On hyperbolic once-punctured-torus bundles IV: automata for lightning curves.

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Abstract. Let $\mathbf{T}$ be a once-punctured torus, let F be a pseudo-Anosov self-homeomorphism of $\mathbf{T}$, and let $\mathbf{T}_F$ be a bundle over the circle having $\mathbf{T}$ as fiber and F as monodromy. Building on work of Jørgensen, Thurston, Cannon, Minsky and others, McMullen proved the conjecture that there exists a *Cannon-Thurston map associated to* F, $\mathrm{CT}_F\colon \mathbb{S}^1 \twoheadrightarrow \hat{\mathbb{C}}$. In particular, CT_F is a surjective, continuous, $\pi_1(\mathbf{T}_F)$-equivariant map from an oriented topological circle $\mathbb{S}^1$ to the Riemann sphere $\hat{\mathbb{C}} \coloneqq \mathbb{C} \cup \{\infty\}$, and $\pi_1(\mathbf{T}_F)$ acts by self-homeomorphisms on $\mathbb{S}^1$, and by conformal automorphisms on $\hat{\mathbb{C}}$. Moreover, by composing with a conformal automorphism of $\hat{\mathbb{C}}$, we may assume that some rank-two free-abelian subgroup of $\pi_1(\mathbf{T}_F)$ fixes ∞. Here, Cannon-Dicks, building on work of Cannon-Thurston and Bowditch, showed that if, as x moves around $\mathbb{S}^1$, the point $\mathrm{CT}_F(x)$ moving around $\hat{\mathbb{C}}$ is made to switch back and forth between gray and white each time it passes through ∞, then $\mathrm{CT}_F(x)$ draws a fractal gray-and-white CW-structure $\mathrm{CW}(F)$ on the plane $\mathbb{C}$. They expressed the one-skeleton of $\mathrm{CW}(F)$ as the union of planar translates of $\partial^- A \cup \partial^+ B - \{\infty\}$, where $\partial^- A$ and $\partial^+ B$ are certain fractal, or lightning, curves, in $\hat{\mathbb{C}}$, each of which arises as the limit set of a finitely generated Kleinian semigroup.

In this paper, we continue the study of $\partial^- A$, $\partial^+ B$ and $\mathrm{CW}(F)$.

There is a natural injective map to $\pi_1(\mathbf{T}_F)$ from the finite set of two-cells of the standard Markov partition of $\mathbf{T}$ with respect to F. The resulting elements of $\pi_1(\mathbf{T}_F)$ generate a subsemigroup whose limit set in $\hat{\mathbb{C}}$ is $\partial^- A$. The inverses of these elements generate a subsemigroup whose limit set is $\partial^+ B$. We deduce that the Hausdorff dimensions of $\partial^- A$ and $\partial^+ B$ are equal. We use the elements to produce automata which codify $\partial^- A$ and $\partial^+ B$ in terms of trees. This is our main result and it permits depth-first tree-searches which yield great improvements in the computer drawings of the lightning curves and, hence, $\mathrm{CW}(F)$. We illustrate this with an example analyzed by Helling.

We record a proof of the folklore implication that if $\mathbf{T}_F$ admits an orientation-reversing symmetry, then the colored columns of $\mathrm{CW}(F)$ are completely vertical.

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1 Introduction

In this section, we give some background on Cannon-Thurston maps for complete hyperbolic once-punctured-torus bundles, and outline the structure of this article.

Let $\mathbb{H} := \{x + y\mathbf{i} + z\mathbf{j} + t\mathbf{k} \mid x, y, z, t \in \mathbb{R}\}$, the quaternion division ring; for $w = x + y\mathbf{i} + z\mathbf{j} + t\mathbf{k}$, $\overline{w} := x - y\mathbf{i} - z\mathbf{j} - t\mathbf{k}$ and $|w|^2 := x^2 + y^2 + z^2 + t^2 = w\overline{w}$. For any division ring $\mathbb{K}$, we let $\hat{\mathbb{K}} := \mathbb{K} \cup \{\infty\} := (\mathbb{K}^2 - \{(0,0)\})/(\mathbb{K} - \{0\})$, the projective line over $\mathbb{K}$, and we let $\mathrm{PGL}_2(\mathbb{K}) := \mathrm{GL}_2(K)/(\mathrm{center}(\mathbb{K}) - \{0\})$, the projective linear group over $\mathbb{K}$. We let $\mathrm{PGL}_2(\mathbb{K})$ have the left (Möbius) action induced on $\hat{\mathbb{K}}$ by matrix multiplication, where the elements of $\mathbb{K}^2$ are thought of as 2×1 matrices that are transposed to save space. Then $\hat{\mathbb{R}}$ is a circle, $\hat{\mathbb{C}}$ is the Riemann two-sphere, $\hat{\mathbb{H}}$ is a four-sphere, and we view $\hat{\mathbb{R}} \subseteq \hat{\mathbb{C}} \subseteq \hat{\mathbb{H}}$. We also view $\mathrm{PSL}_2(\mathbb{R}) \leqslant \mathrm{PSL}_2(\mathbb{C}) \leqslant \mathrm{PGL}_2(\mathbb{H})$.

Except in some figures, we understand that *hyperbolic three-space* is

$$\mathbf{H}^3 := \{x + y\mathbf{i} + z\mathbf{j} \in \mathbb{H} \mid z > 0\}$$

with the metric given by

$$\mathrm{dist}(x_1 + y_1\mathbf{i} + z_1\mathbf{j},\ x_2 + y_2\mathbf{i} + z_2\mathbf{j}) = \operatorname{arccosh}\left(\tfrac{(x_1-x_2)^2+(y_1-y_2)^2+z_1^2+z_2^2}{2z_1z_2}\right); \tag{1.0.1}$$

see, for example, [29, Theorem 4.6.1]. For $w = x + y\mathbf{i} + z\mathbf{j}$ and any complex numbers a, b, c, d, we have

$$(aw + b)(\overline{w}\,\overline{c} + \overline{d}) = a\,|w|^2\,\overline{c} + aw\overline{d} + b\overline{w}\,\overline{c} + b\overline{d} \in \mathbb{C} + az\mathbf{j}\overline{d} - bz\mathbf{j}\overline{c} = \mathbb{C} + (ad - bc)z\mathbf{j};$$

it follows that $\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathbf{H}^3$. Moreover, $\mathrm{PSL}_2(\mathbb{C})$ is the (orientation-preserving) isometry group of $\mathbf{H}^3$; see, for example, [29, Corollary 4.6.2 and Exercise 4.3.5]. The boundary of $\mathbf{H}^3$ in $\hat{\mathbb{H}}$ is $\hat{\mathbb{C}}$. Except in some figures, we understand that *the hyperbolic plane* is $\mathbf{H}^2 := \{x + z\mathbf{j} \in \mathbb{H} \mid z > 0\}$. Then $\mathrm{PSL}_2(\mathbb{R})$ acts on $\mathbf{H}^2$ giving the isometry group, and the boundary of $\mathbf{H}^2$ in $\hat{\mathbb{H}}$ is $\hat{\mathbb{R}}$. Here, $\mathbf{H}^3$ is a trivial bundle over $\mathbb{R}\mathbf{i}$ with fiber $\mathbf{H}^2$.

In $\mathrm{SL}_2(\mathbb{Z})$, let $D := \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $R := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, and $L := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Let p be a positive integer, let $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ be a sequence of $2p$ positive integers, and let $F := R^{a_1}L^{b_1}R^{a_2}L^{b_2}\cdots R^{a_p}L^{b_p} \in \mathrm{SL}_2(\mathbb{Z})$.

Let $\mathbf{T} := (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2$, a once-punctured torus. We let $\mathrm{SL}_2(\mathbb{Z})$ have the left action on $\mathbf{T}$ that is induced by matrix multiplication, and then F acts as a linear, pseudo-Anosov self-homeomorphism, and D acts as a linear involution.

Let $\mathbf{T}_F := (\mathbf{T} \times \mathbb{R})/((s, r) \sim (F(s), r + 1))$, the mapping torus.

Since the actions of D and F on $\mathbf{T}$ commute, D induces a fiber-preserving involution of $\mathbf{T}_F$, which we again denote by D. Up to fiber-preserving homeomorphism, any orientable once-punctured-torus bundle over the circle with pseudo-Anosov monodromy can be expressed as $\mathbf{T}_F$ or its sister $\mathbf{T}_{DF} = \mathbf{T}_{F^2}/\langle\, DF \,\rangle$, for some sequence $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$.

Let $\mathbf{O} \coloneqq \mathbf{T}/\langle\, D\,\rangle$, a sphere with three double points and a puncture.

The quotient $\mathbf{O}_F \coloneqq \mathbf{T}_F/\langle\, D\,\rangle$ is the mapping torus for the self-homeomorphism of $\mathbf{O}$ induced by F.

We view the (orbifold) fundamental groups $\pi_1(\mathbf{T}_F)$, $\pi_1(\mathbf{T})$, and $\pi_1(\mathbf{O})$ as subgroups of $\pi_1(\mathbf{O}_F)$, and we view D and F as elements of $\pi_1(\mathbf{O}_F)$; here, $\langle\, D, F\,\rangle$ is a rank-two free-abelian subgroup of $\pi_1(\mathbf{O}_F)$.

Let $\widetilde{\mathbf{T}}$ denote the universal cover of $\mathbf{T}$ and $\mathbf{O}$. Let $\widetilde{\mathbf{T}_F}$ denote the universal cover of $\mathbf{T}_F$ and $\mathbf{O}_F$. Then $\widetilde{\mathbf{T}_F}$ is a bundle over $\mathbb{R}$ with fiber $\widetilde{\mathbf{T}}$.

It is a classical result that the topological space $\mathbf{T}$ admits complete hyperbolic structures of finite area, in which the puncture then becomes an ideal vertex. Let us choose any such structure. This gives a discrete faithful copy of the fundamental group $\pi_1(\mathbf{T})$ in $\mathrm{PSL}_2(\mathbb{R})$, and a $\pi_1(\mathbf{T})$-tessellation of $\mathbf{H}^2$ by ideal quadrilaterals, and an identification of $\widetilde{\mathbf{T}}$ with $\mathbf{H}^2$. In particular, $\hat{\mathbb{R}}$ is a $\pi_1(\mathbf{T})$-space and is the boundary of $\widetilde{\mathbf{T}}$. The ideal vertices of $\widetilde{\mathbf{T}}$ are then elements of $\hat{\mathbb{R}}$. Nielsen showed that the $\pi_1(\mathbf{T})$-action on $\mathbf{H}^2 \cup \hat{\mathbb{R}}$ extends to a continuous $\pi_1(\mathbf{O}_F)$-action. We can view $\hat{\mathbb{R}}$ as an abstract oriented topological circle $\mathbb{S}^1$ that does not depend on the choice of hyperbolic structure.

In a famous unfinished work, Jørgensen [20] showed that the topological space $\mathbf{T}_F$ admits a complete hyperbolic structure of finite volume; here, the punctures of all the fibers meld into a single ideal vertex whose link is a torus. Rigorous treatments of part of Jørgensen's results were given in [4], [7], [17], [18], [21] and [28]. Thurston [31] gave a general uniformization theorem for surface bundles; see also [23] and [27]. Jørgensen's results give us a discrete faithful copy of $\pi_1(\mathbf{T}_F)$ in $\mathrm{PSL}_2(\mathbb{C})$, and a $\pi_1(\mathbf{T}_F)$-tessellation of $\mathbf{H}^3$ by ideal tetrahedra, and an identification of $\widetilde{\mathbf{T}_F}$ with $\mathbf{H}^3$ which endows $\mathbf{H}^3$ with the structure of a trivial bundle over $\mathbb{R}$ with fiber $\mathbf{H}^2$. In particular, $\hat{\mathbb{C}}$ is a $\pi_1(\mathbf{T}_F)$-space and is the boundary of $\widetilde{\mathbf{T}_F}$. The ideal vertices of $\widetilde{\mathbf{T}_F}$ are then elements of $\hat{\mathbb{C}}$.

By Mostow's Rigidity Theorem, the discrete embedding of $\pi_1(\mathbf{T}_F)$ in $\mathrm{PSL}_2(\mathbb{C})$ is unique modulo conjugation by an element of $\mathrm{PSL}_2(\mathbb{C})$ and complex conjugation. In particular, the action of D on $\pi_1(\mathbf{T}_F)$ is realized by conjugation by some element of $\mathrm{PSL}_2(\mathbb{C})$, and we obtain a discrete faithful copy of $\pi_1(\mathbf{O}_F)$ in $\mathrm{PSL}_2(\mathbb{C})$, and an action of $\pi_1(\mathbf{O}_F)$ on $\mathbf{H}^3 \cup \hat{\mathbb{C}}$. There exists some $q \in \hat{\mathbb{C}}$ which is fixed by D and F, and, after conjugating the copy of $\pi_1(\mathbf{O}_F)$ in $\mathrm{PSL}_2(\mathbb{C})$ by a suitable element of $\mathrm{PSL}_2(\mathbb{C})$, we have $q = \infty$.

Alperin-Dicks-Porti [8] and Cannon-Dicks [12] introduced, for $F = RL$ and for $F = R^{a_1}L^{b_1}\cdots R^{a_p}L^{b_p}$, respectively, two fractal simple curves, or 'lightning curves', in $\hat{\mathbb{C}}$, denoted $\partial^- A$ and $\partial^+ B$, which have the property that $\partial^- A \cup \partial^+ B$ is the theta-shaped graph which marks the fracturing of $\hat{\mathbb{C}}$ induced by the base-point-deletion break-up of a $\pi_1(\mathbf{O})$-tree that is inside $\mathbf{H}^3$; see Figure 1.0.1. They showed that both $\partial^- A$ and $\partial^+ B$ are limit sets of finitely generated Kleinian semigroups.

McMullen [24], building on work of Minsky [25], proved that there exists a (unique) continuous (surjective) $\pi_1(\mathbf{O}_F)$-map from $\hat{\mathbb{R}}$ to $\hat{\mathbb{C}}$; the map is denoted $\mathrm{CT}_F\colon \hat{\mathbb{R}} \twoheadrightarrow \hat{\mathbb{C}}$ and is called the *Cannon-Thurston map associated to F*. We think of CT_F as a cyclic movie of a pen drawing a curve that passes through every point of $\hat{\mathbb{C}}$. The map CT_F gives a continuous extension to the boundaries for the embedding of any fiber $\widetilde{\mathbf{T}}$ in $\widetilde{\mathbf{T}_F}$. Bowditch [11] showed that CT_F has a certain quotient-map structure conjectured in [14] called *the Cannon-Thurston model*.

Cannon-Dicks [13] introduced the *Cannon-Thurston tessellation of* $\mathbb{C}$ *associated to F*, denoted $\mathrm{CW}(F)$, which is a colored $\langle D, F\rangle$-CW-structure on $\mathbb{C}$ that arises by letting the Cannon-Thurston pen switch ink-color between gray and white each time the pen passes through ∞. They showed how to read the bi-infinite word

$$\cdots\cdots R^{a_1}L^{b_1}R^{a_2}L^{b_2}\cdots R^{a_p}L^{b_p}R^{a_1}L^{b_1}R^{a_2}L^{b_2}\cdots R^{a_p}L^{b_p}\cdots\cdots$$

from $\mathrm{CW}(F)$, and they calculated the color-preserving-or-reversing symmetry group of $\mathrm{CW}(F)$. They also proved that the one-skeleton of $\mathrm{CW}(F)$ is made up of two alternating sequences of simple curves, with one sequence of simple curves being given by the $\langle D, F\rangle$-translates of $\partial^- A - \{\infty\}$, and the other sequence being given by the $\langle D, F\rangle$-translates of $\partial^+ B - \{\infty\}$.

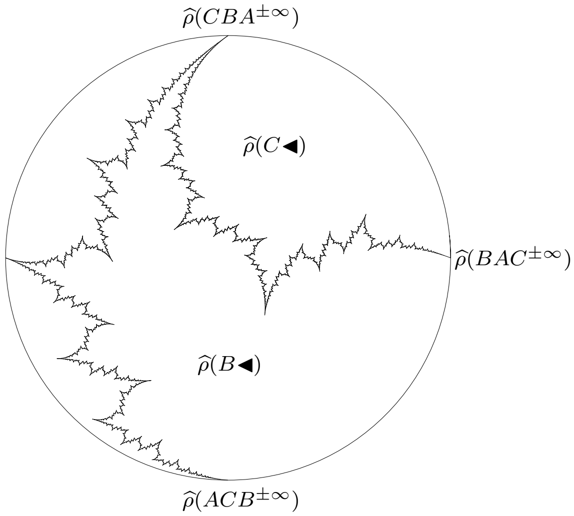
Dicks-Sakuma [16] showed that, for all sufficiently large t_0, the CW-structure of $\mathrm{CW}(F)$ is closely related to the triangulation of $\mathbb{C} + t_0\mathbf{j}$ that is induced by Jørgensen's tessellation of $\mathbf{H}^3 = \widetilde{\mathbf{T}_F}$ by ideal tetrahedra; in particular, it was shown that the two CW-structures on $\mathbb{C}$ share the same vertex set.

In this article, we continue the study of $\partial^- A$, $\partial^+ B$ and $\mathrm{CW}(F)$.

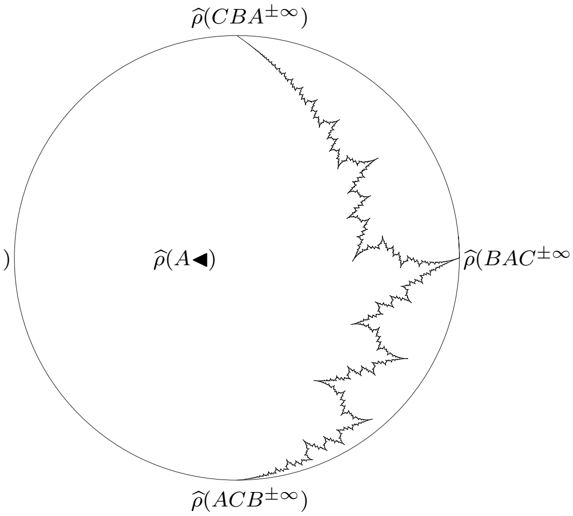
In Section 2, we fix notation that will be used throughout and impose Jørgensen's normalization which specifies a mapping from a certain five-term generating sequence of $\pi_1(\mathbf{O}_F)$ to $\mathrm{PSL}_2(\mathbb{C})$. In particular, $q = \infty$, $\langle D, F\rangle$ acts by translations on $\mathbb{C}$, $D(0) = 1$, and the imaginary part of $F(0)$ is positive. We recall Nielsen's action of $\pi_1(\mathbf{O}_F)$ on $\widetilde{\mathbf{T}}$ and its ideal vertices. We give more details about CT_F, $\partial^- A$, $\partial^+ B$ and $\mathrm{CW}(F)$.

In Section 3, we review work of Helling [19] which explicitly describes, for each positive integer n, the normalized five-term sequence in $\mathrm{PSL}_2(\mathbb{C})$ which corresponds to $F = RL^n$; we draw $\mathrm{CW}(RL^{100})$ in Figure 3.3.2. Further, Helling showed that, as n tends to ∞, the five-term sequence converges to a simple five-term sequence in $\mathrm{PSL}_2(\mathbb{C})$ and the resulting hyperbolic orbifold has a two-fold cover which is the complement of a Whitehead link in a three-sphere. Here, we create an associated *groupoid* whose limit set in $\hat{\mathbb{C}}$ is a stack of Apollonian gaskets, which we superimpose on $\mathrm{CW}(RL^{100})$ in Figure 3.3.2.

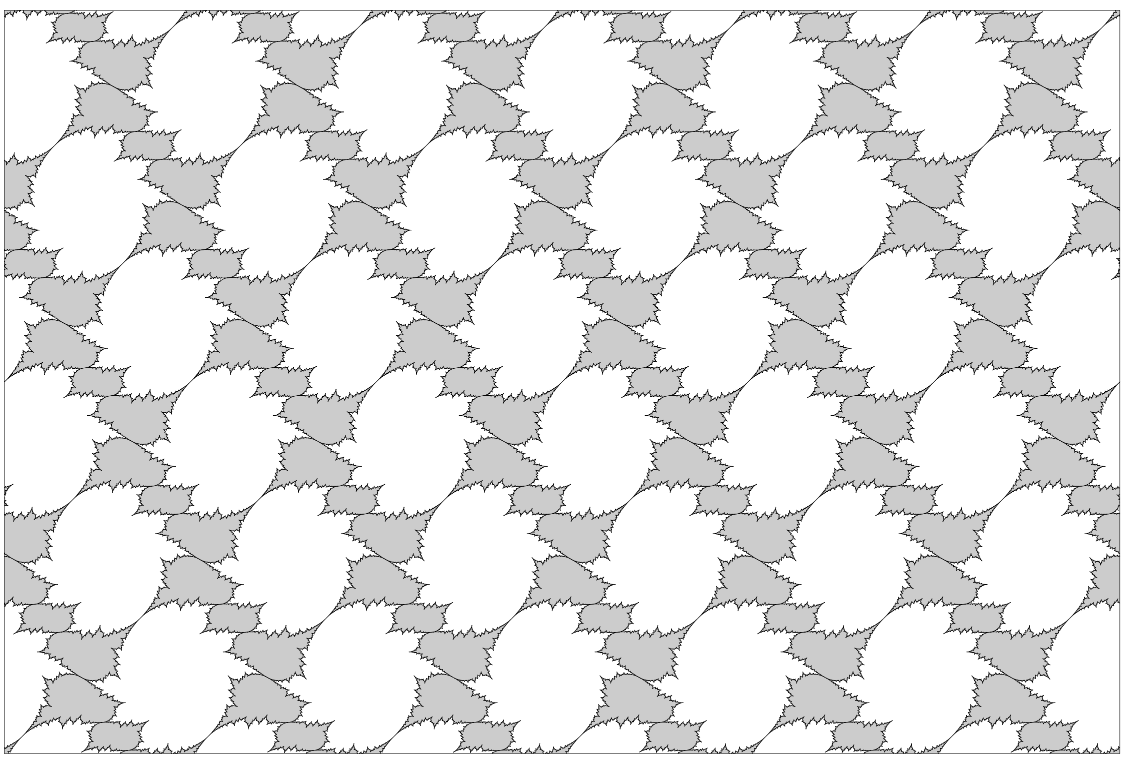
In Section 4, we digress to record a proof of the folklore result that the colored columns of $\mathrm{CW}(F)$ are strictly vertical whenever some odd-length cyclic shift carries $(a_1, b_1, a_2, \ldots, a_p, b_p)$ to itself and/or some even-length cyclic shift carries $(a_1, b_1, a_2 \ldots, a_p, b_p)$ to its reverse, $(b_p, a_p, \ldots, a_2, b_1, a_1)$. In [13], it was seen that these conditions correspond to $\mathrm{CW}(F)$ having a color-reversing symmetry. In [22], it was seen that these conditions correspond to $\mathbf{T}_F$ having an orientation-reversing symmetry.



(1) Hemisphere of $\hat{\mathbb{C}}$ with $\partial^+\mathcal{B}$



(2) Anticonformal hemisphere of $\hat{\mathbb{C}}$ with $\partial\mathcal{A}$



(3) white $= R =$ down and gray $= L =$ up

Figure 1.0.1: Theta-shaped graph and CW(F), for $F = RL^3$

In Sections 5 and 6, we recall a well-known Markov partition of $\mathbf{T}$ associated with F, thought of as the Adler-Weiss CW-structure on a fundamental domain for the $\pi_1(\mathbf{T})$-action on $\widetilde{\mathbf{T}}$ and its ideal vertices. Each two-cell is carried back into the fundamental domain by a unique element of $\pi_1(\mathbf{T})F \subseteq \pi_1(\mathbf{T}_F)$ acting on $\widetilde{\mathbf{T}} \cup \hat{\mathbb{R}}$. These elements, one for each two-cell, are called the ∂^+-*syllables*, and they generate a subsemigroup, called the ∂^+-*semigroup*, in $\pi_1(\mathbf{T}_F)$ whose limit set acting on $\widetilde{\mathbf{T}_F} \cup \hat{\mathbb{C}}$ is $\partial^+ B$. The inverses of the ∂^+-syllables, called the ∂^--*syllables*, generate a subsemigroup, called the ∂^--*semigroup*, whose limit set is $\partial^- A$. Thus, a Markov partition associated to the action of $\pi_1(\mathbf{T}_F)$ on $\mathbf{H}^2$ is giving lightning curves associated to the action of $\pi_1(\mathbf{O})$ on $\mathbf{H}^3$. This contrasts with [16, Theorem 5.3] in which the action of $\pi_1(\mathbf{O})$ on $\hat{\mathbb{C}}$ is used to describe tetrahedra associated to the action of $\pi_1(\mathbf{T}_F)$ on $\mathbf{H}^3$.

We use the Markov partition to order the set of ∂^--syllables and then we create an automaton which outputs a tree of words in the ∂^--syllables; there is a map from the ends of the tree to the points of $\partial^- A$, and the map is bijective except on a countable set, where it is two-to-one. The tree structure then admits depth-first searches which are useful for graphical representation.

Similar results hold for the ∂^+-semigroup.

We also construct an automaton that outputs, in the limit, the boundary of a gray region which is a fundamental domain for the $\langle\, D, F\,\rangle$-action on the gray portion of $\mathrm{CW}(F)$. Then $\mathrm{CW}(F)$ is recovered by (affine) $\langle\, D, F\,\rangle$-translation.

In Section 7, we observe that the ∂^--semigroup and the ∂^+-semigroup have equal critical exponents. We then use techniques of Bishop-Jones and Beardon-Maskit to prove that these critical exponents equal the Hausdorff dimensions of $\partial^- A$ and $\partial^+ B$, respectively, and, hence, these two curves have equal Hausdorff dimensions.

2 Notation

In this section, we fix terminology that will be used throughout, and relate it to the terminology used in Section 1.

2.1 Notation. For two subsets $\mathbb{A}$, $\mathbb{B}$ of a set $\mathbb{X}$, the complement of $\mathbb{A} \cap \mathbb{B}$ in $\mathbb{A}$ will be denoted by $\mathbb{A} - \mathbb{B}$ (and not by $\mathbb{A} \setminus \mathbb{B}$ since, when we have a multiplicative group π acting on a set $\mathbb{Y}$ on the left, we let $\pi \backslash \mathbb{Y}$ denote the set of π-orbits in $\mathbb{Y}$).

We will find it useful to have notation for intervals in $\mathbb{Z}$ that is different from the notation for intervals in $\mathbb{R}$. Let i, $j \in \mathbb{Z}$. We write

$$[i{\uparrow}j] \coloneqq \begin{cases} (i, i+1, \ldots, j-1, j) \in \mathbb{Z}^{j-i+1} & \text{if } i \leqslant j, \\ () \in \mathbb{Z}^0 & \text{if } i > j. \end{cases}$$

Also, $[i{\uparrow}\infty[\ \coloneqq (i, i+1, i+2, \ldots)$. We define $[j{\downarrow}i]$ to be the reverse of the sequence $[i{\uparrow}j]$, that is, $(j, j-1, \ldots, i+1, i)$.

Let v be a symbol.

For each $k \in \mathbb{Z}$, we let v_k denote the ordered pair (v, k).

We let $v_{[i\uparrow j]} := \begin{cases} (v_i, v_{i+1}, \cdots, v_{j-1}, v_j) & \text{if } i \leqslant j, \\ () & \text{if } i > j. \end{cases}$

Also, $v_{[i\uparrow\infty[} := (v_i, v_{i+1}, v_{i+2}, \ldots)$. We define $v_{[j\downarrow i]}$ to be the reverse of the sequence $v_{[i\uparrow j]}$.

Now suppose that $v_{[i\uparrow j]}$ is a sequence *in* a multiplicative group π, that is, there is specified a map of sets $v_{[i\uparrow j]} \to \pi$, and we treat the elements of $v_{[i\uparrow j]}$ as elements of π (possibly with repetitions). We let

$$\Pi v_{[i\uparrow j]} := \begin{cases} v_i v_{i+1} \cdots v_{j-1} v_j \in \pi & \text{if } i \leqslant j, \\ 1 \in \pi & \text{if } i > j. \end{cases}$$

$$\Pi v_{[j\downarrow i]} := \begin{cases} v_j v_{j-1} \cdots v_{i+1} v_i \in \pi & \text{if } j \geqslant i, \\ 1 \in \pi & \text{if } j < i. \end{cases}$$

□

2.2 Notation. Let $\langle\, A, B, C \,\rangle := \langle\, A, B, C, D \mid A^2 = B^2 = C^2 = ABCD = 1 \,\rangle$.

The elements $X := CB$ and $Y := AB$ freely generate an index-two subgroup of $\langle\, A, B, C \,\rangle$, and $XYX^{-1}Y^{-1} = (CBA)^2 = D^2$. It will sometimes be useful to work with other free-generating sequences, such as (BC, BA) and (CA, BA).

By an *end* $E_{[1\uparrow\infty[}$ of $\langle\, A, B, C \,\rangle$, we mean a function $[1\uparrow\infty[\ \to \{A, B, C\}$, $i \mapsto E_i$, such that, for each $i \in [1\uparrow\infty[$, $E_{i+1} \neq E_i$. We then think of $E_{[1\uparrow\infty[}$ as an infinite reduced product $\prod E_{[1\uparrow\infty[}$, and, for each $n \in [0\uparrow\infty[$, we say that $E_{[1\uparrow\infty[}$ *begins with* $\prod E_{[1\uparrow n]} \in \langle\, A, B, C \,\rangle$. The set of ends of $\langle\, A, B, C \,\rangle$ is denoted $\mathfrak{E}\langle\, A, B, C \,\rangle$.

For each $W \in \langle\, A, B, C \,\rangle$, we define the *shadow* of W, denoted $(W\blacktriangleleft)$, as the set of elements of $\mathfrak{E}$ that begin with W; for example, $(CBA)^\infty$ is in the shadow of $CBAC$. Then $\mathfrak{E}\langle\, A, B, C \,\rangle$ is a topological space in which the shadows form a basis of the open sets, and, in a natural way, $\langle\, A, B, C \,\rangle \cup \mathfrak{E}\langle\, A, B, C \,\rangle$ is a compactification of $\langle\, A, B, C \,\rangle$ with the discrete topology.

Let $\operatorname{Aut}\langle\, A, B, C \,\rangle$ denote the automorphism group of $\langle\, A, B, C \,\rangle$ acting on the left as exponents. Thus, for $G \in \operatorname{Aut}\langle\, A, B, C \,\rangle$, we write $G\colon \langle\, A, B, C \,\rangle \to \langle\, A, B, C \,\rangle$, $W \mapsto {}^G W$. We shall use the triple $({}^G A, {}^G B, {}^G C)$ to denote G.

In a natural way, the space $\langle\, A, B, C \,\rangle \cup \mathfrak{E}\langle\, A, B, C \,\rangle$ is an $\operatorname{Aut}\langle\, A, B, C \,\rangle$-space.

We view $\langle\, A, B, C \,\rangle$ as a subgroup of $\operatorname{Aut}\langle\, A, B, C \,\rangle$ with ${}^U W := UWU^{-1}$ for all U, W in $\langle\, A, B, C \,\rangle$. Any $\operatorname{Aut}\langle\, A, B, C \,\rangle$-space is then an $\langle\, A, B, C \,\rangle$-space by restriction.

In $\operatorname{Aut}\langle\, A, B, C \,\rangle$, let $R := (A, BCB, B)$, $L := (B, BAB, C)$. Fix $p \in [1\uparrow\infty[$, fix symbols a, b, and fix sequences $a_{[1\uparrow p]}$, $b_{[1\uparrow p]}$ in $[1\uparrow\infty[$. Let $F := \prod\limits_{i\in[1\uparrow p]} (R^{a_i} L^{b_i}) \in \operatorname{Aut}\langle\, A, B, C \,\rangle$. Then the subgroup $\langle\, A, B, C, F \,\rangle$ of $\operatorname{Aut}\langle\, A, B, C \,\rangle$ can be expressed as a semidirect product, $\langle\, A, B, C \,\rangle \rtimes \langle F \rangle$. □

Work of Nielsen gives an action of $\mathrm{Aut}\langle\, A,B,C\,\rangle$ on a closed disk; see, for example, [12, Section 3]. Here we limit ourselves to a brief review of the main points.

2.3 Review. There exists a (unique) (affine) action of $\mathrm{Aut}\langle\, A,B,C\,\rangle$ on $\mathbb{R}^2$ such that, for all $(x,y)\in\mathbb{R}^2$, $A(x,y)=(-1-x,-y)$, $B(x,y)=(-1-x,-1-y)$, and $C(x,y)=(-x,-1-y)$. Then D acts as multiplication by -1, and D^2 acts trivially. Also, $CB(x,y)=(x+1,y)$, $AB(x,y)=(x,y+1)$, $R(x,y)=(x+y,y)$ and $L(x,y)=(x,x+y)$.

Then $F(x,y)=(f_{1\,1}x+f_{1\,2}y,\, f_{2\,1}x+f_{2\,2}y)$ where $\left(\begin{smallmatrix} f_{1\,1} & f_{1\,2}\\ f_{2\,1} & f_{2\,2}\end{smallmatrix}\right) := \prod\limits_{i\in[1\uparrow p]}\left(\left(\begin{smallmatrix} 1 & a_i\\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0\\ b_i & 1\end{smallmatrix}\right)\right)\in \mathrm{SL}_2(\mathbb{Z})$. In $\mathbb{R}$, let

$$\lambda := \tfrac{f_{1\,1}+f_{2\,2}+\sqrt{(f_{1\,1}+f_{2\,2})^2-4}}{2},\quad \mu_+ := \tfrac{f_{1\,1}-f_{2\,2}+\sqrt{(f_{1\,1}+f_{2\,2})^2-4}}{2f_{2\,1}},\quad \mu_- := \tfrac{f_{1\,1}-f_{2\,2}-\sqrt{(f_{1\,1}+f_{2\,2})^2-4}}{2f_{2\,1}}.$$

Then $\lambda\in\,]1,\infty[\,$, and λ, $\frac{1}{\lambda}$ are the eigenvalues for the (linear) action of F on $\mathbb{R}^2$, and the eigenlines have slopes $\frac{1}{\mu_+}$ and $\frac{1}{\mu_-}$. It is straightforward to check that $\mu_+\in\,]1,\infty[$ and $\mu_-\in\,]-1,0[\,$; see, for example, [12, Proposition 2.5]. The straight lines of slope $\frac{1}{\mu_+}$ will be called *plus-lines*, and the straight lines of slope $\frac{1}{\mu_-}$ will be called *minus-lines*. These give two foliations of $\mathbb{R}^2$ that will be very important in the theory. We will use terms such as *width* and *row* and *from right to left* for the plus-line directions, and terms such as *height* and *column* and *from top to bottom* for the minus-line directions.

In $\mathbb{R}^2$, let $\boldsymbol{\Delta} := \mathrm{hull}((0,0),(-1,0),(0,-1))$. Then $\boldsymbol{\Delta}$ is a fundamental domain for the $\langle\, A,B,C\,\rangle$-action, and, hence, the $\langle\, A,B,C\,\rangle$-translates of $\boldsymbol{\Delta}$ triangulate $\mathbb{R}^2$. We label all the $\langle\, A,B,C\,\rangle$-translates of the line segment $\mathrm{hull}((0,0),(-1,0))$ with the letter A, all the $\langle\, A,B,C\,\rangle$-translates of $\mathrm{hull}((-1,0),(0,-1))$ with the letter B, and all the $\langle\, A,B,C\,\rangle$-translates of $\mathrm{hull}((0,-1),(0,0))$ with the letter C. These labels will allow us to codify elements of $\langle\, A,B,C\,\rangle \cup \mathfrak{E}\langle\, A,B,C\,\rangle$ as sequences of adjacent triangles in $\mathbb{R}^2$ starting at the base triangle. In $\mathbb{R}^2$, for any $W\in\langle\, A,B,C\,\rangle$, starting at $\boldsymbol{\Delta}$ and crossing the edges indicated by reading the letters of W leads to the triangle $W\boldsymbol{\Delta}$.

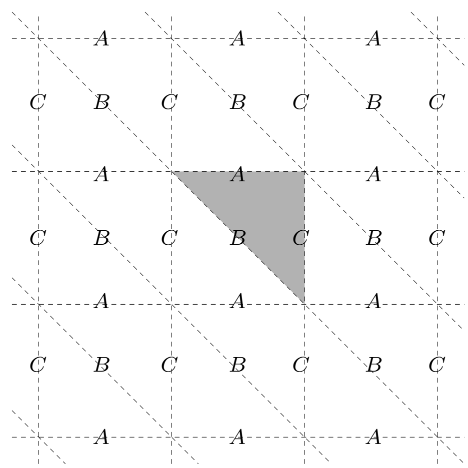
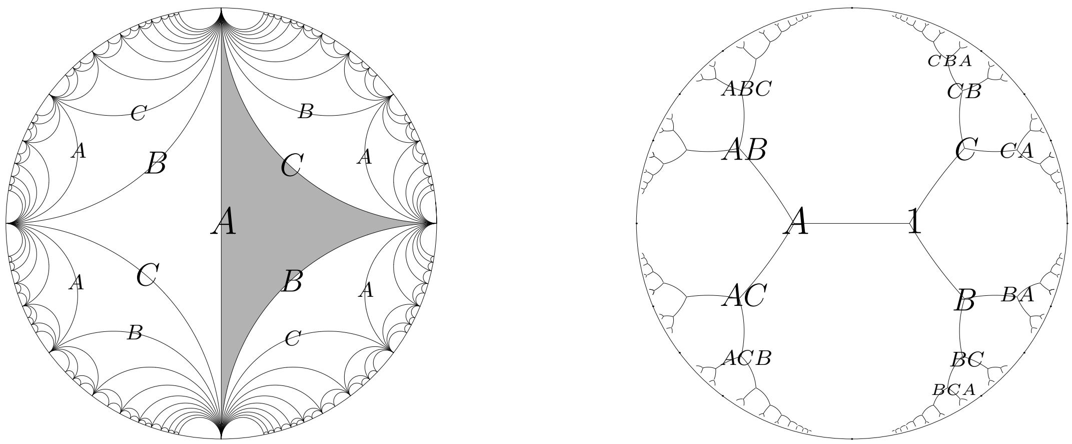


Figure 2.3.1: Labelled edges

We now delete $\mathbb{Z}^2$ from $\mathbb{R}^2$. The universal cover of $\mathbb{R}^2-\mathbb{Z}^2$ is an open disk, denoted $\mathbf{D}^\circ$.

Deleting $\mathbb{Z}$ from $\mathbb{R}^2$ converts some of the plus-lines into two half-lines called *half-plus-lines*. We define *half-minus-lines* similarly. A *half-plus-eigenline* is a half-plus-line incident to the origin, and similarly for the other concepts. The foliation of $\mathbb{R}^2-\mathbb{Z}^2$ by minus-lines and half-minus-lines lifts to a foliation of $\mathbf{D}^\circ$ called the *minus foliation*. Similarly, the foliation of $\mathbb{R}^2-\mathbb{Z}^2$ by plus-lines and half-plus-lines lifts to a foliation of $\mathbf{D}^\circ$ called the *plus foliation*.



(1) Universal cover $\mathbf{D}^\circ$ of $\mathbb{R}^2 - \mathbb{Z}^2$ (2) $\mathfrak{E}\langle A, B, C \rangle \twoheadrightarrow \mathbb{S}^1$

Figure 2.3.2: $\mathrm{Aut}\langle A, B, C \rangle$ acts on a closed disk

Deleting $\mathbb{Z}$ from $\mathbb{R}^2$ deletes the end-points from the labelled line segments. The resulting segments lift to labelled curves in $\mathbf{D}^\circ$; see Figure 2.3.2(1). Let $\mathbf{\Delta}' \coloneqq \mathbf{\Delta} - \mathbb{Z}^2$, an ideal triangle in $\mathbb{R}^2 - \mathbb{Z}^2$. We choose a lift $\tilde{\mathbf{\Delta}}'$ of $\mathbf{\Delta}'$ in $\mathbf{D}^\circ$ bounded by labelled curves. We can now codify elements of $\langle A, B, C \rangle \cup \mathfrak{E}\langle A, B, C \rangle$ as sequences of adjacent ideal triangles in $\mathbf{D}^\circ$; for example, for any $W \in \langle A, B, C \rangle$, starting at $\tilde{\mathbf{\Delta}}'$ and crossing the edges indicated by reading the letters of W leads to $W\tilde{\mathbf{\Delta}}'$. In a similar way, each element of $\mathfrak{E}\langle A, B, C \rangle$ corresponds to a unique infinite sequence of ideal triangles in $\mathbf{D}^\circ$, and it is known that we can compactify $\mathbf{D}^\circ$ by attaching to each such chain a limiting point with the appropriate label in $\mathfrak{E}\langle A, B, C \rangle$, and then identifying, for each $W \in \langle A, B, C \rangle$, $WD^{-\infty}$ and WD^{∞} to form a single point, denoted $WD^{\pm\infty}$. We have then defined a boundary $\mathbb{S}^1$ of $\mathbf{D}^\circ$ as the quotient space obtained from $\mathfrak{E}\langle A, B, C \rangle$ by identifying $WD^{-\infty}$ and WD^{∞}, for each $W \in \langle A, B, C \rangle$; see Figure 2.3.2(2). It follows that $\mathbb{S}^1$ is an $\mathrm{Aut}\langle A, B, C \rangle$-space.

The compact space $\mathbf{D}^\circ \cup \mathbb{S}^1$ will be denoted $\mathbf{D}$; it is a topological closed disk. It can be shown that $\mathbf{D}$ has a (unique) $\mathrm{Aut}\langle A, B, C \rangle$-action that simultaneously agrees with the action on $\mathbb{S}^1$ and lifts to $\mathbf{D}^\circ$ the affine $\mathrm{Aut}\langle A, B, C \rangle$-action on $\mathbb{R}^2 - \mathbb{Z}^2$. The set of ideal vertices of $\mathbf{D}$ is $\langle A, B, C \rangle D^{\pm\infty}$. In $\mathbf{D}$, we let $\tilde{\mathbf{\Delta}}$ denote $\tilde{\mathbf{\Delta}}'$ together with its three ideal vertices, $(ABC)^{\pm\infty}$, $(BCA)^{\pm\infty}$, and $(CAB)^{\pm\infty}$. We give $\mathbb{S}^1$ the orientation that keeps $\mathbf{D}^\circ$ to the left of $\mathbb{S}^1$. □

2.4 Notation. We have altered our notation, but in a compatible way.

In Section 1, we used the four symbols D, R, L and F to denote elements of $\mathrm{SL}_2(\mathbb{Z})$, and then lifted D and F to elements of $\pi_1(\mathbf{O}_F)$.

In Notation 2.2, we used the same four symbols, D, R, L and F, to denote elements of

$\mathrm{Aut}\langle A, B, C \rangle$.

In Review 2.3, we then fixed an action of $\mathrm{Aut}\langle A, B, C \rangle$ on $\mathbb{R}^2$. The actions of D, R, L and F on $\mathbb{R}^2$ are linear and have the matrices that were assigned to them in Section 1.

The action of $\langle CB, AB \rangle = \langle X, Y \rangle$ on $\mathbb{R}^2$ is that of $\mathbb{Z}^2$ by translation. We have

$$\mathbf{T} \coloneqq (\mathbb{R}^2 - \mathbb{Z}^2)/\mathbb{Z}^2 = \langle X, Y \rangle \backslash (\mathbb{R}^2 - \mathbb{Z}^2),$$

and $\pi_1(\mathbf{T}) = \langle X, Y \rangle \leqslant \langle A, B, C \rangle$. Also, $\mathbf{O} \coloneqq \mathbf{T}/\langle D \rangle = \langle A, B, C \rangle \backslash (\mathbb{R}^2 - \mathbb{Z}^2)$, and $\pi_1(\mathbf{O}) = \langle A, B, C \rangle$. Now $\pi_1(\mathbf{T}_F) = \langle X, Y, F \rangle$ and $\pi_1(\mathbf{O}_F) = \langle A, B, C, F \rangle$. □

2.5 Notation. Following Jørgensen, as in [7, Lemma 2.3.7(2)], any element $(s, x, y, z) \in \mathbb{C}^4$ satisfying

$$x^2 + y^2 + z^2 = xyz, \quad y \neq 0, \quad \mathrm{Im}(s) > 0, \tag{2.5.1}$$

determines five elements of $\mathrm{SL}_2(\mathbb{C})$

$$\tilde{A} \coloneqq \left(\begin{smallmatrix} -z/y & (x-yz)/y^2 \\ x & z/y \end{smallmatrix}\right),\ \tilde{B} \coloneqq \left(\begin{smallmatrix} 0 & -1/y \\ y & 0 \end{smallmatrix}\right),\ \tilde{C} \coloneqq \left(\begin{smallmatrix} x/y & (z-xy)/y^2 \\ z & -x/y \end{smallmatrix}\right),\ \tilde{D} \coloneqq \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right),\ \tilde{F} \coloneqq \left(\begin{smallmatrix} 1 & s \\ 0 & 1 \end{smallmatrix}\right), \tag{2.5.2}$$

which then satisfy $\tilde{A}^2 = \tilde{B}^2 = \tilde{C}^2 = \tilde{A}\tilde{B}\tilde{C}\tilde{D} = -\mathbf{I}_2, \quad {}^{\tilde{F}}\tilde{D} = \tilde{D}$.

For $\tilde{X} \coloneqq \tilde{C}\tilde{B}^{-1}$, $\tilde{Y} \coloneqq \tilde{A}\tilde{B}^{-1}$, we have $(\mathrm{tr}(\tilde{X}), \mathrm{tr}(\tilde{X}\tilde{Y}), \mathrm{tr}(\tilde{Y})) = (x, y, z)$.

By results of Jørgensen and Mostow mentioned in Section 1, together with another result of Jørgensen [7, Lemma 2.3.7(2)], there exists a unique discrete faithful representation $\rho\colon \langle A, B, C, F \rangle \to \mathrm{PSL}_2(\mathbb{C})$ such that there exists some $(s, x, y, z) \in \mathbb{C}^4$ satisfying (2.5.1) and, with (2.5.2), $\rho(W) = \pm\tilde{W}$ for each $W \in \{A, B, C, D, F\}$. In $\mathrm{SL}_2(\mathbb{C})$, $\tilde{F}$ normalizes $\langle \tilde{A}, \tilde{B}, \tilde{C} \rangle$ and we may assume that $\tilde{F}$ acts on $\langle \tilde{A}, \tilde{B}, \tilde{C} \rangle$ as $\prod_{i \in [1 \uparrow p]} (\tilde{R}^{a_i} \tilde{L}^{b_i})$ where ${}^{\tilde{R}}(\tilde{A}, \tilde{B}, \tilde{C}) = (\tilde{A}, \tilde{B}\tilde{C}\tilde{B}^{-1}, \tilde{B})$ and ${}^{\tilde{L}}(\tilde{A}, \tilde{B}, \tilde{C}) = (\tilde{B}, \tilde{B}^{-1}\tilde{A}\tilde{B}, \tilde{C})$, as in Artin's braid action.

Throughout, we use ρ to endow $\hat{\mathbb{C}}$ with the structure of an $\langle A, B, C, F \rangle$-space. When we wish to emphasize F, we shall denote this ρ by ρ_F. □

2.6 Notation. McMullen [24], using results of Minsky [25], showed that there exists a (unique) (surjective) continuous $\langle A, B, C, F \rangle$-map $\widehat{\rho}\colon \mathfrak{E}\langle A, B, C \rangle \twoheadrightarrow \hat{\mathbb{C}}$. See also [8] for the case $F = RL$. Moreover, $\widehat{\rho}$ factors through the surjective continuous $\mathrm{Aut}\langle A, B, C \rangle$-map $\mathfrak{E}\langle A, B, C \rangle \twoheadrightarrow \mathbb{S}^1$ of Review 2.3 giving a surjective continuous $\langle A, B, C, F \rangle$-map $\widehat{\rho}\colon \mathbb{S}^1 \twoheadrightarrow \hat{\mathbb{C}}$, a sphere-filling curve. Choosing a complete, finite-area, hyperbolic structure on the orbifold $\mathbf{O}$ determines an identification of $\mathbf{D}^\circ$ with $\mathbf{H}^2$, and an identification of $\mathbb{S}^1$ with $\hat{\mathbb{R}}$, and we obtain the Cannon-Thurston map associated to F, $\mathrm{CT}_F\colon \hat{\mathbb{R}} \to \hat{\mathbb{C}}$, discussed in Section 1.

By results in [24] and [11], the continuous $\langle A, B, C, F \rangle$-map $\widehat{\rho}\colon \mathbb{S}^1 \to \hat{\mathbb{C}}$ extends to both a continuous $\langle A, B, C, F \rangle$-map $\widehat{\rho}^{\,-}\colon \mathbf{D} \to \hat{\mathbb{C}}$ which is constant on each minus-foliation curve, and a continuous $\langle A, B, C, F \rangle$-map $\widehat{\rho}^{\,+}\colon \mathbf{D} \to \hat{\mathbb{C}}$ which is constant on each plus-foliation curve. In fact, these two maps are involved in the construction of $\widehat{\rho}\colon \mathbb{S}^1 \to \hat{\mathbb{C}}$.

We distinguish three points in $\hat{\mathbb{C}}$:

$\widehat{\rho}((CBA)^{\pm\infty}) = \widehat{\rho}((ABC)^{\pm\infty}) = \infty;$
$\widehat{\rho}((ACB)^{\pm\infty}) = \widehat{\rho}((BCA)^{\pm\infty}) = A(\infty)$; and,
$\widehat{\rho}((BAC)^{\pm\infty}) = \widehat{\rho}((CAB)^{\pm\infty}) = C(\infty).$

Let $W \in \langle\, A, B, C\,\rangle - \{1\}$ and consider the image $\widehat{\rho}(W\blacktriangleleft)$ of the shadow $(W\blacktriangleleft)$ under the map $\widehat{\rho}\colon \mathfrak{E}\langle\, A, B, C\,\rangle \twoheadrightarrow \hat{\mathbb{C}}$. In [12], results of Bowditch, Cannon, Thurston, and others, were used to show that $\widehat{\rho}(W\blacktriangleleft)$ is a closed Jordan disk in $\hat{\mathbb{C}}$. The oriented boundary of $\widehat{\rho}(W\blacktriangleleft)$ which keeps $\widehat{\rho}(W\blacktriangleleft)$ on the left will be denoted ∂W. Let B' denote the last letter of W, let $W' = WB'$ so $W = W'B'$ in reduced form, and let (A', B', C') be the unique cyclic permutation of (A, B, C) which has B' in the middle. It was found that the two points $W'\widehat{\rho}((B'C'A')^{\pm\infty})$, $W'\widehat{\rho}((B'A'C')^{\pm\infty})$ lie on ∂W and, hence, divide ∂W into two simple curves, denoted $\partial^- W$ and $\partial^+ W$. Thus $\partial W = W'\partial B'$, and $\partial B'$ is formed from the sequence $\widehat{\rho}((B'C'A')^{\pm\infty})$, $\partial^- B'$, $\widehat{\rho}((B'A'C')^{\pm\infty})$, $\partial^+ B'$.

It was also shown that $\widehat{\rho}^{\,-}(\tilde{\boldsymbol{\Delta}}) = \partial^- A$, and that $\widehat{\rho}^{\,-}$ induces homeomorphisms from the A-, B-, and C-labelled curves in $\tilde{\boldsymbol{\Delta}}$ to $\partial^- A$, $\partial^- B$, and $\partial^- C$, respectively, if the labelled curves are oriented to keep $\tilde{\boldsymbol{\Delta}}$ on the left. Similarly, $\widehat{\rho}^{\,+}(\tilde{\boldsymbol{\Delta}}) = \partial^+ B$, and $\widehat{\rho}^{\,+}$ induces homeomorphisms from the A-, B- and C-labelled curves in $\tilde{\boldsymbol{\Delta}}$ to reverse$(\partial^+ A)$, reverse$(\partial^+ B)$, reverse$(\partial^+ C)$, respectively, if the labelled curves are oriented to keep $\tilde{\boldsymbol{\Delta}}$ on the left. It then follows that reverse$(\partial^- A)$ is the concatenation of $\partial^- B$, $\partial^- C$, and that reverse$(\partial^+ B)$ is the concatenation of $\partial^+ C$, $\partial^+ A$. In [12, Theorem 5.6(ii)], it was shown that

$$\text{(2.6.1)}\qquad \partial^- A \cap \partial^+ B = \{\widehat{\rho}((CBA)^{\pm\infty}), \widehat{\rho}((ACB)^{\pm\infty}), \widehat{\rho}((BAC)^{\pm\infty})\} = \{\infty, A(\infty), C(\infty)\}.$$

Thus $\partial^- A \cup \partial^+ B$ is a theta-shaped graph that marks out the break-up of $\hat{\mathbb{C}}$ into $\widehat{\rho}(A\blacktriangleleft)$, $\widehat{\rho}(B\blacktriangleleft)$ and $\widehat{\rho}(C\blacktriangleleft)$. See Figure 1.0.1(1), (2) for the case $F = RL^3$.

It was also shown that each of the curves $\partial^- W$, $\partial^+ W$ is the limit set of a certain finitely generated subsemigroup of $\langle\, A, B, C, F\,\rangle$, and this semigroup of Möbius transformations acting on the curve was seen to be isomorphic to a semigroup of affine transformations acting on a line segment. □

2.7 Notation. In the continuation [13] of [12], the pen tracing out the sphere-filling curve CT_F switches ink-color between gray and white each time the pen passes through the point $\widehat{\rho}(D^{\pm\infty}) = \infty$. This gives the *Cannon-Thurston tessellation* $\mathrm{CW}(F)$ of the plane $\hat{\mathbb{C}} - \{\infty\} = \mathbb{C}$. By convention, white regions correspond to the letter R and indicate where the sphere-filling curve is proceeding downwards; a mnemonic is that 'white', 'right' and 'down' are longer than 'gray', 'left' and 'up', respectively; also 'white' and 'right' rhyme, and each has five letters.

It was found in [13] that the $\langle\, D, F\,\rangle$-translates of $\partial^- A \cup \partial^+ B - \{\infty\}$ give the one-skeleton of $\mathrm{CW}(F)$. See Figure 1.0.1(3) for the case $F = RL^3$. □

3 Examples: RL^n

This section, which is not used in any subsequent sections, outlines work of Helling [19] that gives an explicit description of the normalized discrete representation ρ_{RL^n} specified in Notation 2.5, and considers the limit as n tends to ∞. We draw $\mathrm{CW}(RL^{100})$ together with some Apollonian gaskets related to n tending to ∞.

Let us mention that Gueritaud [17, Section 10] gives much useful geometric information about the more general case $F = R^a L^b$.

The following is straightforward algebraically and is mildly different from other treatments that can be found in the literature.

3.1 Lemma. *Let $n \in [1{\uparrow}\infty[$ and let $(s, x, y, z) \in \mathbb{C}^4$. Then the following are equivalent.*

(a) *There exists $w \in \mathbb{C}$ such that*

$$(w^{n+2} + 1)^2 = w^n(w^2 - 1)^2 \tag{3.1.1}$$

and $w \neq -1$, (and, hence, $w(w^4 - 1)(w^{2n} - 1)(w^{n+2} + 1) \neq 0$,) and

$$x = \tfrac{w^{n+2}+w^n+w^2+1}{w^{n+2}+1}, \quad y = z = w + w^{-1}, \quad s = \tfrac{w^{n+2}-w^n-w^2+1}{w^{n+2}+w^n+w^2+1}. \tag{3.1.2}$$

(b) *$x^2 + y^2 + z^2 = xyz$, and $y \neq 0$, and the elements*

$$\tilde{A} := \begin{pmatrix} -z/y & (x-yz)/y^2 \\ x & z/y \end{pmatrix}, \tilde{B} := \begin{pmatrix} 0 & -1/y \\ y & 0 \end{pmatrix}, \tilde{C} := \begin{pmatrix} x/y & (z-xy)/y^2 \\ z & -x/y \end{pmatrix}, \tilde{D} := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \tilde{F} := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix},$$

of $\mathrm{SL}_2(\mathbb{C})$ satisfy $\tilde{A}^2 = \tilde{B}^2 = \tilde{C}^2 = \tilde{A}\tilde{B}\tilde{C}\tilde{D} = -\mathbf{I}_2$, and $\tilde{F}$ acts on $\langle \tilde{A}, \tilde{B}, \tilde{C} \rangle$ as $\tilde{R}\tilde{L}^n$, where ${}^{\tilde{R}}(\tilde{A}, \tilde{B}, \tilde{C}) = (\tilde{A}, \tilde{B}\tilde{C}\tilde{B}^{-1}, \tilde{B})$ and ${}^{\tilde{L}}(\tilde{A}, \tilde{B}, \tilde{C}) = (\tilde{B}, \tilde{B}^{-1}\tilde{A}\tilde{B}, \tilde{C})$, that is, ${}^{\tilde{F}}(\tilde{A}, \tilde{B}, \tilde{C}) = (\tilde{A}(\tilde{A}\tilde{B}\tilde{C}\tilde{B})^n, (\tilde{B}\tilde{C}\tilde{B}\tilde{A})^{n+1}\tilde{A}, \tilde{B})$.

Proof. (a) $\Rightarrow$ (b). Suppose that w is an element of $\mathbb{C}$ such that (3.1.1) holds and $w \neq -1$.

We first show that $w(w^4 - 1)(w^{2n} - 1)(w^{n+2} + 1) \neq 0$. Clearly $w \neq 0$ and $w \neq 1$ (and $w \neq -1$). If $w^2 = -1$, then $w^n(-1 - 1)^2 = (-w^n + 1)^2 = (-1)^n - 2w^n + 1$, which implies $6w^n = 1 + (-1)^n$, a contradiction; hence $w^2 \neq -1$. If $w^n = 1$, then $(w^2+1)^2 = (w^2-1)^2$ which implies $4w^2 = 0$, a contradiction; hence, $w^n \neq 1$. If $w^n = -1$, then $(-w^2 + 1)^2 = -(w^2 - 1)^2$ which implies $2(w^2 - 1)^2 = 0$, a contradiction; hence, $w^n \neq -1$. If $w^{n+2} = -1$, then $w^n(w^2-1)^2 = 0$, a contradiction; hence $w^{n+2} \neq -1$. Thus, $w(w^4-1)(w^{2n}-1)(w^{n+2}+1) \neq 0$, as desired.

By the preceding paragraph, the values in (3.1.2) are well-defined non-zero elements of $\mathbb{C}$.

Let $v := \frac{w^{n+2}+1}{w^2-1} \in \mathbb{C}$.

We see that $v^2 = w^n \neq 0, \pm 1$ and that $v(w^2 - 1) = w^{n+2} + 1 = v^2 w^2 + 1$. Hence, $-\frac{v+1}{v^2-v} = w^2 \neq -1$. In particular, the matrix $P := \begin{pmatrix} v+1 & v-v^2 \\ v^2-2v-1 & v^2-2v-1 \end{pmatrix}$ is invertible.

Using $w^n = v^2$ and $w^2 = -\frac{v+1}{v^2-v}$, we find that $x^2+y^2+z^2 = xyz$, and, hence, with the notation as in (b), $\tilde{A}^2 = \tilde{B}^2 = \tilde{C}^2 = \tilde{A}\tilde{B}\tilde{C}\tilde{D} = -\mathbf{I}_2$, and ${}^{\tilde{F}}\tilde{D} = \tilde{D}$.

We also find that $\tilde{F}\tilde{C} = \tilde{B}\tilde{F}$, and that $\tilde{A}^{-1}\tilde{F}\tilde{A}\tilde{F}^{-1}P = P\left(\begin{smallmatrix} v^2 & 0 \\ 0 & v^{-2}\end{smallmatrix}\right)$, and that $\tilde{A}\tilde{B}\tilde{C}\tilde{B}P = P\left(\begin{smallmatrix} w & 0 \\ 0 & w^{-1}\end{smallmatrix}\right)$. Since $v^2 = w^n$ and P is invertible, we see that $\tilde{A}^{-1}\tilde{F}\tilde{A}\tilde{F}^{-1} = (\tilde{A}\tilde{B}\tilde{C}\tilde{B})^n$.

We now have ${}^{\tilde{F}}\tilde{A} = \tilde{A}(\tilde{A}\tilde{B}\tilde{C}\tilde{B})^n$, ${}^{\tilde{F}}\tilde{C} = \tilde{B}$ and ${}^{\tilde{F}}\tilde{D} = \tilde{D}$, and (b) follows.

(b) $\Rightarrow$ (a). Suppose that (b) holds.

Using $\tilde{F}\tilde{C} = \tilde{B}\tilde{F}$, we find that $z = y$, $x = -sy^2$ and $y^2(s+s^2) = -2$. In particular, $s \neq 0$ and $s \neq -1$.

Let $v := \frac{s+\sqrt{2s^2+2s+1}}{s+1} \in \mathbb{C}$. Then $(s+1)v^2 - 2sv - (s+1) = 0$. If $v = 0$, then $s = -1$, a contradiction; hence, $v \neq 0$. If $v = \pm 1$, then $s = 0$, a contradiction; hence, $v \neq \pm 1$. Notice that $s(v^2-2v-1) = -(v^2-1) \neq 0$, and, hence, $s = -\frac{v^2-1}{v^2-2v-1}$.

Let $w := -\frac{(v+1)y}{v^2-2v-1} \in \mathbb{C}$. Then $y = -\frac{(v^2-2v-1)w}{v+1}$. Since $y^2(s+s^2) = -2$, it follows that $v^2w^2 - vw^2 + v + 1 = 0$, that is, $(v-v^2)w^2 = v+1$. We can now calculate that

$$(s,x,y,z) = \left(-\frac{v^2-1}{v^2-2v-1}, -\frac{v^2-2v-1}{v}, -\frac{(v^2-2v-1)w}{v+1}, -\frac{(v^2-2v-1)w}{v+1}\right). \tag{3.1.3}$$

The matrix $Q := \left(\begin{smallmatrix} v+1 \\ v^2-2v-1\end{smallmatrix}\right)$ satisfies $\tilde{A}^{-1}\tilde{F}\tilde{A}\tilde{F}^{-1}Q = Qv^2$ and $\tilde{A}\tilde{B}\tilde{C}\tilde{B}Q = Qw$, and, hence, $(\tilde{A}\tilde{B}\tilde{C}\tilde{B})^nQ = Qw^n$. Since $\tilde{A}^{-1}\tilde{F}\tilde{A}\tilde{F}^{-1} = (\tilde{A}\tilde{B}\tilde{C}\tilde{B})^n$ and Q is non-zero, we see that $v^2 = w^n$. Since $(v-v^2)w^2 = v+1$, we see that $(v-w^n)w^2 = v+1$, that is, $v(w^2-1) = w^{n+2}+1$. On squaring both sides, we have $(w^{n+2}+1)^2 = v^2(w^2-1)^2 = w^n(w^2-1)^2$, and (3.1.1) holds.

If $w = -1$, then n is odd and $v = \pm\mathbf{i}$. Here, $\tilde{A}\tilde{B}\tilde{C}\tilde{B} = \left(\begin{smallmatrix} -2 & -\frac{1}{2} \\ 2 & 0\end{smallmatrix}\right)$ and $\tilde{A}^{-1}\tilde{F}\tilde{A}\tilde{F}^{-1} = \left(\begin{smallmatrix} 1\pm 2\mathbf{i} & 1\pm\mathbf{i} \\ -4\mp 4\mathbf{i} & -3\mp 2\mathbf{i}\end{smallmatrix}\right)$. Now $\tilde{A}^{-1}\tilde{F}\tilde{A}\tilde{F}^{-1} \neq (\tilde{A}\tilde{B}\tilde{C}\tilde{B})^n$, since the right-hand side has real coordinates and the left-hand side has non-real coordinates. This is a contradiction. Hence, $w \neq -1$.

It is clear that $w \neq 1$. By substituting $v^2 = w^n$ and $v = \frac{w^{n+2}+1}{w^2-1}$ in (3.1.3), we can recover (3.1.2). □

3.2 Review. In this subsection, $\mathbf{e}^{\mathbf{i}\pi} = -1$, that is, here π does not denote a group.

(i). Let $n \in [1{\uparrow}\infty[$ and let $F = RL^n \in \mathrm{Aut}\langle A, B, C\rangle$.

We now describe Helling's choice of solution of (3.1.1).

The solutions of (3.1.1) occur in $n+2$ conjugate-inverse pairs, that is, pairs with equal arguments and mutually inverse norms. If n is odd, one conjugate-inverse pair of solutions of (3.1.1) is given by the repeated root -1 and it gives no representations. Inverting a solution of (3.1.1) produces another solution which gives the same representation in $\mathrm{SL}_2(\mathbb{C})$. Complex-conjugating a solution produces another solution which gives the complex-conjugate representation in $\mathrm{SL}_2(\mathbb{C})$. If n is even, changing the sign of a solution produces another solution which gives the same representation in $\mathrm{PSL}_2(\mathbb{C})$.

The family of solutions of

$$(w^{n+2}+1)^2 = 0 \tag{3.2.1}$$

is $\mathbf{e}^{\frac{\pi\mathbf{i}}{n+2}}\mathbf{e}^{\frac{2\pi\mathbf{i}}{n+2}[0\uparrow(n+1)]}$ with multiplicity 2, the vertex set of a regular $(n+2)$-gon. The set

$$\{w\in\mathbb{C}-\{0\}\mid w^{n+2}+2+w^{-(n+2)}\in(w^2-2+w^{-2})[0,1]\,\}, \tag{3.2.2}$$

equals $\{w\in\mathbb{C}\mid (w^{n+2}+1)^2\in w^n(w^2-1)^2[0,1]\,\}$ which is the union of $n+2$ paths connecting the solutions of (3.2.1) to the solutions of (3.1.1). If n is odd, we have two degenerate paths at $w=-1$. For w near -1, $\frac{w^{n+2}+1}{w^2-1}$ is near $\frac{n+2}{2}$ and $\frac{(w^{n+2}+1)^2}{w^n(w^2-1)^2}$ is near $-\frac{(n+2)^2}{4}$. It follows that any other path is a simple curve.

If w lies on the circle $\mathbf{e}^{\mathbf{i}[0,2\pi]}$, then, in $\mathbb{R}$, $w^{n+2}+w^{-(n+2)}+2\geqslant 0\geqslant w^2+w^{-2}-2$, with equality throughout if and only if $w=-1$ and n is odd. It follows that the paths minus their initial vertices do not meet $\mathbf{e}^{\mathbf{i}[0,2\pi]}$. The conjugate-inverse of any path is another path with the same initial vertex. Hence, deleting $\mathbf{e}^{\mathbf{i}[0,2\pi]}$ from $\mathbb{C}$ separates the paths minus their initial vertices into two conjugate-inverse sets.

Recall that if $w\neq 0$ then $|w|^{n+2}+|w|^{-n-2}\geqslant|w|^2+|w|^{-2}$ with equality if and only if $|w|=1$, since, for $m\in[0\uparrow\infty[$,

$$|w|^{m+1}+|w|^{-m-1}=|w|^m+|w|^{-m}+|w|^{-m-1}(|w|^{2m+1}-1)(|w|-1)\geqslant|w|^m+|w|^{-m}\,.$$

If w lies in the star $\mathbf{e}^{\frac{2\pi\mathbf{i}}{n+2}[0\uparrow(n+1)]}[0,\infty[$, then w does not lie in (3.2.2) since this is clear if $w=0$, and if $w\neq 0$ then, in $\mathbb{R}$,

$$\left|w^{n+2}+2+w^{-n-2}\right|=|w|^{n+2}+2+|w|^{-n-2}\geqslant|w|^2+2+|w|^{-2}\geqslant\left|w^2-2+w^{-2}\right|,$$

with equality throughout if and only if $w=\pm\mathbf{i}$ and $n\equiv 2(\mathrm{mod}\ 4)$, where we are outside (3.2.2).

Deleting $\mathbf{e}^{\mathbf{i}[0,2\pi]}\ \cup\ \mathbf{e}^{\frac{2\pi\mathbf{i}}{n+2}[0\uparrow(n+1)]}[0,\infty[$ from $\mathbb{C}$ leaves $2n+4$ components and deletes the initial vertices of the paths; the preceding paragraphs show that each path minus its initial vertex lies entirely in one of the components, and each component contains at most one path minus its initial vertex, and the closure of each component contains the initial vertex of two paths. In particular, no two paths meet, except that the conjugate-inverse pairs have a common initial vertex.

We define w to be the unique solution of (3.1.1) that lies in (the interior of) the region $\mathbf{e}^{\mathbf{i}\left[0,\frac{2\pi}{n+2}\right]}[1,\infty[$, and we impose (3.1.2), and this specifies a representation.

Let θ denote the argument of w. Thus, $\theta\in\left]0,\frac{2\pi}{n+2}\right[$ and $w=|w|\,\mathbf{e}^{\mathbf{i}\theta}$. Let $w^{\frac{1}{2}}:=|w|^{\frac{1}{2}}\,\mathbf{e}^{\mathbf{i}\frac{\theta}{2}}$. We can express (3.1.1) as

$$(w^{\frac{n+2}{2}}+w^{-\frac{n+2}{2}})^2=(w-w^{-1})^2. \tag{3.2.3}$$

Since $\frac{\theta}{2}\in\left]0,\frac{\pi}{n+2}\right[$, it is not difficult to see that $\mathrm{Im}(w^{\frac{n+2}{2}}+w^{-\frac{n+2}{2}})>0$ and $\mathrm{Im}(w-w^{-1})>0$. Together with (3.2.3) this implies that

$$w^{\frac{n+2}{2}}+w^{-\frac{n+2}{2}}=w-w^{-1}, \tag{3.2.4}$$

and, hence, $w^{n+2}+1 = w^{\frac{n}{2}}(w^2-1)$. It can then be seen that $x = 2-w^{\frac{n}{2}}+w^{-\frac{n}{2}}$, and, hence, $\mathrm{Im}(x)<0$. Moreover, $s=-1+\frac{2}{x}$, and, hence, $\mathrm{Im}(s)>0$.

By equating the real and imaginary parts in (3.2.4), one can show that

$$\text{(3.2.5)}\qquad \frac{\cos\theta}{\cos(\frac{n+2}{2}\theta)} = \frac{|w|^{\frac{n+2}{2}}+|w|^{-\frac{n+2}{2}}}{|w|-|w|^{-1}} > \frac{|w|+|w|^{-1}}{|w|^{\frac{n+2}{2}}-|w|^{-\frac{n+2}{2}}} = \frac{\sin(\frac{n+2}{2}\theta)}{\sin\theta} > 0,$$

and it is then not difficult to deduce that $\theta\in\left]\frac{\pi}{n+4},\frac{\pi}{n+2}\right[$. Then,

$$\text{(3.2.6)}\qquad \text{for } n\geqslant 2,\ |w|-|w|^{-1} \leqslant \frac{|w|^{\frac{n+2}{2}}-|w|^{-\frac{n+2}{2}}}{|w|+|w|^{-1}} = \frac{\sin\theta}{\sin(\frac{n+2}{2}\theta)} \leqslant \frac{\sin\theta}{\sin(\frac{\pi}{2}-\theta)} = \tan\theta.$$

Also, y equals the $\bar z_n$ of [19, Proposition 4], and here Helling [19, Theorem 1] proves that the representation is discrete and faithful. In terms of Notation 2.5, (s,x,y,z) gives the normalized discrete representation ρ_{RL^n}.

As a semi-direct product, $\langle\, A,B,C,F\,\rangle$ can be expressed as an HNN extension

$$\langle\, A,B,C,F\,\rangle = \langle\, A,B,C,D \mid A^2{=}B^2{=}C^2{=}ABCD{=}1\,\rangle \underset{A\mapsto A(ABCB)^n,\, C\mapsto B,\, D\mapsto D}{*} F.$$

The associated Bass-Serre tree is a copy of $\mathbb{R}$ which F shifts by one unit and $\langle\, A,B,C\,\rangle$ fixes. The vertices in the associated Bass-Serre tree have valence two, the subtree fixed by $\langle\, A,B,C\,\rangle$ is the whole tree, and the limit set of $\langle\, A,B,C\,\rangle$ acting on $\hat{\mathbb{C}}$ via ρ_{RL^n} is all of $\hat{\mathbb{C}}$.

(ii). We now consider the limit of ρ_{RL^n} as n tends to ∞, following [19].

It is not difficult to apply (3.2.6) and show that w tends to 1 as n tends to ∞. For large n, $y=z=w+w^{-1}$ is near 2, and $x^2+y^2+z^2-xyz=0$ is near x^2+8-4x. We have seen that $\mathrm{Im}(x)<0$; hence, x is near $2-2\mathbf{i}$. We have seen that $s=-1+\frac{2}{x}$; hence, s is near $\frac12+\frac{\mathbf{i}}{2}$.

As n tends to ∞, $\tilde A = \left(\begin{smallmatrix}-1 & -\frac{2}{x}\\ x & 1\end{smallmatrix}\right)$ tends to $\tilde A_\infty \coloneqq \left(\begin{smallmatrix}-1 & -\frac12-\frac{\mathbf{i}}{2}\\ 2-2\mathbf{i} & 1\end{smallmatrix}\right)$, $\tilde B = \left(\begin{smallmatrix}0 & -\frac{1}{y}\\ y & 0\end{smallmatrix}\right)$ tends to $\tilde B_\infty \coloneqq \left(\begin{smallmatrix}0 & -\frac12\\ 2 & 0\end{smallmatrix}\right)$, $\tilde C = -\tilde B\tilde A\tilde D^{-1}$ tends to $\tilde C_\infty \coloneqq \left(\begin{smallmatrix}1-\mathbf{i} & -\frac12+\mathbf{i}\\ 2 & -1+\mathbf{i}\end{smallmatrix}\right)$, $\tilde D = \tilde D_\infty \coloneqq \left(\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right)$, and $\tilde F$ tends to $\tilde F_\infty \coloneqq \left(\begin{smallmatrix}1 & -\frac12+\frac{\mathbf{i}}{2}\\ 0 & 1\end{smallmatrix}\right)$. Left conjugating these limit matrices by $\left(\begin{smallmatrix}\sqrt2 & 0\\ 0 & \frac{1}{\sqrt2}\end{smallmatrix}\right)\in \mathrm{SL}_2(\mathbb{C})$ carries them into $\mathrm{SL}_2(\mathbb{Z}[\mathbf{i}])$.

For the purposes of this discussion, we deviate from our normal usage and write $A\coloneqq\pm\tilde A_\infty$, $B\coloneqq\pm\tilde B_\infty$, $C\coloneqq\pm\tilde C_\infty$, $D\coloneqq\pm\tilde D_\infty$, $F\coloneqq\pm\tilde F_\infty$ in $\mathrm{PSL}_2(\mathbb{C})$. The subgroup $\langle\, A,B,C,D,F\,\rangle$ of $\mathrm{PSL}_2(\mathbb{C})$ is conjugate to an index-six subgroup of the (clearly discrete) Picard group $\mathrm{PSL}_2(\mathbb{Z}[\mathbf{i}])$ and has a presentation as the HNN extension

$$\text{(3.2.7)}\qquad \langle\, A,B,C,D \mid A^2=B^2=C^2=ABCD=1\,\rangle \underset{C\mapsto B,\, D\mapsto D}{*} F,$$

and the resulting hyperbolic orbifold has a two-fold cover which is the complement of a Whitehead link in a three-sphere; see [19] and [32, Example 1].

In the Bass-Serre tree associated to the HNN extension, all the vertices have infinite valence, the subtree fixed by $\langle A, B, C \rangle$ is a single vertex, the limit set of $\langle A, B, C, F \rangle$ acting on $\hat{\mathbb{C}}$ is all of $\hat{\mathbb{C}}$, and the limit set of $\langle A, B, C \rangle$ acting on $\hat{\mathbb{C}}$ is an Apollonian gasket; this contrasts with the situation at the end of (i). □

3.3 Examples. We apply Review 3.2.

(i). $n = 3$.

The tessellation $\mathrm{CW}(RL^3)$ can be seen in Figure 1.0.1(3).

(ii). $n = 100$.

The tessellation $\mathrm{CW}(RL^{100})$ can be seen in Figure 3.3.2. The gray regions suggest to us chains of fleas preying on successively smaller fleas, therein reversing the well-known arrangement imagined by Jonathan Swift in *On Poetry: A Rhapsody* (1733):

> So, naturalists observe, a flea
> Has smaller fleas that on him prey;
> And these have smaller still to bite 'em;
> And so proceed *ad infinitum*.

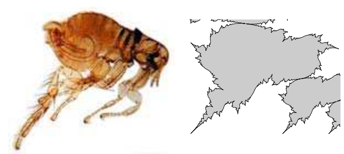


Figure 3.3.1: Flea and fractal

The photograph in Figure 3.3.1 is reproduced with the kind permission of *Flea Information*, http://www.flea-i.com, Copyright 2003–2006, all rights reserved.

(iii). $n \to \infty$.

We use the notation of Review 3.2(ii).

Here, we have no fibration with which to construct a Cannon-Thurston map, and, in order to produce an associated $\langle D, F \rangle$-tessellation of $\mathbb{C}$, we create *ad hoc* the following groupoid: the class of objects is $\mathbb{Z}$, the set of morphisms from i to j is the subset $F^j \langle A, B, C \rangle F^{-i}$ of $\mathrm{PSL}_2(\mathbb{C})$, and the groupoid multiplication is induced from the group multiplication in $\mathrm{PSL}_2(\mathbb{C})$. The set of 'ends' of this groupoid is the set of formal expressions $\langle F \rangle \mathfrak{E} \langle A, B, C \rangle$. The corresponding limit set in $\hat{\mathbb{C}}$ is a stack of Apollonian gaskets indexed by $\mathbb{Z}$, and two of these gaskets are depicted in Figure 3.3.2. It can be seen that the stack of gaskets is traced out by a curve which roughly shadows the Cannon-Thurston map $\mathrm{CT}_{RL^{100}}$, travelling diagonally upward near the gray regions and diagonally downward near the white regions. Here, each horizontal gasket inherits an alternating-gray-and-white-column structure, and we thus recover the Apollonian-gasket case of Sakuma's alternating-gray-and-white-column structure for limit sets of double-cusped groups [30]. □

3.4 Remarks. We use the notation of Review 3.2.

Let us clarify that Helling works not with $\langle A, B, C, F \rangle$-groups but with the index-two subgroups $\langle X, Y, F \rangle$, where $X = CB$ and $Y = AB$, and uses the generating sequence $(YX, F, X^{-1}F^{-1})$.

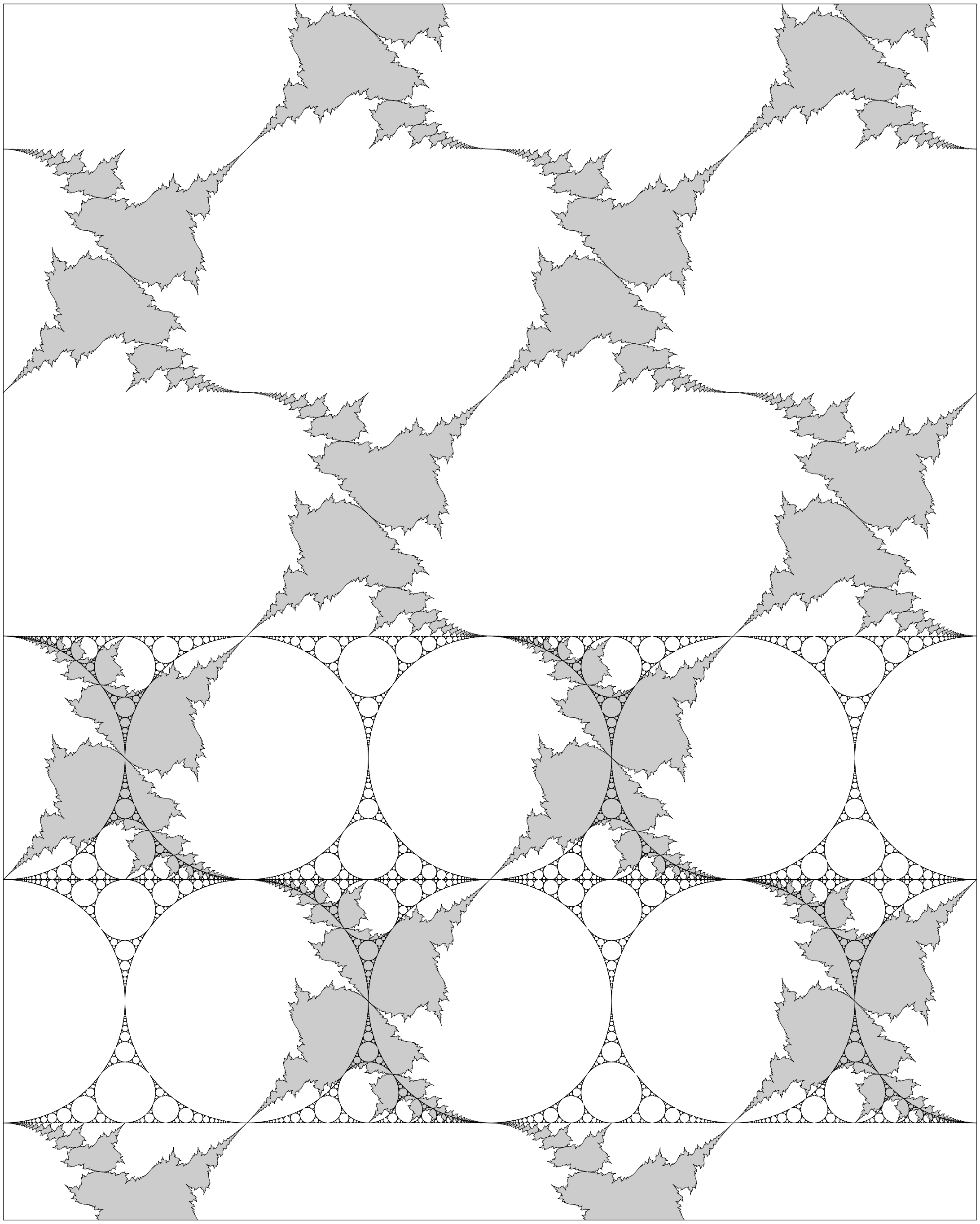


Figure 3.3.2: $\mathrm{CW}(RL^{100})$ and two Apollonian gaskets

For (3.2.7), we get the HNN extension

$$\langle X, Y, Z \mid XYX^{-1}Y^{-1}Z = 1 \rangle \underset{Y \mapsto YX,\, Z \mapsto Z}{*} F,$$

and the resulting hyperbolic manifold is the complement of a Whitehead link in a three-sphere.

For $F = RL^n$, we get the quotient HNN extension

$$\langle X, Y, Z \mid XYX^{-1}Y^{-1}Z = 1 \rangle \underset{Y \mapsto YX,\, Z \mapsto Z,\, X \mapsto X(YX)^n}{*} F$$

and the resulting hyperbolic manifold is an RL^n-once-punctured-torus bundle, the same manifold that arises on performing $\frac{n}{1}$-Dehn surgery on one of the Whitehead links. Helling shows that, for the composition of ρ_{RL^n} with conjugation by the involution

$$\pm\left(\begin{smallmatrix} 0 & (w+w^{-1})^{-\frac{1}{2}} \\ -(w+w^{-1})^{\frac{1}{2}} & 0 \end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{C}),$$

a fundamental domain for $\langle X, Y, F \rangle$ is given by the union of the $n+1$ ideal tetrahedra whose sequence of ideal vertex sets is

$$(3.4.1) \qquad (\{\infty, 0, \tfrac{a_{n-2i-2}}{a_{n-2i}}, \tfrac{a_{n-2i}}{a_{n-2i+2}}\} \mid i \in [n{\downarrow}0]),$$

where, for each $j \in \mathbb{Z}$, $a_j := w^{\frac{j}{2}} + w^{\frac{-j}{2}}$. To obtain a fundamental domain for $\rho_{RL^n}(\langle X, Y, F \rangle)$, we multiply each complex number in (3.4.1) by $-w - w^{-1}$ and then invert the result.

Let us drop down even further, to the infinite-index subgroups $\langle X, Y \rangle$; here the resulting hyperbolic manifold is homeomorphic to $\mathbf{T} \times \mathbb{R}$, and we are on the boundary of the space of Kleinian one-cusped-torus groups. Work of Ahlfors, Bers, Bonahon, Kra, Marden, Maskit, Minsky, Sullivan, Thurston, and others, resulted in a classification of Kleinian one-cusped-torus groups using ending-lamination pairs, where each ending lamination is a point in the closed northern Riemann hemisphere including the equator, $\hat{\mathbb{R}}$, and the pair does not lie in the diagonal of the equator.

When both terms of an ending-lamination pair are irrational numbers, we have a doubly degenerate group, where the limit set is all of $\hat{\mathbb{C}}$.

When both terms of an ending-lamination pair are rational numbers, we have a double-cusped group, where the complement of the limit set consists of infinitely many round open disks.

Since RL^n acts linearly on $\mathbb{R}^2$ with matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ n & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} n+1 & 1 \\ n & 1 \end{smallmatrix}\right)$, RL^n acts on $\hat{\mathbb{R}}$ as $x \mapsto \frac{(n+1)x+1}{nx+1}$. The pair consisting of the repelling fixed point and the attracting fixed point, $(\frac{1}{2} - \frac{1}{2}\sqrt{1+\frac{4}{n}}, \frac{1}{2} + \frac{1}{2}\sqrt{1+\frac{4}{n}})$, both of which are irrational, is 'the' ending-lamination pair for 'the' Kleinian one-cusped-torus group arising from the RL^n-bundle. As n tends to ∞, the pair tends to $(0, 1)$, and this is 'the' ending-lamination pair for 'the' Kleinian one-cusped-torus group whose limit set is the Apollonian gasket. The well-known discontinuity of limit sets is of interest: at each step of the sequence the limit set is all of $\hat{\mathbb{C}}$, while at the end of the sequence the limit set is suddenly not all of $\hat{\mathbb{C}}$. Perhaps better behavior can be expected from suitable tessellations of limit sets. Notice that Helling [19] lifts the convergent sequence of ending-lamination pairs to the level of discrete representations. □

4 Digression: when the columns of $\mathrm{CW}(F)$ are vertical

This section, which is not used in any subsequent sections, amounts to an addendum to [13] recording the fact that the colored columns of $\mathrm{CW}(F)$ are strictly vertical whenever some odd-length cyclic shift carries $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ to itself (for example, $(1,1)$ or $(1,2,3,1,2,3)$), and/or some even-length cyclic shift carries $(a_1, b_1, a_2, b_2, \ldots, a_p, b_p)$ to its reverse, $(b_p, a_p, \ldots, b_2, a_2, b_1, a_1)$ (for example, $(1,1)$ or $(1,1,2,2)$).

4.1 Notation. Let $\{R, L\}^{\mathbb{Z}}$ denote the set of all functions from $\mathbb{Z}$ to $\{R, L\}$.

The action of $\mathbb{Z}$ on itself by translation induces a $\mathbb{Z}$-action on $\{R, L\}^{\mathbb{Z}}$. The resulting quotient set, $\mathbb{Z}\backslash\{R, L\}^{\mathbb{Z}}$, is called the *set of bi-infinite words in R and L*, and we think of the elements as bi-infinite products. Recall that $F = \prod\limits_{i\in[1\uparrow p]} (R^{a_i} L^{b_i}) \in \mathrm{Aut}\langle\, A, B, C\,\rangle$. We define $\mathfrak{b}(F)$ to be the bi-infinite word in R and L given by $\mathfrak{b}(F) := \prod\limits_{n\in\mathbb{Z}} (\prod\limits_{i\in[1\uparrow p]} (R^{a_i} L^{b_i}))$.

The automorphism group of $\mathbb{Z}$ acts on $\{R, L\}^{\mathbb{Z}}$ and on $\mathbb{Z}\backslash\{R, L\}^{\mathbb{Z}}$, and the non-trivial (negation) element gives a map denoted

$$\text{backwards}\colon \mathbb{Z}\backslash\{R, L\}^{\mathbb{Z}} \to \mathbb{Z}\backslash\{R, L\}^{\mathbb{Z}}.$$

Notice that $\text{backwards}(\mathfrak{b}(F)) = \mathfrak{b}(F)$ if and only if there exists some odd-length cyclic shift which carries $(a_1, b_1, a_2, \ldots, a_p, b_p)$ to its reverse, $(b_p, a_p, \ldots, a_2, b_1, a_1)$, for example, $(1,1)$ or $(1,2)$.

The permutation group of $\{R, L\}$ acts on $\{R, L\}^{\mathbb{Z}}$ and on $\mathbb{Z}\backslash\{R, L\}^{\mathbb{Z}}$, and the non-trivial element gives a map denoted

$$\text{interchange}\colon \mathbb{Z}\backslash\{R, L\}^{\mathbb{Z}} \to \mathbb{Z}\backslash\{R, L\}^{\mathbb{Z}}.$$

Notice that $\text{interchange}(\mathfrak{b}(F)) = \mathfrak{b}(F)$ if and only if there exists some odd-length cyclic shift which carries $(a_1, b_1, a_2, \ldots, a_p, b_p)$ to itself, for example, $(1,1)$ or $(1,2,3,1,2,3)$.

The two maps just defined commute, and their composite is denoted

$$\text{transpose}\colon \mathbb{Z}\backslash\{R, L\}^{\mathbb{Z}} \to \mathbb{Z}\backslash\{R, L\}^{\mathbb{Z}}.$$

Notice that $\text{transpose}(\mathfrak{b}(F)) = \mathfrak{b}(F)$ if and only if there exists some even-length cyclic shift which carries $(a_1, b_1, a_2, \ldots, a_p, b_p)$ to its reverse, for example, $(1,1)$ or $(1,1,2,2)$. □

With (2.5.2), D acts on $\mathbb{C}$ by translation by 1 and F acts on $\mathbb{C}$ by translation by s. Hence, the (F-invariant) colored columns of $\mathrm{CW}(F)$ are orthogonal to $\mathbb{R}$ if and only if $s \in \mathbb{R}\mathbf{i}$.

McCullough [22] showed that $\mathbf{T}_F$ has an orientation-reversing symmetry if and only if $\mathrm{interchange}(\mathfrak{b}(F)) = \mathfrak{b}(F)$ and/or $\mathrm{transpose}(\mathfrak{b}(F)) = \mathfrak{b}(F)$. It is folklore that if $\mathbf{T}_F$ has an orientation-reversing symmetry, then $s \in \mathbb{R}\mathbf{i}$; we have not found a statement or proof of this in the literature, and the purpose of this section is to provide a proof.

In [13], it was shown that the colored planar tessellation $\mathrm{CW}(F)$ is of type $\mathbf{p\,1}$ (torus), or of type $\mathbf{p\,2}$ (sphere with four double points), or of type $\mathbf{p\,g}$ (Klein bottle), or of type $\mathbf{p\,g\,g}$ (projective plane with two double points); here the orbifold in parentheses indicates the quotient of $\mathbb{C}$ modulo the color-preserving-or-reversing symmetry group of the colored planar tessellation. Moreover, $\mathrm{CW}(F)$ is of type $\mathbf{p\,g\,g}$ if and only if $\mathrm{interchange}(\mathfrak{b}(F)) = \mathrm{transpose}(\mathfrak{b}(F)) = \mathfrak{b}(F)$. Also, $\mathrm{CW}(F)$ is of type $\mathbf{p\,g}$ if and only if $\mathrm{interchange}(\mathfrak{b}(F)) = \mathfrak{b}(F)$, or $\mathrm{transpose}(\mathfrak{b}(F)) = \mathfrak{b}(F)$, but not both.

The following result is the main step in the standard proof that a planar group of type $\mathbf{p\,g}$ or $\mathbf{p\,g\,g}$ contains two mutually orthogonal planar translations; what we want to show is that if $\mathrm{CW}(F)$ is of type $\mathbf{p\,g}$ or $\mathbf{p\,g\,g}$ then the planar translations D and F are mutually orthogonal.

4.2 Lemma. *Let* $\mathbf{g}$ *and* $\mathbf{t}$ *be symmetries of the Euclidean plane such that* $\mathbf{g}$ *is a glide reflection, and* $\mathbf{t}$ *is a translation, and* $\mathbf{g}^{-1}\mathbf{t}\mathbf{g} = \mathbf{t}^{-1}$. *Then* $\mathbf{g}^2$ *and* $\mathbf{t}$ *are mutually orthogonal translations.*

Proof. We take the complex numbers as our model of the Euclidean plane.

Here, $\mathbf{g}$ is the composite of some reflection $\mathbf{r}$ followed by some translation $\mathbf{t}'$.

Let $\mathrm{axis}(\mathbf{r})$ denote the set of fixed points of $\mathbf{r}$, the axis of reflection. Now $\mathbf{r}$ is of the form $z \mapsto \gamma\overline{z} + \delta$, with γ, δ in $\mathbb{C}$. If 0 and 1 are fixed by $\mathbf{r}$ then $\delta = 0$, $\gamma = 1$, and, hence, $\mathbf{r}$ is complex conjugation.

There exists some affine transformation $\mathbf{a}$ of $\mathbb{C}$ such that $\mathbf{a}(\mathrm{axis}(\mathbf{r})) = \mathbb{R}$. Then $\mathbf{ara}^{-1}$ is a reflection with $\mathrm{axis}(\mathbf{ara}^{-1}) = \mathbb{R}$, and, hence, $\mathbf{ara}^{-1}$ is complex conjugation.

By left conjugating $\mathbf{g}$, $\mathbf{t}'$, $\mathbf{t}$, and $\mathbf{r}$ by $\mathbf{a}$, we may assume that $\mathbf{r}$ is complex conjugation, and, hence, $\mathbf{g}$ is of the form $z \mapsto \overline{z} + \alpha$ for some $\alpha \in \mathbb{C}$. In this event, $\mathbf{g}^2$ is of the form $z \mapsto z + (\alpha + \overline{\alpha})$. Clearly, $(\alpha + \overline{\alpha}) \in \mathbb{R}$.

Also, $\mathbf{t}$ is of the form $z \mapsto z + \beta$ for some $\beta \in \mathbb{C}$.

Now, $\mathbf{tg}(0) = \mathbf{t}(\alpha) = \alpha + \beta$ and $\mathbf{gt}^{-1}(0) = \mathbf{g}(-\beta) = -\overline{\beta} + \alpha$. The equation $\mathbf{g}^{-1}\mathbf{t}\mathbf{g} = \mathbf{t}^{-1}$ then implies that $-\overline{\beta} = \beta$, and, hence, $\beta \in \mathbb{R}\mathbf{i}$.

Now, $\mathbf{g}^2$ is translation by a real number and $\mathbf{t}$ is translation by a purely imaginary number. Thus, $\mathbf{g}^2$ and $\mathbf{t}$ are mutually orthogonal translations. $\square$

The cases of the following result where

$$(a_1, b_1, a_2, \ldots, a_p, b_p) \quad \in \quad \{(1,1),\ (2,2),\ (1,1,2,2),\ (1,2,3,1,2,3)\}$$

are illustrated in [13, Section 14].

4.3 Theorem. *With the above notation, if* $\text{interchange}(\mathfrak{b}(F)) = \mathfrak{b}(F)$ *and/or* $\text{transpose}(\mathfrak{b}(F)) = \mathfrak{b}(F)$*, then* $s \in \mathbb{R}\mathbf{i}$*, and, hence, the (*$F$*-invariant) colored columns of* $\text{CW}(F)$ *are orthogonal to* $\mathbb{R}$*.*

Proof. We use the notation of [13], where $\text{CW}(F)$ is denoted $\text{CW}(\widehat{\rho}, \infty)$.

We work in $\text{Isom}(\text{CW}'(\mathfrak{f}))$ and then translate the information to $\text{Aut}(\hat{\mathbb{C}}, \text{CW}(\widehat{\rho}, \infty))$.

(a) Suppose that $\text{interchange}(\mathfrak{b}) = \mathfrak{b}$.

By Propositions 13.9, 13.6, and 13.5 of [13], there exists a vertical glide reflection G in $\text{Isom}(\text{CW}'(\mathfrak{f}))$ such that the following hold:

G has the form $(x, y) \mapsto (-x, y + d)$, for some $d \in \mathbb{R}$,

G^2 and F have the form $(x, y + 2d)$, and

D has the form $(x, y) \mapsto (x + 2, y)$.

Now, in $\mathbb{R}^2$,

$$DGD(x, y) = DG(x + 2, y) = D(-x - 2, y + d) = (-x, y + d) = G(x, y).$$

Thus $G^{-1}DG = D^{-1}$ in $\text{Isom}(\text{CW}'(\mathfrak{f}))$. The latter equation continues to hold in $\text{Aut}(\hat{\mathbb{C}}, \text{CW}(\widehat{\rho}, \infty))$, by [13, Corollary 14.6]. By Lemma 4.2, D and $G^2 = F$ act on $\mathbb{C}$ as mutually orthogonal translations, as desired.

(b) Suppose that $\text{transpose}(\mathfrak{b}(F)) = \mathfrak{b}(F)$.

By Propositions 13.8, 13.5, and 13.6 of [13], there exists a horizontal glide reflection G in $\text{Isom}(\text{CW}'(\mathfrak{f}))$ such that the following hold:

G has the form $(x, y) \mapsto (x + 1, -y + d_1)$, for some $d_1 \in \mathbb{R}$,

G^2 and D have the form $(x + 2, y)$, and

F has the form $(x, y) \mapsto (x, y + d_2)$, for some $d_2 \in \mathbb{R}$.

Now, in $\mathbb{R}^2$,

$$FGF(x, y) = FG(x, y + d_2) = F(x + 1, -y - d_2 + d_1) = (x + 1, -y + d_1) = G(x, y).$$

Thus $G^{-1}FG = F^{-1}$ in $\text{Isom}(\text{CW}'(\mathfrak{f}))$. The latter equation continues to hold in $\text{Aut}(\hat{\mathbb{C}}, \text{CW}(\widehat{\rho}, \infty))$, by [13, Corollary 14.6]. By Lemma 4.2, F and $G^2 = D$ act on $\mathbb{C}$ as mutually orthogonal translations, as desired. □

4.4 Question. With the above notation, if $s \in \mathbb{R}\mathbf{i}$, is it necessarily true that $\text{interchange}(\mathfrak{b}(F)) = \mathfrak{b}(F)$ and/or $\text{transpose}(\mathfrak{b}(F)) = \mathfrak{b}(F)$?

5 Results: profiles of $\partial^- A$ and $\partial^+ B$ and $\mathrm{CW}(F)$

In this section, we present the main results that will be proved in later sections. We describe tree structures for $\partial^- A$, for $\partial^+ B$, and for the one-skeleton of $\mathrm{CW}(F)$. We show how they are suitable for programming. We discuss the example $F = RL^3$.

5.1 Notation. We use the notation of Section 2.

Recall that we have positive integers $p, a_{[1\uparrow p]}, b_{[1\uparrow p]}$, and that, in $\mathrm{Aut}\langle A, B, C \rangle$, $R \coloneqq (A, BCB, B)$, $L \coloneqq (B, BAB, C)$, and $F \coloneqq \prod\limits_{i\in[1\uparrow p]} (R^{a_i} L^{b_i})$, and $\langle A, B, C, F \rangle$ acts on $\hat{\mathbb{C}}$ via ρ_F, and, in $\mathrm{SL}_2(\mathbb{Z})$, $\left(\begin{smallmatrix} f_{1\,1} & f_{1\,2} \\ f_{2\,1} & f_{2\,2} \end{smallmatrix}\right) \coloneqq \prod\limits_{i\in[1\uparrow p]} \left(\left(\begin{smallmatrix} 1 & a_i \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ b_i & 1 \end{smallmatrix}\right)\right)$.

For $x \in \mathbb{R}$, $[x]$ denotes the greatest integer in the interval $]-\infty, x]$.

A simple curve is said to be *non-degenerate* if it contains infinitely many points. For a given oriented, non-degenerate, simple curve $\mathfrak{a}$, by a *fracturing* of $\mathfrak{a}$ we mean any finite sequence $\mathfrak{b}_{[1\uparrow n]}$ of non-degenerate oriented subcurves of $\mathfrak{a}$ such that $\mathfrak{a} = \bigcup \mathfrak{b}_{[1\uparrow n]}$ and, for each $i \in [2\uparrow n]$, the intersection of $\mathfrak{b}_{i-1}$ and $\mathfrak{b}_i$ consists of a single point which is the terminal point of $\mathfrak{b}_{i-1}$ and the initial point of $\mathfrak{b}_i$. By abuse of notation, we will then say that the expression $\mathfrak{a} = \bigcup \mathfrak{b}_{[1\uparrow n]}$ is a fracturing of $\mathfrak{a}$. □

5.2 Profile of $\partial^- A$. We use Notation 5.1.

Let $\mathfrak{a}_1 \coloneqq \partial^- A$, $\mathfrak{a}_2 \coloneqq \mathrm{reverse}(\partial^- C)$, $X_1 \coloneqq BC$, $X_2 \coloneqq BA$, $f_1 \coloneqq f_{1\,1} + f_{2\,1}$, and $f_2 \coloneqq f_{1\,2} + f_{2\,2}$.

In Notation 6.1, we shall see that $f_2 < f_1$, and that there exists a sequence $\mathbf{s}_{[1\uparrow f_1]}$ in $[1\uparrow 2]$ such that ${}^F X_1 = \prod X_{\mathbf{s}_{[1\uparrow f_1]}}$ and ${}^F X_2 = \prod X_{\mathbf{s}_{[1\uparrow f_2]}}$.

For each $\ell \in [1\uparrow f_1]$, let $U_\ell \coloneqq F^{-1} \prod X_{\mathbf{s}_{[1\uparrow(\ell-1)]}}$.

We call $U_{[1\uparrow f_1]}$ the *∂^--syllable sequence.*

The subsemigroup of $\langle A, B, C, F \rangle$ generated by the ∂^--syllable sequence is called the *∂^--semigroup.*

In Theorem 6.4, we shall see that the limit set of the ∂^--semigroup is $\mathfrak{a}_1$, and that, for each $i \in [1\uparrow 2]$, $(U_\ell \mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [1\uparrow f_i])$, called the sequence of *columns* of $\mathfrak{a}_i$, is a fracturing of $\mathfrak{a}_i$. Thus, we have fracturings

$$\mathfrak{a}_1 = \bigcup_{\ell\in[1\uparrow f_1]} U_\ell \mathfrak{a}_{\mathbf{s}_\ell} \quad \text{and} \quad \mathfrak{a}_2 = \bigcup_{\ell\in[1\uparrow f_2]} U_\ell \mathfrak{a}_{\mathbf{s}_\ell}. \tag{5.2.1}$$

Recall that $\mu_+ = \frac{f_{1\,1} - f_{2\,2} + \sqrt{(f_{1\,1}+f_{2\,2})^2 - 4}}{2 f_{2\,1}}$. In Lemma 6.3, we shall see that for each $\ell \in [1\uparrow f_1]$, $\mathbf{s}_\ell = [\frac{\ell+1}{1+\mu_+}] - [\frac{\ell}{1+\mu_+}] + 1$. This is useful for numerical programming. □

5.3 Profile of $\partial^+ B$. We use Notation 5.1.

Let $\mathfrak{a}_{-1} := \text{reverse}(\partial^+ B)$, $\mathfrak{a}_{-2} := \text{reverse}(\partial^+ BC)$, $G := AF^{-1}A$, $X_{-1} := CA$, $X_{-2} := BA$, $f_{-1} := f_{1\,1} + f_{2\,1}$, and $f_{-2} := f_{1\,1}$.

In Notation 6.1, we shall see that $f_{-2} < f_{-1}$, and that there exists a sequence $\mathbf{t}_{[1\uparrow f_{-1}]}$ in $[(-1)\downarrow(-2)]$ such that ${}^G X_{-1} = \prod X_{\mathbf{t}_{[1\uparrow f_{-1}]}}$ and ${}^G X_{-2} = \prod X_{\mathbf{t}_{[1\uparrow f_{-2}]}}$.

For each $\ell \in [1\uparrow f_{-1}]$, let $V_\ell := G^{-1} \prod X_{\mathbf{t}_{[1\uparrow(\ell-1)]}}$.

We call $V_{[1\uparrow f_{-1}]}$ the ∂^+-*syllable sequence*. The subsemigroup of $\langle\, A, B, C, F \,\rangle$ generated by the ∂^+-syllable sequence is called the ∂^+-*semigroup*.

In Theorem 6.6, we shall see that the limit set of the ∂^+-semigroup is $\mathfrak{a}_{-1}$, and that, for each $i \in [(-1)\downarrow(-2)]$, $(V_\ell \mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [1\uparrow f_i])$, called the sequence of *rows* of $\mathfrak{a}_i$, is a fracturing of $\mathfrak{a}_i$, that is, we have fracturings

$$\mathfrak{a}_{-1} = \bigcup_{\ell \in [1\uparrow f_{-1}]} V_\ell \mathfrak{a}_{\mathbf{s}_\ell} \quad \text{and} \quad \mathfrak{a}_{-2} = \bigcup_{\ell \in [1\uparrow f_{-2}]} V_\ell \mathfrak{a}_{\mathbf{s}_\ell}.$$

Recall that $\mu_- := \frac{f_{1\,1} - f_{2\,2} - \sqrt{(f_{1\,1}+f_{2\,2})^2 - 4}}{2 f_{2\,1}}$. In Notation 6.1, we shall see that $\mathbf{t}_{f_{-1}} = -1$, and, in Lemma 6.5, we shall see that, for each $\ell \in [1\uparrow(f_{-1}-1)]$, $\mathbf{t}_\ell = [(\ell)(-\mu_-)] - [(\ell-1)(-\mu_-)] - 2$. This is useful for numerical programming. □

5.4 Profile of the syllables. We use the notation of Profiles 5.2 and 5.3.

Clearly $f_{-1} = f_1$. In Notation 6.1, we shall see that $f_2 f_{-2} \equiv 1 (\text{mod } f_1)$.

In Proposition 6.7, we shall see that, for each $\ell \in [1\uparrow f_1]$, $V_\ell = U^{-1}_{(1-\ell)f_2\,(\text{mod } f_1)}$ and, hence, $U_\ell = V^{-1}_{1-(\ell f_{-2})\,(\text{mod } f_1)}$.

In particular, the sequences $V_{[1\uparrow f_1]}$ and $U^{-1}_{[1\uparrow f_1]}$ have the same underlying set. In [12, Section 6.1], it was seen that the limit set of the semigroup generated by $U^{-1}_{[1\uparrow f_1]}$ is $\partial^+ B$. In [12, Section 6.4], it was seen that the limit set of the semigroup generated by $V^{-1}_{[1\uparrow f_1]}$ is $\partial^- A$.

In Proposition 6.7, we shall see that $U_1 = V^{-1}_{f_{2\,1}+1}$, $U_{f_2} = V^{-1}_{f_1}$ and $U_{f_1} = V^{-1}_1$ are the parabolic ∂^--syllables, where an element of $\langle\, A, B, C, F \,\rangle$ is *parabolic* if it has a unique fixed point for the action on $\hat{\mathbb{C}}$. In Corollary 7.7, we shall see that the parabolic elements of the ∂^--semigroup are the positive powers of the three parabolic ∂^--syllables. □

5.5 Example. Let $F = RL^3 = (A(ABCB)^3, (BCBA)^4 A, B)$.

(i). We calculate

$${}^F(BC, BA) = ((BCBA)^3 BC, BCBA).$$

Here, $f_1 = 7$, $f_2 = 2$, $\mathbf{s}_{[1\uparrow 7]} = (1,2,1,2,1,2,1)$, and $\mathfrak{a}_1 = \partial^- A$, $\mathfrak{a}_2 = \text{reverse}(\partial^- C)$ have fracturings

$$\mathfrak{a}_1 = U_1\mathfrak{a}_1 \cup U_2\mathfrak{a}_2 \cup U_3\mathfrak{a}_1 \cup U_4\mathfrak{a}_2 \cup U_5\mathfrak{a}_1 \cup U_6\mathfrak{a}_2 \cup U_7\mathfrak{a}_1,$$
$$\mathfrak{a}_2 = U_1\mathfrak{a}_1 \cup U_2\mathfrak{a}_2,$$

where we have

$U_1\mathfrak{a}_1 = F^{-1}\mathfrak{a}_1$	$= F^{-1}\mathfrak{a}_1$	$\mathfrak{a}_2$	$\mathfrak{a}_1$
$U_2\mathfrak{a}_2 = F^{-1}X_1\mathfrak{a}_2$	$= F^{-1}BC\mathfrak{a}_2$	$\mathfrak{a}_2$	$\mathfrak{a}_1$
$U_3\mathfrak{a}_1 = F^{-1}X_1X_2\mathfrak{a}_1$	$= F^{-1}(BCBA)\mathfrak{a}_1$		$\mathfrak{a}_1$
$U_4\mathfrak{a}_2 = F^{-1}X_1X_2X_1\mathfrak{a}_2$	$= F^{-1}(BCBA)BC\mathfrak{a}_2$		$\mathfrak{a}_1$
$U_5\mathfrak{a}_1 = F^{-1}X_1X_2X_1X_2\mathfrak{a}_1$	$= F^{-1}(BCBA)^2\mathfrak{a}_1$		$\mathfrak{a}_1$
$U_6\mathfrak{a}_2 = F^{-1}X_1X_2X_1X_2X_1\mathfrak{a}_2$	$= F^{-1}(BCBA)^2BC\mathfrak{a}_2$		$\mathfrak{a}_1$
$U_7\mathfrak{a}_1 = F^{-1}X_1X_2X_1X_2X_1X_2\mathfrak{a}_1$	$= F^{-1}(BCBA)^3\mathfrak{a}_1$		$\mathfrak{a}_1$

(ii). We calculate $F^{-1} = ((AB)^3A, C, C(AB)^2AC)$ and, hence,

$$^{F^{-1}}(AC, AB) = ((AB)^3AC(AB)^2AC, (AB)^3AC),$$

and, hence, for $G = AF^{-1}A$,

$$^{G}(CA, BA) = ((BA)^3CA(BA)^2CA, (BA)^3CA).$$

Here, $f_{-1} = 7$, $f_{-2} = 4$, $\mathfrak{t}_{[1\uparrow 7]} = (-2,-2,-2,-1,-2,-2,-1)$, and $\mathfrak{a}_{-1} = \text{reverse}(\partial^+B)$, $\mathfrak{a}_{-2} = \text{reverse}(\partial^+BC)$ have fracturings

$$\mathfrak{a}_{-1} = V_1\mathfrak{a}_{-2} \cup V_2\mathfrak{a}_{-2} \cup V_3\mathfrak{a}_{-2} \cup V_4\mathfrak{a}_{-1} \cup V_5\mathfrak{a}_{-2} \cup V_6\mathfrak{a}_{-2} \cup V_7\mathfrak{a}_{-1},$$
$$\mathfrak{a}_{-2} = V_1\mathfrak{a}_{-2} \cup V_2\mathfrak{a}_{-2} \cup V_3\mathfrak{a}_{-2} \cup V_4\mathfrak{a}_{-1},$$

where we have

$V_1\mathfrak{a}_{-2} = G^{-1}\mathfrak{a}_{-2}$	$= AFA\mathfrak{a}_{-2}$	$\mathfrak{a}_{-2}$	$\mathfrak{a}_{-1}$
$V_2\mathfrak{a}_{-2} = G^{-1}X_{-2}\mathfrak{a}_{-2}$	$= AFA(BA)\mathfrak{a}_{-2}$	$\mathfrak{a}_{-2}$	$\mathfrak{a}_{-1}$
$V_3\mathfrak{a}_{-2} = G^{-1}X_{-2}X_{-2}\mathfrak{a}_{-2}$	$= AFA(BA)^2\mathfrak{a}_{-1}$	$\mathfrak{a}_{-2}$	$\mathfrak{a}_{-1}$
$V_4\mathfrak{a}_{-1} = G^{-1}X_{-2}X_{-2}X_{-2}\mathfrak{a}_{-1}$	$= AFA(BA)^3\mathfrak{a}_{-2}$	$\mathfrak{a}_{-2}$	$\mathfrak{a}_{-1}$
$V_5\mathfrak{a}_{-2} = G^{-1}X_{-2}X_{-2}X_{-2}X_{-1}\mathfrak{a}_{-2}$	$= AFA(BA)^3BC\mathfrak{a}_{-2}$		$\mathfrak{a}_{-1}$
$V_6\mathfrak{a}_{-2} = G^{-1}X_{-2}X_{-2}X_{-2}X_{-1}X_{-2}\mathfrak{a}_{-2}$	$= AFA(BA)^3BC(BA)\mathfrak{a}_{-2}$		$\mathfrak{a}_{-1}$
$V_7\mathfrak{a}_{-1} = G^{-1}X_{-2}X_{-2}X_{-2}X_{-1}X_{-2}X_{-2}\mathfrak{a}_{-1}$	$= AFA(BA)^3BC(BA)^2\mathfrak{a}_{-1}$		$\mathfrak{a}_{-1}$

(iii). We have

$$\begin{aligned}
V_1 &= AFA &&= (ABCB)^3F = U_7^{-1}\\
V_2 &= AFA(BA) &&= (ABCB)^2F = U_5^{-1}\\
V_3 &= AFA(BA)^2 &&= (ABCB)F = U_3^{-1}\\
V_4 &= AFA(BA)^3 &&= F = U_1^{-1}\\
V_5 &= AFA(BA)^3CA &&= CB(ABCB)^2F = U_6^{-1}\\
V_6 &= AFA(BA)^3CA(BA) &&= CB(ABCB)F = U_4^{-1}\\
V_7 &= AFA(BA)^3CA(BA)^2 &&= CBF = U_2^{-1}.
\end{aligned}$$

If we interpret the index set as $\mathbb{Z}/7\mathbb{Z}$, then $f_2 f_{-2} = (2)(4) = 1$ and for each $\ell \in \mathbb{Z}/7\mathbb{Z}$, $V_\ell = U^{-1}_{(1-\ell)2}$, $U_\ell = V^{-1}_{1-(4\ell)}$. □

Starting from the symbol $\mathfrak{a}_1$, recursive substitution using (5.2.1) generates an ordered set of infinite words in the ∂^--syllable sequence $U_{[1\uparrow f_1]}$ which we now discuss.

5.6 Definitions. Let $\mathbf{z}_{[1\uparrow\infty[}$ be an infinite sequence in $[1\uparrow f_1]$.

Then $((\prod U_{\mathbf{z}_{[1\uparrow\ell]}})\mathfrak{a}_1 \mid \ell \in [0\uparrow\infty[\,)$ is a strictly decreasing sequence of closed subcurves of $\mathfrak{a}_1$. We temporarily assign a metric to $\mathfrak{a}_1$ by using the homeomorphism that is induced by restricting the continuous $\langle A, B, C, F\rangle$-map $\widehat{\rho}^{\,-}\colon \mathbf{D} \to \hat{\mathbb{C}}$ to a map with domain the A-labelled curve in $\tilde{\mathbf{\Delta}}$ and with codomain $\mathfrak{a}_1$, and then using the metric induced on $\tilde{\mathbf{\Delta}}$ from $\mathbf{\Delta} \subseteq \mathbb{R}^2$. Then it is not difficult to show that the length of $(\prod U_{\mathbf{z}_{[1\uparrow\ell]}})\mathfrak{a}_1$ is $\lambda^{-\ell}$. It follows that $\bigcap\limits_{\ell\in[0\uparrow\infty[} ((\prod U_{\mathbf{z}_{[1\uparrow\ell]}})\mathfrak{a}_1)$ consists of a single element of $\mathfrak{a}_1$. We then have a map to $\mathfrak{a}_1$ from the set of infinite sequences in $[1\uparrow f_1]$.

Let us say that $\mathbf{z}_{[1\uparrow\infty[}$ is $\mathfrak{a}_1$-*acceptable* if, for each $\ell \in [1\uparrow\infty[$ such that $\mathbf{s}_{\mathbf{z}_\ell} = 2$, we have $\mathbf{z}_{\ell+1} \in [1\uparrow f_2]$. In this event, by (5.2.1), $((\prod U_{\mathbf{z}_{[1\uparrow\ell]}})\mathfrak{a}_{\mathbf{s}_{\mathbf{z}_\ell}} \mid \ell \in [1\uparrow\infty[\,)$ is decreasing, and, hence, $\bigcap\limits_{\ell\in[1\uparrow\infty[} ((\prod U_{\mathbf{z}_{[1\uparrow\ell]}})\mathfrak{a}_{\mathbf{s}_{\mathbf{z}_\ell}})$ is non-empty and, hence, it must equal the one-point superset $\bigcap\limits_{\ell\in[0\uparrow\infty[} ((\prod U_{\mathbf{z}_{[1\uparrow\ell]}})\mathfrak{a}_1)$. Since (5.2.1) gives fracturings, the map to $\mathfrak{a}_1$ from the set of $\mathfrak{a}_1$-acceptable infinite sequences in $[1\uparrow f_1]$ is surjective, and one-to-one on the non break-points, and two-to-one on the break-points. The lexicographic order on the set of $\mathfrak{a}_1$-acceptable infinite sequences in $[1\uparrow f_1]$ agrees with the curve order on $\mathfrak{a}_1$. This type of codification of points of a curve by sequences is often compared to the codification by decimal expansion of the interval $[0, 1]$.

In a natural way, the set of $\mathfrak{a}_1$-acceptable infinite sequences in $[1\uparrow f_1]$ is the set of ends of a tree generated by a finite-state automaton constructed from (5.2.1). From state 1 we can go to state 1 in $f_{1\,1}$ different ways, and we can go to state 2 in $f_{2\,1}$ different ways. From state 2 we can go to state 1 in $f_{1\,2}$ different ways, and we can go to state 2 in $f_{2\,2}$ different ways.

A significant advantage of the tree structure is that it permits depth-first searches that are very useful for drawing. Let us recall this well-known method following [26, pp. 113–117]. We use some Möbius transformation to move $\mathfrak{a}_1$ into $\hat{\mathbb{C}} - \{\infty\}$ and then use the $\mathbb{R}^2$-metric on $\mathbb{C}$. We choose an $\varepsilon \in \,]0, 1[$. For $(M, i) \in \mathrm{PSL}_2(\mathbb{C}) \times [1\uparrow 2]$, we define the *weight of* (M, i), denoted weight(M, i), to be the distance between the two end-points of $M\mathfrak{a}_i$, and we define the *sequence of descendants of* (M, i) to be $((M\rho(U_\ell), \mathbf{s}_\ell) \mid \ell \in [1\uparrow f_i])$. We now construct certain finite sequences $((M_m, i_m) \mid m \in [1\uparrow n])$ in $\mathrm{PSL}_2(\mathbb{C}) \times [1\uparrow 2]$, $n \in [1\uparrow\infty[$. The starting sequence is $((\pm\mathbf{I}_2, 1))$. Suppose that we have recursively constructed some sequence $((M_m, i_m) \mid m \in [1\uparrow n])$. If weight$(M_m, i_m) < \varepsilon$ for all $m \in [1\uparrow n]$, then the recursion terminates. Otherwise, we let m be the smallest possible element of $[1\uparrow n]$ such that weight$(M_m, i_m) \geqslant \varepsilon$, and we redefine our sequence by replacing the pair (M_m, i_m) with the terms of its sequence of descendants. Now we have a new sequence of pairs and we repeat the process. This procedure terminates with a well-defined sequence $((M_m, i_m) \mid m \in [1{\uparrow}n])$ and all of its terms then have weight less than ε. By (5.2.1), $(M_m \mathfrak{a}_{i_m} \mid m \in [1{\uparrow}n])$ is a fracturing of $\mathfrak{a}_1$. Approximating each $M_m \mathfrak{a}_{i_m}$ with a straight line segment joining its end-points gives a piecewise linear approximation of $\mathfrak{a}_1$ which is quite reasonable if ε is small enough. This procedure is more efficient than the techniques previously available for drawing these curves; compare Figure 1.0.1(1) with [12, Figure 5] and [26, Figure 10.14].

For the case $F = RL$, [8, Lemma 8.1] gave a more efficient, simpler automaton, that used orientation-reversing maps; see [8, Fig. 13]. □

5.7 Definitions. Let us now construct CW(F) using the proof of [13, Lemma 7.10(ii)].

Recall that $\bigcup\limits_{\ell \in [1{\uparrow}f_1]} U_\ell \mathfrak{a}_{\mathbf{s}_\ell} = \mathfrak{a}_1 = \partial^- A$ is a simple oriented subcurve of the Jordan curve ∂_0 in $\hat{\mathbb{C}}$, and $\mathfrak{a}_1 = \partial^- A$ joins ∞ to $A(\infty)$. Observe that $U_1 \mathfrak{a}_{\mathbf{s}_1} = F^{-1}\mathfrak{a}_1$ is the oriented subcurve of $\mathfrak{a}_1$ joining $F^{-1}(\infty) = \infty$ to $F^{-1}A(\infty)$. Let $\mathfrak{a}'_1 := \bigcup\limits_{\ell \in [2{\uparrow}f_1]} U_\ell \mathfrak{a}_{\mathbf{s}_\ell}$. Then $\mathfrak{a}'_1$ is the oriented subcurve of $\mathfrak{a}_1$ joining $F^{-1}A(\infty)$ to $A(\infty)$. Hence, $\mathfrak{a}'_1$ is a fundamental domain for the $\langle\, F \,\rangle$-action on $\partial_0 - \{\infty\}$, and, therefore, $\mathfrak{a}'_1$ is a fundamental domain for the $\langle\, D, F \,\rangle$-action on the union of the even-numbered curves in the one-skeleton of CW(F). Hence, $F\mathfrak{a}'_1 = \bigcup\limits_{\ell \in [2{\uparrow}f_1]} FU_\ell \mathfrak{a}_{\mathbf{s}_\ell}$ is an oriented subcurve of $F\partial_0 = \partial_0$ joining $FF^{-1}A(\infty) = A(\infty)$ to $FA(\infty)$, and it too is a fundamental domain for the $\langle\, D, F \,\rangle$-action on the even part of the one-skeleton of CW(F).

Recall that $\bigcup\limits_{\ell \in [1{\uparrow}f_1]} V_\ell \mathfrak{a}_{\mathbf{t}_\ell} = \mathfrak{a}_{-1} = \text{reverse}(\partial^+ B)$ is a simple oriented subcurve of the union of the Jordan curves ∂_{-1} and reverse(∂_1) in $\hat{\mathbb{C}}$, and $\mathfrak{a}_{-1} = \text{reverse}(\partial^+ B)$ joins $A(\infty)$ to $C(\infty)$. Now $V_{f_{2\,1}+1} \mathfrak{a}_{\mathbf{t}_{f_{2\,1}+1}} = F\mathfrak{a}_{-1}$ is the oriented subcurve of $\mathfrak{a}_{-1}$ joining $FA(\infty)$ to $FC(\infty)$. Let $\mathfrak{a}'_{-1} = \bigcup\limits_{\ell \in [1{\uparrow}f_{2\,1}]} V_\ell \mathfrak{a}_{\mathbf{t}_\ell}$. Then $\mathfrak{a}'_{-1}$ is the oriented subcurve of $\mathfrak{a}_{-1}$ joining $A(\infty)$ to $FA(\infty)$. Hence $\mathfrak{a}'_{-1}$ is a fundamental domain for the $\langle\, F \,\rangle$-action on $\partial_{-1} - \{\infty\}$, and, therefore, $\mathfrak{a}'_{-1}$ is a fundamental domain for the $\langle\, D, F \,\rangle$-action on the odd part of the one-skeleton of CW(F).

Now reverse$(\mathfrak{a}'_{-1}) \cup F\mathfrak{a}'_1$ is a non-simple closed curve that is a fundamental domain for the $\langle\, D, F \,\rangle$-action on the one-skeleton of CW(F). The string of Jordan domains which has reverse$(\mathfrak{a}'_{-1}) \cup F\mathfrak{a}'_1$ as oriented boundary is a fundamental domain for the $\langle\, D, F \,\rangle$-action on the L-region, that is, the gray region, of CW(F).

By using the automata for $(\mathfrak{a}_{-1}, \mathfrak{a}_{-2})$ and $(\mathfrak{a}_1, \mathfrak{a}_2)$, it is straightforward to create an automaton to generate an ordered list of points on the self-intersecting closed curve

$$\text{reverse}\Big(\bigcup_{\ell \in [1{\uparrow}f_{2\,1}]} V_\ell \mathfrak{a}_{\mathbf{t}_\ell}\Big) \cup \bigcup_{\ell \in [2{\uparrow}f_1]} FU_\ell \mathfrak{a}_{\mathbf{s}_\ell} = \text{reverse}(\mathfrak{a}'_{-1}) \cup F\mathfrak{a}'_1.$$

Since this curve already lies in $\hat{\mathbb{C}} - \{\infty\}$, we can use the $\mathbb{R}^2$-metric and generate an acceptable piecewise-linear approximation. □

6 Proofs: the Markov partition

We now study the action of F on $\mathbf{D}$ by using the lift of a famous Markov partition described in the 1960's by Adler-Weiss.

6.1 Notation. Recall that we have positive integers $p, a_{[1\uparrow p]}, b_{[1\uparrow p]}$, and that, in $\mathrm{Aut}\langle\, A, B, C\,\rangle$, $R \coloneqq (A, BCB, B)$, $L \coloneqq (B, BAB, C)$, $F \coloneqq \prod\limits_{i\in[1\uparrow p]} (R^{a_i} L^{b_i})$, and $G = AF^{-1}A$. Let $X_1 \coloneqq BC$, $X_2 \coloneqq BA$, $X_{-1} \coloneqq CA$, $X_{-2} \coloneqq BA$. It is straightforward to calculate that, for any a, b in $[1\uparrow\infty[$,

$$(6.1.1)\qquad {}^{R^aL^b}(X_1, X_2) = ((X_1^a X_2)^b X_1, X_1^a X_2),$$

$$(6.1.2)\qquad {}^{AL^{-b}R^{-a}A}(X_{-1}, X_{-2}) = (X_{-2}(X_{-2}^{b-1}X_{-1})^{a+1}, X_{-2}(X_{-2}^{b-1}X_{-1})^{a}).$$

By (6.1.1), F acts on the semigroup freely generated by $X_{[1\uparrow 2]}$, ${}^FX_2$ is an initial segment of ${}^FX_1$, ${}^FX_1$ begins and ends in X_1, and ${}^FX_2$ begins in X_1 and ends in X_2. By abelianizing, we see that the matrix of F with respect to $X_{[1\uparrow 2]}$ is

$$\left(\begin{smallmatrix} f_{1\,1} & f_{1\,2} \\ f_{2\,1} & f_{2\,2}\end{smallmatrix}\right) \coloneqq \prod_{i\in[1\uparrow p]} \left(\left(\begin{smallmatrix} 1 & a_i \\ 0 & 1\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ b_i & 1\end{smallmatrix}\right)\right).$$

Let $f_1 \coloneqq f_{1\,1} + f_{2\,1}$, $f_2 \coloneqq f_{1\,2} + f_{2\,2}$, and let $\mathbf{s}_{[1\uparrow f_1]}$ denote the sequence in $[1\uparrow 2]$ such that ${}^FX_1 = \prod X_{\mathbf{s}_{[1\uparrow f_1]}}$. Then ${}^FX_2 = \prod X_{\mathbf{s}_{[1\uparrow f_2]}}$, $\mathbf{s}_1 = 1$, $\mathbf{s}_{f_2} = 2$, and $\mathbf{s}_{f_2} = 1$. For each $\ell \in [1\uparrow f_1]$, we define the ℓth ∂^-*-syllable* to be

$$U_\ell \coloneqq F^{-1} \prod X_{\mathbf{s}_{[1\uparrow(\ell-1)]}} \in \langle\, A, B, C, F\,\rangle.$$

We call $U_{[1\uparrow f_1]}$ the ∂^-*-syllable sequence*, and we call the subsemigroup of $\langle\, A, B, C, F\,\rangle$ generated by $U_{[1\uparrow f_1]}$ the ∂^-*-semigroup.*

By (6.1.2), G acts on the semigroup freely generated by $X_{[(-1)\downarrow(-2)]}$, ${}^GX_{-2}$ is an initial segment of ${}^GX_{-1}$, and ${}^GX_{-1}$ and ${}^GX_{-2}$ both begin in X_{-2} and end in X_{-1}. By abelianizing, we see that the matrix of F^{-1} and G with respect to $X_{[(-1)\downarrow(-2)]}$ is

$$\begin{pmatrix} -1 & 0 \\ 1 & 1\end{pmatrix}\begin{pmatrix} f_{2,2} & -f_{1\,2} \\ -f_{2\,1} & f_{1\,1}\end{pmatrix}\begin{pmatrix} -1 & 0 \\ 1 & 1\end{pmatrix} = \begin{pmatrix} f_{1\,2}+f_{2\,2} & f_{1\,2} \\ f_{1\,1}-f_{1\,2}+f_{2\,1}-f_{2\,2} & f_{1\,1}-f_{1\,2}\end{pmatrix}.$$

Let $f_{-1} \coloneqq f_{1\,1} + f_{2\,1} = f_1$, $f_{-2} \coloneqq f_{1\,1}$, and let $\mathbf{t}_{[1\uparrow f_{-1}]}$ denote the sequence in $[(-1)\downarrow(-2)]$ such that ${}^GX_{-1} = \prod X_{\mathbf{t}_{[1\uparrow f_{-1}]}}$. Then ${}^GX_{-2} = \prod X_{\mathbf{t}_{[1\uparrow f_{-2}]}}$, $\mathbf{t}_1 = -2$, $\mathbf{t}_{f_{-2}} = -1$, and $\mathbf{t}_{f_{-1}} = -1$. For each $\ell \in [1\uparrow f_{-1}]$, we define the ℓth ∂^+*-syllable* to be

$$V_\ell \coloneqq G^{-1} \prod X_{\mathbf{t}_{[1\uparrow(\ell-1)]}} \in \langle\, A, B, C, F\,\rangle.$$

We call $V_{[1\uparrow f_{-1}]}$ the ∂^+*-syllable sequence*, and we call the subsemigroup of $\langle\, A, B, C, F\,\rangle$ generated by $V_{[1\uparrow f_{-1}]}$ the ∂^+*-semigroup.*

Notice that

$$f_{-2}f_2 - f_{12}f_1 = f_{11}(f_{12}+f_{22}) - f_{12}(f_{11}+f_{21}) = f_{11}f_{22} - f_{12}f_{21} = 1.$$

For each $\ell \in [1\uparrow f_1]$, let $F(\ell)$ denote the unique element of $[1\uparrow f_1]$ such that $1 - \ell f_{-2} \equiv F(\ell) (\bmod\ f_1)$. For example, $F(1) = f_{21}+1$, $F(f_2) = f_1$, $F(f_1) = 1$. Let $F^{-1}(\ell)$ denote the unique element of $[1\uparrow f_1]$ such that $(1-\ell)f_2 \equiv F^{-1}(\ell) (\bmod\ f_1)$. Then F and F^{-1} have mutually inverse actions on $[1\uparrow f_1]$.

We now tessellate $\mathbb{R}^2$ with an L-shaped region as in Figure 6.1.1(4). Deleting from $\mathbb{R}^2$ the (three) minus-lines which pass through $(-1,0)$, $(0,0)$ and $(0,-1)$, and the (three) plus-lines which pass through $(-1,0)$, $(-1,-1)$, and $(0,-1)$, leaves a subspace of $\mathbb{R}^2$ with sixteen components, four of which are interiors of closed parallelograms. Let $\mathbf{P}_{[1\uparrow 4]}$ denote these four closed parallelograms in clockwise order with $\mathbf{P}_1$ being the parallelogram containing $(-1,0)$. We let $\mathbf{L} \coloneqq \bigcup \mathbf{P}_{[1\uparrow 3]}$; see Figure 6.1.1(1). It is not difficult to see that $\mathbf{L}$ is a fundamental domain for the $\mathbb{Z}^2$-action on $\mathbb{R}^2$; see Figure 6.1.1(4). We let $\mathbf{L}' \coloneqq \bigcup \mathbf{P}_{[1\uparrow 4]}$, the smallest parallelogram containing $\mathbf{L}$.

We let $\mathbf{L}_1 \coloneqq \bigcup \mathbf{P}_{[1\uparrow 2]}$, colored gray, and $\mathbf{L}_2 \coloneqq \mathbf{P}_3$, colored white; see Figure 6.1.1(3). Hence, we get an $\mathbf{L}_{[1\uparrow 2]}$-tessellation of $\mathbb{R}^2$; see Figure 6.1.1(4).

We let $\mathbf{L}_{-2} \coloneqq \mathbf{P}_1$, colored white, and $\mathbf{L}_{-1} \coloneqq \bigcup \mathbf{P}_{[2\uparrow 3]}$, colored gray; see Figure 6.1.2(2). Hence, we get an $\mathbf{L}_{[(-1)\downarrow(-2)]}$-tessellation of $\mathbb{R}^2$; see Figure 6.1.2(4).

We let $\tilde{\mathbf{L}}'$ denote the lift of $\mathbf{L}'$ in $\tilde{\mathbf{\Delta}} \cup A\tilde{\mathbf{\Delta}} \cup AB\tilde{\mathbf{\Delta}} \cup B\tilde{\mathbf{\Delta}} \cup BA\tilde{\mathbf{\Delta}} \cup C\tilde{\mathbf{\Delta}} \subseteq \mathbf{D}$, and similarly for $\tilde{\mathbf{L}}$, $\tilde{\mathbf{L}}_1$, $\tilde{\mathbf{L}}_2$, $\tilde{\mathbf{L}}_{-1}$, and $\tilde{\mathbf{L}}_{-2}$. Notice that $\tilde{\mathbf{L}}$ is a fundamental domain for the $\langle\, CB, AB \,\rangle$-action in $\mathbf{D}^\circ \cup \langle\, A,B,C \,\rangle D^{\pm\infty}$, and we get an $\tilde{\mathbf{L}}_{[1\uparrow 2]}$-tessellation and an $\tilde{\mathbf{L}}_{[(-1)\downarrow(-2)]}$-tessellation of $\mathbf{D}^\circ \cup \langle\, A,B,C \,\rangle D^{\pm\infty}$.

For each $\ell \in [1\uparrow f_1]$, the *ℓth column* of $\tilde{\mathbf{L}}$ is defined as $\mathrm{col}_\ell(\tilde{\mathbf{L}}) \coloneqq \tilde{\mathbf{L}} \cap U_\ell \tilde{\mathbf{L}}$ and the *ℓth row* of $\tilde{\mathbf{L}}$ is defined as $\mathrm{row}_\ell(\tilde{\mathbf{L}}) \coloneqq \tilde{\mathbf{L}} \cap V_\ell \tilde{\mathbf{L}}$; see Figure 6.1.3. For $W \in \langle\, A,B,C,F \,\rangle$, $W(\mathrm{col}_\ell(\tilde{\mathbf{L}}))$ will be denoted $\mathrm{col}_\ell(W\tilde{\mathbf{L}})$ and $W(\mathrm{row}_\ell(\tilde{\mathbf{L}}))$ will be denoted $\mathrm{row}_\ell(W\tilde{\mathbf{L}})$.

Using the plus-foliation, we read $\tilde{\mathbf{L}}$ from right to left, and using the minus-foliation, we define $\mathfrak{a}_1 \coloneqq \widehat{\rho}^{\,-}(\tilde{\mathbf{L}}_1) = \widehat{\rho}^{\,-}(\tilde{\mathbf{L}}) = \widehat{\rho}^{\,-}(\tilde{\mathbf{\Delta}}) = \partial^- A$ and $\mathfrak{a}_2 \coloneqq \widehat{\rho}^{\,-}(\tilde{\mathbf{L}}_2) = \mathrm{reverse}(\partial^- C)$.

Using the minus-foliation, we read $\tilde{\mathbf{L}}$ from top to bottom, and using the plus-foliation, we define $\mathfrak{a}_{-1} \coloneqq \widehat{\rho}^{\,+}(\tilde{\mathbf{L}}_{-1}) = \widehat{\rho}^{\,+}(\tilde{\mathbf{L}}) = \widehat{\rho}^{\,+}(\tilde{\mathbf{\Delta}}) = \mathrm{reverse}(\partial^+ B)$ and $\mathfrak{a}_{-2} \coloneqq \widehat{\rho}^{\,+}(\tilde{\mathbf{L}}_{-2}) = B\,\mathrm{reverse}(\partial^+ C) = \mathrm{reverse}(\partial^+ BC)$. □

6.2 Remarks. The sequence $\mathrm{col}_{[1\uparrow f_1]}(\tilde{\mathbf{L}})$ is called *the Markov partition* for F; see Figure 6.1.3. It was described explicitly by Adler-Weiss [2, Section 5], [3, Figure 14], [1, Figure 6], in the 1960s, in connection with their analysis of the entropy of a hyperbolic automorphism of $\mathbb{R}^2/\mathbb{Z}^2$; see also [5], [6] for earlier, related studies. Subsequently, the Markov partition was lifted to the punctured case in the manner we shall describe in this section; where the novelty of this section lies is in the made-to-measure application to fractal curves.

The sequence $\mathrm{row}_{[1\uparrow f_1]}(\tilde{\mathbf{L}})$ is called *the Markov partition* for F^{-1}.

We shall see the following well-known facts.

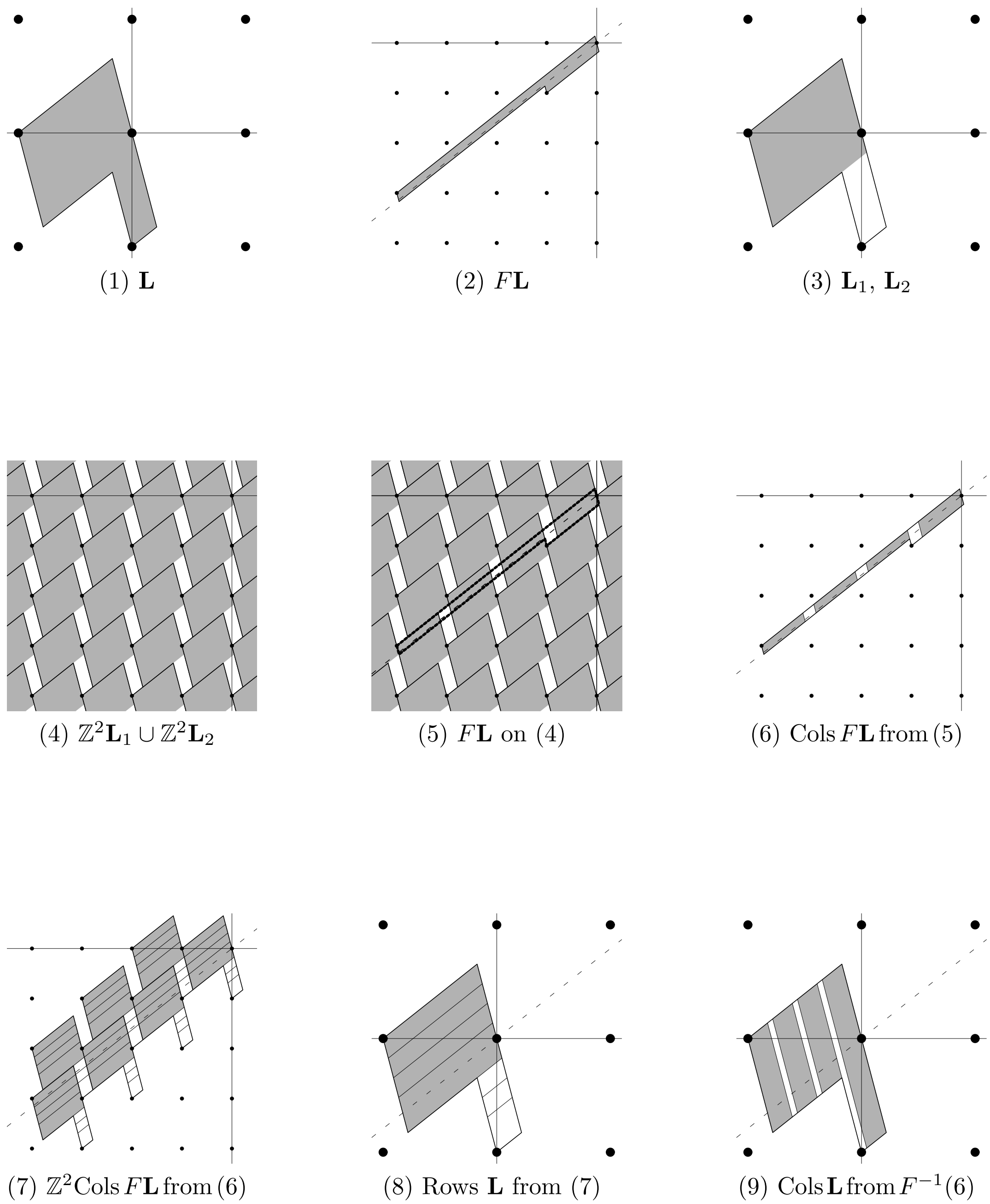
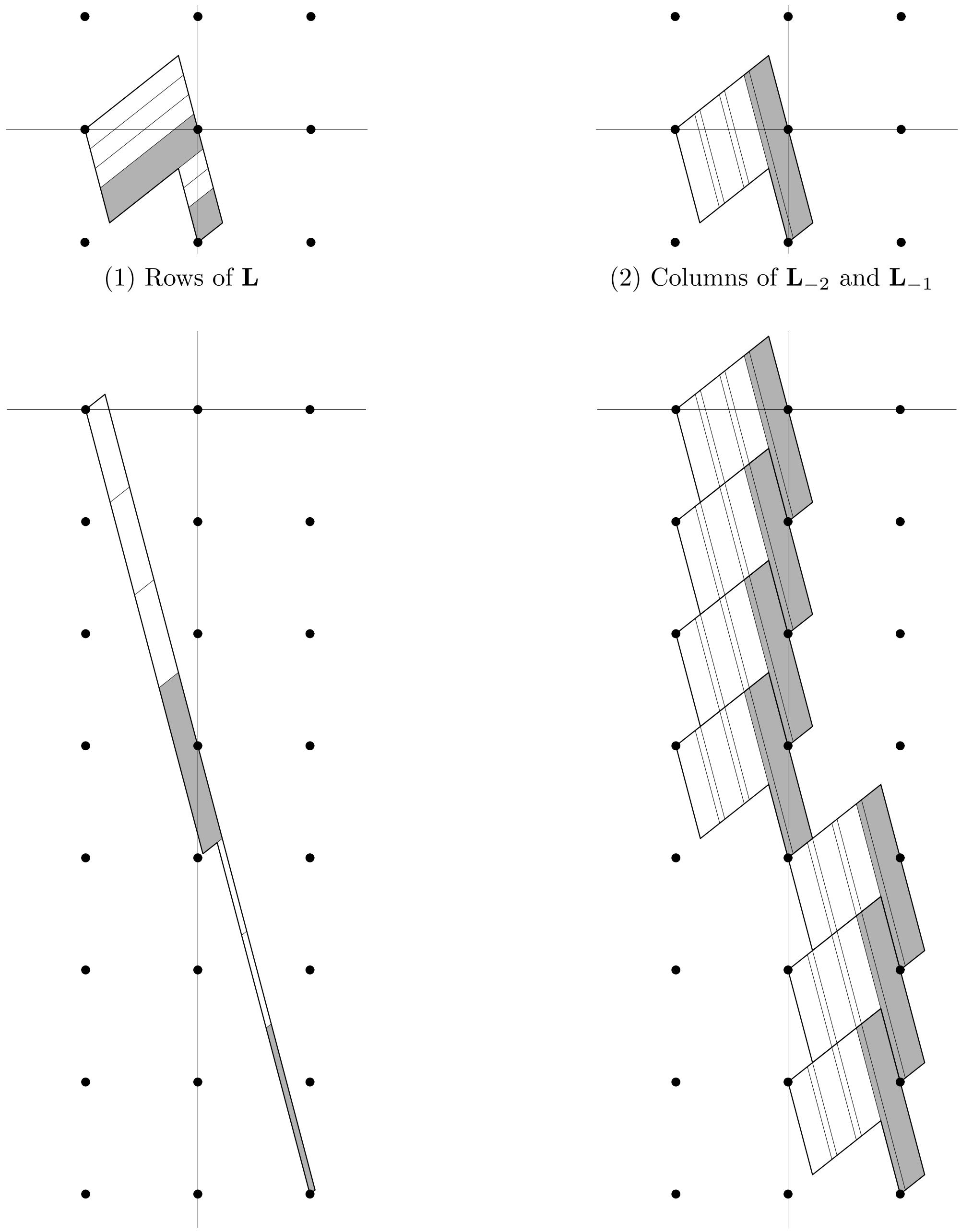


Figure 6.1.1: F, columns to rows: broad = gray = $BC = X_1$, thin = white = $BA = X_2$



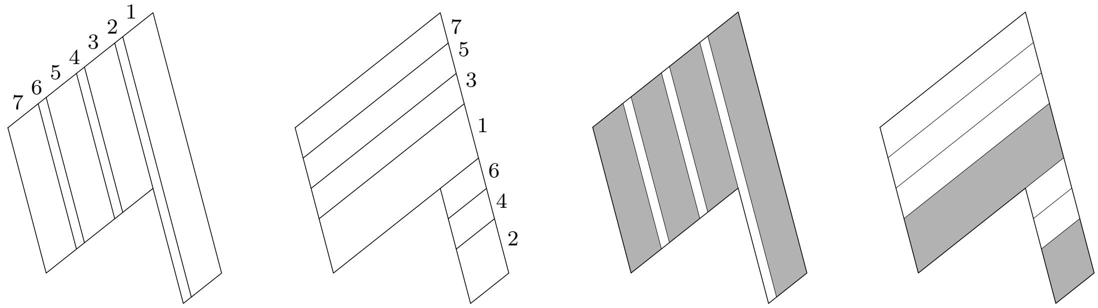
(1) Rows of $\mathbf{L}$

(2) Columns of $\mathbf{L}_{-2}$ and $\mathbf{L}_{-1}$

(3) $G = AF^{-1}A$ applied to (1)

(4) $\mathbb{Z}^2$-translates of (2) or (3)

Figure 6.1.2: F^{-1}, rows to columns: squat = white = $BA = X_{-2}$, tall = gray = $CA = X_{-1}$



(1) The numbering indicates how F transforms columns to rows.

(2) Reading columns from right to left gives ${}^{F}(BC)=(BCBA)^3BC$, ${}^{F}(BA)=BCBA$.
Reading rows from top to bottom gives ${}^{F^{-1}}(AC)=(AB)^3AC(AB)^2AC$, ${}^{F^{-1}}(AB)=(AB)^3AC$.

Figure 6.1.3: The Markov partitions for F and F^{-1} in the case $F = RL^3$

The union of the columns of $\tilde{\mathbf{L}}$ is all of $\tilde{\mathbf{L}}$, the intersections of consecutive columns lie in minus-foliation curves, and distinct non-consecutive columns are disjoint.

Each column of $\tilde{\mathbf{L}}$ is mapped by F into a translate $W\tilde{\mathbf{L}}$ for a unique $W \in \langle A, B, C \rangle$ and translating back by W^{-1} gives us a map carrying the disjoint union of the columns of $\tilde{\mathbf{L}}$ bijectively to the disjoint union of the rows of $\tilde{\mathbf{L}}$. This map then induces the action of F on $\mathbf{T}$, which carries columns to rows expanding in the plus-direction and contracting in the minus-direction. □

We now describe the tessellation of $F\tilde{\mathbf{L}}$ that is induced by the $\tilde{\mathbf{L}}_{[1\uparrow 2]}$-tessellation of $\mathbf{D}^\circ \cup \langle A, B, C \rangle D^{\pm\infty}$; see Figure 6.1.1(5) and (6).

6.3 Lemma. *Let* Notation 6.1 *hold.*

(i) *Let* $\mathrm{CW}(F\tilde{\mathbf{L}})$ *denote* $F\tilde{\mathbf{L}}$ *endowed with the CW-structure that is induced by the* $\tilde{\mathbf{L}}_{[1\uparrow 2]}$*-tessellation of* $\mathbf{D}^\circ \cup \langle A, B, C \rangle D^{\pm\infty}$. *Here, the two-cells of* $\mathrm{CW}(F\tilde{\mathbf{L}})$ *are the terms of the sequence* $(F\tilde{\mathbf{L}} \cap FU_\ell\tilde{\mathbf{L}} \mid \ell \in [1\uparrow f_1])$, *consecutive terms overlap on a minus-foliation one-cell of* $\mathrm{CW}(F\tilde{\mathbf{L}})$, *and non-consecutive distinct terms are disjoint. Each minus-foliation one-cell of* $\mathrm{CW}(F\tilde{\mathbf{L}})$ *lies in the boundary of an* $\tilde{\mathbf{L}}$*-tile. Each plus-foliation one-cell of* $\mathrm{CW}(F\tilde{\mathbf{L}})$ *lies in the boundary of* $F\tilde{\mathbf{L}}$.

(ii) *Let* $\ell \in [1\uparrow f_1]$ *and define* $i_\ell := -2$ *for* $\ell \in [1\uparrow f_2]$, *and* $i_\ell := -1$ *for* $\ell \in [(f_2+1)\uparrow f_1]$. *Then* $F\tilde{\mathbf{L}} \cap FU_\ell\tilde{\mathbf{L}}$ *equals* $F\tilde{\mathbf{L}}_{i_\ell} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}$ *and is a lift of a parallelogram. The left and right borders of* $F\tilde{\mathbf{L}}_{i_\ell} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}$ *lie in the left and right borders of* $FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}$, *respectively. The top and bottom borders of* $F\tilde{\mathbf{L}}_{i_\ell} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}$ *lie in the top and bottom borders of* $F\tilde{\mathbf{L}}_{i_\ell}$, *respectively. The bottom border of* $F\tilde{\mathbf{L}}_{i_\ell} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}$ *lies in the bottom border of* $FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}$ *if and only if* $\ell \in [(f_2-1)\uparrow f_2]$. *The top border of* $F\tilde{\mathbf{L}}_i \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}$ *lies in the top border of* $FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}$ *if and only if* $\ell \in [(f_1-1)\uparrow f_1]$. *Also,* $\mathbf{s}_\ell = [\frac{\ell+1}{1+\mu_+}] - [\frac{\ell}{1+\mu_+}] + 1$.

Proof. Let $\mathfrak{l}_+$ denote the half-plus-line incident to $(0,0)$ in the lower-half-plane, oriented from right to left.

We define a region $\mathbf{M}$ in $\mathbb{R}^2$ as follows. We start by discarding the region of $\mathbb{R}^2$ which is above the minus-eigenline, but do not discard the minus-eigenline itself. In the portion of $\mathbb{R}^2$ which is below the minus-eigenline, we discard the half-minus-lines that do not meet $\mathfrak{l}_+$, but we do not discard the associated elements of $\mathbb{Z}^2$. This completes the description of $\mathbf{M}$.

It is straightforward to see that $\mathbf{M}$ is F-invariant, since F fixes the two half-plus-eigenlines, and fixes the minus-eigenline, and permutes the half-minus-lines.

Notice that $\mathbf{M}$ contains $\mathbf{L}'$ and $\mathfrak{l}_+$ and the minus-lines emanating from $\mathfrak{l}_+$ in both directions, cut off if they meet an element of $\mathbb{Z}^2$.

We now lift the subset $\mathbf{M}$ of $\mathbb{R}^2$ to a copy $\tilde{\mathbf{M}}$ in the compactification $\mathbf{D}$ of the universal cover of $\mathbb{R}^2 - \mathbb{Z}^2$ by first lifting the origin of $\mathbb{R}^2$ to $D^{\pm\infty} \in \mathbf{D}$, and then extending to a lift of all of $\mathbf{M}$. Then $\tilde{\mathbf{M}}$ lies in $\mathbf{D}^\circ \cup \langle A, B, C \rangle D^{\pm\infty}$, and $\tilde{\mathbf{M}}$ is F-invariant, and $\tilde{\mathbf{M}}$ contains $\tilde{\mathbf{L}}'$. For the current purpose of studying the $\tilde{\mathbf{L}}_{[1\uparrow 2]}$-tessellation of $F\tilde{\mathbf{L}}$, we shall treat $\mathbf{M}$ and $\tilde{\mathbf{M}}$ as interchangeable. We understand that the following discussion takes place in $\mathbf{D}$ but with the notation of $\mathbb{R}^2$.

The half-line $\mathfrak{l}_+$ starts from within $\mathbf{\Delta}$, and the first labelled line segment it crosses has the label B. Let $\mathbf{s}'_{[1\uparrow\infty[}$ denote the infinite sequence in $[1\uparrow 2]$ with the property that the element of $\mathfrak{C}\langle A, B, C \rangle$ read by $\mathfrak{l}_+$ is $\prod X_{\mathbf{s}'_{[1\uparrow\infty[}}$.

Let $\ell \in [1\uparrow\infty[$. Define $W_\ell := \prod X_{\mathbf{s}'_{[1\uparrow(\ell-1)]}}$.

We will now show that $\mathbf{s}'_\ell = [\frac{\ell+1}{1+\mu_+}] - [\frac{\ell}{1+\mu_+}] + 1$. Consider the line in $\mathbb{R}^2$ with equation $x + y = -\ell$, which is a concatenation of B-labelled line segments. Consider also the line with equation $x = \mu_+ y$, which is the plus-eigenline. Let (x_ℓ, y_ℓ) denote the point where these two lines intersect, that is, the point where $\mathfrak{l}_+$ reads B for the ℓth time. Then $(x_\ell, y_\ell) = (-\ell - \frac{-\ell}{1+\mu_+}, \frac{-\ell}{1+\mu_+})$. As $\mathfrak{l}_+$ passes through (x_ℓ, y_ℓ) it reads B, and then either $\mathfrak{l}_+$ reads C and reaches $(x_{\ell+1}, y_{\ell+1})$ and we have $[y_{\ell+1}] = [y_\ell]$, or $\mathfrak{l}_+$ reads A and reaches $(x_{\ell+1}, y_{\ell+1})$ and we have $[y_{\ell+1}] = [y_\ell] - 1$; see Figures 2.3.1 and 6.1.1(3). Thus, either $[y_\ell] - [y_{\ell+1}] = 0$ and $\mathbf{s}'_\ell = 1$, or $[y_\ell] - [y_{\ell+1}] = 1$ and $\mathbf{s}'_\ell = 2$. It follows that $\mathbf{s}'_\ell = [y_\ell] - [y_{\ell+1}] + 1 = -[\frac{\ell}{1+\mu_+}] + [\frac{\ell+1}{1+\mu_+}] + 1$, as desired.

We shall now prove that the sequence of $\mathbf{L}_{[1\uparrow 2]}$-tiles cut through by $\mathfrak{l}_+$ from right to left is $(W_\ell \mathbf{L}_{\mathbf{s}'_\ell} \mid \ell \in [1\uparrow\infty[\ )$.

Now, either $\ell = 1$ and $\mathfrak{l}_+$ originates in $W_\ell\mathbf{\Delta}$, or $\ell \geqslant 2$ and $\mathfrak{l}_+$ enters $W_\ell\mathbf{\Delta}$. Then, in either event, $\mathfrak{l}_+$ exits $W_\ell\mathbf{\Delta}$ reading a B-labelled line segment. And then, either $\mathbf{s}'_\ell = 1$ and $\mathfrak{l}_+$ reads a C-labelled line segment and enters $W_\ell BC\mathbf{\Delta} = W_\ell X_1 \mathbf{\Delta} = W_{\ell+1}\mathbf{\Delta}$, or $\mathbf{s}'_\ell = 2$ and $\mathfrak{l}_+$ reads an A-labelled line segment and enters $WBA\mathbf{\Delta} = W_\ell X_2 \mathbf{\Delta} = W_{\ell+1}\mathbf{\Delta}$. From Figures 2.3.1 and 6.1.1(3), we see that, in both cases, the initial point of $\mathfrak{l}_+ \cap W_\ell\mathbf{\Delta}$ lies in $W_\ell \mathbf{L}_{\mathbf{s}'_\ell}$, and $\mathfrak{l}_+$ cuts through $W_\ell \mathbf{L}_{\mathbf{s}'_\ell}$ from right to left, even for $\ell = 1$. It follows that $W_\ell \mathbf{L}_{\mathbf{s}'_\ell}$ is the ℓth $\mathbf{L}_{[1\uparrow 2]}$-tile cut by $\mathfrak{l}_+$, as desired. Since $X_1(x,y) = (x-1, y)$ and $X_2(x,y) = (x, y-1)$, we see that if $W_\ell X_{\mathbf{s}'_\ell}(0,0) = (-p,-q)$ then $p+q=\ell$. Notice that the left border of $W_\ell \mathbf{L}_{\mathbf{s}'_\ell}$ contains $W_\ell X_{\mathbf{s}'_\ell}(0,0)$ as one of its end-points.

We shall now prove that $F\mathbf{L}'$ cuts through $(W_\ell \mathbf{L}_{\mathbf{s}'_\ell} \mid \ell \in [1{\uparrow}f_1])$ in the same way that $\mathfrak{l}_+$ does.

Notice that $\mathbf{L}' \cap \mathbb{Z}^2$, which is $\{(0,0),(-1,0),(0,-1)\}$, lies in the boundary of $\mathbf{L}'$. Hence the boundary of $F\mathbf{L}'$ contains $F(\mathbf{L}' \cap \mathbb{Z}^2) = F(\mathbf{L}') \cap F(\mathbb{Z}^2) = F(\mathbf{L}') \cap \mathbb{Z}^2$. In particular, the interior of $F\mathbf{L}'$ does not meet $\mathbb{Z}^2$.

Since $\mathfrak{l}_+$ cuts through $\mathbf{L}'$ from right to left, we see that $F\mathfrak{l}_+ = \mathfrak{l}_+$ cuts through $F\mathbf{L}'$ from right to left. Since the interior of $F\mathbf{L}'$ does not meet $\mathbb{Z}^2$, we see that any labelled line segment that cuts through $\mathfrak{l}_+ \cap F\mathbf{L}'$ from top to bottom must cut through $F\mathbf{L}'$ from top to bottom.

Notice that the top left corner of $\mathbf{L}'$ is $(-1,0)$; see Figure 6.1.1(1). Hence the top left corner of $F\mathbf{L}'$ is $F(-1,0) = (-f_{11}, -f_{21})$. The A-labelled line segment joining $F(-1,0) = (-f_{11}, -f_{21})$ to $(-f_{11}+1, -f_{21})$ cuts through an $\mathbf{L}_1$-tile incident to $F(-1,0)$ from left to right, and cuts through $F\mathbf{L}'$ from top to bottom, and cuts through $\mathfrak{l}_+$. Hence $\mathfrak{l}_+$ cuts through this $\mathbf{L}_1$-tile from right to left, and this tile must be the $(f_{11}+f_{21})(=f_1)$th tile cut by $\mathfrak{l}_+$, that is, the tile $W_{f_1}\mathbf{L}_{\mathbf{s}'_{f_1}}$. In particular, $\mathbf{s}'_{f_1} = 1$ and the top left corner of $F\mathbf{L}_1$ is $W_{f_1}(-1,0)$. It follows that the top boundary of $F\mathbf{L}' \cap W_{f_1}\mathbf{L}_{\mathbf{s}_{f_1}}$ lies in both the top boundary of $F\mathbf{L}'$ and the top boundary of $W_{f_1}\mathbf{L}_{\mathbf{s}_{f_1}}$. Also the left boundary of $F\mathbf{L}' \cap W_{f_1}\mathbf{L}_{\mathbf{s}_{f_1}}$ lies in both the left boundary of $F\mathbf{L}'$ and the left boundary of $W_{f_1}\mathbf{L}_{\mathbf{s}_{f_1}}$. It can be seen that the bottom boundary of $F\mathbf{L}' \cap W_{f_1}\mathbf{L}_{\mathbf{s}_{f_1}}$ lies in the bottom boundary of $F\mathbf{L}'$ and does not touch the bottom boundary of $W\mathbf{L}_{\mathbf{s}_{f_1}}$. It can also be seen that the right boundary of $F\mathbf{L}' \cap W_{f_1}\mathbf{L}_{\mathbf{s}_{f_1}}$ lies in the right boundary of $W_{f_1}\mathbf{L}_{\mathbf{s}_{f_1}}$. See Figure 6.1.1(5).

Now consider $\ell \in [(f_1-1){\downarrow}1]$. We shall show that the left and right borders of $F\mathbf{L}' \cap W_\ell \mathbf{L}_{\mathbf{s}_\ell}$ lie in the left and right borders of $W_\ell \mathbf{L}_{\mathbf{s}_\ell}$, respectively, and that the top and bottom borders of $F\mathbf{L}' \cap W_\ell \mathbf{L}_{\mathbf{s}_\ell}$ lie in the top and bottom borders of $F\mathbf{L}'$, respectively. We shall also show that if the top border of $F\mathbf{L}' \cap W_\ell \mathbf{L}_{\mathbf{s}_\ell}$ lies in the top border of $F\mathbf{L}'$ then $\ell = f_1 - 1$ and $\mathbf{s}_{f_1-1} = 2$. We consider two cases.

Case 1. $\mathbf{s}_\ell = 1$.

Here, $\mathfrak{l}_+$ cuts through $W_\ell \mathbf{L}_{\mathbf{s}'_\ell} = W_\ell \mathbf{L}_1$ from right to left and subsequently crosses the incident C-labelled line segment joining $W_\ell(-1,0)$ to $W_\ell(-1,-1)$; see Figure 6.1.1(3). Hence this C-labelled line segment cuts through $F\mathbf{L}'$ from top to bottom. It follows that the left and right borders of $F\mathbf{L}' \cap W_\ell \mathbf{L}_{\mathbf{s}_\ell}$ lie in the left and right borders of $W_\ell \mathbf{L}_{\mathbf{s}_\ell}$, respectively, and that the top and bottom borders of $F\mathbf{L}' \cap W_\ell \mathbf{L}_{\mathbf{s}_\ell}$ lie in the top and bottom borders of $F\mathbf{L}'$, respectively; see Figure 6.1.1(3).

If the top border of $W_\ell \mathbf{L}_1$ meets (the top border of) $F\mathbf{L}'$, then $W_\ell(-1,0)$ lies in $F\mathbf{L}'$ and this implies that $\ell = f_1$, a contradiction; hence the top border of $F\mathbf{L}'$ does not meet $W_\ell \mathbf{L}_1$.

Case 2. $\mathbf{s}_\ell = 2$.

Here, $\mathfrak{l}_+$ cuts through $W_\ell \mathbf{L}_{\mathbf{s}_\ell} = W_\ell \mathbf{L}_2$ from right to left and subsequently crosses the incident A-labelled line segment joining $W_\ell(-1,-1)$ to $W_\ell(0,-1)$; see Figure 6.1.1(3). Hence this A-labelled line segment cuts through $F\mathbf{L}'$ from top to bottom. It follows that the left and right borders of $F\mathbf{L}' \cap W_\ell \mathbf{L}_{\mathbf{s}_\ell}$ lie in the left and right borders of $W_\ell \mathbf{L}_{\mathbf{s}_\ell}$, respectively, and that the top and bottom borders of $F\mathbf{L}' \cap W_\ell \mathbf{L}_{\mathbf{s}_\ell}$ lie in the top and bottom borders of $F\mathbf{L}'$, respectively; see Figure 6.1.1(3).

If the top border of $W_\ell \mathbf{L}_2$ meets (the top border of) $F\mathbf{L}'$, then $W_\ell(-1,-1)$ lies in $F\mathbf{L}'$, and this implies that $\ell = f_1 - 1$.

Thus the desired result holds in both cases.

Notice that $(0,-1)$ is the bottom left corner of $\mathbf{L}_2$; see Figure 6.1.1(3). It follows that the bottom left corner of $F\mathbf{L}_2$ coincides with the bottom left corner of an $\mathbf{L}_2$-tile incident to $F(0,-1) = (-f_{1\,2}, -f_{2\,2})$, and it must be the $(f_{1\,2}+f_{2\,2})(=f_2)$th tile cut by $F\mathbf{L}'$ and $\mathfrak{l}_+$, that is, the tile $W_{f_2}\mathbf{L}_{\mathbf{s}'_{f_2}}$. In particular, $\mathbf{s}'_{f_2} = 2$ and the bottom left corner of $F\mathbf{L}_2$ is $W_{f_2}(0,-1)$, that is, $F(0,-1) = (-f_{1\,2}, -f_{2\,2}) = W_{f_2}(0,-1)$.

It is now clear that, for all $\ell \in [1{\uparrow}f_1]$, $F\mathbf{L} \cap W_\ell \mathbf{L} = F\mathbf{L}_{i_\ell} \cap W_\ell \mathbf{L}_{\mathbf{s}_\ell}$, and the borders are as claimed.

It remains to show that $\mathbf{s}_{[1\uparrow f_1]} = \mathbf{s}'_{[1\uparrow f_1]}$, and here we use the notation of $\mathbf{D}$. The A-labelled line segment in $\mathbf{D}$ joining $D^{\pm\infty}$ to $X_1(D^{\pm\infty})$ lies in $\tilde{\mathbf{L}}_1$ and is carried by F to a path from $F(D^{\pm\infty}) = D^{\pm\infty}$ to $FX_1(D^{\pm\infty})$. If the path reads ABC after reaching $FX_1(D^{\pm\infty})$, then the path reads $W_{f_1}X_1$; see Figure 6.1.1(6). Then $FX_1(D^{\pm\infty})$ is the vertex of $W_{f_1}X_1\tilde{\mathbf{\Delta}}$ that lies between a C-labelled edge and an A-labelled edge, that is, $W_{f_1}X_1(D^{\pm\infty})$. Thus,

$$(\textstyle\prod X_{\mathbf{s}'_{[1\uparrow f_1]}})(D^{\pm\infty}) = W_{f_1}X_1(D^{\pm\infty}) = FX_1(D^{\pm\infty}) = ({}^{F}X_1)(D^{\pm\infty}) = (\textstyle\prod X_{\mathbf{s}_{[1\uparrow f_1]}})(D^{\pm\infty}).$$

Since $\mathbf{s}'_{f_1} = 1 = \mathbf{s}_{f_1}$, it follows that $\prod X_{\mathbf{s}'_{[1\uparrow f_1]}} = \prod X_{\mathbf{s}_{[1\uparrow f_1]}}$, and, hence, $\mathbf{s}'_{[1\uparrow f_1]} = \mathbf{s}_{[1\uparrow f_1]}$, as desired. □

The following gives all the claims in Profile 5.2.

6.4 Theorem. *Let* Notation 6.1 *hold. For each* $i \in [1{\uparrow}2]$, $(U_\ell \mathfrak{a}_{\mathbf{s}_\ell})_{\ell\in[1\uparrow f_i]}$ *is a fracturing of* $\mathfrak{a}_i$, *the elements of* $\mathfrak{a}_1$ *are codified as the ends of the* $\mathfrak{a}_1$*-accepting tree of* ∂^-*-syllables, and the limit set of the* ∂^-*-semigroup acting on* $\hat{\mathbb{C}}$ *is* $\mathfrak{a}_1$.

Proof. Notice that $\widehat{\rho}^{\,-}(F\tilde{\mathbf{L}}) = F\widehat{\rho}^{\,-}(\tilde{\mathbf{L}}) = F\mathfrak{a}_1$ is an oriented simple curve.

Lemma 6.3(i) implies that $(\widehat{\rho}^{\,-}(F\tilde{\mathbf{L}} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}) \mid \ell \in [1{\uparrow}f_1])$ is a fracturing of $\widehat{\rho}^{\,-}(F\tilde{\mathbf{L}}) = F\mathfrak{a}_1$. Since $\widehat{\rho}^{\,-}$ is constant on minus-foliation curves, Lemma 6.3(ii) implies that, for each $\ell \in [1{\uparrow}f_1]$, $\widehat{\rho}^{\,-}(F\tilde{\mathbf{L}} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}) = \widehat{\rho}^{\,-}(FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}) = FU_\ell\widehat{\rho}^{\,-}(\tilde{\mathbf{L}}_{\mathbf{s}_\ell}) = FU_\ell\mathfrak{a}_{\mathbf{s}_\ell}$. Hence $(FU_\ell\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [1{\uparrow}f_1])$ is a fracturing of $F\mathfrak{a}_1$. On applying F^{-1}, we see that $(U_\ell\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [1{\uparrow}f_1])$ is a fracturing of $\mathfrak{a}_1$.

Hence, $(FU_\ell\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [1{\uparrow}f_2])$ is a fracturing of $\bigcup\limits_{\ell\in[1\uparrow f_2]} \widehat{\rho}^{\,-}(F\tilde{\mathbf{L}} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell})$. Lemma 6.3(ii) implies that

$$\bigcup_{\ell\in[1\uparrow f_2]} \widehat{\rho}^{\,-}(F\tilde{\mathbf{L}} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}) = \bigcup_{\ell\in[1\uparrow f_2]} \widehat{\rho}^{\,-}(F\tilde{\mathbf{L}}_{-1} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}) \subseteq \widehat{\rho}^{\,-}(F\tilde{\mathbf{L}}_{-1}),$$

$$\bigcup_{\ell\in[(f_2+1)\uparrow f_1]} \widehat{\rho}^{\,-}(F\tilde{\mathbf{L}} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}) = \bigcup_{\ell\in[1\uparrow f_2]} \widehat{\rho}^{\,-}(F\tilde{\mathbf{L}}_{-2} \cap FU_\ell\tilde{\mathbf{L}}_{\mathbf{s}_\ell}) \subseteq \widehat{\rho}^{\,-}(F\tilde{\mathbf{L}}_{-2}).$$

Since the subcurves $\widehat{\rho}^{\,-}(F\tilde{\mathbf{L}}_{-1})$ and $\widehat{\rho}^{\,-}(F\tilde{\mathbf{L}}_{-2})$ overlap in a single point, it follows that $(FU_\ell\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [1\uparrow f_2])$ is a fracturing of $\widehat{\rho}^{\,-}(F\tilde{\mathbf{L}}_{-1}) = \widehat{\rho}^{\,-}(F\tilde{\mathbf{L}}_2) = F\mathfrak{a}_2$. Thus $(U_\ell\mathfrak{a}_{\mathbf{s}_\ell} \mid \ell \in [1\uparrow f_2])$ is a fracturing of $\mathfrak{a}_2$.

As in Definitions 5.6, we can use a tree to codify the elements of $\mathfrak{a}_1$ as infinite words in the ∂^--syllable sequence.

To show that the limit set of the ∂^--semigroup is $\mathfrak{a}_1$, it remains to show that, for each $\ell \in [1\uparrow f_1]$, $U_\ell\mathfrak{a}_1 \subseteq \mathfrak{a}_1$, or, equivalently, $FU_\ell\mathfrak{a}_1 \subseteq F\mathfrak{a}_1$. If $s_\ell = 1$, then

$$FU_\ell\mathfrak{a}_1 = FU_\ell\mathfrak{a}_{\mathbf{s}_\ell} \subseteq F\mathfrak{a}_1.$$

Notice that $\mathfrak{a}_2 \cup X_2\mathfrak{a}_1 \supseteq \mathfrak{a}_1$ since $\widehat{\rho}^{\,-}(\tilde{\mathbf{L}}_2 \cup X_2\tilde{\mathbf{L}}) \supseteq \widehat{\rho}^{\,-}(\tilde{\mathbf{L}})$; see Figure 6.1.1(5). If $s_\ell = 2$, then $\ell \in [(f_1-1)\downarrow 1]$, and

$$FU_\ell\mathfrak{a}_1 \subseteq FU_\ell\mathfrak{a}_2 \cup FU_\ell X_2\mathfrak{a}_1 = FU_\ell\mathfrak{a}_{\mathbf{s}_\ell} \cup FU_{\ell+1}\mathfrak{a}_1,$$

and, by a reverse induction hypothesis, this lies in $F\mathfrak{a}_1$. □

We now turn to $\mathfrak{a}_{-1}$ and apply a very similar argument.

6.5 Lemma. *Let* Notation 6.1 *hold.*

(i) *Let* $\mathrm{CW}(G\tilde{\mathbf{L}})$ *denote* $G\tilde{\mathbf{L}}$ *endowed with the CW-structure that is induced by the* $\tilde{\mathbf{L}}_{[(-1)\downarrow(-2)]}$*-tessellation of* $\mathbf{D}^\circ \cup \langle A, B, C \rangle D^{\pm\infty}$*. Here, the two-cells of* $\mathrm{CW}(G\tilde{\mathbf{L}})$ *are the terms of the sequence* $(G\tilde{\mathbf{L}} \cap GV_\ell\tilde{\mathbf{L}} \mid \ell \in [1\uparrow f_{-1}])$*, consecutive terms overlap on a plus-foliation one-cell, and non-consecutive distinct terms are disjoint. Each plus-foliation one-cell of* $\mathrm{CW}(G\tilde{\mathbf{L}})$ *lies in the boundary of an* $\tilde{\mathbf{L}}$*-tile. Each minus-foliation one-cell of* $\mathrm{CW}(G\tilde{\mathbf{L}})$ *lies in the boundary of* $G\tilde{\mathbf{L}}$*.*

(ii) *Let* $\ell \in [1\uparrow f_{-1}]$ *and define* $j_\ell := 1$ *for* $\ell \in [1\uparrow f_{-2}]$*, and* $j_\ell := 2$ *for* $\ell \in [(f_{-2}+1)\uparrow f_{-1}]$*. Then* $G\tilde{\mathbf{L}} \cap GV_\ell\tilde{\mathbf{L}}$ *equals* $G\tilde{\mathbf{L}}_{j_\ell} \cap GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}$ *and is a lift of a parallelogram. The top and bottom borders of* $G\tilde{\mathbf{L}}_{j_\ell} \cap GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}$ *lie in the top and bottom borders of* $GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}$*, respectively. The left and right borders of* $G\tilde{\mathbf{L}}_{j_\ell} \cap GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}$ *lie in the left and right borders of* $G\tilde{\mathbf{L}}_{j_\ell}$*, respectively. If* $\ell \in [1\uparrow(f_{-1}-1)]$*, then* $\mathbf{t}_\ell = [\ell(-\mu_-)] - [(\ell-1)(-\mu_-)] - 2$*; also,* $\mathbf{t}_{f_{-1}} = -1$*.*

Proof. Let $\mathfrak{l}_-$ denote the half-minus-line incident to $(-1, 0)$ in the lower-half-plane, oriented from top to bottom.

We define a region $\mathbf{M}$ in $\mathbb{R}^2$ as follows. We start by discarding the region of $\mathbb{R}^2$ which is above the plus-line through $(-1,0)$, but do not discard the plus-line itself. In the region of $\mathbb{R}^2$ which is below the plus-line, we discard the half-plus-lines that do not meet $\mathfrak{l}_-$, but we do not discard the associated elements of $\mathbb{Z}^2$. This completes the description of $\mathbf{M}$.

It is straightforward to see that $\mathbf{M}$ is G-invariant, since G fixes the two half-minus-eigen-lines incident to $(-1,0)$, and G fixes the plus-line, and G permutes the half-plus-lines.

Notice that $\mathbf{M}$ contains $\mathfrak{l}_-$ and the plus-lines emanating from $\mathfrak{l}_-$ in both directions, cut off if they meet an element of $\mathbb{Z}^2$. Also, $\mathbf{L}' - \mathbf{M}$ lies in the bottom border of $\mathbf{L}_{-1}$.

We now lift the subset $\mathbf{M}$ of $\mathbb{R}^2$ to a copy $\tilde{\mathbf{M}}$ in the compactification $\mathbf{D}$ of the universal cover of $\mathbb{R}^2 - \mathbb{Z}^2$ by first lifting the origin of $\mathbb{R}^2$ to $D^{\pm\infty} \in \mathbf{D}$, and then extending to a lift of all of $\mathbf{M}$. Then $\tilde{\mathbf{M}}$ lies in $\mathbf{D}^\circ \cup \langle A, B, C \rangle D^{\pm\infty}$, and $\tilde{\mathbf{M}}$ is G-invariant, and the closure of $\tilde{\mathbf{M}}$ contains $\tilde{\mathbf{L}}'$. For the current purpose of studying the $\tilde{\mathbf{L}}_{[(-1)\downarrow(-2)]}$-tessellation of $G\tilde{\mathbf{L}}$, we shall treat $\mathbf{M}$ and $\tilde{\mathbf{M}}$ as interchangeable. We understand that the following discussion takes place in $\mathbf{D}$ but with the notation of $\mathbb{R}^2$.

The half-line $\mathfrak{l}_-$ starts from within $\mathbf{\Delta}$ and exits $\mathbf{\Delta}$ through the infinitesimal beginning of a B-labelled line segment. Let $\mathbf{t}''_{[1\uparrow\infty[}$ denote the sequence in $[(-1)\downarrow(-2)]$ with the property that the element of $\mathfrak{E}\langle A, B, C \rangle$ read by $\mathfrak{l}_-$ is $\prod X_{\mathbf{t}''_{[1\uparrow\infty[}}$. Thus $\mathbf{t}''_1 = -2$.

We will now show that,

$$\text{(6.5.1)} \qquad \text{for each } \ell \in [1\uparrow\infty[\,, \quad \mathbf{t}''_\ell = [\ell(-\mu_-)] - [(\ell-1)(-\mu_-)] - 2.$$

Let $\ell \in [0\uparrow\infty[\,$. Consider the line in $\mathbb{R}^2$ with equation $y = -\ell$, which is a concatenation of A-labelled line segments. Consider also the line with equation $x = \mu_- y - 1$, which is the minus-line through $(-1,0)$. Let (x_ℓ, y_ℓ) denote the point where these two lines intersect. Then $(x_\ell, y_\ell) = (-\ell\mu_- - 1, -\ell)$. Thus $\mathfrak{l}_-$ begins at (x_0, y_0), and, if $\ell \geqslant 1$, then (x_ℓ, y_ℓ) is the point where $\mathfrak{l}_-$ reads A for the ℓth time. After leaving (x_ℓ, y_ℓ), either $\mathfrak{l}_-$ reads B and $\mathfrak{l}_-$ reaches $(x_{\ell+1}, y_{\ell+1})$ and reads A and we have $[x_{\ell+1}] = [x_\ell]$, or $\mathfrak{l}_-$ reads C and $\mathfrak{l}_-$ reaches $(x_{\ell+1}, y_{\ell+1})$ and reads A and we have $[x_{\ell+1}] = [x_\ell] + 1$; see Figures 2.3.1 and 6.1.2(2). Thus, either $[x_{\ell+1}] - [x_\ell] = 0$ and $\mathbf{t}''_{\ell+1} = -2$, or $[x_{\ell+1}] - [x_\ell] = 1$ and $\mathbf{t}''_{\ell+1} = -1$; hence, $\mathbf{t}''_{\ell+1} = [x_{\ell+1}] - [x_\ell] - 2$. Now (6.5.1) follows.

Let $\ell \in [1\uparrow\infty[$ and define $W_\ell := \prod X_{\mathbf{t}''_{[1\uparrow(\ell-1)]}}$.

We shall now prove that the sequence of $\mathbf{L}_{[(-1)\downarrow(-2)]}$-tiles cut through by $\mathfrak{l}_-$ from top to bottom is $(W_\ell \mathbf{L}_{\mathbf{t}''_\ell} \mid \ell \in [1\uparrow\infty[\,)$.

Now, either $\ell = 1$ and $\mathfrak{l}_-$ originates in $W'_\ell \mathbf{\Delta}$, or $\ell \geqslant 2$ and $\mathfrak{l}_-$ enters $W_\ell \mathbf{\Delta}$ reading an A-labelled line segment. In either event, if $\mathbf{t}''_\ell = -2$, then $\mathfrak{l}_-$ exits $W_\ell \mathbf{\Delta}$ reading a B-labelled line segment, and if $\mathbf{t}''_\ell = -1$, then $\mathfrak{l}_-$ exits $W_\ell \mathbf{\Delta}$ reading a C-labelled line segment. From Figures 2.3.1 and 6.1.2(2), we see that, in all cases, the initial point of $\mathfrak{l}_- \cap W_\ell \mathbf{\Delta}$ lies in $W_\ell \mathbf{L}_{\mathbf{t}''_\ell}$, and $\mathfrak{l}_-$ cuts through $W_\ell \mathbf{L}_{\mathbf{t}''_\ell}$ from top to bottom. It follows that $W_\ell \mathbf{L}_{\mathbf{t}''_\ell}$ is the ℓth $\mathbf{L}_{[(-1)\downarrow(-2)]}$-tile cut through by $\mathfrak{l}_-$, as desired. Since $X_{-1}(x,y) = (x+1, y-1)$ and $X_{-2}(x,y) = (x, y-1)$, we see that both $W_\ell(-1,0)$ and $W_\ell(0,0)$ have $1 - \ell$ as the second coordinate.

Define $\mathbf{t}'_{[1\uparrow f_1]} := \mathbf{t}''_{[1\uparrow(f_1-1)]} \vee (-1)$.

We shall now prove that $G\mathbf{L}'$ cuts through $(W_\ell \mathbf{L}_{\mathbf{t}'_\ell} \mid \ell \in [1\uparrow f_1])$.

Notice that $\mathbf{L}' \cap \mathbb{Z}^2$, which is $\{(0,0),(-1,0),(0,-1)\}$, lies in the boundary of $\mathbf{L}'$. Hence the boundary of $G\mathbf{L}'$ contains $G(\mathbf{L}' \cap \mathbb{Z}^2) = G(\mathbf{L}') \cap G(\mathbb{Z}^2) = G(\mathbf{L}') \cap \mathbb{Z}^2$. In particular, the interior of $G\mathbf{L}'$ does not meet $\mathbb{Z}^2$.

Since $\mathfrak{l}_-$ contains the left border of $\mathbf{L}'$, we see that $G\mathfrak{l}_- = \mathfrak{l}_-$ contains the left border of $G\mathbf{L}'$. Since the interior of $G\mathbf{L}'$ does not meet $\mathbb{Z}^2$, we see that any labelled line segment whose interior cuts through $\mathfrak{l}_- \cap G\mathbf{L}'$ from right to left must cut through $G\mathbf{L}'$ from right to left.

Notice that the bottom border of $\mathbf{L}'$ contains $(0,-1)$; see Figure 6.1.2(2). Hence, the bottom border of $G\mathbf{L}'$ contains $G(0,-1) = (f_2-1,-f_1)$ which is an end-point of the B-labelled line segment joining $(f_2,-f_1+1)$ to $(f_2-1,-f_1)$. The other end-point of the B-labelled line segment does not lie in $G\mathbf{L}'$, and, hence, the B-labelled line segment must exit the left border of $G\mathbf{L}'$. Hence, the left border of $G\mathbf{L}'$ cuts through an $\mathbf{L}_{-2}$-tile incident to $(f_2,-f_1+1)$, and this must then be the f_1st $\tilde{\mathbf{L}}_{[(-1)\downarrow(-2)]}$-tile cut by $\mathfrak{l}_-$, that is, $W_{f_1}\mathbf{L}_{\mathbf{t}''_{f_1}}$, In particular, $\mathbf{t}''_{f_1} = -2$. Now $G(0,-1)$ is the lower left corner of $W_{f_1}\mathbf{L}_{-1} = W_{f_1}\mathbf{L}_{\mathbf{t}'_{f_1}}$, and is also the lower left corner of $G\mathbf{L}'$. Hence $G\mathbf{L}' \cap W_{f_1}\mathbf{L} = G\mathbf{L}_{-1} \cap W_{f_1}\mathbf{L}_{\mathbf{t}'_{f_1}}$, and the bottom border of $W_{f_1}\mathbf{L}_{\mathbf{t}'_{f_1}}$ contains the bottom border of $G\mathbf{L}_{-1}$. Moreover, the top and bottom borders of $G\mathbf{L}_{-1} \cap W_{f_1}\mathbf{L}_{\mathbf{t}'_{f_1}}$ lie in the top and bottom borders of $W_{f_1}\mathbf{L}_{\mathbf{t}'_{f_1}}$, respectively, and the right and left borders of $G\mathbf{L}_{-1} \cap W_{f_1}\mathbf{L}_{\mathbf{t}'_{f_1}}$ lie in the right and left borders of $G\mathbf{L}_{-1}$, respectively; see Figure 6.1.2(3).

Now consider $\ell \in [(f_1-1)\downarrow 1]$. We shall show that the left and right borders of $G\mathbf{L}' \cap W_\ell \mathbf{L}_{\mathbf{t}''_\ell}$ lie in the left and right borders of $W_\ell\mathbf{L}_{\mathbf{t}''_\ell}$, respectively, and that the top and bottom borders of $G\mathbf{L}' \cap W_\ell\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the top and bottom borders of $G\mathbf{L}'$, respectively. We consider two cases.

Case 1. $\mathbf{t}''_\ell = -1$.

Here, $\mathfrak{l}_-$ cuts through $W_\ell\mathbf{L}_{\mathbf{t}''_\ell} = W_\ell\mathbf{L}_{-1}$ from top to bottom and crosses the embedded C-labelled line segment joining $W_\ell(0,0)$ to $W_\ell(0,-1)$; see Figure 6.1.2(2). Hence this C-labelled line segment cuts through $G\mathbf{L}'$ from right to left. It follows that the right and left borders of $G\mathbf{L}' \cap W_\ell\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the right and left borders of $G\mathbf{L}'$, respectively, and that the top and bottom borders of $G\mathbf{L}' \cap W_\ell\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the top and bottom borders of $W_\ell\mathbf{L}_{\mathbf{t}''_\ell}$, respectively; see Figure 6.1.2(2).

Case 2. $\mathbf{t}''_\ell = -2$.

Here, $\mathfrak{l}_-$ cuts through $W_\ell\mathbf{L}_{\mathbf{t}''_\ell} = W_\ell\mathbf{L}_{-2}$ from top to bottom and crosses the incident B-labelled line segment joining $W_\ell(-1,-1)$ to $W_\ell(0,-1)$; see Figure 6.1.2(2). Hence this B-labelled line segment cuts through $G\mathbf{L}'$ from right to left. It follows that the top and bottom borders of $G\mathbf{L}' \cap W_\ell\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the top and bottom borders of $W_\ell\mathbf{L}_{\mathbf{t}''_\ell}$, respectively, and that the right and left borders of $G\mathbf{L}' \cap W_\ell\mathbf{L}_{\mathbf{t}''_\ell}$ lie in the right and left borders of $G\mathbf{L}'$, respectively; see Figure 6.1.1(3).

Thus the desired result holds in both cases.

Notice that the lower left corner of $BC\mathbf{L}_{-1}$ is $(-1,-1)$ and the right border of $BC\mathbf{L}_{-1}$ is contained in $\mathfrak{l}_{-}$. Hence, the lower left corner of $GBC\mathbf{L}_{-1}$ is $G(-1,-1) = (f_{1\,2}-1, -f_{1\,1})$ and the right border of $GBC\mathbf{L}_{-1}$ is contained in $G\mathfrak{l}_{-} = \mathfrak{l}_{-}$. Hence, the C-labelled line segment joining $(f_{1\,2}-1, -f_{1\,1}+1)$ to $(f_{1\,2}-1, -f_{1\,1})$ crosses $GBC\mathbf{L}_{-1}$. It follows that the $\mathbf{L}_{-1}$-tile containing the C-labelled line segment is the $f_{1\,1} = f_{-2}$th $\mathbf{L}_{[(-1)\downarrow(-2)]}$-tile cut by $\mathfrak{l}_{-}$, that is, $W_{f_{-2}}\mathbf{L}_{\mathbf{t}''_{f_{-2}}}$. The minus-line through $(-1,-1)$ contains the bottom borders of $BC\mathbf{L}_{-1}$ and $\mathbf{L}_1$. Hence, the minus-line through $G(-1,-1)$ contains the bottom borders of $GBC\mathbf{L}_{-1}$ and $G\mathbf{L}_1$. Hence $G\mathbf{L}_1$ lies in $\bigcup\limits_{[1\uparrow f_{-2}]} W_\ell \mathbf{L}_{\mathbf{t}'_\ell}$ and $G\mathbf{L}_2$ lies in $\bigcup\limits_{[(f_{-2}+1)\uparrow f_{-1}]} W_\ell \mathbf{L}_{\mathbf{t}'_\ell}$.

It is now clear that, for all $\ell \in [1\uparrow f_{-1}]$, $G\mathbf{L} \cap W_\ell\mathbf{L} = G\mathbf{L}_{j_\ell} \cap W_\ell \mathbf{L}_{\mathbf{t}'_\ell}$, and the borders are as claimed.

It remains to show that $\mathbf{t}_{[1\uparrow f_1]} = \mathbf{t}'_{[1\uparrow f_1]}$, and here we use the notation of $\mathbf{D}$. The B-labelled line segment in $\mathbf{D}$ joining $AD^{\pm\infty}$ to $X_{-1}A(D^{\pm\infty})$ lies in $\tilde{\mathbf{L}}_1$ and is carried by G to a path from $GA(D^{\pm\infty}) = AD^{\pm\infty}$ to $GX_{-1}A(D^{\pm\infty})$. If the path reads B at the initial vertex $AD^{\pm\infty}$ and CA at the terminal vertex $GX_{-1}A(D^{\pm\infty})$, then the path reads $W_{f_1}X_{-1}$; see Figure 6.1.2(3). Thus $GX_{-1}A(D^{\pm\infty})$ equals the vertex of $W_{f_1}X_{-1}\tilde{\mathbf{\Delta}}$ that lies between an A-labelled edge and a B-labelled edge, that is, $({}^{G}X_{-1})A(D^{\pm\infty}) = W_{f_1}X_{-1}A(D^{\pm\infty})$. Thus,

$$(\textstyle\prod X_{\mathbf{t}'_{[1\uparrow f_1]}})A(D^{\pm\infty}) = W_{f_1}X_{-1}A(D^{\pm\infty}) = ({}^{G}X_{-1})A(D^{\pm\infty}) = (\textstyle\prod X_{\mathbf{t}_{[1\uparrow f_1]}})A(D^{\pm\infty}).$$

Since $\mathbf{t}'_{f_1} = -1 = \mathbf{t}_{f_1}$, it follows that $\prod X_{\mathbf{t}'_{[1\uparrow f_1]}} = \prod X_{\mathbf{t}_{[1\uparrow f_1]}}$, and, hence, $\mathbf{t}'_{[1\uparrow f_1]} = \mathbf{t}_{[1\uparrow f_1]}$, as desired. □

The following gives all the claims in Profile 5.2.

6.6 Theorem. *Let* Notation 6.1 *hold. For each $j \in [(-1)\downarrow(-2)]$, $(V_\ell \mathfrak{a}_{\mathbf{t}_\ell})_{\ell\in[1\uparrow f_j]}$ is a fracturing of $\mathfrak{a}_j$, the elements of $\mathfrak{a}_{-1}$ are codified as the ends of some $\mathfrak{a}_{-1}$-accepting tree of ∂^{+}-syllables, and the limit set of the ∂^{+}-semigroup acting on $\hat{\mathbb{C}}$ is $\mathfrak{a}_{-1}$.*

Proof. Notice that $\widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}) = G\widehat{\rho}^{\,+}(\tilde{\mathbf{L}}) = G\mathfrak{a}_{-1}$ is an oriented simple curve.

Lemma 6.5(i) implies that $(\widehat{\rho}^{\,+}(G\tilde{\mathbf{L}} \cap GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}) \mid \ell \in [1\uparrow f_{-1}])$ is a fracturing of $\widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}) = G\mathfrak{a}_{-1}$. Since $\widehat{\rho}^{\,+}$ is constant on plus-foliation curves, Lemma 6.5(ii) implies that, for each $\ell \in [1\uparrow f_{-1}]$, $\widehat{\rho}^{\,+}(G\tilde{\mathbf{L}} \cap GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}) = \widehat{\rho}^{\,+}(GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}) = GV_\ell\widehat{\rho}^{\,+}(\tilde{\mathbf{L}}_{\mathbf{t}_\ell}) = GV_\ell\mathfrak{a}_{\mathbf{t}_\ell}$. Hence, $(GV_\ell\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [1\uparrow f_{-1}])$ is a fracturing of $F\mathfrak{a}_{-1}$. On applying G^{-1}, we see that $(V_\ell\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [1\uparrow f_{-1}])$ is a fracturing of $\mathfrak{a}_{-1}$.

Hence, $(GV_\ell\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [1\uparrow f_{-2}])$ is a fracturing of $\bigcup\limits_{\ell\in[1\uparrow f_{-2}]} \widehat{\rho}^{\,+}(G\tilde{\mathbf{L}} \cap GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell})$. Lemma 6.5(ii) implies that

$$\bigcup_{\ell\in[1\uparrow f_{-2}]} \widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}\cap GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}) = \bigcup_{\ell\in[1\uparrow f_{-2}]} \widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}_1\cap GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}) \subseteq \widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}_1),$$

$$\bigcup_{\ell\in[(f_{-2}+1)\uparrow f_{-1}]} \widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}\cap GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}) = \bigcup_{\ell\in[1\uparrow f_{-2}]} \widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}_2\cap GV_\ell\tilde{\mathbf{L}}_{\mathbf{t}_\ell}) \subseteq \widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}_2).$$

Since the subcurves $\widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}_1)$ and $\widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}_2)$ overlap in a single point, it follows that $(GV_\ell\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [1\uparrow f_{-2}])$ is a fracturing of $\widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}_1) = \widehat{\rho}^{\,+}(G\tilde{\mathbf{L}}_{-2}) = G\mathfrak{a}_{-2}$. Thus $(V_\ell\mathfrak{a}_{\mathbf{t}_\ell} \mid \ell \in [1\uparrow f_{-2}])$ is a fracturing of $\mathfrak{a}_{-2}$.

As in Definitions 5.6, we can use a tree to codify the elements of $\mathfrak{a}_{-1}$ as infinite words in the ∂^+-syllable sequence. To show that the limit set of the ∂^+-semigroup is $\mathfrak{a}_{-1}$, it remains to show that, for each $\ell \in [1\uparrow f_{-1}]$, $V_\ell\mathfrak{a}_{-1} \subseteq \mathfrak{a}_{-1}$, or, equivalently, $GV_\ell\mathfrak{a}_{-1} \subseteq G\mathfrak{a}_{-1}$. If $\mathbf{t}_\ell = -1$, then

$$GV_\ell\mathfrak{a}_{-1} = GV_\ell\mathfrak{a}_{\mathbf{t}_\ell} \subseteq G\mathfrak{a}_{-1}.$$

Notice that $\mathfrak{a}_{-2} \cup X_{-2}\mathfrak{a}_{-1} \supseteq \mathfrak{a}_{-1}$ since $\widehat{\rho}^{\,+}(\tilde{\mathbf{L}}_{-2} \cup X_{-2}\tilde{\mathbf{L}}) \supseteq \widehat{\rho}^{\,+}(\tilde{\mathbf{L}})$; see Figure 6.1.2(4). If $\mathbf{t}_\ell = -2$, then $\ell \in [(f_{-1}-1)\downarrow 1]$, and

$$GV_\ell\mathfrak{a}_{-1} \subseteq GV_\ell\mathfrak{a}_{-2} \cup GV_\ell X_{-2}\mathfrak{a}_{-1} = GV_\ell\mathfrak{a}_{\mathbf{t}_\ell} \cup GV_{\ell+1}\mathfrak{a}_{-1},$$

and, by a reverse induction hypothesis, this lies in $G\mathfrak{a}_{-1}$. □

We now verify most of Profile 5.4.

6.7 Proposition. *Let* Notation 6.1 *hold. Then the following hold.*

(i). *As a set,* $U_{[1\uparrow f_1]} = \{W \in \langle A,B,C\rangle F^{-1} \mid$ *the interior of* $W\tilde{\mathbf{L}}$ *meets the interior of* $\tilde{\mathbf{L}}\}$.

(ii). *As a set* $V_{[1\uparrow f_1]} = \{W \in \langle A,B,C\rangle F \mid$ *the interior of* $W\tilde{\mathbf{L}}$ *meets the interior of* $\tilde{\mathbf{L}}\}$.

(iii). *The inversion map on* $\langle A,B,C,F\rangle$ *carries* $U_{[1\uparrow f_1]}$ *to a permutation of* $V_{[1\uparrow f_1]}$, *and carries the* ∂^-*-semigroup bijectively to the* ∂^+*-semigroup.*

(iv). *For each* $\ell \in [1\uparrow f_1]$, $U_\ell^{-1} = V_{F(\ell)}$ *and* $U_\ell^{-1}(\mathrm{col}_\ell(\tilde{\mathbf{L}})) = \mathrm{row}_{F(\ell)}(\tilde{\mathbf{L}})$.

(v). $U_1 = V_{f_2 1+1}^{-1} = F^{-1}$, $U_{f_2} = V_{f_1}^{-1} = CF^{-1}C$ *and* $U_{f_1} = V_1^{-1} = AF^{-1}A$; *these are parabolic on* $\hat{\mathbb{C}}$ *with fixed points* $\widehat{\rho}((ABC)^{\pm\infty}) = \infty$, $\widehat{\rho}((CAB)^{\pm\infty}) = C(\infty)$, *and* $\widehat{\rho}((BCA)^{\pm\infty}) = A(\infty)$, *respectively.*

Proof. (i)–(iii) are clear from the preceding results of this section.

(iv). It can be seen that $\tilde{\mathbf{L}}_{-1} = \bigcup_{\ell\in[1\uparrow f_2]} \mathrm{col}_\ell(\tilde{\mathbf{L}})$ and $\tilde{\mathbf{L}}_{-2} = \bigcup_{\ell\in[(f_2+1)\uparrow f_1]} \mathrm{col}_\ell(\tilde{\mathbf{L}})$; see Figure 6.1.2(2). The top border of $\tilde{\mathbf{L}} = \tilde{\mathbf{L}}_{-1} \cup \tilde{\mathbf{L}}_{-2}$ is equal to the bottom border of $X_{-2}^{-1}\tilde{\mathbf{L}}_{-2} \cup X_{-1}^{-1}\tilde{\mathbf{L}}_{-1}$. Hence, the top border of $\bigcup_{\ell\in[1\uparrow f_1]} \mathrm{col}_\ell(\tilde{\mathbf{L}})$ is equal to the bottom border of $\bigcup_{\ell\in[(f_2+1)\uparrow f_1]} \mathrm{col}_\ell(X_{-2}^{-1}\tilde{\mathbf{L}}) \cup \bigcup_{\ell\in[1\uparrow f_2]} \mathrm{col}_\ell(X_{-1}^{-1}\tilde{\mathbf{L}})$, and we can see that there is a cyclic shift by f_2. It follows from Lemma 6.3(ii) that the top borders of the first f_1-2 columns of $\tilde{\mathbf{L}}$ equal the bottom borders of the columns of $X_{-2}^{-1}\tilde{\mathbf{L}}_{-2}$ and $X_{-1}^{-1}\tilde{\mathbf{L}}_{-1}$. For each $\ell \in [1\uparrow(f_1-f_2-1)]$, the top border of $\mathrm{col}_\ell(\tilde{\mathbf{L}})$ is equal to the bottom border of $\mathrm{col}_{\ell+f_2}(X_{-2}^{-1}\tilde{\mathbf{L}})$, and, for each $\ell \in [(f_1-f_2)\uparrow(f_1-2)]$, the top border of $\mathrm{col}_\ell(\tilde{\mathbf{L}})$ is equal to the bottom border of $\mathrm{col}_{\ell+f_2-f_1}(X_{-1}^{-1}\tilde{\mathbf{L}})$, and the top border of $\mathrm{col}_{f_1-1}(\tilde{\mathbf{L}})$ is equal to a right segment of the bottom border of $\mathrm{col}_{f_2-1}(X_2^{-1}X_1\tilde{\mathbf{L}})$.

Let $\ell \in [1\uparrow f_1]$. Then U_ℓ^{-1} carries $\mathrm{col}_\ell(\tilde{\mathbf{L}}) = \tilde{\mathbf{L}} \cap U_\ell \tilde{\mathbf{L}}$ bijectively to $U_\ell^{-1}\tilde{\mathbf{L}} \cap \tilde{\mathbf{L}} \subseteq \tilde{\mathbf{L}}$, and this must be one of the rows of $\tilde{\mathbf{L}}$. In particular, $U_{f_1}^{-1} = AFA$ carries the f_1st column to a row containing $A(D^{\pm\infty})$, and this is the first row. The bottom border of the f_1th column of $\mathbf{L}$ is the top border of the (f_1-f_2)th column of $X_{-2}\tilde{\mathbf{L}}$. Hence, $U_{f_2}^{-1}$ carries the (f_1-f_2)th column of $\tilde{\mathbf{L}}$ to the second row. At each step, we increase the number of the row by one and decrease the number of the column by f_2 mod f_1. Hence, if $\ell' = F^{-1}(\ell) \equiv -f_2(\ell-1)(\text{mod } f_1)$, then $U_{\ell'}^{-1}$ carries the ℓ'th column of $\tilde{\mathbf{L}}$ to the ℓth row of $\tilde{\mathbf{L}}$; see Figure 6.1.1(9) and (8). This proves (iv).

(v). Notice that

$$\begin{aligned} U_1 &= F^{-1}, \\ U_{f_2} &= F^{-1}({}^F X_2)X_2^{-1} = BAF^{-1}AB = CF^{-1}C, \\ U_{f_1} &= F^{-1}({}^F X_1)X_1^{-1} = BCF^{-1}CB = AF^{-1}A. \end{aligned}$$

□

7 Applications: Hausdorff dimensions

In this section, we adapt for general F some techniques of Dicks-Porti [15] that were developed for the case $F = RL$. We use results of Bishop-Jones and Beardon-Maskit to prove that the critical exponent of the ∂^--semigroup equals the Hausdorff dimension of $\partial^- A$. Similarly, the critical exponent of the ∂^+-semigroup equals the Hausdorff dimension of $\partial^+ B$. These four numbers are then equal.

7.1 Definitions. (i). The Möbius action of $\left(\begin{smallmatrix} -1 & \mathbf{j} \\ \mathbf{j} & -1 \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{H})$ on $\hat{\mathbb{H}}$ determines a map

$$\mathbf{H}^3 \cup \hat{\mathbb{C}} \to \mathbb{R}^3, \quad x + y\mathbf{i} + z\mathbf{j} \mapsto \tfrac{(2x,\,2y,\,x^2+y^2+z^2-1)}{x^2+y^2+z^2+2z+1}, \quad \infty \mapsto (0,0,1),$$

which identifies $\mathbf{H}^3 \cup \hat{\mathbb{C}}$ with the closed ball in $\mathbb{R}^3$ of radius one centered at the origin. The Euclidean metric on $\mathbb{R}^3$ then induces a metric on $\hat{\mathbb{C}}$ for which the distance between elements w and v of $\hat{\mathbb{C}}$ is

$$\mathrm{chord}(w,v) := \tfrac{2|w-v|}{\sqrt{(1+|w|^2)(1+|v|^2)}}\,,$$

with the natural interpretations if one of the points is ∞; see, for example, [29, Theorem 4.2.1].

The stabilizer of $\mathbf{j}$ for the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\hat{\mathbb{H}}$ is $\mathrm{PSU}_2(\mathbb{C}) \simeq \mathrm{SO}_3(\mathbb{R})$, and the action of $\mathrm{PSU}_2(\mathbb{C})$ on $\hat{\mathbb{C}}$ gives the (orientation-preserving) isometry group; see, for example, [29, Exercises 5.2.11–12].

By (1.0.1), for any $x \in \,]0,1]$, $\operatorname{dist}(\mathbf{j}, x\mathbf{j}) = \operatorname{arccosh}(\frac{1+x^2}{2x}) = \operatorname{arccosh}(\frac{2}{\operatorname{chord}(-x,x)})$, and, moreover, $x\mathbf{j}$ is the point closest to $\mathbf{j}$ on the geodesic curve in $\mathbf{H}^3$ which has limiting end-points x and $-x$.

It then follows that, for any distinct w, v in $\hat{\mathbb{C}}$, if σ denotes the geodesic curve in $\mathbf{H}^3$ which has limiting end-points w and v, then $\operatorname{dist}(\mathbf{j}, \sigma) = \operatorname{arccosh}(\frac{2}{\operatorname{chord}(w,v)})$.

(ii) Let $\mathbf{X}$ be a subset of $\mathbb{R}^3$ endowed with the induced metric, for example, $\mathbf{X}$ could be the image in $\mathbb{R}^3$ of a subset of $\hat{\mathbb{C}}$ with the chord metric.

Let $\mathcal{R}\mathbf{X}$ denote the set consisting of those infinite sequences $\mathbf{r}_{[1\uparrow\infty[}$ in $[0,\infty[$ for which there exists some infinite sequence $z_{[1\uparrow\infty[}$ in $\mathbb{R}^3$ such that the balls with center z_n and radius $\mathbf{r}_n$ cover $\mathbf{X}$, that is, for each $x \in \mathbf{X}$, there exists some $n \in [1\uparrow\infty[$ such that $\operatorname{dist}_{\mathbb{R}^3}(z_n, x) \leqslant \mathbf{r}_n$. The *Hausdorff dimension of* $\mathbf{X}$ is defined as

$$\begin{aligned}\operatorname{Hdim}(\mathbf{X}) &:= \inf\{t \in [0,\infty[\ : \forall \varepsilon \in\,]0,\infty[\ \ \exists\, \mathbf{r}_{[1\uparrow\infty[} \in \mathcal{R}\mathbf{X} \ \text{ such that } \textstyle\sum \mathbf{r}^t_{[1\uparrow\infty[} < \varepsilon\} \\ &= \sup\{t \in [0,\infty[\ : \exists\, \varepsilon \in\,]0,\infty[\ \ \text{ such that } \forall \mathbf{r}_{[1\uparrow\infty[} \in \mathcal{R}\mathbf{X} \ \textstyle\sum \mathbf{r}^t_{[1\uparrow\infty[} \geqslant \varepsilon\}.\end{aligned}$$

(iii). Let $\mathbf{X}$ be any infinite discrete subset of $\mathbf{H}^3$. If we enumerate the elements of $\mathbf{X}$, we can think of $\mathbf{X}$ as an infinite sequence.

Let $z \in \hat{\mathbb{C}}$, and let $j \in \mathbf{H}^3$, and let σ denote the geodesic curve in $\mathbf{H}^3$ which starts at j and has limiting end-point z. We say that z is a *conical limit point* for $\mathbf{X}$ if the sequence $(\operatorname{dist}(x,\sigma) \mid x \in \mathbf{X})$ in $[0,\infty[$ does not diverge to ∞; this definition is independent of the choice of j. The set of conical limit points for $\mathbf{X}$ is denoted $\operatorname{conical}(\mathbf{X})$. Every conical limit point for $\mathbf{X}$ is a limit point for $\mathbf{X}$, that is, an accumulation point of $\mathbf{X}$. The limit points for $\mathbf{X}$ that are not conical limit points for $\mathbf{X}$ are called *non-conical limit points* for $\mathbf{X}$.

Let $t \in [0,\infty]$, and let $j \in \mathbf{H}^3$, and define

$$\operatorname{Poincaré}(\mathbf{X}, j)(t) := \sum_{x\in\mathbf{X}} \mathbf{e}^{-\operatorname{dist}(x,j)\cdot t} \in [0,\infty],$$

with the convention that $\mathbf{e}^{-\operatorname{dist}(x,j)\cdot\infty} := 0$. Thus $\operatorname{Poincaré}(\mathbf{X},j)(\infty) = 0$, $\operatorname{Poincaré}(\mathbf{X},j)(0) = \infty$, and we have a decreasing function $\operatorname{Poincaré}(\mathbf{X},j)\colon [0,\infty] \to [0,\infty]$, called the *Poincaré series* of $\mathbf{X}$ at j. The *critical exponent of* $\mathbf{X}$ is defined as

$$\begin{aligned}\operatorname{critical}(\mathbf{X}) &:= \inf\{t \in [0,\infty] : \operatorname{Poincaré}(\mathbf{X},j)(t) < \infty\} \\ &= \sup\{t \in [0,\infty] : \operatorname{Poincaré}(\mathbf{X},j)(t) = \infty\};\end{aligned}$$

this definition is independent of the choice of j.

(iv). Let S be an infinite discrete subset of $\mathrm{PSL}_2(\mathbb{C})$.

Let $j \in \mathbf{H}^3$. Let $S(j)$ denote the orbit $\{s(j) \mid s \in S\}$, an infinite discrete subset of $\mathbf{H}^3$.

We define the set of conical limit points of S to be the set of conical limit points of $S(j)$, and write $\operatorname{conical}(S) := \operatorname{conical}(S(j))$; this definition is independent of the choice of j.

We define the critical exponent of S to be the critical exponent of $S(j)$ and write $\operatorname{critical}(S) := \operatorname{critical}(S(j))$; this definition is independent of the choice of j.

If $k \in \mathbf{H}^3$, and $t \in [0,\infty]$, then $\operatorname{Poincaré}(S(j),k)(t) = \operatorname{Poincaré}(S^{-1}(k),j)(t)$, and, hence, $\operatorname{critical}(S(j)) = \operatorname{critical}(S^{-1}(k))$, and, hence, $\operatorname{critical}(S) = \operatorname{critical}(S^{-1})$. □

Recall the following result from [9].

7.2 The Beardon-Maskit Theorem. *Let π be a discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$*. Suppose that π is geometrically finite, that is, there exists a fundamental domain for the π-action in $\mathbf{H}^3$ whose closure in $\mathbf{H}^3 \cup \hat{\mathbb{C}}$ is a polyhedron with finitely many faces. Then the non-conical limit points of π are precisely the fixed points of the parabolic elements of π.* □

We cannot apply the Beardon-Maskit Theorem directly to semigroups. The following two lemmas will be useful for our semigroups.

7.3 Lemma. *Let* Notation 6.1 *hold. Suppose that $\ell \in [1{\uparrow}f_1]$ and that x and y are elements of $\{\infty, A(\infty), C(\infty)\}$.*

If $V_\ell(x) = y$, then either $(x, y, \ell) = (C(\infty), \infty, f_{2\,1})$, or $y = x$ and V_ℓ is parabolic.

If $U_\ell(x) = y$, then either $(x, y, \ell) = (\infty, C(\infty), f_2 + 1)$, or $y = x$ and U_ℓ is parabolic.

Proof. By Proposition 6.7 and Theorem 6.6, the following hold:

$A(\infty)$ is the initial point of the first row $V_1\mathfrak{a}_{-2}$ and the terminal point of the last column $U_{f_1}\mathfrak{a}_1$;

∞ is an interior point of the $(f_{2\,1}+1)$th row $V_{f_{2\,1}+1}\mathfrak{a}_{-1}$ and the initial point of the first column $U_1\mathfrak{a}_1$; and,

$C(\infty)$ is the terminal point of the last row $V_{f_1}\mathfrak{a}_{-1}$ and the terminal point of the f_2th column $U_{f_2}\mathfrak{a}_2$.

If $x = A(\infty)$ and $y = V_\ell(x)$, then y is the initial point of the ℓth row $V_\ell\mathfrak{a}_{\mathbf{t}_\ell}$, and we see that $\ell = 1$, and $y = x$.

If $x = \infty$ and $y = V_\ell(x)$, then y is an interior point of $\mathfrak{a}_{-1}$, and we see that $y = x$. Then, $U_{F^{-1}(\ell)}(y) = V_\ell^{-1}(y) = x$ is the initial point of the $F^{-1}(\ell)$th column $U_{F^{-1}(\ell)}\mathfrak{a}_{\mathbf{s}_{F^{-1}(\ell)}}$ and $F^{-1}(\ell) = 1$. Hence $\ell = F(1) = f_{2\,1} + 1$.

If $x = C(\infty)$ and $y = V_\ell(x)$, then y is the terminal point of $V_\ell\mathfrak{a}_{-1}$. If $\ell \neq f_1$, then we must have $y = \infty$. Here, $U_{F^{-1}(\ell)}(y) = V_\ell^{-1}(y) = x$ is the initial point of the $F^{-1}(\ell)$th column $U_{F^{-1}(\ell)}\mathfrak{a}_{\mathbf{s}_{F^{-1}(\ell)}}$. Hence $F^{-1}(\ell) = f_2 + 1$ and $\ell = F(f_2+1) = f_{2\,1}$.

The result is now clear. □

7.4 Lemma. *Let* Notation 6.1 *hold. Let $\mathbf{z}_{[1\uparrow\infty[}$ be a sequence in $[1{\uparrow}f_1]$ that does not converge to an element of $\{1, f_2, f_1\}$ in the discrete topology of $[1{\uparrow}f_1]$. Let z_1 denote the element of $\mathfrak{a}_1$ such that $\bigcap\limits_{\ell\in[0\uparrow\infty[} ((\prod U_{\mathbf{z}_{[1\uparrow\ell]}})\mathfrak{a}_1) = \{z_1\}$. Then z_1 is a conical limit point of the infinite discrete subset $\{\rho(\prod U_{\mathbf{z}_{[1\uparrow\ell]}}) \mid \ell \in [0{\uparrow}\infty[\ \}$ of* $\mathrm{PSL}_2(\mathbb{C})$*.*

Proof. Let us choose some element z_{-1} of $\mathfrak{a}_{-1} - \{\infty, A(\infty), C(\infty)\}$. By (2.6.1), $z_{-1} \notin \mathfrak{a}_1$. By Theorem 6.4, z_{-1} is not a limit point of the ∂^--semigroup.

Let S denote the set of accumulation points of the infinite sequence

$$((\textstyle\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(z_{-1}) \mid \ell \in [1{\uparrow}\infty[\ ).$$

Notice that S is nonempty.

We show first that there exists some $z_{-2} \in S - \mathfrak{a}_1$.

By Proposition 6.7, $\{(\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1} \mid \ell \in [1\uparrow\infty[\ \}$ lies in the ∂^+-semigroup, and, by Theorem 6.6, the ∂^+-semigroup acts on $\mathfrak{a}_{-1}$. Hence, $\{(\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(z_{-1}) \mid \ell \in [1\uparrow\infty[\ \} \subseteq \mathfrak{a}_{-1}$. Since $\mathfrak{a}_{-1}$ is closed, $S \subseteq \mathfrak{a}_{-1}$.

By (2.6.1), $S - \mathfrak{a}_1 = S - \{\infty, A(\infty), C(\infty)\}$.

Fact 1. If $\infty \in S$, then there exists some $z_{-2} \in S - \{\infty, A(\infty), C(\infty)\}$.

Proof of Fact 1. Here, there exists an infinite subset N_1 of $[1\uparrow\infty[$ with the property that the sequence $((\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(z_{-1}) \mid \ell \in N_1)$ converges to ∞ in $\mathfrak{a}_{-1}$.

Now U_1^{-1} acts on $\mathfrak{a}_{-1}$, and U_1^{-1} is parabolic with fixed point ∞. It follows that, for each $n \in [0\uparrow\infty[$, $(((\prod U_{\mathbf{z}_{[1\uparrow\ell]}})U_1^{n})^{-1}(z_{-1}) \mid \ell \in N_1)$ converges to $U_1^{-n}(\infty) = \infty$, and does so more and more rapidly, in the $\mathfrak{a}_{-1}$ curve order, as n increases.

For each $\ell \in [1\uparrow\infty[$, let ℓ' denote the least element of $[(\ell+1)\uparrow\infty[$ such that $\mathbf{z}_{\ell'} \neq 1$; this is well-defined since $\mathbf{z}_{[1\uparrow\infty[}$ does not converge to 1.

It then follows that $((\prod U_{\mathbf{z}_{[1\uparrow(\ell'-1)]}})^{-1}(z_{-1}) \mid \ell \in N_1)$ also converges to ∞.

There exists some $m \in [2\uparrow f_1]$ such that $N_2 \coloneqq \{\ell \in N_1 \mid \ell' = m\}$ is infinite.

Then $((\prod U_{\mathbf{z}_{[1\uparrow\ell']}})^{-1}(z_{-1}) \mid \ell \in N_2)$ converges to $z_{-2} \coloneqq U_m^{-1}(\infty) \in S$.

By Lemma 7.3, $z_{-2} \in S - \{\infty, A(\infty), C(\infty)\}$, and Fact 1 is proved. □

By the same type of argument, we prove the following.

Fact 2. If $A(\infty) \in S$, then there exists some $z_{-2} \in S - \{\infty, A(\infty), C(\infty)\}$. □

Fact 3. If $C(\infty) \in S$, then there exists some $z_{-2} \in S - \{\infty, A(\infty), C(\infty)\}$.

Proof of Fact 3. If we apply the same type of argument again, we find there exists some $z_{-3} \in S - \{A(\infty), C(\infty)\}$. Together with Fact 1, this implies Fact 3. □

In summary, we have proved that there exists some $z_{-2} \in S - \mathfrak{a}_1$.

Let σ denote the geodesic curve in $\mathbf{H}^3$ with limiting end-points z_{-1} and z_1.

Let $\varepsilon = \operatorname{chord}(z_{-2}, \mathfrak{a}_1)$; see Definitions 7.1(i). Since $z_{-2} \notin \mathfrak{a}_1$, $\varepsilon > 0$.

Let $N_3 \coloneqq \{\ell \in [0\uparrow\infty[\ : \operatorname{chord}(z_{-2}, (\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(z_{-1})) < \frac{\varepsilon}{2}\}$. Since $z_{-2} \in S$, N_3 is infinite.

Let $\ell \in N_3$. By the choice of ε, $\operatorname{chord}((\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(z_{-1}), \mathfrak{a}_1) > \frac{\varepsilon}{2}$. We no longer need z_{-2}, and we argue as in [9]. We know that $z_1 \in (\prod U_{\mathbf{z}_{[1\uparrow\ell]}})\mathfrak{a}_1$. Hence, $(\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(z_1) \in \mathfrak{a}_1$. In particular,

$$\operatorname{chord}(\ (\textstyle\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(z_{-1}),\ (\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(z_1)\ ) > \frac{\varepsilon}{2}.$$

The geodesic curve in $\mathbf{H}^3$ with limiting end-points $(\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(z_{-1})$ and $(\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(z_1)$ is $(\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(\sigma)$. By Definitions 7.1(i),

$$\operatorname{arccosh}(\tfrac{4}{\varepsilon}) \quad > \quad \operatorname{dist}(\mathbf{j}, (\textstyle\prod U_{\mathbf{z}_{[1\uparrow\ell]}})^{-1}(\sigma)) \quad = \quad \operatorname{dist}((\prod U_{\mathbf{z}_{[1\uparrow\ell]}})(\mathbf{j}), \sigma).$$

Since N_3 is infinite, the foregoing implies that z_{-1} and/or z_1 is a conical limit point of $\{(\prod U_{\mathbf{z}_{[1\uparrow\ell]}})(\mathbf{j}) \mid \ell \in [0\uparrow\infty[\ \}$. Since z_{-1} is not a limit point of the ∂^--semigroup, the result follows. □

7.5 Theorem. *Let* Notation 6.1 *hold. The set of conical limit points of the* ∂^-*-semigroup equals* $\mathfrak{a}_1 - \langle A, B, C \rangle(\infty)$.

Proof. Jørgensen showed that $\rho(\langle A, B, C, F \rangle)$ is geometrically finite. By the Beardon-Maskit Theorem 7.2, or simply [9, Proposition 3], every fixed point of a parabolic element of $\langle A, B, C, F \rangle$ is a non-conical limit point of $\langle A, B, C, F \rangle$, and, in particular, is not a conical limit point of the ∂^--semigroup. Thus, every conical limit point of the ∂^--semigroup lies in $\mathfrak{a}_1 - \langle A, B, C \rangle(\infty)$.

It remains to show that if $z \in \mathfrak{a}_1 - \langle A, B, C \rangle(\infty)$, then z is a conical limit point of the ∂^--semigroup. Let $\mathbf{z}_{[1\uparrow\infty[}$ be a sequence in $[1\uparrow f_1]$ such that $\bigcap\limits_{\ell\in[1\uparrow\infty[} ((\prod U_{\mathbf{z}_{[1\uparrow(\ell-1)]}})\mathfrak{a}_1) = \{z\}$. Since $z \notin \langle A, B, C \rangle(\infty)$, $\mathbf{z}_{[1\uparrow\infty[}$ does not converge to 1 or f_2 or f_1. By Lemma 7.4, z is a conical limit point of the ∂^--semigroup. □

A similar argument shows the following.

7.6 Theorem. *Let* Notation 6.1 *hold. The set of conical limit points of the* ∂^+*-semigroup equals* $\mathfrak{a}_{-1} - \langle A, B, C \rangle(\infty)$. □

Using Lemma 7.4 again, we can deduce the following.

7.7 Corollary. *Let* Notation 6.1 *hold, let* $N \in [1\uparrow\infty[$ *and let* $\mathbf{z}_{[1\uparrow N]}$ *be a sequence in* $[1\uparrow f_1]$.

If $\prod U_{\mathbf{z}_{[1\uparrow N]}}$ *is parabolic, then* $\mathbf{z}_{[1\uparrow N]}$ *is a constant sequence in* $\{1, f_2, f_1\}$.

If $\prod V_{\mathbf{z}_{[1\uparrow N]}}$ *is parabolic, then* $\mathbf{z}_{[1\uparrow N]}$ *is a constant sequence in* $\{1, f_{21}+1, f_1\}$. □

In [10, Section 2], a succinct, self-contained proof of the following result is given, although the result itself is not stated; the argument uses the description of Hausdorff dimension given in Definitions 7.1(ii).

7.8 The Bishop-Jones Theorem. *If* S *is a subsemigroup of a discrete subgroup of* $\mathrm{PSL}_2(\mathbb{C})$ *such that* S *does not have a global fixed point in* $\hat{\mathbb{C}}$, *then* $\mathrm{critical}(S) = \mathrm{Hdim}(\mathrm{conical}(S))$. □

7.9 Theorem. *Let* Notation 6.1 *hold. The following four numbers are equal:*

the Hausdorff dimension of $\partial^- A$;

the critical exponent of the ∂^-*-semigroup;*

the critical exponent of the ∂^+*-semigroup;*

the Hausdorff dimension of $\partial^+ B$.

Proof. By Theorem 7.5, $\mathrm{Hdim}(\partial^- A)$ equals the Hausdorff dimension of the set of conical limit points of the ∂^--semigroup which, by the Bishop-Jones Theorem 7.8, equals the critical exponent of the ∂^--semigroup. By Theorem 7.6, $\mathrm{Hdim}(\partial^+ B)$ equals the Hausdorff dimension of the set of conical limit points of the ∂^+-semigroup which, by Theorem 7.8, equals the critical exponent of the ∂^+-semigroup. By Proposition 6.7(iii) and Definitions 7.1(iv), the critical exponent of the ∂^--semigroup equals the critical exponent of the ∂^+-semigroup. □

For $F = RL$, the four equal numbers seem to be near 1.27910. For the sequence $b_{[0\uparrow\infty[}$ of [15, Calculations 5.2], Jaume Amoros and Javier Vindel very kindly wrote and ran a program that gave $b_{[21\uparrow 25]}$ as

$$(547206912858, 2002063012565, 7324919842341, 26799555593731, 98051175064382);$$

in particular, $\log \frac{b_{24}-b_{23}}{b_{23}-b_{22}} = 1.2971027\ldots$ and $\log \frac{b_{25}-b_{24}}{b_{24}-b_{23}} = 1.2971046\ldots$.

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