(s), s\right], s\right\} \text{ for all } i.$$

The widely used egalitarian solution can be captured also as a special case with the following definitions of adequate social choice functions.

Definition 16 *A feasible social choice function x, given ψ, is ex post adequate if*

$$u^i\left[x^i(s), s\right] = u^i\left\{\psi^i\left[x(s), s\right], s\right\} \text{ for all } i \text{ and } s.$$

Definition 17 *A feasible social choice function x, given ψ, is ex ante adequate if*

$$\sum_{s\in S} q^i(s)\cdot u^i\left[x^i(s), s\right] = \sum_{s\in S} q^i(s)\cdot u^i\left\{\psi^i\left[x(s), s\right], s\right\} \text{ for all } i.$$

The above definitions are natural generalizations of the ones in Corchón and Iturbe-Ormaetxe (2001). The generalized version of Proposition 2, describing the relation between ex post and ex ante terms, holds using either the satisfactory or the adequate fairness concepts.

Proposition 7 *If a social choice function is ex post satisfactory (adequate), then it is also ex ante satisfactory (adequate).*

Proof. If one weights the inequalities (equalities) in the definition for ex post satisfactory (adequate) social choice functions by the prior probabilities and sum the results up for every possible state, one gets the requirement stated in the definition for ex ante satisfactory (adequate) social choice function. ■

Naturally I could define the corresponding interim concepts as well, and state the inclusion result in a similar proposition. Since I am not particularly interested in the interim stage in this section these parts are omitted.

Intertemporal fairness now can be captured by ex post satisfactory (or adequate) social choice functions that are ex ante Pareto efficient. This point of view allows for the same arguments as presented before for the envy-free case. Unfortunately with the general form of aspirations I cannot prove existence of the satisfactory nor the adequate social choice functions, neither ex post or ex ante. Therefore let us introduce the concept of unbiased social choice functions that do exist under some mild assumptions on preferences and the feasible consumption set in the certain case, as shown in Corchón and Iturbe-Ormaetxe (2001).

Definition 18 *A feasible social choice function x, given ψ, is ex post unbiased if for any state s any of the following statements holds:*
a) $u^i [x^i (s), s] \geq u^i \{\psi^i [x (s), s], s\}$ *for all i, or*
b) $u^i [x^i (s), s] < u^i \{\psi^i [x (s), s], s\}$ *for all i.*

Note that I can not define ex post biasedness in such a way that requires inequality a) or inequality b) to hold for all possible s. Vaguely speaking this is because aspirations are now state-dependent. There might be states in which aspirations are too high to inequality a) to hold, while in some other might be so low that b) is impossible. The next example, even if it is an extreme case, illustrates this statement.

Example 5 *Suppose that in state s_1, independently from the social choice function, every agent is satisfied with the indivisible object she has been assigned to, but aspires to some extra amount of money, $\varepsilon > 0$. This falls clearly under case b) in Definition 18, as personal aspirations can not be reached. If s_1 was the only possible state of nature, I would have unbiasedness. Let us suppose that in some other state s_2 aspirations are humble in the sense that every agent is satisfied with her indivisible object and does not aspires to any amount of money. This and $M > 0$ give case a) in Definition 18.*

With my definition an ex post unbiased and ex ante Pareto efficient social choice functions exists if and only if Condition 1 holds. This is a direct consequence of the results for the certain case in Corchón and Iturbe-Ormaetxe (2001) and the one that I discussed before according to which Condition 1 is needed for ex ante Pareto efficiency.

Definition 19 *A feasible social choice function x, given ψ, is ex ante unbiased if for any state s any of the following statements holds:*
a) $\sum_{s\in S} q^i (s) \cdot u^i [x^i (s), s] \geq \sum_{s\in S} q^i (s) \cdot u^i \{\psi^i [x (s), s], s\}$ *for all i, or*
b) $\sum_{s\in S} q^i (s) \cdot u^i [x^i (s), s] < \sum_{s\in S} q^i (s) \cdot u^i \{\psi^i [x (s), s], s\}$ *for all i.*

There is no relation between the above ex post and ex ante unbiasedness concepts of the inclusion type, like I had before for the envy-free case. Therefore the question whether an ex ante unbiased and ex ante efficient social choice function exists is not trivial. The following propositions states that in fact, under Condition 1, there exist social choice functions that are ex ante unbiased and ex ante Pareto efficient.

Proposition 8 *Given the aspiration function ψ, there exists an ex ante unbiased and ex ante Pareto social choice function if and only if Condition 1 holds.*

Proof. This proof for the uncertainty case I present here goes parallel with the proof for certainty from Corchón and Iturbe-Ormaetxe (2001), and it is tailored to the specific set-up I study. For the sake of this proof let us introduce a technical change in the definition of the consumption set and extend the set of possible money consumption in the bundles to the set of extended real numbers: $A = (O \times \mathbb{R}^*)^N$ where $\mathbb{R}^*$ denotes the set of extended real numbers that is compact, non-empty and convex. Let S^{n-1} be the $(n-1)$ dimensional simplex. In my set-up a social choice function is ex ante Pareto efficient if for a given $\lambda \in S^{n-1}$ it solves the following maximization problem:

$$\max_{x \in X} \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot u^i\left[x^i(s), s\right].$$

I can split the above problem into two part: finding the way of distributing the indivisible objects among agents and then the distribution of the perfectly divisible one.

$$\max_{x \in X} \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot u^i\left[o^{xi}(s), s\right] + \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot \phi^i(s) \cdot m^{xi}(s),$$
$$\max_{o^x} \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot u^i\left[o^{xi}(s), s\right] + \max_{m^x} \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot \phi^i(s) \cdot m^{xi}(s).$$

Since the sets S, N, O are finite the first maximization problem has a solution with some finite value. The second one has a solution too, because $m^{xi}(s) \in \mathbb{R}^*$ for all i and s, and $q^i(s) \cdot \phi^i(s) \cdot m^{xi}(s)$ is continuous, strictly increasing in $m^{xi}(s)$. Let the solution of the second maximization problem be $\rho : S^{n-1} \to (\mathbb{R}^*)^{N \cdot S}$. It is convex-valued ($\mathbb{R}^*$ is convex and utilities are quasilinear in money) and upper hemicontinuous (by Berge's Maximum Theorem). Now let us define

$$D^i[x(s), s] = D^i\left\{x^i(s), \psi^i[x(s), s], s\right\} = u^i\left\{\psi^i[x(s), s], s\right\} - u^i\left[x^i(s), s\right]$$

that can be separated into two parts: utility difference from the indivisible goods and the difference from money. Note that $D^i[x(s), s]$ is continuous in money. Consider now the following maximization problem for a fixed social choice function x:

$$\max_{\lambda \in S^{n-1}} \sum_{i \in N} \lambda^i \cdot \sum_{s \in S} q^i(s) \cdot D^i\left\{x^i(s), \psi^i[x(s), s], s\right\}.$$

I define $\varphi : (\mathbb{R}^*)^{N \cdot S} \to S^{n-1}$ as the solution function for the money allocation part of the above maximization problem. This correspondence is also convex-valued and upper hemicontinuous. Therefore the mapping $\varphi \circ \rho : S^{n-1} \to S^{n-1}$ has a fixed point λ^* with $m^* \in \lambda^*$. And there is some x^* that belongs to that m^*. Now let us show that I must have ex ante unbiasedness by contradiction. Suppose that there are i and j such that

$$\sum_{s \in S} q^i(s) \cdot D^i[x^*(s), s] \geq 0,$$
$$\sum_{s \in S} q^j(s) \cdot D^j[x^*(s), s] < 0.$$

Then I must have $\lambda^{*j} = 0$, and in the ex ante Pareto program (according to x^*) agent j will be assigned the amount of $-\infty$ of money (the worst possible amount) in every state. But that, with my assumptions would be in contradiction with $\sum_{s\in S} q^j(s) \cdot D^j[x^*(s), s] < 0$. This completes the proof. ■

Condition 1 turns out to be crucial for existence results, because I have required Pareto efficiency for every fairness concept. Condition 1, in fact, arises due to this fact, and it is possible to show that any non-wasteful social choice function that induces an ex post optimal assignment of indivisible objects is ex ante Pareto efficient if and only if Condition 1 holds. For the formal proof check Appendix B.

When considering implementation of some social choice set an important topic of monotonicity arises. In the certainty case Maskin monotonicity can be guaranteed by some rationality requirements on aspirations.[11] my goal now is to study Bayesian monotonicity, because that is the property needed for Bayesian implementability. In order to do so more concepts and some pieces of notation that generalize those from Corchón and Iturbe-Ormaetxe (2001) for the uncertainty case have to be introduced .

Definition 20 *The social choice set F is attainable in an ex ante satisfactory way if there is ψ such that the social choice function x belongs to F if and only if x is an ex ante satisfactory social choice function for ψ.*

Let $Z^i : X \longrightarrow X^i$ be a correspondence. The set $Z^i(x)$ is interpreted as the set of state-contingent allocations that agent i thinks she is entitled to, given the social choice function x.

Definition 21 *The aspiration function ψ is called ex ante rational if, for all i,*

$$\psi^i = \arg\max \sum_{s\in S} q^i(s) \cdot u^i\left[x^i(s), s\right]$$

with $x^i \in Z^i(x)$.

The above maximization problem need not to have a single solution, but ties can be handled by some arbitrary rule. ex ante rationality now requires utility maximization ex ante. For a given social choice function, every agent should choose her personal aspiration in such a way that maximizes her expected utility over the set $Z^i(x)$.

I consider compatible deceptions and Bayesian monotonicity in the form as they appear in Palfrey and Srivastava (1987). I take the definitions from there and present them in order to have a self-contained study of the problem.

[11] For more details see Corchón and Iturbe-Ormaetxe (2001).

Definition 22 *A collection of functions* $\alpha = \left(\alpha^1, ..., \alpha^N\right)$, *with* $\alpha^i : \Pi^i \to \Pi^i$, *is a deception compatible with* $\{\Pi^i\}$ *if for all* $\left(E^1, ..., E^N\right)$ *such that* $E^i \in \Pi^i$ *for all* i, $\underset{i \in N}{\cap} E^i \neq \varnothing$ *implies* $\underset{i \in N}{\cap} \alpha\left(E^i\right) \neq \varnothing$.

Let us introduce the following short-hand notation:

$$\alpha(s) = \underset{i \in N}{\cap} \alpha^i \left[E^i(s)\right], \; x_\alpha(s) = x\left[\alpha(s)\right], \; x_\alpha = \left[x_\alpha(s_1), x_\alpha(s_2), ...\right].$$

It will simplify the following definition of Bayesian monotonicity.

Definition 23 *The social choice set* F *satisfies Bayesian monotonicity if for all* α *compatible with* $\{\Pi^i\}$ *if*
a) $x \in F$,
b) for all agent i, *state* s^* *and social choice function* y,

$$\begin{array}{ccc} \sum\limits_{s \in E^i(\alpha(s^*))} q^i \left\{s \mid E^i\left[\alpha(s^*)\right]\right\} \cdot u^i \left[x^i(s), s\right] & \geq & \sum\limits_{s \in E^i(\alpha(s^*))} q^i \left\{s \mid E^i\left[\alpha(s^*)\right]\right\} \cdot u^i \left[y^i(s), s\right] \\ & \Downarrow & \\ \sum\limits_{s \in E^i(s^*)} q^i \left[s \mid E^i(s^*)\right] \cdot u^i \left[x^i_\alpha(s), s\right] & \geq & \sum\limits_{s \in E^i(s^*)} q^i \left[s \mid E^i(s^*)\right] \cdot u^i \left[y^i_\alpha(s), s\right] \end{array}$$

then $x_\alpha \in F$.

Now I can state the following positive result on Bayesian monotonicity in my set-up. It implies that imposing an extra condition on the social choice function and aspirations, the set of intertemporally fair social choice functions can be implemented at the interim stage, i.e. by Bayesian implementations.

Proposition 9 *Let* F *be a social choice set that is attainable in an ex ante satisfactory way with ex ante rational aspirations.* F *is Bayesian monotonic if*

$$y^i \in Z^i(x) \Longrightarrow y^i_\alpha \in Z^i(x_\alpha) \text{ for all } i. \qquad \text{(Condition 3)}$$

Proof. Let $x \in F$, where F is attainable in an ex ante satisfactory way with rational aspirations. Then for all i and s^* I have that

$$\begin{array}{ccc} \sum\limits_{s \in E^i(\alpha(s^*))} q^i \left\{s \mid E^i\left[\alpha(s^*)\right]\right\} \cdot u^i \left[x^i(s), s\right] & \geq & \sum\limits_{s \in E^i(\alpha(s^*))} q^i \left\{s \mid E^i\left[\alpha(s^*)\right]\right\} \cdot u^i \left[y^i(s), s\right] \\ \text{for all } y^i & \in & Z^i(x). \end{array}$$

By the implication in the definition of Bayesian monotonicity I also have that the above implies

$$\begin{array}{ccc} \sum\limits_{s \in E^i(s^*)} q^i \left[s \mid E^i(s^*)\right] \cdot u^i \left[x^i_\alpha(s), s\right] & \geq & \sum\limits_{s \in E^i(s^*)} q^i \left[s \mid E^i(s^*)\right] \cdot u^i \left[y^i_\alpha(s), s\right] \\ \text{for all } y^i & \in & Z^i(x). \end{array}$$

If Condition 3 holds the latter shows that x_α is satisfactory, because then I have the inequality for all $y_\alpha \in Z^i(x_\alpha)$. Hence $x_\alpha \in F$ and therefore F is Bayesian monotonic. ■

As a special case, the ex post envy-free social choice set satisfies Condition 3. To see this in an intuitive way consider the following. With the concept of envy-freeness, and some social choice function x, the set $Z^i(x)$ contains those state-contingent allocations that in a given state s^* have the other agents' consumption bundles, from the same state s^*, as components. For x to be ex post envy-free every agent i has to choose the best one among these, and that for every state in M. Now let us introduce compatible deceptions, α. With this $Z^i(x_\alpha)$ will contain those state-contingent allocations that in state $\alpha(s^*)$ have the others' consumption bundles, from state $\alpha(s^*)$. Note that $\alpha(s^*)$ contains elements from S. Therefore the implication in Condition 3 is straightforward.

6 Concluding remarks

I have considered the problem of allocating indivisible goods and money among members of an economy in which agents are not perfectly informed on the others' preferences. The set of intertemporally fair social choice functions have been studied that are defined as ex-post envy-free and ex-ante Pareto efficient, as this intersection is the most restrictive among all the possible ones. The appealing features of envy-free allocations explored in the literature on economies without uncertainties extend to the economy with uncertainty. These are the lattice structure of the intertemporally fair set, the consistency and monotonicity results (not studied here in detail) and the fact that envy-freeness implies Pareto efficiency. It is the latter in the ex-ante stage that might make the intertemporally fair set be empty. I have derived a necessary and sufficient condition for existence (non-emptiness) that in economies, in which the marginal utility of money is the same for every agent, requires prior beliefs to be the same for everyone.

Under this conditions and the one of nonexclusiveness of information the implementation of the intertemporally fair set has been studied both in the interim and ex-ante stage. Concrete mechanisms have been proposed to achieve full implementation.

I have also proposed a generalized version of intertemporal fairness based on the aspiration function and Pareto efficiency. Due to the presence of Pareto efficiency the condition on beliefs derived in the first part of the paper for existence remains necessary and sufficient. In the concluding result a condition for Bayesian monotonicity has been derived, i.e. for Bayesian implementation of the generalized intertemporally fair set.

7 Appendix A

Definition 24 *Suppose that agents have reached the interim stage and some state of nature, s_1, has occurred. A social choice function x is interim intertemporally fair if $x \in P_i \cap EF_p$.*

I keep the notation for simplicity, but point out that the definition of P_i slightly differs now, because I only consider the state that in fact has occurred, i.e. when for instance s_1 has occurred. From an interim point of view I shall use the following definition.

Definition 25 *From an interim point of view a non-wasteful social choice function x is interim Pareto optimal if there is no non-wasteful social choice function y such that*

$$\sum_{s \in E^i(s_1)} q^i \left[s \mid E^i(s_1) \right] \cdot u^i \left[y^i(s), s \right] \geq \sum_{s \in E^i(s_1)} q^i \left[s \mid E^i(s_1) \right] \cdot u^i \left[x^i(s), s \right]$$

for all i in N, with strict inequality for at least one i. Let P_i denote the set of interim Pareto optimal social choice functions.

Proposition 10 *Suppose that agents have reached the interim stage, i.e. a given state has occurred. $EF_p \cap P_i \neq \varnothing$ if and only if agents have no uncertainty about the state of nature at the interim stage.*

Proof. The if part has been already shown before, since if there is no uncertainty I am dealing with the intersection of $EF_p \cap P_p$ that is known to be not empty in my set-up with indivisibilities.
For the only if part suppose that s_1 has occurred. Then the result can be proven similarly as Proposition 3. From that proof the condition for non-emptiness that arises is

$$q^i(s \mid s_1) \cdot \phi^i(s) = \gamma(s') \cdot q^i(s' \mid s_1) \cdot \phi^i(s')$$

for all s, $s' \in E^i(s_1)$ and i with $\gamma \in \mathbb{R}^+$. This implies that $E^i(s_1) = E(s_1)$ for all i that is only compatible with the assumption of no redundant states if $E^i(s_1) = E(s_1) = \{s_1\}$ for all i. And of course this should hold for any particular s_1. ■

8 Appendix B

Definition 26 *The assignment of the indivisible objects under the non-wasteful social choice function x is ex post optimal whenever $\sum_{i \in N} u^i \left[o^{xi}(s), s \right] \geq \sum_{i \in N} u^i \left[o^{yi}(s), s \right]$ for every s, and any non-wasteful social choice function y.*

Proposition 11 *Any non-wasteful social choice function that induces an ex post optimal assignment of indivisible objects is ex ante Pareto efficient if and only if Condition 1 holds.*

Proof. The if part: Note that the optimal assignment of the indivisible objects is a necessary condition for any kind of Pareto efficiency. Taking into account the previous proposition it is enough to prove that under Condition 1 ex post unbiasedness implies ex ante Pareto efficiency. Consider the social choice function x that is supposed to be non-wasteful and ex post unbiased, and the following maximization problem whose solutions give the ex ante efficient money transfers.

$$\max_{y\in X^{fnw}} \sum_{i\in N} \tau^i \cdot \sum_{s\in S} q^i(s) \cdot \phi^i(s) \cdot m^{yi}(s)$$

$$\text{with } \tau^i \in (0,1) \text{ for all } i\text{, and } \sum_{i\in N}^{i} \tau^i = 1 \qquad (3)$$

If I can find some weights $\overline{\tau^i}$ such that x solves the above problem then I am done. Now let consider a non-wasteful social choice function y and the positive numbers, $\lambda^i \in \mathbb{R}^+$ for every i, where λ^i is defined such that $\lambda^i \cdot q^i(s_1) \cdot \phi^i(s_1) = \beta$ for all i with $\beta \in \mathbb{R}^+$. Note that fixing s_1 by Condition 1 for any i and s' I have that $\lambda^i \cdot q^i(s') \cdot \phi^i(s') = \frac{\beta}{\gamma(s_1,s')}$

$$\frac{1}{\sum\limits_{i\in N}\lambda^i} \cdot \lambda^i \cdot \sum_{s\in S} q^i(s) \cdot \phi^i(s) \cdot m^{yi}(s) =$$

$$= \frac{1}{\sum\limits_{i\in N}\lambda^i} \cdot \beta \cdot \left(m^{yi}(s_1) + \frac{m^{yi}(s_2)}{\gamma(s_1,s_2)} + \ldots \right)$$

Now let us sum up the above equality for all players.

$$\sum_{i\in N} \frac{1}{\sum\limits_{i\in N}\lambda^i} \cdot \lambda^i \cdot \sum_{s\in S} q^i(s) \cdot \phi^i(s) \cdot m^{yi}(s) =$$

$$= \frac{1}{\sum\limits_{i\in N}\lambda^i} \cdot \beta \cdot \sum_{i\in N} \left(m^{yi}(s_1) + \frac{m^{yi}(s_2)}{\gamma(s_1,s_2)} + \ldots \right) =$$

$$= \frac{1}{\sum\limits_{i\in N}\lambda^i} \cdot \beta \cdot M \cdot \sum_{i\in N} \left(1 + \frac{1}{\gamma(s_1,s_2)} + \ldots \right) = const.$$

The above expression does not depend on the chosen social choice function y, therefore I can take

$$\overline{\tau^i} = \frac{\lambda^i}{\sum\limits_{i\in N}\lambda^i}$$

that will guarantee that the social choice function x solves the maximization problem in (7).

The only if part: The proof is like in Proposition 3. By contradiction one could consider an ex ante Pareto efficient social choice function with optimal assignment and suppose that Condition 1 does not hold. Using the same argument now I can conclude that in this case the social choice function in question can not be ex ante Pareto efficient. It is possible to improve somebody's expected utility without harming anyone else in the way I did for the proof of Proposition 3. ■

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