GROTHENDIECK TOPOLOGIES FROM UNIQUE FACTORISATION SYSTEMS

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ABSTRACT. This article presents a way to associate a Grothendieck site structure to a category endowed with a unique factorisation system of its arrows. In particular this recovers the Zariski and Etale topologies and others related to Voevodsky's cd-structures. As unique factorisation systems are also frequent outside algebraic geometry, the same construction applies to some new contexts, where it is related with known structures defined otherwise. The paper details algebraic geometrical situations and sketches only the other contexts.

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This article is about how certain Grothendieck topologies can be generated from unique factorisation systems (or also unique lifting systems, see below). A particular case of our construction will be the Zariski and Etale topologies of algebraic geometry, and others related to Voevodsky's cd-structures. As unique factorisation systems are also frequent outside algebraic geometry, the same construction applies to some new contexts, where it is often related with known structures defined otherwise. The paper details algebraic geometrical situations and sketch only those other contexts. Most of the results are well known, only a systematic presentation using unique factorisation system is new here.

Topological interpretation of lifting diagrams. In a category $\mathcal{C}$, a lifting diagram is a commutative diagram as follows

$$
\begin{array}{ccc}
P & \longrightarrow & U \\
{\scriptstyle u}\downarrow & {\scriptstyle \ell}\nearrow & \downarrow{\scriptstyle f} \\
N & \longrightarrow & X.
\end{array}
$$

The arrow ℓ, when it exists, is called a *lift of u through f*. The diagram is called a *lifting diagram* if a lift exist, and a *unique lifting diagram* if the lift exists and is unique. In this last case, u (resp. f) is said left (resp. right) orthogonal to f (resp. u). A *lifting system* is defined as two classes of maps $\mathcal{A}, B \subset C$ such that each map of $\mathcal{A}$ is left orthogonal to any of $\mathcal{B}$ and such that $\mathcal{A}$ and $\mathcal{B}$ are saturated for this relation (cf. §1.1). $\mathcal{A}$ is called the left class and $\mathcal{B}$ the right class.

We propose the following topological interpretation of a unique lifting diagram: all objects are to be thought as spaces, the composite map $P \to X$ is a point of X (in the generalized sense of 'family of points'), the map $u \colon P \to N$ is a neighbourhood (or a thickening) of P, the map $N \to X$ say that this neighbourhood is "in" X, the map $f \colon U \to X$ is a open immersion X, and the map $P \to U$ says that the open contains the point P. The unique lifting property then reads: *in a space X, any open U containing a point P contains every neighbourhood N of P contained in X*, which is exactly the fundamental intuition behind the classical definition of open subsets of topological spaces. We proposed here an approach of topology based on this remark.

In the topological setting the notion of neighbourhood is rather complicated as they are objects not defined by some underlying set of points, and they are quickly forgotten and replaced by the more convenient open subsets. Algebraic geometry, on the contrary, possesses through spectra of local rings, a way to make

the neighbourhoods "real" objects and this authorizes in particular an easy local-global movement. The existence of these neighbourhoods will be a consequence of the extra structure of a lifting system, and even a factorisation system, behind the Zariski topology, distinguishing it from the Grothendieck topology of topological spaces for which no such system exists.

Factorisation systems. It is a remarkable fact that lifting systems are related to factorisation systems. A *factorisation system* on a category $\mathcal{C}$ is the data of two classes of maps $\mathcal{A}, B \subset \mathcal{C}$ and a factorisation $X \to \phi(u) \to Y$ of any map $u \colon X \to Y \in \mathcal{C}$ such that $X \to \phi(u) \in \mathcal{A}$ and $\phi(u) \to Y \in \mathcal{B}$. The factorisation is said *unique* if $\phi(u)$ is unique up to a unique isomorphism (cf. §1.2). The two classes (A, B) of a unique factorisation system define always a unique lifting system and the reciprocal is true under some hypothesis of local presentability (cf. prop. 3). The unique lifting systems that will appear in this paper will all be associated to unique factorisation systems.

The Zariski topology has the particularity that Zariski open embeddings between affine schemes are all in the right class of a unique factorisation system $(Cons^o, Loc^o)$ on $CRings^o$ (cf. §3.2). Loc is the class of localisations of rings and $Cons$ the class of conservative maps of rings: a map $u \colon A \to B$ is conservative if $u(a)$ invertible implies a invertible, an example is the map $A \to k$ from a local ring to its residue field. Any map of rings $u \colon A \to B$ factors in a localisation followed by a conservative map $A \to A[S^{-1}] \to B$ where $S = u^{-1}(B^{\times})$. In particular this factorisation applied to a map $u \colon A \to k$ where k is a residue field of A gives $A \to A_p \to k$ where A_p is the local ring of A at the kernel of p of u. Geometrically, $A \to k$ corresponds to a point p of $X = Spec_{Zar}(A)$, $N = Spec_{Zar}(A_p) \to X$ is the germ of X at p and $P = Spec_{Zar}(k) \to N$ is the embedding of a point into some neighbourhood. If $U \to X$ is a Zariski open subset of X containing P, this data define a lifting square as above and the existence of the lift $N \to U$ is a consequence of N being the limit of all $U \to X$ containing P.

With the previous considerations in mind, it is tempting to look at a unique factorisation system (A, B), the following way: the right class $\mathcal{B}$ would be formed of open embeddings and the left class $\mathcal{A}$ of abstract neighbourhoods. But the example of Zariski topology, show us also that not all localisations of rings are to be thought as open embeddings, they contain also germ at some points, so a general map in the right class should rather be thought as a "pro"-open embeddings. Also, it is possible to see using a topological intuition, that a map lifting uniquely the neighbourhood of some point, once given a lift of the point is not in general an open embedding but rather an *etale map*. In the Zariski topology this fact is almost invisible but the Etale topology is another example of the same setting and makes it clear.

So finally, we are going to propose an interpretation of the class $\mathcal{B}$ of a unique lifting system as a class of "pro"-etale maps. A unique lifting systems is then though as a theory of pro-etale maps and a tool to develop abstract analogs of the Etale topology: this class $\mathcal{B}$ will be used to define the covering families of a Grothendieck topology on $\mathcal{C}$ (cf. §2) that in the particular case of the $(Cons^o, Loc^o)$ lifting system on the category $CRings$ of commutative rings will give back Zariski topology (cf. §3.2).

We list here the four unique factorisation systems on the category $CRings$ of commutative rings that we are going to study in the sequel.

Name	Left class	Right class
Zariski	localisations	conservative maps
Etale	ind-etale maps	henselian maps
Domain	surjections	monomorphisms
Proper	ind-proper maps	integrally closed maps

A factorisation system (A, B) on $\mathcal{C}$ define another one $(\mathcal{A}^o, \mathcal{B}^o)$ on $\mathcal{C}^o$ and we will in fact have more interests on the opposite systems of the previous four. To each of them will be associated a Grothendieck topology on the opposite category $CRings^o$ of commutative rings, the third one corresponding to Voevodsky's plain lower cd-topology in [Vo].

Results. From a factorisation system (in a category with finite limits), we built a general scheme associating to it:

- a notion of *etale map* (§2.1),
- a notion of *points* of an object (§2.2),
- a notion of *local objects* (§2.4),
- a Grothendieck topology (called the *factorisation topology*) which covering families are etale families surjective on points (§2.3),
- two *toposes* functorialy associated to any object X and called the *small and big spectra* of X, the big one being always a retraction of the big one (§2.6),
- and a *structural sheaf* on the small spectra of X whose stalks are the "local forms" of X, *i.e.* pro-etale local objects over X (§2.6.2)

such that in the case of the four systems on $CRings^o$ these notions gives:

	Zariski	Etale	Domain	Proper
Etale maps	Zariski open maps	etale maps	Zariski closed embeddings	proper maps
Points	nilpotent extension of fields	nilpotent extension of separably closed fields	fields	algebraically closed fields
Local objects	local rings	strict henselian local rings	integral domains	strict integrally closed domains (cf. §3.6.2)
Small spectrum of A	usual Zariski spectrum (topos classifying all localisations of A)	usual Etale spectrum (topos classifying all strict henselisation of A)	a topos classifying all quotients domains of A	a topos classifying strict integral closure of quotient domains of A
Big spectrum of A	usual big Zariski topos classifying local A-algebras	usual big Etale topos classifying strict henselian local A-algebras	a topos classifying A-algebras that are integral domains	a topos classifying A-algebras that are strict integrally closed domains.

(The structure sheaves are the tautological ones.)

Then the main result of the paper is theorem 2.5 allowing one to compute the categories of global points of the spectra using the local objects. We refer to it as the moduli interpretation of the spectra, but we won't study fully the moduli aspects of our spectra in this paper, such a study would require a much more topossic approach than we have chosen here and will be the subject of another paper [An].

Nisnevich contexts. Nisnevich topology on schemes is defined by etale covering families satisfying a lifting property for maps from spectra of fields. Such a lifting property cannot in general be obtained by a single etale map and this does not distinguish a class of maps $\mathcal{B}$ that could be part a factorisation system. For this reason Nisnevich topology is not a factorisation topology, but it defines an interesting operation on such that we called *Nisnevich forcing*. It consists to force a class of objects to be local objects (cf. §2.5) by selecting the covering families

of the factorisation topology that satisfy an extra lifting condition for maps from the objects of the forcing class. Apply with the Etale topology and the class of fields, this gives the usual Nisnevich topology. But an interesting other case is to apply this, still with the class of field as forcing class, to the Proper topology; the resulting topology is then the lower cd-structure of Voevodsky in [Vo].

The data of a factorisation system and a Nisnevich forcing class is called a *Nisnevich context* (def. 11) and the construction of our spectra (§2.6) as well as theorem 2.5 are defined directly in such a context. The previous table can then be completed by the following one:

	Nisnevich	Proper Nisnevich
Local objects	henselian local rings	integrally closed domains
Small spectrum of A	topos classifying ind-etale henselian local A-algebra	topos classifying ind-proper integraly closed A-algebra
Big spectrum of A	topos classifying henselian local A-algebra	topos classifying integraly closed A-algebra

Other examples. Many example of unique factorisation system exists outside of algebraic geometry and we sketch the details of some examples in §3.9. The two first examples deal with the (Epi, $Mono$) factorisation systems that always exist in a topos or an abelian category, the notion of point correspond to irreducibles objects and the associated spectra are essentielly discrete spaces. Another example study the factorisation systems on the category of small categories given by initial (resp. final) functors and discrete left (resp. right) fibrations. The associated spectra of a category C are respectively the toposes of covariant and of contravariant functors. Moreover this example share a duality of the same flavour of that of etale and proper maps. We study also a factorisation system on the category of simplicial sets left generated by inclusion of faces of simplices, points and local objects are vertices and a more interesting situation is obtained forcing all simplices to be local object. For this topology, the small spectrum of a simplicial set is related to the cellular dual of the usual geometric realisation (where vertices correspond to open).

Finally, we sketch an application to homotopy theory of what should be a natural generalisation of our setting to homotopically unique factorisation on higher categories. The example study Postnikov family of factorisation systems (a.k.a Postnikov towers of morphisms), the associated spectra of a homotopy type X are higher toposes of representations of its fundamental n-groupoid.

Plan of the paper. Section 1 consists in some recollections and lemmas about lifting and factorisation systems, the main result is proposition 3 describing the construction of a unique factorisation from a lifting system, which will be used in §3.3. All this can be skipped at first reading.

Section 2 is the core of the article. It develops the topological interpretations and constructions associated to a factorisation system. It uses an extra structure not mentionned yet called a *finiteness context* that we could have avoided in the context of rings by simply looking at finitely presented objects, but it seems to be an interesting degree of freedom of the theory (cf. §3.9). The notion of a Nisnevich context and the associated small and big spectra are defined in §2.5 and §2.6. The theorem of computation of their points is in §2.6.1 and their expected structure is proven in §2.6 and §2.6.2.

At last, section 3 develops the examples. The first six are the one in algebraic geometry mentioned above and are fully detailled, but our general setting gives also interesting results in a priori less geometrical situations (§3.9).

Motivations and acknowledgments. The origin of this work was to understand why Zariski and Etale topologies where coming with both notions of small and big topos and a class of distinguished maps playing the role of "open embeddings", the classical theory of Grothendieck (pre)topologies being insufficient to explain this extra structure. It is André Joyal that suggested to me that Zariski topology should be related to the $(Loc, Cons)$ factorisation system on commutative rings. Although he won't be satisfied with the way I've chosen to present the ideas here, this paper have been influenced by numerous conversations with him. It is after a conversation with Georges Maltsiniotis, that I had the idea for the notion points, I am particularly grateful to him for listening the first one my raw ideas and for his remarks. I learn first the excellent philosophy of thinking of spectrum of an object X as the moduli spaces of some "local forms" of X from Joseph Tapia, although all this is not fully described here this have been influential, i'm grateful to him for our conversations on the subject. I'm also grateful to Jonathan Pridham for pointing out to me that the orthogonal class of henselian maps should be that of ind-etale ones.

Most of this study has been worked out during the excellent 2007-2008 program on Homotopy Theory and Higher Categories in Barcelona's CRM, I'm very grateful to the organizers for inviting me all year. It has been written while I was staying at Montreal's CIRGET that I thank also for inviting me.

Notations. For an object X of a category $\mathcal{C}$ the category of objects of $\mathcal{C}$ under X is noted $_{X\backslash}\mathcal{C}$ and that of objects over X $\mathcal{C}_{/X}$. For a category $\mathcal{C}$ $Ind(\mathcal{C})$ is its the category of ind-objects. For a class $\mathcal{B}$ of maps of a category $\mathcal{C}$, $Ind - \mathcal{B}$ is the class of maps of $\mathcal{C}$ that can be defined as cofiltered colimits of maps in $\mathcal{B}$. $CRings$ is the category of commutative rings. $\mathcal{S}$ will denote the topos of sets.

1. Lifting properties and factorisation systems

We recall the notion of lifting and factorisation systems from [Bou, Joy].

1.1. **Lifting systems.** In a commutative diagram square

$$
\begin{array}{ccc}
P & \longrightarrow & U \\
{\scriptstyle u}\downarrow & {\scriptstyle \ell}\nearrow & \downarrow{\scriptstyle f} \\
N & \longrightarrow & X.
\end{array}
$$

the map u is said to have the *unique left lifting property* with respect to f and the map f is said to have the *unique right lifting property* with respect to u if it exist a unique diagonal arrow ℓ making the two obvious triangles commutative. The arrow ℓ will be called the lift or the lifting.

Let $\mathcal{B}$ a class of maps of $\mathcal{C}$, a map $u\colon X \to Y \in \mathcal{C}$ is said to be *left (resp. right) orthogonal* to $\mathcal{B}$ iff it has the unique left (resp. right) lifting property with respect to all maps of $\mathcal{B}$. The class of maps left (resp. right) orthogonal to $\mathcal{B}$ is noted ${}^{\perp}\mathcal{B}$ (resp. $\mathcal{B}^{\perp}$). If $\mathcal{A} \subset B$ then $\mathcal{B}^{\perp} \subset \mathcal{A}^{\perp}$ and ${}^{\perp}\mathcal{B} \subset {}^{\perp}\mathcal{A}$.

Definition 1. The data of two classes $\mathcal{A}, B$ of maps of $\mathcal{C}$ such that $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$ is called a *unique lifting system* on $\mathcal{C}$. Such a system is noted (A, B).

For a class G of maps of $\mathcal{C}$ we define $\mathcal{B} = G^{\perp}$ and $\mathcal{A} = {}^{\perp}\mathcal{B}$.

Lemma 1.1. *The previous classes $\mathcal{A}$ and $\mathcal{B}$ form a unique lifting system.*

Proof. We must show $\mathcal{B} = \mathcal{A}^{\perp}$. By construction $G \subset \mathcal{A}$ so $\mathcal{A}^{\perp} \subset G^{\perp} = \mathcal{B}$, and the inverse inclusion is a consequence of $\mathcal{A} = {}^{\perp}\mathcal{B}$. $\qquad\square$

Such a factorisation system will be qualified as *left generated by the set G*. There is a dual notion of right generation.

Here are some properties of the classes of a lifting system.

Proposition 1.   (1) $\mathcal{A}$ and $\mathcal{B}$ are stable by composition.
    (2) $\mathcal{A} \cap \mathcal{B}$ is the class of isomorphisms of $\mathcal{C}$.
    (3) $\mathcal{B}$ is stable by pullback and has the left cancellation property (see proof). In particular any section or retraction of a map in $\mathcal{B}$ is in $\mathcal{B}$. (The dual statement holds for $\mathcal{A}$.)
    (4) In the category of arrows of $\mathcal{C}$, any limit of maps in $\mathcal{B}$ is in $\mathcal{B}$. (The dual statement holds for $\mathcal{A}$.)
    (5) (Codiagonal property) the class $\mathcal{A}$ contains the codiagonals of its morphisms (see proof). (The dual statement holds for $\mathcal{B}$.)

Proof. The first and second properties are left to the reader.

3. Stability by composition and pullback are easy. A class $\mathcal{B}$ of maps in $\mathcal{C}$ has the *left cancellation property* if for any $X \xrightarrow{u} Y \xrightarrow{v} Z$ such that vu and v are in $\mathcal{B}$, so is u. We are going to prove that for a map $a \colon X \to Y \in \mathcal{C}$, the class $a^{\perp}$ has the left cancellation property. Let $u \colon Z \to T$ and $v \colon T \to U \in \mathcal{B}$ such that $vu \in \mathcal{B}$, for any square

$$
\begin{array}{ccc}
X & \longrightarrow & Z \\
a \downarrow & \overset{\ell}{\nearrow} & \downarrow u \\
Y & \underset{q}{\longrightarrow} & T
\end{array}
$$

we are looking for a lift ℓ. Composing at the bottom by v gives

$$
\begin{array}{ccc}
X & \longrightarrow & Z \\
a \downarrow & \nearrow u & \downarrow \\
Y & \overset{s}{\dashrightarrow} & T \quad\Big) vu \\
\| & & \downarrow v \\
Y & \longrightarrow & U
\end{array}
$$

and a lift s of a through vu. We need to show that this is the good one, *i.e.* that $us = q$. This can be seen in

$$
\begin{array}{ccccc}
X & \longrightarrow & Z & \overset{u}{\longrightarrow} & T \\
a \downarrow & \overset{s}{\nearrow} & u \downarrow & /\!/ & \downarrow v \\
Y & \underset{q}{\longrightarrow} & T & \longrightarrow & U
\end{array}
$$

as us and q give two lifts of a through v. The conclusion follows as classes having the cancellation property are stable by intersection.

As for *4.*, let I be the interval category $\{0 \to 1\}$, $\mathcal{C}^I$ is the arrow category of $\mathcal{C}$, $\mathcal{B}$ is a subclass of the class of objects of $\mathcal{C}^I$. If $D \colon \mathcal{D} \to \mathcal{C}$ is a diagram of arrows all in $\mathcal{B}$, then, if the limit of this diagram exists, it is in $\mathcal{B}$. Indeed, let $Z_d \to T_d \in \mathcal{B}$ be the value of the diagram D at d and $Z \to T$ be the limit of D, the existence of a lift ℓ for a square

$$
\begin{array}{ccc}
X & \longrightarrow & Z \\
a \in \mathcal{A} \downarrow & \overset{\ell}{\nearrow} & \downarrow \\
Y & \longrightarrow & T
\end{array}
$$

is equivalent to the existence of lift for all

$$
\begin{array}{ccc}
X & \longrightarrow & Z_d \\
{\scriptstyle a\in\mathcal{A}}\downarrow & {\scriptstyle \ell_d}\nearrow & \downarrow \\
Y & \longrightarrow & T_d
\end{array}
$$

such that for $\delta\colon d \to d' \in \mathcal{D}$

$$
\begin{array}{ccccc}
X & \longrightarrow & Z_d & \xrightarrow{\ \zeta_\delta\ } & Z_{d'} \\
{\scriptstyle a\in\mathcal{A}}\downarrow & {\scriptstyle \ell_d}\nearrow & \downarrow{\scriptstyle \ell_{d'}} & \nearrow & \downarrow \\
Y & \longrightarrow & T_d & \longrightarrow & T_{d'}
\end{array}
$$

$\zeta_\delta \circ \ell_d = \ell_{d'}$, but this is a consequence of the unicity of the lift.

5. The codiagonal of a morphism $A \to B$ is the map $B \cup_A B \to B$. It is a retract of the inclusion $B \to B \cup_A B$ which is a pushout of $A \to B$ so it is in $\mathcal{A}$ is $A \to B$ is. Then the cancellation property for $\mathcal{A}$ ensures $B \cup_A B \to B \in \mathcal{A}$ too. $\qquad\square$

The following lemma gives an interesting equivalence between the right cancellation property and having codiagonals.

Lemma 1.2. *A subcategory G of $\mathcal{C}$ stable by cobase change satisfies the right cancellation iff it contains the codiagonals of all its morphisms.*

Proof. For $u\colon A \to B \in G$, $i_1\colon B \to B \sqcup_A B$ is in G as cobase change of u along itself. If G has right cancellation, $\delta_u\colon B \sqcup_A B \to B$ is in G as $\delta_u \circ i_1 = id_B$. For $u\colon A \to B$ and $v\colon B \to C$ such that $u, vu \in G$, we want to prove that $v \in G$. The square

$$
\begin{array}{ccc}
B \sqcup_A B & \xrightarrow{\ \delta_v\ } & B \\
{\scriptstyle v\sqcup_A id_B}\downarrow & & \downarrow{\scriptstyle u} \\
C \sqcup_A B & \xrightarrow{\ w\ } & C
\end{array}
$$

is a pushout. If G is stable by codiagonals $w \in G$. Then as $i_1\colon C \to C \sqcup_A B$ is in G as a pushout of vu, so is $u = w \circ i_1$. $\qquad\square$

To finish we mention that there is an obvious notion of a general (non unique) lifting system. The following result says that unicity of the lift is a property of a non unique lifting system.

Lemma 1.3. *A general lifting system (A, B) is unique iff the $\mathcal{A}$ is stable by codiagonals iff the $\mathcal{B}$ is stable by diagonals.*

Proof. We are going to work only with the condition on $\mathcal{A}$. Suppose we have a square

$$
\begin{array}{ccc}
A & \longrightarrow & C \\
\downarrow & \ell_1 \nearrow \nearrow & \downarrow \\
& \nearrow \ell_2 & \\
B & \longrightarrow & D
\end{array}
$$

with two lifts. These two lifts agree iff the following square have a lift:

$$
\begin{array}{ccc}
B \cup_A B & \longrightarrow & C \\
\downarrow & \nearrow & \downarrow \\
B & \xrightarrow{(\ell_1,\ell_2)} & D \,.
\end{array}
$$

$\square$

1.2. Factorisation systems.

Definition 2. A *unique factorisation system* on a category $\mathcal{C}$ is the data of two classes $\mathcal{A}, B$ of maps in $\mathcal{C}$ such that any arrow $u\colon X \to Y$ admits a factorisation

$$
\begin{array}{ccc}
& \phi(u) & \\
\alpha(u) \nearrow & & \searrow \beta(u) \\
X & \xrightarrow{\quad u \quad} & Y
\end{array}
$$

with $\alpha(u) \in \mathcal{A}$ and $\beta(u) \in \mathcal{B}$, which is unique up to unique isomorphism, *i.e.* for two such factorisations $X \to \phi(u) \to Y$ and $X \to \varphi(u) \to Y$ there exists a unique isomorphism $\phi(u) \to \varphi(u)$ making the two obvious triangles commuting.

For short, such a factorisation system will be noted $\mathcal{C} = (A, B)$. It is obvious that $(\mathcal{B}^o, \mathcal{A}^o)$ is another factorisation system on $\mathcal{C}^o$.

The definition of a unique factorisation system has many consequences toward the following lemma.

Lemma 1.4. *In $\mathcal{C} = (A, B)$, any commuting square*

$$
\begin{array}{ccc}
X & \longrightarrow & Z \\
\alpha(u) \downarrow & \ell \nearrow & \downarrow b \\
\phi(u) & \xrightarrow{\beta(u)} & Y
\end{array}
$$

where $X \to \phi(u) \to Y$ is a factorisation of some $u\colon X \to Y$ and $b \in \mathcal{B}$, admits a unique lifting ℓ.

Proof. This follows by considering a factorisation of $X \to Z$ and using the uniqueness of the decomposition of u. $\square$

The dual lemma considering a square

$$
\begin{array}{ccc}
X & \longrightarrow & \phi(u) \\
{\scriptstyle a}\downarrow & {\scriptstyle \ell}\nearrow & \downarrow \\
Z & \longrightarrow & Y
\end{array}
$$

with $a \in \mathcal{A}$ is also true.

Corollary 1. *The classes $\mathcal{A}$ and $\mathcal{B}$ of a unique factorisation system define a unique lifting system.*

Proof. Given a commuting square

$$
\begin{array}{ccc}
X & \longrightarrow & Z \\
{\scriptstyle a}\downarrow & & \downarrow{\scriptstyle b} \\
Y & \longrightarrow & T
\end{array}
$$

with $a \in \mathcal{A}$ and $b \in \mathcal{B}$, the result follows by considering a factorisation of the diagonal $X \to T$ and by the above lemma and its dual. $\qquad\square$

We are going to see in §1.3 that the converse is true if $\mathcal{C}$ is nice enough.

Proposition 2. *This factorisation is functorial in the sense that, for any commuting square*

$$
\begin{array}{ccc}
X & \xrightarrow{\;u\;} & Y \\
\downarrow & & \downarrow \\
X' & \xrightarrow[\;v\;]{} & Y'
\end{array}
$$

and any choice of factorisation of u and v is associated a unique map $\varphi(u) \to \varphi(v)$ such that the following diagram commutes:

$$
\begin{array}{ccccc}
X & \xrightarrow{\alpha(u)} & \varphi(u) & \xrightarrow{\beta(u)} & Y \\
\downarrow & & \downarrow & & \downarrow \\
X' & \xrightarrow[\alpha(v)]{} & \varphi(v) & \xrightarrow[\beta(v)]{} & Y'.
\end{array}
$$

Proof. From

$$
\begin{array}{ccccc}
X & \xrightarrow{\alpha(u)} & \varphi(u) & \xrightarrow{\beta(u)} & Y \\
\downarrow & & & & \downarrow \\
X' & \xrightarrow[\alpha(v)]{} & \varphi(v) & \xrightarrow[\beta(v)]{} & Y'.
\end{array}
$$

one can extract the square

$$
\begin{array}{ccc}
X & \xrightarrow{\alpha(u)} & \varphi(u) \\
\downarrow & & \downarrow \\
\varphi(v) & \xrightarrow[\beta(v)]{} & Y' .
\end{array}
$$

Then, the wanted map exists by the previous corollary. $\qquad\square$

1.3. **From lifts to factorisations.** For non unique lifting system, the small object argument is used to construct a non unique factorisation system. This contruction works for unique lifting system (A, B) and the resulting factorisation can be proved to be unique as a consequence of the stability of $\mathcal{A}$ or $\mathcal{B}$ by codiagonals or diagonals (cf. lemma 1.3). In case where $\mathcal{C}$ is a category of ind-objects and the lifting system is left generated (cf. lemma 1.1) by maps between objects of finite presentation we are going to present a more straightforward construction of the associated unique factorisation. This will be used in 3.3.

The idea is the following: for a lifting system (A, B), suppose we have a factorisation of a morphism $X \to Y$ in $X \xrightarrow{\alpha} X' \xrightarrow{\beta} Y$ with $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, then for any square

$$
\begin{array}{ccc}
X & \xrightarrow{\alpha} & X' \\
{\scriptstyle \in \mathcal{A}}\downarrow & {\scriptstyle \exists! s}\nearrow & \downarrow{\scriptstyle \beta} \\
U & \longrightarrow & Y
\end{array}
$$

there exists a section s. This suggests to build $X \to X'$ as a colimit of all $X \to U$.

For the colimit to exist we are going to assume that we have a set G of left generators for (A, B) and we define $\overline{G}$ as the set of all maps of $\mathcal{C}$ obtained as pushouts of maps in G. Then, for $u \colon X \to Y$ we define G_u to be the category whose objects are compositions $X \to X_g \to Y$, where $X \to X_g$ is this pushout of some $g \in G$, and whose morphisms are diagrams

$$
\begin{array}{ccc}
X & \longrightarrow X_g & \longrightarrow Y \\
\| & \downarrow & \| \\
X & \longrightarrow X_h & \longrightarrow Y.
\end{array}
$$

where $X_g \to X_h$ is in $\overline{G}$. It is a small category. The middle object of the factorisation to be of $X \to Y$ is then defined as the colimit X' over G_u of all X_g. The problem is now to check that $X \to X' \in \mathcal{A}$ and $X' \to Y \in \mathcal{B}$. For this we are going to assume that $\mathcal{C} = Ind(\mathcal{D})$, $G \subset \mathcal{D}$ and that G_u is cofiltered. As cofiltered diagrams are connected, the colimit of the constant diagram $X \colon G_u \to \mathcal{C}$ will be X and the natural map $X \to X'$ is the colimit of maps $X \to X_g$, and so is in $\mathcal{A}$

by its stability by colimits. As for $X' \to Y$ being in $\mathcal{B}$, given a square

$$
\begin{array}{ccc}
A & \longrightarrow & X' \\
{\scriptstyle h \in G} \downarrow & & \downarrow \\
B & \longrightarrow & Y
\end{array}
$$

X' being in $Ind(\mathcal{D})$, the map $A \to X'$ will factors through some $A \to X_g$. If $(X_g)_h$ is the pushout of h along $A \to X_g$ and if $X \to (X_g)_h$ is still in G_u (which is true if $\overline{G}$ is stable by composition), a map $(X_g)_h \to X'$ would exist and be unique by construction of X', proving that $X' \to Y$ is in $\mathcal{B}$.

Proposition 3. *If $\mathcal{C} = Pro(\mathcal{D})$ is endowed with a unique lifting system (A, B), left generated by a small category $G \subset \mathcal{D}$ stable by cobase change in $\mathcal{D}$ and having right cancellation, then (A, B) is associated to a unique factorisation system and the factorisation of a map $X \to Y$ is given by $X \to X' \to Y$ as above. In particular $\mathcal{A} = Ind - G$.*

Proof. According to the previous discussion, it is sufficient to prove that G_u is cofiltered and $\overline{G}$ is stable by composition. For the first point, we are going to show that G_u has in fact all finite colimits. G_u is by construction stable by pushout so G_u but it is not clear that the constructed object have the pushout universal property internally in G_u, *i.e.* that the map p in the following diagram is in G_u:

$$
\begin{array}{ccc}
Y & \longrightarrow & Z \\
\downarrow & & \downarrow \\
X & \longrightarrow & X \sqcup_Y Z \\
& & {\scriptstyle p} \searrow \\
& & U \, .
\end{array}
$$

However, this is true if $\overline{G}$ had right cancellation, the following lemma ensures that this is the case.

Lemma 1.5. *If G is a subcategory of $\mathcal{D}$ stable by cobase change and with right cancellation, then its extension by cobase change $\overline{G}$ in $Ind(\mathcal{D})$ has right cancellation too.*

Proof. For $u \colon X \to Y \in \overline{G}$, there exists an $A \to B \in G$ such that $X \to Y$ is the cobase change of $A \to B$ along some $A \to X$. Now for two $X \to Y_1$ and $X \to Y_2$, considering two associated $A_i \to B_i \in G$ as above, the maps $g_i \colon A_1 \sqcup A_2 \to B_i \sqcup A_j$ (where $\{i, j\} = \{1, 2\}$) are still in $\mathcal{D}$ and $X \to Y_i$ is the cobase change of g_i along the map $A_1 \sqcup A_2 \to X$. This proves that in the two maps $A_i \to B_i$ the sources can be chosen to be the same.

If we have now a map $Y_1 \to Y_2$ under X, we want to prove that there exists a diagram $A \to B_1 \to B_2 \in G$ such that $X \to Y_1 \to Y_2$ is its cobase change along some $A \to X$. To prove so, we first consider two $A \to B_i \in G$ as above giving

$X \to Y_i$ by cobase change, now they may not exist a map $A_1 \to A_2$ over B as we look for, but this map exists over B' for some $B \to B' \in \mathcal{D}$. Indeed $A \to X$ is the colimit of all A' indexed by the cofiltered diagram of maps $A \to A' \in \mathcal{D}$ factoring $A \to X$, so Y_2 is the colimit of all $A_2 \sqcup_A A'$ indexed by the same diagram. Now by property of ind-objects, the map $A_1 \to Y_2$ must factor through one of the $A_2 \sqcup_A A'$. $X \to Y_1 \to Y_2$ is then a pushout of $A \to A_1 \sqcup_A A' \to A_2 \sqcup_A A'$ and because G has right cancellation the map $A_1 \sqcup_A A' \to A_2 \sqcup_A A'$ is in G so $Y_1 \to Y_2$ is in $\overline{G}$. $\square$

Now G_u has amalgamated sums and an obvious initial objet, so it has all finite colimits and is cofiltered.

To finish the proof, it remains to show the stability by composition of $\overline{G}$: given two pushouts

$$
\begin{array}{ccc}
A \longrightarrow X \\
\left\downarrow{\scriptstyle g}\right. \qquad \left\downarrow\right. \\
B \longrightarrow Y
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
C \longrightarrow Y \\
\left\downarrow{\scriptstyle h}\right. \qquad \left\downarrow\right. \\
D \longrightarrow Z
\end{array}
$$

with g and h in G, we can build a diagram

$$
\begin{array}{ccc}
A \cup C \longrightarrow X \\
{\scriptstyle g \cup C}\left\downarrow\right. \qquad \left\downarrow\right. \\
B \cup C \longrightarrow Y \\
{\scriptstyle B \cup h}\left\downarrow\right. \qquad \left\downarrow\right. \\
B \cup D \longrightarrow Z
\end{array}
$$

where the two squares are still pushouts, and so their composition. Now $g \cup C$ and $B \cup h$ are in G by its stability by pushout and so is $B \cup h \circ g \cup C$ by its stability by composition, making $X \to Z$ in $\overline{G}$.

As for the last remark about $\mathcal{A} = Ind - G$, the inclusion $\mathcal{A} \subset Ind - G$ is clear by construction and $Ind - G \subset \mathcal{A}$ comes from the stability of $\mathcal{A}$ by colimits. $\square$

Remark. If $\mathcal{C} = Pro(\mathcal{D})$ and $G \subset \mathcal{D}$ is a set of left generators for (A, B), then G can always be completed in a subcategory as in proposition 3, so only the left generation is really important as an hypothesis on (A, B).

From now on, all unique lifting systems that we are going to consider will always be associated to some unique factorisation sytems and we are going to denoted them the same way.

2. Topology

This section presents the topological interpretation of factorisation systems sketched in the introduction. Let $\mathcal{C} = (A, B)$ be a category with a unique factorisation system, as the purpose of this section is to transform the objects of $\mathcal{C}$ into topological objects, we will need and assume some basic "geometric" properties: $\mathcal{C}$ will be taken with finite limits and with a strict initial object (any map to it is an isomorphism) if it exists. The initial object will be called *empty* and noted $\emptyset$ and the terminal one will be noted $*$.

The construction will take the form of a covariant functor

$$
\begin{aligned}
Spec : \mathcal{C} &\longrightarrow \mathcal{T}opos \\
X &\longmapsto Spec(X)
\end{aligned}
$$

where $\mathcal{T}opos$ is the category of toposes and geometric morphisms up to natural isomorphisms.

2.1. Finiteness contexts and Etale maps.

As explained in the introduction, the basic idea is to think maps in the class $\mathcal{B}$ as defining some kind of etale topology, but in algebraic geometry, where $\mathcal{B}$ can be a subclass of formally etale maps, one do not want to take them all as etale and a finiteness conditions is required. This suggest to consider a subcategory $\mathcal{C}^f$ of $\mathcal{C}$ of maps morally satisfying some finiteness conditions.

Definition 3. A subcategory $\mathcal{C}^f \subset \mathcal{C}$ is called a *finiteness context* for $\mathcal{C}$ if

a. it contains all isomorphisms (and therefore has the same objects as $\mathcal{C}$),
b. it is stable by base change along morphisms of $\mathcal{C}$,
c. (left cancellation) for all $X \in \mathcal{C}$, $\mathcal{C}^f_{/X} \subset \mathcal{C}_{/X}$ is full.

Maps in $\mathcal{C}^f$ will be called of *finite presentation*. An object in $\mathcal{C}^f_{/*}$ will be called of *finite presentation*, by cancellation, $\mathcal{C}^f_{/*}$ is a full subcategory of $\mathcal{C}$.

A particular example of $\mathcal{C}^f$ is of course the whole $\mathcal{C}$. Another example, when $\mathcal{C} = CRings^o$ is the opposite of the category of commutative rings, is to take $\mathcal{C}^f = (CRings^f)^o$ the opposite of the category of morphisms of finite presentation. These are the two examples we will use.

Definition 4. The data $(\mathcal{C} = (A, B), \mathcal{C}^f)$ of a category $\mathcal{C}$ with a unique factorisation system (A, B) and a choice of a finiteness context $\mathcal{C}^f$ is called a *factorisation context*.

Definition 5. Given a factorisation context $(\mathcal{C} = (A, B), \mathcal{C}^f)$:

a. A map $U \to X$ in $\mathcal{B}$ will be called *f-etale* and a *f-etale open* of X (the name is chosen to recall formally etale maps of algebraic geometry).

b. A ind-etale map $U \to X$ in called an *etale map* and an *etale open* of X if it is in $\mathcal{C}^f$.

The intersections $\mathcal{A} \cap \mathcal{C}^f$ and $\mathcal{B} \cap \mathcal{C}^f$ will be noted $\mathcal{A}^f$ and $\mathcal{B}^f$. $\mathcal{B}^f$ is the category of etale maps and $\mathcal{B}^f_{/X}$ the category of etale opens of X. By left cancellation of $\mathcal{B}$ and $\mathcal{C}^f$, $\mathcal{B}^f_{/X}$ is a full subcategory of $\mathcal{C}_{/X}$. It is this category that we are going to endow with a topology to create the small spectrum of X.

2.2. **Points.** Now in order to extract from the class $\mathcal{B}$ some covering families, we need a notion of surjectivity. Algebraic geometrical examples motivate the following definitions.

Definition 6. Given a factorisation context $(\mathcal{C} = (A, B), \mathcal{C}^f)$:

a. An object P of $\mathcal{C}$ is called a (A, B)-*point* (or only a *point* if the context is clear) if it is not empty and if any map $U \to P \in \mathcal{B}^f$ where U is non empty has a section (non necessarily unique).

b. A *point of an object X* is a map $x \colon P \to X$ from a point P.

c. The *category of points of an objet* $X \in \mathcal{C}$, noted $\mathcal{P}t_{\mathcal{B}^f}(X)$, is the subcategory of $\mathcal{C}_{/X}$ span by objects $P \to X$, where P is a point. In particular, $\mathcal{P}t_{\mathcal{B}^f}(\mathcal{C}) := \mathcal{P}t_{\mathcal{B}^f}(*)$, is the subcategory of $\mathcal{C}$ spanned by all points.

d. The *set of points* of an object X, noted $pt_{\mathcal{B}^f}(X)$ is defined as the set of connected components of $\mathcal{P}t_{\mathcal{B}^f}(X)$.

In topological terms, points are those objects such that any etale map has a section, if one thinks monomorphic etale maps as open embeddings, a point will have no non trivial opens as any monomorphism with a section is an isomorphism. This is one argument for the name 'point' for this notion. Also, in the study of rings, our points will correspond to various kinds of rings closed under some operations (inverses, algebraic elements...) extracting the classes of fields, separably closed fields... which are indeed the 'points' of algebraic geometry.

2.3. **Point covering families.**

Proposition 4. *Given a family of etale maps* $\{U_i \to X\}$, *the following two properties are equivalent:*

(1) *Any point* $P \to X$ *lift to one of the* U_i

$$
\begin{array}{ccc}
 & & U_i \\
 & \nearrow & \downarrow {\scriptstyle \exists i} \\
P & \longrightarrow & X \, .
\end{array}
$$

(2) *The induced map of sets* $\sqcup_i pt_{\mathcal{B}^f}(U_i) \longrightarrow pt_{\mathcal{B}^f}(X)$ *is surjective.*

Proof. It is clear that *1.* implies *2.* Reciprocally, *2.* says that for any $P \to X$, there exists an i and a morphism $P' \to P$ from another point P' such that $P' \to X$ lift to U_i. But this forces $U_i \times_X P$ to be non empty and as $\mathcal{B}$ is stable by base change, $U_i \times_X P \to P$ must then have a section. $\qquad\square$

We are now able to define our covering families.

Definition 7. A family $\{U_i \to X\}$ in $\mathcal{B}^f$ is a *point covering family of X* if it satisfies one of the above two conditions.

Proposition 5. *Point covering families of X define a pretopology on $\mathcal{B}^f_{/X}$ and $\mathcal{C}^f_{/X}$.*

Proof. Our definition of pretopology is taken from [SGA4-1, II.1.3.]. Maps in $\mathcal{B}$ are stable by pullbacks in $\mathcal{B}$ or in $\mathcal{C}$, and so are maps surjective on points (easy from the definition): any pullback of a point covering family is again a point covering family. Identities are in $\mathcal{B}$ and surjective on points. And finally, for $\{U_i \to X, i\}$ and for $\{V_{ij} \to U_i, j\}$ all covering families, all $V_{ij} \to X$ are etale by composition and $\sqcup V_{ij} \to X$ is still surjective on points. $\qquad\square$

The associated topology will be called the *factorisation topology.*

2.4. **Local objects.** We defined our covering families such that any points would lift through them, but many more objects have this lifting property, this is the idea of a local object. Topologically, they correspond to germs. This is nothing as the notion of points of a topos and has nothing to do with factorisation systems, but in the particular case of factorisation topologies, it gives back many known classes of objects (such as local rings). We will define in fact two notions of local objects with respect to a factorisation system. These notions should not be equivalent in general, but they will coincide in all our examples.

A family $\{U_i \to L\}$ is said to *have a section* if there exists an i and a section of $U_i \to L$. A family $\{U_i \to X\}$ is said to *have a section along $L \to X$* (or to *lift through $\{U_i \to X, i\}$*) if there exists an i and a section of $U_i \times_X L \to L$.

Proposition 6. *For $L \in \mathcal{C}$, the following assertions are equivalent:*
  (1) *L is such that every point covering family $\{U_i \to L\}$ admit a section.*
  (2) *L is such that for every point covering family $\{U_i \to X\}$ has a section along any $L \to X$.*
  (3) *L define a point of the topos $\widetilde{\mathcal{C}}$ of sheaves on $\mathcal{C}$ for the factorisation topology.*

Proof. The equivalence of *1.* and *2.* is trivial. As $\mathcal{C}$ is assumed to have finite limits, a point of $\widehat{\mathcal{C}}$ is a left exact functor $\mathcal{C} \longrightarrow \mathcal{S}$. An object $L \in \mathcal{C}$ define a point of $\widehat{\mathcal{C}}$ via $\mathcal{C}(L, -) \colon \mathcal{C} \longrightarrow \mathcal{S}$. Now a point of $\widehat{\mathcal{C}}$ is a point of $\widetilde{\mathcal{C}}$ iff it send point covering families to jointly surjective families of sets, *i.e.* iff for any $\{U_i \to X\}$,

$\sqcup_i \mathcal{C}(L, U_i) \to \mathcal{C}(L, X)$ is surjective. But this is a reformulation of equivalent to *2*. $\square$

Definition 8. Any object L satisfying those conditions wil be called *local*.

Lemma 2.1. *If L is local and $L \to L' \in \mathcal{A}$, then L' is local.*

Proof. For $L \to L' \in \mathcal{A}$, let $\{U_i \to L'\}$ be a point covering family of L', the pulled-back cover $U_i' \to L$ has a section by assumption on L and this give a square where one can use property of the lifting system (A, B):

$$\begin{array}{ccc} L & \xrightarrow{\exists i} & U_i \\ {\scriptstyle \in \mathcal{A}} \downarrow & \nearrow & \downarrow {\scriptstyle \in \mathcal{B}} \\ L' & = \!\!= & L'. \end{array}$$
$\square$

As it is clear that points are local objects, the previous lemma authorizes the construction of local objects by considering target of maps $P \to L \in \mathcal{A}$ where P is a point.

Definition 9. A *pointed local object* of $\mathcal{C}$ is an object L such that there exist a point $P \to L \in \mathcal{A}$.

It is not clear in general that any local object can be pointed or that this point would be unique, nonetheless this will be the case in our main examples.

Lemma 2.2. *If $\mathcal{C}$ has a strict initial object $\emptyset$, it can never be a local object.*

Proof. $\emptyset$ is strict if any map $X \to \emptyset$ is an isomorphism, so, as points are supposed not initial, the set of points of $\emptyset$ is empty. This prove that the empty family is a point covering family of $\emptyset$ and such a family cannot have a section. $\square$

In algebraic geometrical examples this will prove that the zero ring is never a local object for the factorisation topologies.

2.5. **Nisnevich forcing.** Although the definition of the factorisation topology (def. 7) will give back known topologies in algebraic geometry, it will be in general too fine for other examples. In particular, the Nisnevich topology cannot be defined as a factorisation topology and this example suggests to use the following general construction, that we will call *Nisnevich forcing*. Starting with a topology given by some covering families, the idea is to select some of those families satisfying an extra lifting condition. This is completely independent of the existence of any factorisation system.

Definition 10. Let $\mathcal{C}$ be a category with a topology τ defined via some covering families $U_i \to X$, and $\mathcal{L}$ a class of objects of $\mathcal{C}$.

a. A covering family $U_i \to X$ is said $\mathcal{L}$-*localising* if for any object $L \in \mathcal{L}$ and any map $L \to X$ lift through the cover, *i.e.* if the pulled-back cover $U_i' \to L$ has a section. It is clear that such covers are stable by base change.

b. The $\mathcal{L}$-*Nisnevich forcing* of τ (refered to for short as the *Nisnevich topology*), noted $\tau_{\mathcal{L}}$, is the topology generated by $\mathcal{L}$-localising covering families. This topology is coarser than τ.

The class $\mathcal{L}$ will be called the *forcing class*. The *saturation* of $\mathcal{L}$, noted $\overline{\mathcal{L}}$, is defined as the subcategory of $\mathcal{C}$ of local objects (def. 8) for the topology $\tau_{\mathcal{L}}$, these objects will be called *Nisnevich local objects*. $\tau_{\overline{\mathcal{L}}} = \tau_{\mathcal{L}}$ and $\overline{\mathcal{L}}$ is maximal for this property. If $\mathcal{L} = \emptyset$ then $\tau_{\mathcal{L}} = \tau$ and $\overline{\emptyset} = \mathcal{L}oc$; the category of local objects. If $\mathcal{L}' \subset \mathcal{L}$ then $\overline{\mathcal{L}'} \subset \overline{\mathcal{L}}$ so one has always $\mathcal{L}oc \subset \overline{\mathcal{L}}$.

Definition 11. The data $\mathcal{N} = (\mathcal{C} = (A, B), \mathcal{C}^f, \mathcal{L})$ where $(\mathcal{C} = (A, B), \mathcal{C}^f)$ is a factorisation context (def. 4) and $\mathcal{L}$ a full subcategory of $\mathcal{C}$ is called a *Nisnevich context*. Two Nisnevich contexts are said *equivalent* if they have the same underlying factorisation context and if both localising classes have the same saturation.

Nisnevich contexts will be our basic data to generate spectra, but when the localising class is trivial, we'll refer to them simply as *factorisation context*.

Lemma 2.3. *For a Nisnevich context* $\mathcal{N} = (\mathcal{C} = (A, B), \mathcal{C}^f, \mathcal{L})$, *if a map* $L \to L' \in \mathcal{A}$ *is such that* $L \in \overline{\mathcal{L}}$ *then* $L' \in \overline{\mathcal{L}}$.

Proof. If $\{U_i \to X\}$ is a Nisnevich covering family, by hypothesis any $L \to X$ lift though on of the $U_i \to X$. If the map $L \to X$ is coming from a map $L' \to X$, this give a lifting square and a map $L' \to U_i$. □

Definition 12. A *distinguished class of Nisnevich covering families* is defined as a class of Nisnevich covering families $U_i \to X$ with X (and therefore the U_i) in $\mathcal{C}^f_{/*}$ such that an object $L \in \mathcal{C}$ is Nisnevich local iff it lift through any distinguished Nisnevich covering family.

The Nisnevich topology can be restricted to $\mathcal{C}^f_{/*}$ to define a topos $\widetilde{\mathcal{C}^f_{/*}}$. The previous condition can be stated as: points of $\widetilde{\mathcal{C}^f_{/*}}$ are points of $\widetilde{\mathcal{C}}$, *i.e.* Nisnevich local objects.

The following definition will be used in theorem 2.5.

Definition 13. A Nisnevich context $(\mathcal{C} = (A, B), \mathcal{C}^f, \mathcal{L})$ is said *compatible* if
a. for any X, $\mathcal{C}_{/X} = Pro(\mathcal{C}^f_{/X})$ and $\mathcal{B}_{/X} = Pro(\mathcal{B}^f_{/X})$
b. and there exists a distinguished class of point covering families.

2.6. **Spectra.** For $\mathcal{N} = (\mathcal{C} = (A, B), \mathcal{C}^f, \mathcal{L})$ a Nisnevich context.

Definition 14. a. $\mathcal{B}^f_{/X}$ endowed with the Nisnevich topology is called the *small site* of X. The associated topos is noted $Spec_{\mathcal{N}}(X)$ and called the *small $\mathcal{N}$-spectrum* of X.

b. $\mathcal{C}^f_{/X}$ endowed with the Nisnevich topology is called the *big site* of X. The associated topos is noted $SPEC_{\mathcal{N}}(X)$ and called the *big $\mathcal{N}$-spectrum* of X.

When $X = *$ is the terminal object of $\mathcal{C}$, $SPEC_{\mathcal{B}^f}(*)$ is simply noted $\widetilde{\mathcal{C}}$.

Let $\mathcal{T}opos$ be the category whose objects are toposes and morphisms equivalence classes of geometric morphisms for natural isomorphisms.

Lemma 2.4. *If $\mathcal{C}$ as finite limits, the category $\mathcal{B}^f_{/X}$ has all finite limits and for $u\colon X \to Y \in \mathcal{C}$, the base change functor $u^*\colon \mathcal{B}^f_{/Y} \to \mathcal{B}^f_{/X}$ is left exact.*

Proof. As $\mathcal{B}^f_{/X}$ has a terminal object, it is sufficient to prove that is has fiber products. But, using the cancellation property as in prop. 3, they can be computed independently of the base X in $\mathcal{C}$ (which will also imply the exactness of u) and as $\mathcal{B}$ and $\mathcal{C}^f$ are stable by pullback the resulting diagram it is in $\mathcal{B}^f$. $\square$

Proposition 7. *$Spec_{\mathcal{N}}(-)$ and $SPEC_{\mathcal{N}}(-)$ are functors $\mathcal{C} \to \mathcal{T}opos$. Moreover, maps in $\mathcal{B}^f$ are send to etale maps of toposes.*

Proof. We detail only the functoriality of the small spectrum. A map $u\colon X \to Y \in \mathcal{C}$ induces a base change functor $u^*\colon \mathcal{B}^f_{/Y} \to \mathcal{B}^f_{/X}$ is left exact by lemma 2.4 and clearly preserve covering families, so it is continuous [SGA4-1, III.1.6] and defines a geometric morphism $(u^*, u_*)\colon Spec_{\mathcal{N}}(X) \longrightarrow Spec_{\mathcal{N}}(Y)$. Now the problem of compatibility with composition is taken care of in the definition of $\mathcal{T}opos$ as a 1-category. (It could also be defined as a pseudo-functor from $\mathcal{C}$ to the 2-category of toposes.) As for the second statement, recall that a geometric morphism $u\colon \mathcal{E} \to F$ is etale (local homeomorphism in [Jo2, C.3.3.4]) iff there exists an $F \in \mathcal{F}$ and an isomorphism $\mathcal{F}/F \simeq \mathcal{E}$ such that u is equivalent to the geometric morphism $\mathcal{F}/F \longrightarrow \mathcal{F}$. It is then clear by construction that any $X \to Y \in \mathcal{B}^f$ will give such a map. $\square$

Proposition 8. *For $X \in \mathcal{C}$, if $\mathcal{C}^f_{/X}$ is small, there exists two geometric morphisms (natural in X) $r_X = (r^*_X, r^X_*)\colon SPEC_{\mathcal{N}}(X) \to Spec_{\mathcal{N}}(X)$ and $s_X = (s^*_x, s^X_*)\colon Spec_{\mathcal{N}}(X) \to SPEC_{\mathcal{N}}(X)$, such that*
   - *$r^X_* = s^*_X$,*
   - *r^*_X and s^X_* are fully faithful, in particular $rs \simeq id$.*

In other terms
   - *r_X is left adjoint to s_X in the 2-category of toposes,*
   - *r_X is a quotient with connected fiber,*
   - *and s_X is a subtopos embedding and a section of r_X, i.e. the adjunction (r_X, s_X) is a reflexion of $SPEC_{\mathcal{N}}(X)$ on $Spec_{\mathcal{N}}(X)$.*

Proof. The morphism of small sites $\iota_X\colon \mathcal{B}^f_{/X} \to \mathcal{C}^f_{/X}$ commute to finite limits, and the topology of $\mathcal{B}^f$ is induced by that of $\mathcal{C}^f_{/X}$, so ι is continous and cocontinuous

by [SGA4-1, III.3.4], and induces three adjoint functors $\iota_!^X \dashv \iota_X^* \dashv \iota_*^X$:

$$SPEC_{\mathcal{N}}(X) \underset{\iota_*^X}{\overset{\iota_!^X}{\underset{\longleftarrow}{\overset{\longrightarrow}{\longleftarrow}}}} Spec_{\mathcal{N}}(X).$$

ι_X being fully faithful, so are $\iota_!^X$ and ι_*^X. r_X is defined as the adjonction (ι_X^*, ι_*^X) and s_X is defined as the adjonction $(\iota_!^X, \iota_X^*)$. For s_X to be a geometric morphism, we need to check that $\iota_!^X$ is left exact, but this is a consequence of ι being left exact. $\qquad\square$

Corollary 2. *The category of points of $Spec_{\mathcal{N}}(X)$ is a reflexive full subcategory of that of $SPEC_{\mathcal{N}}(X)$.*

We study now the functoriality of our spectra with respect to the factorisation system. We are going to focus only on *Spec* but the results are the same for *SPEC*. Unique factorisation systems on $\mathcal{C}$ are entirely characterized by their right classes $\mathcal{B}$. It is then possible to put an order of them by looking at the inclusion relation of the right classes. We say that $(\mathcal{A}_1, \mathcal{B}_1)$ is *finer than* or *a refinement of* $(\mathcal{A}_2, \mathcal{B}_2)$ if $\mathcal{B}_2 \subset \mathcal{B}_1$. This order admit an initial and a terminal element that are detailed in §3.1. More generaly, a Nisnevich context $\mathcal{N} = (\mathcal{C} = (\mathcal{A}_1, \mathcal{B}_1), \mathcal{C}^f, \mathcal{L}_1)$ will be said *finer than* (or *a refinement of*) $\mathcal{N}' = (\mathcal{C} = (\mathcal{A}_2, \mathcal{B}_2), \mathcal{C}^f, \mathcal{L}_2)$ if the underlying finiteness context are the same, if $(\mathcal{A}_1, \mathcal{B}_1)$ is finer than $(\mathcal{A}_2, \mathcal{B}_2)$ and if $\mathcal{L}_1 \subset \overline{\mathcal{L}}_2$.

Proposition 9. *For two Nisnevich contexts $\mathcal{N}_1 = (\mathcal{C} = (\mathcal{A}_1, \mathcal{B}_1), \mathcal{C}^f, \mathcal{L}_1)$ and $\mathcal{N}_2 = (\mathcal{C} = (\mathcal{A}_2, \mathcal{B}_2), \mathcal{C}^f, \mathcal{L}_2)$, if $\mathcal{N}_1$ is refinement of $\mathcal{N}_2$, there is a natural transformation of functors $Spec_{\mathcal{N}_1}(-) \to Spec_{\mathcal{N}_2}(-)$.*

Proof. $(\mathcal{A}_1, \mathcal{B}_1)$ is finer than $(\mathcal{A}_2, \mathcal{B}_2)$. This implies $\mathcal{P}t_{\mathcal{B}_1^f} \subset \mathcal{P}t_{\mathcal{B}_2^f}$ and so $\mathcal{B}_2$ point covering families are $\mathcal{B}_1$ point covering families. The functor

$$\mathcal{B}_{2/X} \longrightarrow \mathcal{B}_{1/X}$$

is then continuous and gives a geometric morphism

$$Spec_{\mathcal{N}_1}(X) \longrightarrow Spec_{\mathcal{N}_2}(X),$$

i.e. Spec is covariant with respect to the refinement relation for factorisation systems. As for the Nisnevich forcing, $\mathcal{B}_{2/X} \longrightarrow \mathcal{B}_{1/X}$ will send covering families to covering families iff the forcing class $\mathcal{L}_1$ is contained in $\overline{\mathcal{L}}_2$. $\qquad\square$

2.6.1. *Moduli interpretation.* We investigate a computation of the categories of points of the two spectra. Theorem 2.5 establishes that under some hypothesis they can be described as local objects. A complete study of the moduli aspects of our spectral theory would ask to compute not only global points but all categories of points of our spectra, but this would require to develop more the topos theoretic aspects which we'll do in another paper [An].

If $P \to X$ is a point of an object X, we already interpreted the factorisation $P \to L \to X$ as the germ of the point in X. This suggest the following definitions.

A *local form* of an object X is a map $L \to X \in \mathcal{B}$ where L is a local object, it is *pointed* if L is. Any point of X define a (pointed) local form of X. Let $\mathcal{L}oc(X)$ be the full subcategory of $\mathcal{C}_{/X}$ generated by local forms of X, the left cancellation property of $\mathcal{B}$ (prop. 1) ensures that all morphisms between local forms of X are in $\mathcal{B}$. More generally, for a Nisnevich forcing class $\mathcal{L}$ with saturation $\overline{\mathcal{L}}$, a $\mathcal{L}$-*local form* of X is a map $L \to X \in \mathcal{B}$ where $L \in \overline{\mathcal{L}}$ and the category $\overline{\mathcal{L}}(X)$ of $\mathcal{L}$-local forms of X is defined as the subcategory of $\overline{\mathcal{L}}_{/X}$ generated by object whose structural map is in $\mathcal{B}$. Again, all morphisms of $\overline{\mathcal{L}}(X)$ are in $\mathcal{B}$.

Let's recall the characterization of points of a site.

Proposition 10. *Let $\mathcal{D}$ be a site with a topology given by some covering families, the category of points of the associated topos $\widetilde{\mathcal{D}}$ is the full subcategory of $Pro(\mathcal{D})$ of those pro-objects of $\mathcal{D}$ that have the lifting property through any covering family.*

Proof. Briefly (see [MM] for details). The category of points of $\widehat{\mathcal{D}}$ is $Pro(\mathcal{D})$ the category of pro-objects of $\mathcal{D}$. In $Pro(D)$, an object P is a point of $\widetilde{\mathcal{D}} \subset \widehat{\mathcal{D}}$ iff it transforms covering families into epimorphic families. This last part is equivalent to have in $Pro(\mathcal{D})$ a diagram

$$
\begin{array}{ccc}
 & & U_i \\
 & \overset{\exists i}{\nearrow} & \downarrow \\
P & \longrightarrow & X
\end{array}
$$

hence the statement of the result. $\qquad\square$

Theorem 2.5. *For a compatible Nisnevich context $\mathcal{N} = (\mathcal{C} = (A,B), \mathcal{C}^f, \mathcal{L})$ (def. 13):*

(1) *the category of points of $SPEC_{\mathcal{N}}(X)$ is that $\overline{\mathcal{L}}_{/X}$ of local objects over X*

(2) *and the category of points of $Spec_{\mathcal{N}}(X)$ is that $\overline{\mathcal{L}}(X)$ of $\mathcal{L}$-local forms of X.*

Proof. The lifting condition for points of the two spectra is weaker than the one used to define local objects so local objects will define points as soon as they are pro-objects in the good category, which is what ensures the two conditions $Pro(\mathcal{C}^f_{/X}) \simeq \mathcal{C}_{/X}$ and $Pro(\mathcal{B}^f_{/X}) \simeq \mathcal{B}_{/X}$. Reciprocally, a point of $SPEC_{\mathcal{N}}(X)$ is a local object by existence of a distinguished class of covering families for the Nisnevich context, and so are points of $Spec_{\mathcal{N}}(X)$ by cor. 2. The category of points of $SPEC_{\mathcal{N}}(X)$ is then that of local objects of $\mathcal{N}$ over X. And that of $Spec_{\mathcal{N}}(X)$ is the subcategory of those local objects that are in $\mathcal{B}_{/X}$. $\qquad\square$

Remark. For a factorisation context $\mathcal{F} = (\mathcal{C} = (A, B), \mathcal{C}^f)$ and a Nisnevich context $\mathcal{N} = (\mathcal{C} = (A, B), \mathcal{C}^f, \mathcal{L})$ if $\mathcal{L}oc \subsetneq \overline{\mathcal{L}}$, the set of points of $Spec_{\mathcal{N}}(X)$ contains strictly that of $Spec_{\mathcal{F}}(X)$ and the same is true for big spectra. In particular not all Nisnevich local objects will be pointed (as those that are are local for $\mathcal{F}$).

To finish, we recall the definition of the *set of points of a topos* $\mathcal{T}$ as the set of equivalence classes of geometric morphisms $\mathcal{S} \to T$ for natural isomorphisms, and that a topos is said *spatial* if can be written as the topos of sheaves of a topological space. The category of points of such a topos is at most a poset, this remark will be used to prove that most of our example of spectra are not spaces.

2.6.2. *Structure sheaf.* The naturality of s_X and r_X gives a diagram

$$
\begin{array}{ccccc}
Spec_{\mathcal{N}}(X) & \xrightarrow{\;s_X\;} & SPEC_{\mathcal{N}}(X) & \xrightarrow{\;r_Y\;} & Spec_{\mathcal{N}}(X) \\
{\scriptstyle u}\downarrow & (1) & {\scriptstyle U}\downarrow & (2) & \downarrow{\scriptstyle u} \\
Spec_{\mathcal{N}}(Y) & \xrightarrow{\;s_Y\;} & SPEC_{\mathcal{N}}(Y) & \xrightarrow{\;r_Y\;} & Spec_{\mathcal{N}}(Y)
\end{array}
$$

of which we are going to study the commutation properties.

Proposition 11. *The square (2) is commutative up to a natural isomorphism, and the square (1) up to a natural transformation $\alpha(u)\colon Us_X \to s_Y u$. Moreover, under the hypothesis of theorem 2.5, for each point $\mathcal{S} \to Spec_{\mathcal{N}}(X)$ the morphism induced by $\alpha(u)$ on points of $SPEC_{\mathcal{N}}(Y)$ is in $\mathcal{A}$.*

Proof. For the square (2), it is sufficient to check it at the level of the inverse images functors restricted to the generating sites, and it a consequence of the stability of $\mathcal{B}^f$ by pullback in $\mathcal{C}^f$. The result on (1) is then a consequence: there is a natural isomorphism $r_Y U s_X \simeq r_Y s_Y u (\simeq u)$, composing by s_Y and using the unit and counit of the adjunction (r_Y, s_Y), we obtain the wanted map $\alpha(u)\colon Us_X \to s_Y u$.

For the second part, points of a topos can be viewed as some pro-objects and the effect on points of a geometric morphism $(u^*, u_*)\colon \mathcal{E} \to \mathcal{F}$ is understood looking at the left pro-adjoint $u_!$ of u^*. If $\mathcal{E}$ is a topos, the category $\underline{Pro}(\mathcal{E})$ of internal pro-objects of $\mathcal{E}$ is defined as the category of $\mathcal{E}$-enriched left-exact (accessible) endofunctors of $\mathcal{E}$. In particular, it contains fully faithfully the category $Pro(\mathcal{E})$ of pro-objects of $\mathcal{E}$ view as a category

The left pro-adjoint of $u^*\colon \mathcal{F} \to \mathcal{E}$ is defined the following way: every object $X \in \mathcal{E}$ defined a geometric morphism $i_X = (i_X^*, i_*^X)\colon \mathcal{E}_{/X} \to \mathcal{E}$, and by composition an endofunctor $u_* i_*^X i_X^* u^*$ of $\mathcal{F}$, this endofunctor is a composition of left exact functors so it is itself left exact and define an internal pro-object $u_!(X)$ of F. This construction is functorial in X and define a functor $u_!\colon \mathcal{E} \to \underline{Pro}(\mathcal{F})$.

As for the adjunction property:

$$
\begin{array}{rcl}
X & \longrightarrow & u^*Y \\
X \simeq i_X^*(X) & \longrightarrow & i_X^* u^*Y \\
* \simeq i_*^X(X) & \longrightarrow & i_*^X i_X^* u^*Y \\
* \simeq u_*(*) & \longrightarrow & u_* i_*^X i_X^* u^*Y \simeq \underline{Hom}_{\underline{Pro}(\mathcal{F})}(u_!(X), Y)
\end{array}
$$

where $\underline{Hom}_{\underline{Pro}(\mathcal{F})}(-,-)$ is the $\mathcal{F}$-enriched hom of $\underline{Pro}(\mathcal{F})$.

We will now compute the pro-adjoints of the following diagram and their action on the categories of points.

$$
\begin{array}{ccc}
\underline{Pro}(\widetilde{\mathcal{B}_{/X}^f}) & \xrightarrow{\check{s}_!^X} & \underline{Pro}(\widetilde{\mathcal{C}_{/X}^f}) \\
\check{u}_! \downarrow & & \downarrow \check{U}_! \\
\underline{Pro}(\widetilde{\mathcal{B}_{/Y}^f}) & \xrightarrow{\check{s}_!^Y} & \underline{Pro}(\widetilde{\mathcal{C}_{/Y}^f})
\end{array}
$$

where, for a geometric morphism $(u^*, u_*)\colon \mathcal{E} \to \mathcal{F}$, $\check{u}_!\colon \underline{Pro}(\mathcal{E}) \to \underline{Pro}(\mathcal{F})$ denotes the (internal) right Kan extension of $u_!$. $\check{u}_!$ is left adjoint to the right Kan extension $\check{u}^*$ of u^*. The diagram is still commutative up to a natural transformation constructed the same way as before (in a sense this is the same natural transformation).

To extract the action on points we'll use implicitly the following lemma.

Lemma 2.6. *If in a diagram of functors*

$$
\begin{array}{ccc}
C & \xrightarrow{\gamma} & C' \\
v \downarrow \uparrow u & & v' \downarrow \uparrow u' \\
D & \xrightarrow{\delta} & D'
\end{array}
$$

v is left adjoint to u, v' left adjoint to u', γ and δ are dense in the sense that any object of C' (resp. D') is a limit of objects of C (resp. D) and $\gamma u = u'\delta$, then $\delta v = v'\gamma$, i.e. v is the restriction of v' to C.

Proof. Any $y \in D'$ can be written $y = \lim_i \delta(y_i)$, so for all $x \in C, y \in D$:
$D'(\delta v(x), y) \simeq \lim_i D'(\delta v(x), \delta(y_i)) \simeq \lim_i C(x, u(y_i)) \simeq \lim_i C'(\gamma(x), \gamma u(y_i)) \simeq \lim_i C'(\gamma(x), u'\delta(y_i)) \simeq \lim_i D'(v'\gamma(x), \delta(y_i)) \simeq D'(v'\gamma(x), y)$. $\qquad\qquad\square$

The functor $\check{s}_!^X$ is the extension of the inclusion $\mathcal{B}_{/X}^f \to \mathcal{C}_{/X}^f$, so we have a diagram:

$$
\begin{array}{ccccccc}
\overline{\mathcal{L}}(X) & \longrightarrow & \mathcal{B}_{/X} & \overset{\simeq}{\longrightarrow} & Pro(\widetilde{\mathcal{B}_{/X}^f}) & \longrightarrow & \underline{Pro}(\widetilde{\mathcal{B}_{/X}^f}) \\
\downarrow & & \downarrow & & & & \downarrow {\scriptstyle \check{s}_!^X} \\
\overline{\mathcal{L}}_{/X} & \longrightarrow & \mathcal{C}_{/X} & \overset{\simeq}{\longrightarrow} & Pro(\widetilde{\mathcal{C}_{/X}^f}) & \longrightarrow & \underline{Pro}(\widetilde{\mathcal{C}_{/X}^f})
\end{array}
$$

where the horizontal arrows are fully faithful and the vertical arrows are all restrictions of $\check{s}_!^X$. The morphism induced on points is simply the inclusion of $\overline{\mathcal{L}}(X)$ in $\overline{\mathcal{L}}_{/X}$. The analysis is analog for $s_!^Y$.

For $\check{U}_!$ we have a diagram

$$
\begin{array}{ccccccc}
\overline{\mathcal{L}}_{/X} & \longrightarrow & \mathcal{C}_{/X} & \overset{\simeq}{\longrightarrow} & Pro(\widetilde{\mathcal{C}_{/Y}^f}) & \longrightarrow & \underline{Pro}(\widetilde{\mathcal{C}_{/X}^f}) \\
\downarrow & & \downarrow & & & & \downarrow {\scriptstyle \check{U}_!} \\
\overline{\mathcal{L}}_{/Y} & \longrightarrow & \mathcal{C}_{/Y} & \overset{\simeq}{\longrightarrow} & Pro(\widetilde{\mathcal{C}_{/Y}^f}) & \longrightarrow & \underline{Pro}(\widetilde{\mathcal{C}_{/Y}^f}) .
\end{array}
$$

$u^* \colon \mathcal{C}_{/Y} \to \mathcal{C}_{/X}$ has a left adjoint $u_!$ given by composing with u, which is the restriction of $\check{U}_!$.

For $\check{u}_!$ we have a diagram

$$
\begin{array}{ccccccc}
\overline{\mathcal{L}}(X) & \longrightarrow & \mathcal{B}_{/X} & \overset{\simeq}{\longrightarrow} & Pro(\widetilde{\mathcal{B}_{/X}^f}) & \longrightarrow & \underline{Pro}(\widetilde{\mathcal{B}_{/X}^f}) \\
\downarrow & & \downarrow {\scriptstyle v} & & & & \downarrow {\scriptstyle \check{u}_!} \\
\overline{\mathcal{L}}(Y) & \longrightarrow & \mathcal{B}_{/Y} & \overset{\simeq}{\longrightarrow} & Pro(\widetilde{\mathcal{B}_{/Y}^f}) & \longrightarrow & \underline{Pro}(\widetilde{\mathcal{B}_{/Y}^f}) .
\end{array}
$$

We will prove that the functor $u^* = - \times_Y X \colon \mathcal{B}_{/Y} \to \mathcal{B}_{/X}$ has a left adjoint given by sending $b \colon U \to X$ to the $\phi(ub) \to Y$ where $U \to \phi(ub) \to Y$ is the factorisation of $ub \colon U \to X \to Y$. Indeed, given a choice of (A, B) factorisation for any arrow of $\mathcal{C}$, a map $b \colon U \to X$ defines a unique square

$$
\begin{array}{ccc}
U & \overset{\alpha}{\longrightarrow} & \phi(ub) \\
{\scriptstyle b} \downarrow & & \downarrow {\scriptstyle \beta} \\
X & \underset{u}{\longrightarrow} & Y
\end{array}
$$

where $U \to \phi(ub) \to Y$ is defined as the factorisation of $ub \colon U \to Y$. From this we deduce a bijection between the set of squares

$$(*) = \begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X & \xrightarrow{\ u\ } & Y \end{array}$$

and that of morphisms of $\mathcal{B}_{/Y}$:

$$\begin{array}{ccc} \phi(ub) & \longrightarrow & V \\ & \searrow & \downarrow \\ & & Y \end{array}$$

(the map $\phi(ub) \to V$ comes from the factorisation of $X \to V$). But squares $(*)$ are also in bijection with morphisms in $\mathcal{B}_{/X}$:

$$\begin{array}{ccc} U & \longrightarrow & V \times_Y X \\ \downarrow & \swarrow & \\ X & & \end{array}$$

which gives us the adjonction. Now, the restriction to $\overline{\mathcal{L}}(X)$ clearly takes its values in $\overline{\mathcal{L}}(Y)$ and is the morphism induced by u between the categories of points.

Finally the situation is the following: a point $b \colon L \to X$ is send on one side to $ub \colon L \to Y$ and on the other to $\beta \colon \phi(ub) \to Y$ and the natural transformation $\alpha(u)$ is given by the factorisation

$$\begin{array}{ccc} L & \xrightarrow{\ \alpha\ } & \phi(ub) \\ & \searrow_{bu} & \downarrow_{\beta} \\ & & Y \,. \end{array}$$

This is what we meant saying that it was given by a map in $\mathcal{A}$. $\qquad\square$

Definition 15. The composition $O_X^{\mathcal{N}} \colon Spec_{\mathcal{N}}(X) \to SPEC_{\mathcal{N}}(X) \to SPEC_{\mathcal{N}}(*)$ is called the *structural sheaf* of X. For every point $x \colon \mathcal{S} \to Spec_{\mathcal{B}^f}(X)$, the *stalk of* $\mathcal{O}_X^{\mathcal{N}}$ *at* x is the induced point $\mathcal{O}_{X,x}^{\mathcal{N}} \colon \mathcal{S} \to SPEC_{\mathcal{N}}(*)$.

Proposition 12. *For a point* $x \colon \mathcal{S} \to Spec_{\mathcal{N}}(X)$ *corresponding to a local form* $L \to X$, *the stalk* $O_{X,x}^{\mathcal{N}}$ *is the objet* L.

Proof. $O_X^{\mathcal{N}}$ is the composition $Spec_{\mathcal{N}}(X) \to SPEC_{\mathcal{N}}(X) \to SPEC_{\mathcal{N}}(*)$ and the action of these morphisms on the points have been explained inside the proof of proposition 11. $\qquad\square$

As a corollary of proposition 11, a map $u\colon X \to Y \in \mathcal{C}$ induces a diagram of toposes

$$
\begin{array}{ccc}
Spec_{\mathcal{N}}(X) & \xrightarrow{\;s_X\;} & SPEC_{\mathcal{N}}(X) \\
\downarrow{\scriptstyle u} & & \downarrow{\scriptstyle U} \qquad \searrow \\
Spec_{\mathcal{N}}(Y) & \xrightarrow[\;s_Y\;]{} & SPEC_{\mathcal{N}}(Y) \longrightarrow SPEC_{\mathcal{N}}(*)
\end{array}
$$

and $\alpha(u)$ induces a natural transformation $\mathcal{O}(u)\colon \mathcal{O}_X \to \mathcal{O}_Y \circ u$ such that, for every point of $x\colon \mathcal{S} \to Spec_{\mathcal{B}^f}(X)$, the induced map on the stalk $\mathcal{O}(u)_x\colon \mathcal{O}_{X,x} \to \mathcal{O}_{Y,u(x)}$ is in $\mathcal{A}$.

Remark. The category of points of $SPEC_{\mathcal{N}}(*)$ is that $\overline{\mathcal{L}}$ of local objects and the factorisation system of $\mathcal{C}$ restrict to $\overline{\mathcal{L}}$. This is in fact a general phenomenon and for every topos $\mathcal{T}$ the category of morphisms from $\mathcal{T}$ to $SPEC_{\mathcal{N}}(*)$ will inherit a unique factorisation system. The point of view chosen for the exposition in this article makes the details of this factorisation system a bit complicated to explicit and we won't explain this here. We won't explain either the nice adjunction property of the small spectrum implying that it is a universal localisation. We will treat these questions in a better context in [An].

We study now the functoriality of the map $Spec_{\mathcal{N}}(X) \to SPEC_{\mathcal{N}}(X)$ with respect to the Nisnevich context.

Proposition 13. *For two Nisnevich contexts $\mathcal{N} = (\mathcal{C} = (\mathcal{A}_1, \mathcal{B}_1), \mathcal{C}^f, \mathcal{L}_1)$ and $\mathcal{N}' = (\mathcal{C} = (\mathcal{A}_2, \mathcal{B}_2), \mathcal{C}^f, \mathcal{L}_2)$, if $\mathcal{N}$ is finer than $\mathcal{N}'$, there is a diagram of geometric morphisms*

$$
\begin{array}{ccccc}
Spec_{\mathcal{N}}(X) & \xrightarrow{\;s_X\;} & SPEC_{\mathcal{N}}(X) & \xrightarrow{\;r_X\;} & Spec_{\mathcal{N}}(X) \\
\downarrow{\scriptstyle r} & (1) & \downarrow{\scriptstyle R} & (2) & \downarrow{\scriptstyle r} \\
Spec_{\mathcal{N}'}(X) & \xrightarrow[\;s'_X\;]{} & SPEC_{\mathcal{N}'}(X) & \xrightarrow[\;r'_X\;]{} & Spec_{\mathcal{N}'}(X)
\end{array}
$$

where R is a subtopos embedding, (2) commutes up to a natural isomorphism and (1) commutes up to a natural transformation β. At the level of the category of points, β is given by a map in $\mathcal{B}_1 \cap \mathcal{A}_2$.

Proof. The assertion of R is due to the facts that $SPEC_{\mathcal{N}}(X)$ and $SPEC_{\mathcal{N}'}(X)$ have the same underlying site and the topology of $SPEC_{\mathcal{N}}(X)$ is finer. The commutation of (2) can be seen at the level of inverse images restricted to the sites. From this we deduced natural isomorphisms $r_X \simeq rr_X s_X \simeq r_X R s_X$ and composing by s_X and using unit of (r_X, s_X) we have a transformation $\beta\colon R s_X \to s_X r$. As for the action of β on points we need only to study the pro-adjoint $r_!$. We'll use again lemma 2.6. By a reasoning analog to that in the proof of proposition 11, the map $(\mathcal{B}_2)_{/X} \to (\mathcal{B}_1)_{/X}$ admits a left adjoint given

by the $(\mathcal{A}_2, \mathcal{B}_2)$ factorisation:

$$U \xrightarrow{\ \beta\ } \phi(b)$$
$$\begin{array}{ccc} & & \\ & \searrow_{b} \quad \Big\downarrow \beta' & \\ & X & \end{array}$$

$\beta \in \mathcal{A}_2$ by definition, and as both b and β' are in $\mathcal{B}_1$, so is β by cancellation. The map $\overline{\mathcal{L}}_1(X) \to \overline{\mathcal{L}}_2(X)$ between the categories of points is then given also by this factorisation. $\qquad \square$

This result will be used in particular when $\mathcal{N}'$ is the Indiscrete factorisation context (cf. §3.1) to defined the structural map of the structural sheaf.

3. EXAMPLES

This part deals with examples of the previous setting. After a short part on the two trivial factorisation systems that always exist on a category, we present how Zariski and Etale topology are associated to unique factorisation systems according to the scheme of the previous section and how the general notion of point and local objects, gives back known classes of objects. The Nisnevich topology is also considered as an illustration and a motivation of the Nisnevich forcing.

Then, we study a sort of dual systems where Zariski closed sets play the role of opens and proper maps that of etale maps. There is also a notion of Nisnevich topology in this context. This material has some flavour of Voevodsky cdh topologies and, again, the general framework gives known classes of objects. Section 3.8 contains some remarks about these two dual settings, but raises more question than it gives answers.

The last section study very rapidly the situation of some other factorisation systems outside of algebraic geometry, such as the $(Epi, Mono)$ system of a topos or an abelian category.

The opposite of any category with a factorisation system is of the same kind but the new factorisation system has no reason to be compatible if its opposite was. For this reason, and because the caracterisation of points is not straightforward, we do not present here a study of the opposite of Zariski of Etale systems (or their duals).

3.1. Extremal examples.
Every category $\mathcal{C}$ admits a two canonical unique factorisation systems $\mathcal{C} = (Iso(\mathcal{C}), \mathcal{C})$ and $\mathcal{C} = (\mathcal{C}, Iso(\mathcal{C}))$ where $Iso(\mathcal{C})$ is the subcategory of isomorphisms. The factorisation of a map is then given by composing with the identity of the source or of the target. These two systems will be called respectively *discrete* and *indiscrete* because they behave like the discrete and indiscrete topologies, being somehow the finest and the coarsest factorisation systems.

We fix a finiteness context $\mathcal{C}^f$ for $\mathcal{C}$.

Discrete factorisation system. $\mathcal{C} = (Iso(\mathcal{C}), \mathcal{C})$ is the discrete factorisation system. Points are objets P splitting every map $U \to P$, their full subcategory in $\mathcal{C}$ is a groupoid. Little can be said in general, beside that they will be points of any factorisation system on $\mathcal{C}$. Little can be said also about covering families or local objects. The only remark is that the small and big toposes agree in this case (and are noted $SPEC_{Dis}(X) = Spec_{Dis}(X)$).

The Nisnevich context $Dis = (\mathcal{C} = (Iso(\mathcal{C}), \mathcal{C}), \mathcal{C}^f, \emptyset)$ is the finest Nisnevich context.

In the case where $\mathcal{C} = CRings^o$ is the opposite category of that of commutative rings, the set of discrete points is empty. Indeed it would correspond to the set of rings A such that any map $A \to B$ has a retraction, *i.e.* an affine scheme such that any scheme over it has a rational point. If B is a quotient of A it has to be isomorphic to A, so A need to be a field, but now no non trivial field extension $A \to K$ has a retraction. This imply that the set of points of any object is empty and so that the empty family will cover any object, collapsing $Spec$ and $SPEC$ to the empty topos.

Indiscrete factorisation system. $\mathcal{C} = (\mathcal{C}, Iso(\mathcal{C}))$ is the indiscrete factorisation system. Every object is a point, $\mathcal{P}t_{(\mathcal{C},Iso(\mathcal{C}))}(X) = \mathcal{C}_{/X}$ and the essentially only point covering family of $X \in \mathcal{C}$ is the identity of X. The Nisnevich context $Ind = (\mathcal{C} = (Iso(\mathcal{C}), \mathcal{C}), \mathcal{C}^f, \emptyset)$ is the coarsest Nisnevich context. The small site of X is reduced to a ponctual category and $Spec_{Ind}(X)$ is the ponctual topos. As for the big topos $SPEC_{Ind}(X) := SPEC_{Iso(\mathcal{C})^f}(X)$ it is the topos of presheaves over $\mathcal{C}^f_{/X}$.

If $X \in Pro(\mathcal{C}^f)$, the structural sheaf $X \colon \mathcal{S} \simeq Spec_{Ind}(X) \longrightarrow \widehat{\mathcal{C}^f}$ is simply X view as a point of $\widehat{\mathcal{C}^f}$ (hence the notation).

Comparisons. For any other Nisnevich context $\mathcal{N}$, proposition 13 gives a diagram

$$
\begin{array}{ccc}
Spec_{\mathcal{N}}(X) & \xrightarrow{O^{\mathcal{N}}_X} & SPEC_{\mathcal{N}}(X) \\
{\scriptstyle r_{Ind}} \downarrow & & \downarrow {\scriptstyle R_{Ind}} \\
\mathcal{S} & \xrightarrow{\quad X \quad} & SPEC_{Ind}(*)
\end{array}
$$

and a natural transformations $\beta_{Ind} \colon O^{\mathcal{N}}_X \to X \circ r_{Ind}$, called the *structural map* of the structure sheaf.

Proposition 14. *For a point* $x \colon \mathcal{S} \to Spec_{\mathcal{N}}(X)$ *corresponding to a local form* $L \to X$, *the map* $\beta_{Ind,x} \colon O^{\mathcal{N}}_{X,x} \to X \circ r_{Ind} \circ x$ *is that map* $L \to X$.

Proof. This is can be deduced from the proof of proposition 13.   □

$X \circ r_{Ind}$ is the constant sheaf on $Spec_{\mathcal{N}}(X)$ with value X and if $x \colon \mathcal{S} \to Spec_{\mathcal{N}}(X)$ is a point corresponding to a local object $L \to X$ over X, the stalk of $X \circ r_{Ind}$ at x is X and the map β evaluated at x gives tautologically the map $L \to X$.

3.2. **Zariski topology.** The category $\mathcal{C}$ is the opposite of that of commutative unital rings, but to simplify the manipulation we are going to work in $\mathcal{C}^o = CRings$. All definitions of points and local objects will have to be opposed, and the role of left and right class of maps are interchanged: the $(Loc, Cons)$ factorisation system that we'll construct on $CRings$ has to be though as $(Cons^o, Loc^o)$ in $CRings^o$. We apologize to the reader for this inconvenience, but we felt that

it was better to develop the general framework with the geometric intuition, as sketched in the introduction, rather than the algebraic one.

3.2.1. *Factorisation system.* A map $A \to B$ in $CRings$ is called a *localisation* if there exists a set $S \in A$ and $B \simeq A[\{x_s, s \in S\}]/(\{sx_s - 1, s \in S\})$. The class of localisation maps is noted Loc. A map $u \colon A \to B$ in $CRings$ is called a *conservative* if any $a \in A$ is invertible iff $u(a)$ is. The class of conservative maps is noted $Cons$.

The following lemma is a reformulation of the definition.

Lemma 3.1. *A map is conservative iff it has the right lifting property with respect to $\mathbb{Z}[x] \longrightarrow \mathbb{Z}[x, x^{-1}]$.*

Proposition 15. *The classes of maps Loc and $Cons$ are the left and right class of a unique factorisation system.*

Proof. For a map $u \colon A \to B$, we define $S := u^{-1}(B^\times)$ and $A[S^{-1}]$ the associated localisation. u factors $A \to A[S^{-1}] \to B$, the first map is a localisation by construction, it remains to prove that $v \colon A[S^{-1}] \to B$ is conservative. Let $a/s \in A[S^{-1}]$ such that $v(a/s) = u(a)u(s)^{-1}$ has an inverse $b \in B$, this is equivalent to the fact that $u(a)$ has an inverse, *i.e.* to $a \in S$. Elements of $A[S^{-1}]$ invertible in B are therefore fractions of elements of S, which are precisely the invertible elements of $A[S^{-1}]$. $\square$

Lemma 3.1 shows that the $(Loc, Cons)$ factorisation system can also be defined as left generated by the single map $\mathbb{Z}[x] \longrightarrow \mathbb{Z}[x, x^{-1}]$. But it happens that the construction of the middle object is quite simple here.

Finiteness context. The finiteness context $\mathcal{C}^f = (CRings^f)^o$ is taken to be the opposite of subcategory of $CRings$ of finitely presented maps. That $\mathcal{C}^f$ satisfies the condition to be a finiteness context is classical and we will only focus on its compatibility with the factorisation system (def. 13). First, any A-algebra is the colimit of its finitely generated subalgebras and this poset if cofiltered, so $_A\backslash CRings = Ind(_A\backslash CRings^f)$. To check that $Loc = Ind - (Loc^f)$ it is enough to remark that the $(Loc, Cons)$ system is generated by a map of finite presentation between rings of finite presentation (cf. remark at the end of §1.3).

The compatibility will be proven when a distinguished class of coverings families will be extracted, this will be the point of lemma 3.3.

We'll use implicitly the following lemma in the sequel.

Lemma 3.2. *A localisation is of finite presentation iff it can be define by inverting a single element.*

Proof. A localisation $A \to A[S^{-1}]$ is always the cofiltered colimit of $A \to A[F^{-1}]$ where F run through all finite subsets of S. Now if $A \to A[S^{-1}]$ is of finite presentation, the identity of $A[S^{-1}]$ factors through one of the $A[F^{-1}]$ and this gives a section s of $r \colon A[F^{-1}] \to A[S^{-1}]$. Now by cancellation both s is an

epimorphism and $srs = s$ implies also $sr = 1$, so r is an isomorphism. Finally if $F = \{f_1, \ldots, f_n\}$, $A[F^{-1}] = A[(f_1 \ldots f_n)^{-1}]$. $\hfill\square$

3.2.2. *Points.* The opposite of the condition for a point gives the following: a ring A corresponds to a point iff for any non zero localisation $\ell \colon A \to A[a^{-1}]$ there exists s a retraction of ℓ.

A *nilpotent extension* of a ring A is a map $B \to A$ such that any element in the kernel is nilpotent.

Proposition 16. *A ring A correspond to a point of the $(Cons^o, Loc^o)$ factorisation system iff it is a nilpotent extensions of a field.*

Proof. As a localisation is zero iff it inverses a nilpotent element of A, the condition of being a point says that any non nilpotent element of A is invertible, so A_{red} is a field. $\hfill\square$

For short we are going to refer to these objects as *fat fields*. Any field is a fat field and the reduction of any fat field is a field. Any fat field is a local ring, the unique maximal ideal being given by the nilradical.

Proposition 17. *The set of points of a ring A is in bijection with the set of prime ideals of A.*

Proof. The set of points of A is defined as a the set of all maps $A \to K$ with K a fat field quotiented by the relation generated by $A \to K \sim A \to K'$ if there exists $K \to K'$ such that $A \to K \to K' = A \to K'$. Any $A \to K$ can be replaced by one where the target is a field ($A \to K_{red}$) and K'' above can always be taken to be a field too. This ensure that instead of fat fields one can use only fields to define the same set. The result is then classical: the kernel of a map to a field is a prime ideal and every prime ideal is the kernel of the map to its residue field. $\hfill\square$

3.2.3. *Covering families.* It should be already clear that our covering families are exactly Zariski covering families, but we'll need the following result to compute local objects.

Proposition 18. *Finite presentation point covers are families $A \to A[a_i^{-1}]$ such that 1 is a linear combinaison of the a_i. As a consequence all point covering families admits a finite point covering subfamily.*

Proof. For K a field, and a given $A \to K$, a_i is either in the kernel of invertible in K, *i.e.* $A \to K$ factors through $A[a_i^{-1}]$ or A/a_i. So $A \to A[a_i^{-1}]$ is a cover iff no non zero $A \to K$ factors through $A/(a_i; i)$ iff $A/(a_i; i) = 0$. For the last equivalence if $A/(a_i; i) \neq 0$ it has at least one residue field giving a map $A \to A/(a_i; i) \to K$ and if such a factorisation $A \to A/(a_i; i) \to K$ exists as $A \to K$ is non zero, $A/(a_i; i)$ has to be non trivial. The conclusion is now deduced from $1 \in (a_i; i) \iff A/(a_i; i) = 0$. $\hfill\square$

3.2.4. *Local objects.* A ring B is a *local ring* iff for any $x, y \in B$ satisfying $x+y = 1$ ($\iff$ $x + y$ invertible), x or y is invertible. This condition can be read as: a A-algebra B is a local ring iff for any $x, y \in B$, the map $A[x, y, (x + y)^{-1}] \to B$ factors through $A[x, x^{-1}]$ or $A[y, y^{-1}]$. Now as 1 is a linear combinaison of x and y in $A[x, y, (x + y)^{-1}]$, the two maps $A[x, y, (x + y)^{-1}] \to A[x, x^{-1}]$ and $A[x, y, (x + y)^{-1}] \to A[y, y^{-1}]$ form a covering family and B and this gives the following lemma.

Lemma 3.3. *A A-algebra B is a local ring iff for any $x, y \in B$ such that $x + y$ is invertible, B lift through the point covering family $A[x, y, (x + y)^{-1}] \to A[x, x^{-1}]$ and $A[x, y, (x + y)^{-1}] \to A[y, y^{-1}]$ of $A[x, y, (x + y)^{-1}]$.*

Proposition 19. *A ring A corresponds to a pointed local object for the $(Cons^o, Loc^o)$ system iff it is a local ring.*

Proof. In a local ring (A, m), elements not in m are invertible so $A \to A/m$ is a conservative map. Conversely, let $u: A \to K$ be a conservative map with target a fat field, and $x, y \in A$ such that $x + y = 1$, then the same equation holds in K and K being a local ring, either $u(x)$ or $u(y)$ is invertible in K. But u being conservative the same is true in A. $\qquad\square$

Proposition 20. *A ring A corresponds to a local object for the $(Cons^o, Loc^o)$ system iff it is a local ring.*

Proof. Any local ring is a local object by prop. 19. Now, let A be a a local object and $x, y \in A$ such that $x + y = 1$. The family $\{A \to A[x^{-1}], A \to A[y^{-1}]\}$ is then a cover by 18 and the existence of a section of this cover says that either x or y is invertible in A. $\qquad\square$

In this setting, the fact that pointed local and local objects coincide is a sophisticated way to say that any local ring has a residue field.

3.2.5. *Spectra and moduli interpretation.* It is clear that the topology given by the general theory coincide with the Zariski topology for affine schemes.

Proposition 21. *For $A \in CRings^o$, $Spec_{Zar}(A)$ is the usual small Zariski spectrum of A and $SPEC_{Zar}(A)$ is the usual big Zariski topos of A.*

For the factorisation context $Zar = (CRings^o = (Cons^o, Loc^o), (CRings^f)^o)$ to be compatible, we need to show that the condition of being a local A-algebra can be tested using only finitely presented covering families, but this is exactly lemma 3.3, so we can apply theorem 2.5.

Proposition 22. *$SPEC_{Zar}(A)$ classifies A-algebras that are local rings, such algebras can have automorphisms so $SPEC_{Zar}(A)$ is not a spatial topos. $Spec_{Zar}(A)$ classifies localisations of A that are local rings.*

The following result is highly classical but not obvious from our definition.

Proposition 23. $Spec_{Zar}(A)$ *is a topological space.*

Proof. The topos $Spec_{Zar}(A)$ is generated by the category $({}_{A\backslash}Loc^f)^o$ which is a poset, so it is localic. This poset is formed of compact objects and we would like to apply the result of [Jo1, II.3.] to deduced the local is coherent and then spatial. To do that we have to check that the topology on $({}_{A\backslash}Loc^f)^o$ is the jointly surjective topology. First, $({}_{A\backslash}Loc^f)^o$ is a distributive lattice: the intersection of $A[a^{-1}]$ and $A[b^{-1}]$ is $A[ab^{-1}]$ and the union is the middle object C of the $(Loc, Cons)$ factorisation of $A \to A[a^{-1}] \oplus A[b^{-1}]$ (indeed C will add to A all elements invertible both in $A[a^{-1}]$ and $A[b^{-1}]$, such C will be some $A[c^{-1}]$); and to prove the distributive law the lemma is the following: if $B \to C \to D$ is a $(Loc, Cons)$ factorisation, for any $b \in B$, $B[b^{-1}] \to C[b^{-1}] \to D[b^{-1}]$ is still a $(Loc, Cons)$ factorisation, *i.e.* $C[b^{-1}] \to D[b^{-1}]$ is still conservative but as new invertible elements in B are fractions of denominator b with invertible numerator, they can be lifted to $C[b^{-1}]$.

As for the topology on $({}_{A\backslash}Loc^f)^o$: for a finite family $a_i \in A$, $c \in A$ is invertible in all the $A[a_i^{-1}]$ iff $(a_i; i) \subset \sqrt{c}$, in particular there is an equivalence $(a_i; i) = A$ iff c is invertible, so $A[a_i^{-1}]$ is a joint covering family iff $(a_i; i) = A$, which is also the characterisation of point covering families. The same reasoning work relatively to any $B \in ({}_{A\backslash}Loc^f)^o$ and this proves that the factorisation topology is the jointly surjective one. $\square$

Also in this case the two notions of points (of the factorisation system and of the spectrum) agree.

Proposition 24. *For $A \in CRings$, the category of points of $Spec_{Zar}(A)$ is a poset equivalent to the opposite of that of prime ideals of A. In particular the set of point of $Spec_{Zar}(A)$ is in bijection with $pt_{Zar}(A)$.*

Proof. We need to prove that this set is in bijection with that of prime ideals of A. This is well known: any prime ideal $p \subset A$ defines a point of $Spec_{Zar}(A)$ by $A \to A_p = A[(A \setminus p)^{-1}]$. And given a localisation of $A \to B$ where B is a local ring, the inverse image of the maximal ideal of B is a prime ideal p of A and $B \simeq A_p$. $\square$

3.2.6. *Remark on a variation.* A class L of maps in a site $(\mathcal{C}, \tau)$ is said to be *local* if, for $u \colon X \to Y$, for any covering $V_i \to Y$ and any covering of $u_{ij} \colon U_{ij} \to V_i \times_Y X$, the map u is in L iff all u_{ij} are in L. Such classes are stable by intersection, so it is always possible to saturate any class L into a local class L^{loc} of maps locally (after pullback) maps in L. If the class L had moreover the property that covering sieves of τ can be generated by families of maps in L, it is clear that covering families in L^{loc} will generate the same topology.

We claim that the class Loc^o is not local for the Zariski topology on $CRings^o$ and its saturation is the class Zet^o of etale maps that are locally trivial for

the Zariski topology (called Zariski etale maps). We claim also that, remarkably, the class Zet is again the left class of a unique factorisation system $(Zet, Conv)$ on $CRings$ where $Conv$ is the class of conservative maps having an extra unique lifting property for idempotents, *i.e.* $(Zet, Conv)$ is left generated by $\mathbb{Z}[x] \to \mathbb{Z}[x, x^{-1}]$ and $\mathbb{Z} \to \mathbb{Z}[x]/(x^2 - x) \simeq \mathbb{Z} \times \mathbb{Z}$. Maps in $Conv$ have connected fiber and are thus proposed to be called *connervative maps*.

We could replace in the previous study the factorisation system $(Loc, Cons)$ by $(Zet, Conv)$ to generate the same factorisation topology and the same spectra, but with different sites. Only the proof of the spatiality of $Spec_{Zar}(X)$ is less straightforward.

3.3. **Etale topology.** The category $\mathcal{C}$ is again $CRings^o$ and we keep the same convention of opposing everything as in the Zariski case.

3.3.1. *Factorisation system.* A map of rings is said *etale* if it flat and unramified [Mi, §3]. The class of etale maps of finite presentation is noted Et^f, that of etale maps between rings of finite presentation is noted Et^f_*. A map of rings is said *henselian* if it has the right lifting property with respect to Et^f_*. The class of henselian maps is noted $Hens$.

Proposition 25. $^{\perp}Hens$ *is the class* $indEt = Ind - Et^f_*$ *and the classes* $indEt$ *and* $Hens$ *are respectively the left and right classes of a unique factorisation system on* $CRings$.

Proof. Et^f_* satisfies hypothesis of proposition 3: the compactness of objects is clear, the stability by cobase change also and the right cancellation comes from the fact that the codiagonal of a finite presentation unramified (and thus etale) map is an open immersion [Mi, prop. 3.5] and lemma 1.2. $\square$

The factorisation of $A \to B$ is not explicit but morally it consists in a separable closure of A relatively to B: one needs to add an element to A for every simple root in B of a polynomial of $A[X]$.

Lemma 3.4.   *(1)* $Hens \subset Cons$ *and* $Loc \subset indEt$.
    (2) Loc^f *point covering families are* Et^f *point covering families.*

Proof. 1. From properties of lifting systems that the two inclusions are equivalent. Any map lifting $u \colon \mathbb{Z}[X] \to \mathbb{Z}[X, X^{-1}]$ is conservative, so as u is etale any henselian morphism in conservative.

2. As $Loc \subset indEt$, points of the $(Hens^o, indEt^o)$ system are points of the $(Cons^o, Loc^o)$ system. $\square$

Let Nil be the class of maps in $CRings$ that are extensions by a nilpotent ideal. The class $Nil^{\perp}$ is the class fEt of *formally etale maps* and if $\overline{Nil} = fEt^{\perp}$, $(fEt, \overline{Nil})$ is a unique lifting system that we are going to compare to $(Hens, indEt)$.

Lemma 3.5. $Nil \subset Hens$ *and* $indEt \subset fEt$.

Proof. indEt is the class of ind-etale maps of finite presentation, now as fEt contains Et^f and is stable by any colimit, $indEt \subset fEt$. □

Proposition 26. *The inclusion $indEt \subset fEt$ is strict.*

Proof. Let A be a noetherian henselian local ring with residue field k and $\widehat{A}$ its completion for its maximal ideal, the residue field of $\widehat{A}$ is still k. As $\widehat{A}$ is also henselian,both maps $A \to k$ and $\widehat{A} \to k$ are henselian and so is $A \to \widehat{A}$ by cancellation. This implies that $A \to \widehat{A}$ is ind-etale iff it is an isomorphism. Now $A \to \widehat{A}$ is always formally smooth but not always an isomorphism. □

Lemma 3.6. *If $CRing^f$ is the category of maps of finite presentation between rings, $indEt \cap CRing^f = Et^f$*

Proof. Clearly $Et^f \subset indEt \cap CRing^f$, and as $indEt \subset fEt$ and $fEt \cap CRing^f = Et$, $indEt \cap CRing^f \subset Et^f$. □

Remark. The unique lifting system $(fEt, \overline{Nil})$ induces another unique factorisation system different from $(indEt, Hens)$ that we won't study here as it is not compatible: $(fEt)^f = Et^f$ but $ind - Et^f = indEt \neq fEt$. Nonetheless, taking all $CRings^o$ as finiteness context, the big spectrum of the $(fEt, \overline{Nil})$ factorisation context should be related to the topos classifying complete local rings with separably closed residue field, and the Nisnevich forcing along fields should relate to the classifying topos of complete local rings.

Finiteness context. The finiteness context $\mathcal{C}^f = (CRings^f)^o$ is still taken to be the opposite of subcategory of $CRings$ of finitely presented maps. The compatibility with ind-etale maps, is clear by construction. The class of distinguished covering family will be extracted in §3.3.4.

3.3.2. *Points.* A ring A corresponds to a point if any map $A \to B \in Et^f$ admits a retraction.

Proposition 27. *A ring A is a point for the $(Hens^o, indEt^o)$ system iff it is a nilpotent extension of a separably closed field.*

Proof. It is sufficient to prove that A_{red} is a separably closed field. First, A_{red} is a field from the fact that a localisation $A_{red} \to A_{red}[a^{-1}]$ is an etale map, so any non zero element of A_{red} has to be invertible. Then a field is separably closed if, embedded in an algebraic closure, it contains all elements which minimal polynomial has simple roots. Any such polynomial P being irreducible, it defines a normal extension N of A_{red} containing all roots of P; the map $A_{red} \to N$ is etale and the lifting property of A gives a retraction, ensuring that all roots of P are in A_{red}.

Reciprocally, if A_{red} is a separably closed field, it is in particular en henselian local ring (§37). Now for a henselian local ring (B, m) with residue field $B/m = k$, an etale extension $B \to C$ has a retraction iff there exists a maximal ideal n of

C sent to m which residue field is also k [Mi, thm 4.2]. As $A_{red} = B = k$ in our case, for any $A_{red} \to C$ etale, a maximal ideal n over m always exist and as k is separably closed the residue field at n has to be k, so a retraction exists. □

Proposition 28. *The set of points of a ring A is in bijection with that of prime ideals of A.*

Proof. Lemma 3.4 implies $pt_{Et}(A) \subset pt_{Et}(A)$. Now the same reasoning as in prop. 16 proves that separably closed fields are enough to compute points, and the inverse inclusion is then a consequence of the fact that any field has a separable closure. □

3.3.3. *Covering families and local objects.*

Proposition 29. *Point covering families of Et are ordinary etale covers.*

Proof. By lemma 3.6 $Et^f = indEt \cap (\mathcal{C}^f)^o$. Then, by prop. 28, a family of $A \to A_i$ of finitely presented etale maps is a cover iff it induces a surjective family on the set of prime ideals, which is the ordinary definition. □

A local ring (A, m) is called *henselian* ([Mi, thm. 4.2.d]) if any etale map $A \to B$ such that there exists a maximal ideal n of B lifting m with the same residue field has a section.

Proposition 30. *A local ring (A, m) is henselian iff $A \to A/m$ is an henselian map.*

Proof. Etale maps being stable by pushout, it is sufficient to prove the lifting property of $A \to A/m$ for squares

$$\begin{array}{ccc} A & =\!\!=\!\!= & A \\ \downarrow & & \downarrow \\ B & \longrightarrow & A/m \end{array}$$

where $A \to B$ is etale. As $A \to B$ is etale, $B \otimes_A A/m$ is separable extension of k, sum of the residue fields of maximal ideals of B over m. If k is one of these fields, k is an extension of A/m and the map $B \to A/m$ gives a map $k \to B \otimes_A A/m \to A/m$ so in fact $k \simeq A/m$. So any $A \to B$ entering such square is of the kind of extension used in the definition of a henselian ring. And reciprocally any such extension define a square like above. Hence the equivalence. □

A henselian local ring (A, m) is called *strictly henselian* if moreover A/m is a separably closed field.

Proposition 31. *A ring A is a pointed local object for the $(Hens^o, indEt^o)$ system iff it is a strictly henselian local ring.*

Proof. A point K is a nilpotent extension of a separably closed field, so by lemma 3.5 $K \to K_{red}$ is a henselian map. Therefore a map $A \to K$ is henselian iff $A \to K_{red}$ is (the necessary condition uses the cancellation property). So a ring A is pointed local iff there exists a henselian map $A \to K$ with K a separably closed field. As henselian maps are conservative, prop. 19 tells us that A is a local ring. Then, if m is the maximum ideal of A, $A \to K$ factors through $(A/m)^{sep}$, the separable closure of A/m in K. Now, by construction, $(A/m)^{sep} \to K$ is henselian and the cancellation property says that so is $A \to (A/m)^{sep}$. $\qquad\square$

Proposition 32. *A ring A is a local object for the $(Hens^o, indEt^o)$ system iff it is a strictly henselian local ring.*

Proof. Local objects correspond to rings A such that any etale cover $\{A \to A_i\}$ as a retraction of one of the $A \to A_i$. As etale covers contain Zariski covers, A is local by prop. 20.

Now we are going to prove that $A \to k$ (k residue field of A) is a henselian map. Let $A \to B$ be an etale map lifting the residue field k, we need to show that it admits a section (necessary unique). To prove this we consider an affine Zariski cover $\{A \to A_i, i\}$ of the complement of the closed point of A, the family $\{A \to B\} \cup \{A \to A_i; i\}$ is an etale cover (if fact even a Nisnevich cover, this will be useful to prove prop. 37). So there exists a map of this family admitting a retraction, and because all $A \to A_i$ are strict open embeddings it can only be $A \to B$. It remains to prove that k is separably closed. We are going to prove that any separable (*i.e.* etale) extension $k \to k'$ admits a retraction. A being henselian there is a bijection between finite etale A-algebras and finite etale k-algebras, so k' defines an etale A-algebra A' which is an etale covering family (or can be completed as such in the same way as before), and so admit a retraction from A, proving the same for $k \to k'$. $\qquad\square$

3.3.4. *Distinguished covering families.* In order to apply theorem 2.5 we need to show that the condition of being a strict henselian ring can be tested using only finitely presented point covering families.

A point covering family $\{B \to B_i, i\}$ of an A-algebra B is said *distinguished* if every $B \to B_i$ is of finite presentation over A and if it satisfies one of the following two conditions

a. it is a Zariski covering family,
b. or it consists of single etale map (such map will be called an *etale covering map*).

Lemma 3.7. *Any finitely presented etale map $B \to C$ between finitely presented A-algebras can be factored into a finitely presented localisation followed by a finitely presented etale covering map $B \to D \to C$.*

Proof. The etale map $B \to C$ defines a degree function which associate to each point p of B the dimension of $C \otimes_B \kappa(p)$ as a $\kappa(p)$-vector space. This dimension

is finite because the map is finitely presented and it is a lower semi-continuous function [EGA4-4, 18.2.8]. The level set of value zero is a finitely presented closed Zariski subset whose complement is a localisation D' of B. The natural map $C \to C \otimes_B D'$ is an isomorphism of B-algebras as it can be checked at every prime ideal of B, this gives a factorisation $B \to D' \to C$ of $B \to C$. We use the $(Loc, Cons)$ factorisation on $B \to C$ to obtain a localisation D of B. As $B \to D'$ is another intermediate localisation, the universal property of D gives a localisation $D' \to D$. Geometrically the Zariski spectrum of D' contains that of D, which means that every prime ideal of D has a non empty fiber over it. Reciprocally, if K is a separably closed field and if $B \to C \to K$ is a point of B factoring through C, it gives a map $B \to D \to K$ whose first map is a localisation, so D has a map to the middle object A_p of the $(Loc, Cons)$ factorisation of $B \to K$. This means that any prime ideal that has a non empty fiber is in D, and so $D = D'$. Finally, the map $D \to C$ is ind-etale and of finite presentation by cancellation. $\qquad\square$

Proposition 33. *A A-algebra B is a strictly henselian local ring iff it lifts through any distinguished covering families.*

Proof. The necessary condition is obvious by characterisation of local objects as strictly henselian rings. Reciprocally, the lifting condition with respect to finitely presented Zariski covering families says that B is a local ring (lemma 3.3). If m is the maximal ideal of B and $\kappa(m)^{sep}$ some separable closure of its residue field, we are going to prove that the map $B \to \kappa(m)^{sep}$ is henselian. It has to have the left lifting property with respect to finitely presented etale maps $C \to D$ between finitely presented A-algebras, we are going to transform this problem into a lifting through an etale covering map. We can complete the lifting diagram as
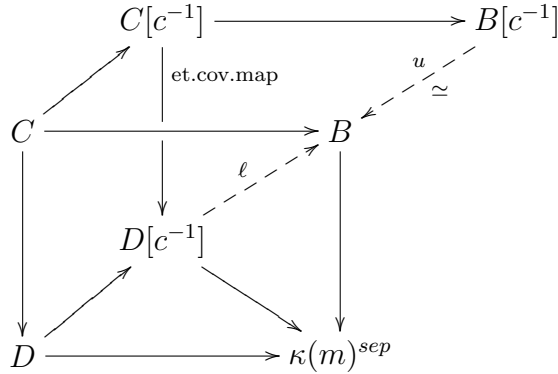
where $C \to C[c^{-1}] \to D$ is the localisation of lemma 3.7. The map u exists and is an isomorphism because $B[c^{-1}]$ is a localisation of B still containing the maximal ideal. Now the lift ℓ exists by property of B. $\qquad\square$

3.3.5. *Spectra and moduli interpretation.* The factorisation context will be called $Et = (CRings^o = (indEt, Hens), (CRing^f)^o)$, the results of §3.3.1 and §3.3.4

says that it is compatible. Prop. 29 ensures that the topology given by the general theory coincide with the usual etale topology for affine schemes.

Proposition 34. *For $A \in CRings^o$, $Spec_{Et}(A)$ is the usual etale spectrum (small etale topos) of A and $SPEC_{Et}(A)$ is the usual big etale topos of A.*

As for the moduli interpretation of etale spectra, theorem 2.5 gives us something well known.

Proposition 35. *$SPEC_{Et}(A)$ classifies A-algebras that are strictly henselian local rings and $Spec_{Et}(A)$ classifies ind-etale A-algebras that are strictly henselian local rings. In particular those A-algebras can have automorphisms and neither of $SPEC_{Et}(A)$ or $Spec_{Et}(A)$ is a spatial topos.*

Again in this case, the two notions of points agree.

Proposition 36. *For $A \in CRings$, the set of points of $Spec_{Et}(A)$ is in bijection with $pt_{Et}(A)$.*

Proof. We need to construct a bijection between the set of points of $Spec_{Et}(A)$ and the set of prime ideals of A. First, for p a prime ideal, we have the map $A \to A_p \to \kappa(p) \to \kappa(p)^{sep}$ where $\kappa(p)^{sep}$ is a separable closure of $\kappa(p)$. If $A \to A_p^{sh} \to \kappa(p)^{sep}$ is the $(indEt, Hens)$ factorisation of the previous map, A_p^{sh} is a strictly henselian local ring (as a pointed local object) called a *strict henselisation of A at p* (it depends up to a unique iso of the choice of $\kappa(p)^{sep}$). To prove that p can be recover from $A \to A_p^{sh}$ we are going to show that the composition $A \to A_p \to A_p^{sh}$ is the $(Loc, Cons)$ factorisation of $A \to A_p^{sh}$, so A_p (and then p) will be uniquely determine by A_p^{sh}. We only need to prove that $h \colon A_p \to A_p^{sh}$ is conservative: in the square

$$
\begin{array}{ccc}
A_p & \xrightarrow{\ Cons\ } & \kappa(p) \\
{\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \iota} \\
A_p^{sh} & \xrightarrow[Hens \subset Cons]{} & \kappa(p)^{sep}
\end{array}
$$

the map ι is conservative (as any map between fields is a monomorphism) then h is conservative by cancellation. All this creates an injective map from the set of prime ideals of A to that of points of $Spec_{Et}(A)$. We prove now that this map is surjective. If B is a stricly henselian local ring with residue field K separably closed, and $A \to B$ an ind-etale map, the $(Loc, Cons)$-factorisation of $A \to B$ give a local ring A_p. The map $A_p \to K$ factors through some separable closure of $\kappa(p)$ and defines a strict henselisation A_p^{sh} of A at p. With the above notations,

we have the diagram

$$
\begin{array}{ccc}
A \xrightarrow{\ indEt\ } B \xrightarrow{\ Hens\ } K \\
\end{array}
$$

Then the map $A \to B \to K$ admits another $(indEt, Hens)$ factorisation $A \to A_p^{sh} \to K$ so $B \simeq A_p'$. $\square$

This proof gives the following construction of the ind-etale henselian local A-algebra at a prime $p \subset A$: it is the middle object A_p^{sh} of the $(indEt, Hens)$ factorisation of the map $A \to \kappa(p)^{sep}$ where $\kappa(p)^{sep}$ is a separable closure of the residue field at p.

3.3.6. *Remark.* The two factorisation systems $(Loc, Cons)$ and $(indEt, Hens)$ are related by the inclusion $Loc \subset indEt$. For a map $A \to B$, this constructs in fact a triple factorisation system

$$
A \xrightarrow{\ Loc\ } C \xrightarrow{\ indEt\ \&\ Cons\ } D \xrightarrow{\ Hens\ } B
$$

where $A \to C \to B$ is the $(Loc, Cons)$ factorisation and $A \to D \to B$ the $(indEt, Hens)$ factorisation. As shown in lemma 3.7, the map $A \to C$ is the "open support" of the etale map $A \to D$ and the map $C \longrightarrow D$ is an etale covering.

This triple factorisation will be inspire the construction of the $(IntSurj, IntClo)$ factorisation system in §3.6.

3.4. **Nisnevich topology.** The Nisnevich topology on $CRings^o$ is not associated to a factorisation system, but will be constructed from the etale factorisation system by Nisnevich forcing (§2.5), more precisely by forcing fields, to be local objects. The setting is the same as in §3.3.

An etale point covering family $A \to A_i$ is a *Nisnevich covering family* if for any field K and any map $A \to K$

$$
\begin{array}{ccc}
& A_i & \\
& \uparrow \quad \diagdown\ ^{\exists i} & \\
& & \searrow \\
A & \longrightarrow & K\,.
\end{array}
$$

This is equivalent to the condition that the pull-back (in $CRings^o$) of $A \to A_i$ to any field admit a global section. If $\mathcal{F}$ is the subcategory of $CRings$ generated by fields, $(CRings^o = (Hens^o, indEt^o), \mathcal{F})$ is a Nisnevich context.

The following lemma is a consequence of lemma 3.4 and of the definition of Nisnevich covering families.

Lemma 3.8. *Zariski point covering families are Nisnevich covering families.*

3.4.1. *Local objects.*

Proposition 37. *A ring is a Nisnevich local objects iff it is a henselian local ring.*

Proof. Let A be a local object. Zariski covering families are Nisnevich covering families so prop. 20 shows that A is a local ring. Let k be the residue field of A, we need to prove that $A \to k$ is a henselian map. The argument is the one use in the proof of prop. 32. □

Let $f\mathcal{F}$ be the category of fat fields, *i.e.* nilpotent extension of fields (§3.2.2).

Corollary 3. $(CRings^o = (Hens^o, indEt^o), \mathcal{F})$ *and* $(CRings^o = (Hens^o, indEt^o), f\mathcal{F})$ *are two equivalent Nisnevich contexts.*

Proof. As $\mathcal{F} \subset f\mathcal{F}$, localising by $f\mathcal{F}$ selects less covering families so more local objects: $\overline{\mathcal{F}} \subset \overline{f\mathcal{F}}$. The reciprocal inclusion is equivalent to fat fields being henselian rings, *i.e.* that the map $K \to K_{red}$ is henselian. This is a consequence of lemma 3.5. □

This corollary is interesting as $f\mathcal{F}$ is exactly the category of points of the $(Loc, Cons)$ factorisation system (§3.2.2), which is a way to say that this Nisnevich localisation is not arbitrary (see §3.8).

3.4.2. *Distinguished covering families.* The finiteness context $\mathcal{C}^f = (CRings^f)^o$ is still taken to be the opposite of subcategory of $CRings$ of finitely presented maps. Compatibility conditions have been checked in §3.3.4, we need only to extract a class of distinguished coverings sufficient to detect henselian rings.

A Nisnevich point covering family $\{B \to B_i, i\}$ of an A-algebra B is said *distinguished* if it is of finite presentation over A (*i.e.* there exist $A \to B' \to B$ where $A \to B'$ is of finite presentation and all $B \to B_i$ are pushout of some $B' \to B_i'$) and satisfy one of the following two conditions

a. it is a Zariski covering family,

b. or there exist a radical ideal I of B' such that $A \to A/I$ factors through one of the $B' \to B_i'$ and the others $B \to B_i$ are localisations of B' covering the complement of I. In particular, this implies that the B_i' factoring $B' \to B'/I$ is unique.

Geometrically (for the Zariski topology), this last condition says that the covering family is distinguished if it covers the complement of a finitely presented closed set Z by Zariski opens and has another etale map covering Z that moreover has a section over Z.

Proposition 38. *A A-algebra B is a henselian local ring iff it lifts through any distinguished Nisnevich covering families.*

Proof. We need to prove only the sufficient part. Lifting through finitely presented Zariski covering families says that B is a local ring (lemma. 3.3), we need

then to show that, if m is the maximal ideal of B and $\kappa(m)$ its residue field, the map $B \to \kappa(m)$ is henselian. This is true if it has the left lifting property with respect finitely presented etale maps $C \to D$ between finitely presented A-algebras, we can use the same trick as in prop. 33 and replace $C \to D$ by an etale covering map. We are now going to transform $C \to D$ into a distinguished Nisnevich covering of the second kind. The Zariski closed set involved will be the closure $\overline{p}$ of the image p of the ideal m by $C \to B$, but we need to show that $C \to D$ has a section over it. The finitely presented etale map $\kappa(p) \to D \otimes_C \kappa(p)$ has a section which furnishes an idempotent of $D \otimes_C \kappa(p)$ [Mi, cor. 3.12], this idempotent can be lifted as some element $d \in D$ and the composition $C \to D[d^{-1}]$ is still finitely presented etale covering map but is now of degree exactly one over p. The set Z of prime ideals of C over for which $C \to D$is of degree exactly 1 is a closed Zariski subset, over which $C \to D$ is even an isomorphism. Then, the wanted section exists as Z contains on $\overline{p}$. Completing $C \to D$ by a Zariski covering of the complement of $\overline{p}$, and pushing forward to B, there exists a retraction of one of the covering maps and it can be only of $B \to D \otimes_C B$ as all other maps misses m in their image by construction. $\qquad\square$

3.4.3. *Spectra and moduli interpretation.* Let $\mathcal{F}$ be the full subcategory of $\mathcal{C} = CRings^o$ generated by fields. $Nis := (\mathcal{C} = (Hens^o, indEt^o), \mathcal{C}^f, \mathcal{F})$ is a Nisnevich context (def. 11) and prop. 37 says that $\overline{\mathcal{F}}$ is the category of henselian rings.

§3.4.2 finishes the proof of the compatibility of this context, we can use theorem 2.5 to compute the points of our spectra.

Proposition 39. *For $A \in CRings^o$, $SPEC_{Nis}(A)$ classifies A-algebras that are henselian local rings and $Spec_{Nis}(A)$ classifies ind-etale A-algebras that are henselian local rings. In particular those A-algebras can have automorphisms and neither of $SPEC_{Nis}(A)$ or $Spec_{Nis}(A)$ is a spatial topos.*

To any prime ideal p of A is associated two points of $Spec_{Nis}(A)$: first, $Spec_{Et}(A)$ being a subtopos of $Spec_{Nis}(A)$, the strict henselisation of A at p is also a point of $Spec_{Nis}(A)$; the second one is the *henselisation* of A at p: it is the middle object A_p^h of the $(indEt, Hens)$ factorisation of the map $A \to \kappa(p)$ where $\kappa(p)$ is the residue field at p.

3.4.4. *Context Comparisons.* We have three Nisnevich contexts $Zar = (CRings^o = (Cons^o, Loc^o), (CRings^f)^o, \emptyset)$, $Et = (CRings^o = (Hens^o, indEt^o), (CRings^f)^o, \emptyset)$, and $Nis = (CRings^o = (Hens^o, indEt^o), (CRings^f)^o, \mathcal{F})$. Et is clearly a refinement of Nis and of Zar and as objects of $\mathcal{F}$ are local for Zar, Nis is also a refinement of Zar. This give the following diagram

$$\begin{array}{ccccccc}
Spec_{Et}(X) & \longrightarrow & Spec_{Nis}(X) & \longrightarrow & Spec_{Zar}(X) & \longrightarrow & Spec_{Ind}(X) \\
\downarrow{\scriptstyle s_X} & & \downarrow{\scriptstyle s_X} & & \downarrow{\scriptstyle s_X} & & \downarrow{\scriptstyle s_X} \\
SPEC_{Et}(X) & \longrightarrow & SPEC_{Nis}(X) & \longrightarrow & SPEC_{Zar}(X) & \longrightarrow & SPEC_{Ind}(X)
\end{array}$$

and associated natural transformations of structural sheaves $\mathcal{O}_X^{Et} \rightarrow \mathcal{O}_X^{Nis} \rightarrow \mathcal{O}_X^{Zar} \rightarrow X$ (pulled-back on $Spec_{Et}(X)$. The bottom row of the diagram consists in inclusions of subtoposes, and reads at the level of points: strict henselian local rings are henselian local rings which are local rings which are rings.

The example of a field. The Etale topos is the classifying topos of the galois group of k, its category of points is the groupoid of separable closure of k. The Zariski topos of a field k is a point, but the Nisnevich topos of a field is not, its category of points is the opposite of that of algebraic extensions of k. (As k is henselian, an ind-etale k-algebra A is a product of local k-algebras A_i and if A is henselian so are the A_i. If k_i is the residue field of A_i, as both maps $k \rightarrow k_i$ and $k \rightarrow A_i$ are ind-etale so is $A_i \rightarrow k_i$. It is then an isomorphism if A_i is henselian.) This category has an terminal object (k itself) and geometrically, the Nisnevich spectrum can be thought as a sort of cone interpolating between k and the groupoid of its separable closures. Homotopically, unless the etale spectrum, it will be contractible.

3.5. **Domain topology.** We are now going to investigate the obvious $(Surj, Mono)$ factorisation system on $CRings$ with the same convention as before, *i.e.* thinking of the opposite factorisation system $(Mono^o, Surj^o)$ on $CRings^o$. Let $u\colon A \rightarrow B \in CRings$ with kernel I, the $(Surj, Mono)$ factorisation of u is $A \rightarrow A/I \rightarrow B$. A map $A \rightarrow A/I$ is called a *surjection* or a *quotient* and a map $A \rightarrow B$ with 0 kernel is called a *monomorphism*.

The following lemma gives a set of left generators.

Lemma 3.9. *A map is a monomorphism iff it has the right lifting property with respect to* $\mathbb{Z}[x] \longrightarrow \mathbb{Z}\colon x \mapsto 0$.

It is interesting to remark that this map $\mathbb{Z}[x] \longrightarrow \mathbb{Z}$ is the "complement" of the generator $\mathbb{Z}[x] \longrightarrow \mathbb{Z}[x, x^{-1}]$ of the $(Loc, Cons)$ system. This simple fact seems to be the source of an unclear duality between the $(Surj, Mono)$ and $(Loc, Cons)$ systems (cf. §3.8).

3.5.1. *Finiteness context and points.* The finiteness context $\mathcal{C}^f = (CRings^f)^o$ is still taken to be the opposite of subcategory of $CRings$ of finitely presented maps. The fact that every surjection $A \rightarrow B$, of kernel I, is a ind-object in finitely presented surjection can be seen by writing B as the limit of the filtered diagram of quotients of A by a finite number of elements of I. Distinguished families will be extracted from lemma 3.10.

Proposition 40. *A ring corresponds to a points of the* $(Mono^o, Surj^o)$ *factorisation system iff it is a field.*

Proof. A ring A corresponds to a point if any quotient $A \rightarrow A/I$ by a finitely presented ideal admits a retraction. But this forces q to be a monomorphism and then an isomorphism. An element $a \in A$ is either zero, invertible or non-zero

and non invertible. In the first case the quotient by a is A, in the second 0 and in the third something non isomorphic to A. This third case is excluded by the previous remark, so every element in A as to be either zero or invertible. $\square$

The same classical argument as in prop. 17 gives the following.

Proposition 41. *The set of point of a ring A for the $(Mono^o, Surj^o)$ system is that of prime ideals of A.*

3.5.2. *Covering families and local objects.* Point covering families of the $(Mono^o, Surj^o)$ system are families of quotients $A \to A/I_i$ by finitely generated ideals such that any residue field of A factors through one of the A/I. Using the geometric intuition coming from the Zariski topology, this correspond to cover a scheme by non reduced closed subschemes of finite codimension.

A ring B is an integral domain iff for any $x, y \in B$, $xy = 0$ iff $x = 0$ or $y = 0$. If B is an A-algebra this can be read as, for any $x, y \in B$, the map $A[x,y]/(xy) \to B$ factors through $A[x,y]/(xy) \to A[y]$ or $A[x,y]/(xy) \to A[x]$. Those two maps form a covering family of $A[x,y]/(xy)$: for any map $A[x,y]/(xy) \to K$ to some field, either x or y has to be zero in K.

This proves the following lemma dual to lemma 3.3.

Lemma 3.10. *A A-algebra B is a integral domain iff for any $x, y \in B$ such that $xy = 0$ is invertible, B lift through the point covering family $A[x,y]/(xy) \to A[x]$ and $A[x,y]/(xy) \to A[y]$ of $A[x,y]/(xy)$.*

The following results justify the name chosen for this topology.

Proposition 42. *A ring is a pointed local object of the $(Mono^o, Surj^o)$ system iff it is an integral domain.*

Proof. If $A \to K$ is a monomorphism with target a field, then A is an integral domain, and reciprocally for any such ring is associated a monomorphism $A \to K(A)$ into the fraction field. $\square$

Proposition 43. *A ring is a local object of the $(Mono^o, Surj^o)$ system iff it is an integral domain.*

Proof. Let A be a domain and $\{A \to A/I_i\}$ a cover, then in order to cover the generic point of A it must contain a copy of A itself. Reciprocally, if A is a ring such that any cover $\{A \to A/I_i\}$ has a retraction, the family of inclusions of irreducible components, *i.e.* $A \to A/p_i$ where p_i's are minimal prime ideals, defines a point covering family of A and then must have a retraction. So 0 is one (and the only) of the primes p_i. $\square$

3.5.3. *Spectra and moduli interpretation.* For the factorisation context $Dom = (CRings^o = (Mono^o, Surj^o), (CRing^f)^o)$ to be compatible, we need to show that the condition of being an integral domain can be tested using only finitely presented point covering families, but this is lemma 3.10. So we can apply theorem 2.5.

Proposition 44. *For $A \in CRings$, points of $SPEC_{Dom}(A)$ are A-algebras that are integral domains and points of $Spec_{Dom}(A)$ are quotients of A that are domains.*

Proposition 45. *For $A \in CRings$, the set of points of $Spec_{Dom}(A)$ is in bijection with $pt_{Dom}(A)$.*

Proof. This a way to say that the set of points of $Spec_{Dom}(A)$ is that of prime ideals of A: it is well known that a quotient of A that is an integral domain iff the kernel is a prime ideal. □

We are now going to prove that $Spec_{Dom}(A)$ is a topological space. We would like to apply the same argument as in prop. 23 but the equivalence between the jointly surjective topology and the point covering topology will fail without a slight modification of the site defining $Spec_{Dom}(A)$ (cf. end of proof of prop. 47).

For a ring A, the subset $\sqrt{0}$ of all nilpotent elements is also the intersection of all prime ideal of A.

Lemma 3.11. *For a ring A, $A \to A/\sqrt{0}$ is point covering family and any sheaf for the factorisation topology send such a map to an isomorphism.*

Proof. Any field K is an integral domain so any $A \to K$ factors through $A/\sqrt{0}$. For the second part, any sheaf as to send $A \to A/\sqrt{0}$ to the kernel of $A/\sqrt{0} \rightrightarrows A/\sqrt{0} \otimes_A A/\sqrt{0}$, but as $A/\sqrt{0} \otimes_A A/\sqrt{0} = A/\sqrt{0}$ this kernel is the identity of $A/\sqrt{0}$. □

Corollary 4. *The domain topology is not subcanonical.*

Proof. Both A and A_{red} will have the same spectra, this will be developed further below. □

Lemma 3.12. *A family $B \to B/I_i$ in $_{A\backslash}Surj^f$ corresponds to a point covering family iff $B \to B/(\cap I_i)$ is a point covering family iff $\cap I_i \subset \sqrt{0}$.*

Proof. $B \to B/(\cap I_i)$ factors every $B \to B/I_i$, so it has the joint of the lifting properties of all $B \to B/I_i$ and so is a point covering family. Reciprocally, if $I_i = (a_i^1, \ldots a_i^{k_i})$, $\cap I_i$ is generated by products $\prod_i a_i^{k(i)}$ for some function $i \mapsto 1 \le k(i) \le k_i$, we want to prove that for any point $A \to K$ factoring through $A \to A/(\cap I_i)$, there exists an i such that all a_i^k are send to zero in K. If this is not the case, for all i there would exist a $a_i^{k(i)}$ not sent to zero in K, and so their product will not either, contradicting the fact that $A \to K$ factors through $A \to A/(\cap I_i)$.

As for the second equivalence, if p is a prime ideal of B with residue field $\kappa(p)$, the existence of a lift $B/(\cap I_i) \to \kappa(p)$ of $B \to \kappa(p)$ proves that p has not become the zero ideal in $B/(\cap I_i)$ so $\cap I_i \subset p$. This says that $(\cap I_i)$ is contained in every prime ideal of B. □

$Spec_{Dom}(A)$ is the topos associated to $(_{A\backslash}Surj^f)^o$ with the factorisation topology, it depends only on A_{red}. If $_{A\backslash}RedSurj^f$ is the sub-category of $_{A\backslash}Surj^f$ formed of reduced finitely presented quotients of A, the factorisation topology restrict to it. The inclusion $\iota : {}_{A\backslash}RedSurj^f \subset_{A\backslash} Surj^f$ has a left adjoint red given by $A \to A_{red} = A/\sqrt{0}$ which if continuous (the reduction of a covering family is still a covering family).

Lemma 3.13. *A family $B \to B/sqrtI_i$ in $(_{A\backslash}RedSurj^f)^o$ is point covering family iff $\cap\sqrt{I_i} = \sqrt{0}$.*

Proof. This is a consequence of lemma 3.12 and of $\cap\sqrt{I_i} = \sqrt{\cap I_i}$. □

Proposition 46. *The continuous functor $red \colon (_{A\backslash}Surj^f)^o \longrightarrow (_{A\backslash}RedSurj^f)^o$ is an equivalence of sites.*

Proof. Recall that a continuous functor is an equivalence of sites if the geometric map (red^*, red_*) induced on the toposes is a equivalence. We have a diagram

$$
\begin{array}{ccc}
(_{A\backslash}\widehat{RedSurj}^f)^o & \overset{red^*}{\underset{red_*}{\rightleftarrows}} & (_{A\backslash}\widehat{Surj}^f)^o \\[2mm]
a \big\Updownarrow & & a \big\Updownarrow \\[2mm]
(_{A\backslash}\widetilde{RedSurj}^f)^o & \overset{red^*}{\underset{red_*}{\rightleftarrows}} & (_{A\backslash}\widetilde{Surj}^f)^o
\end{array}
$$

where the a's are the sheafification functors. We have to prove that a presheaf on $(_{A\backslash}Surj^f)^o$ is a sheaf iff its restriction to $(_{A\backslash}RedSurj^f)^o$ is a sheaf. It is enough to check it on the level of generators where $red_* = \iota^*$. The unit and counit of (red^*, red_*) are those of (red, ι): the counit is always an isomorphism and lemma 3.11 prove that the unit of (red, ι) is transform in an isomorphism by sheafification. □

Proposition 47. *$Spec_{Dom}(A)$ is a topological space whose poset of points is equivalent to that of prime ideal of A.*

Proof. We are going to apply the same argument as in prop. 23. $Spec_{Dom}(A)$ is generated by the category $(_{A\backslash}RedSurj^f)^o$ which is a poset of compact object, so it is a localic topos. [Jo1, II.3.] will say it is coherent and spatial as soon as the topology on $(_{A\backslash}Surj^f)^o$ is the jointly surjective topology. $(_{A\backslash}Surj^f)^o$ is a distributive lattice: the intersection of $A/\sqrt{I}$ and $A/\sqrt{J}$ is $A/\sqrt{I+J}$ and the union is $A/\sqrt{I \cap J}$; the distributivity law is the lemma: for I, J, K three finitely generated ideals of A, $K + (I \cap J) = K \cap I + K \cap J$. As for the topology, a family $A \to A/\sqrt{I_i}$ is jointly surjective iff $\sqrt{\cap I_i} = \sqrt{0}$ but this is the characterisation of point covering families of lemma 3.13. (This last equivalence is in fact the whole reason of considering the site $(_{A\backslash}RedSurj^f)^o$.) □

The poset of points of $Spec_{Dom}(A)$ is the opposite of that of $Spec_{Zar}(A)$, in particular generic points of one are closed point of the other. We can think

those two space as as "opposite" as categories can be opposed. In fact the two sites $_{A\backslash}RedSurj^f$ and $_{A\backslash}Loc^f$ are opposite categories and this duality between $Spec_{Dom}(A)$ and $Spec_{Zar}(A)$ is part of a general duality on compactly generated spaces exposed in [Jo1] ($Spec_{Dom}(A)$ is the domain spectrum of [Jo1, V.3.11]).

3.5.4. *Remark.* The same remark as in §3.2.6 is true: the class $Surj^o$ is not local for the Domain topology on $CRings^o$. Its saturation is the class $EtSurj^o$ opposite to that of integrally closed maps (cf. §3.6) that are locally trivial for the Domain topology (called *etale-surjective* maps). Again, we claim that $EtSurj$ is the left class of a unique factorisation system $(EtSurj, MIdem)$ on $CRings$ where $MIdem$ is the class of monomorphisms having the extra unique lifting property for idempotents, *i.e.* $(EtSurj, MIdem)$ is left generated by $\mathbb{Z}[x] \to \mathbb{Z}$ and $\mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$. Replacing the factorisation system $(Loc, Cons)$ by $(EtSurj, MIdem)$ in the previous study would generate the same factorisation topology and the same spectra.

3.6. **Proper topology.** For a inclusion of rings $A \subset B$ an element $b \in B$ is said *integral over* A if there exists a monic polynomial P with coefficients in A such that b is a root of P. In particular every element of A is integral. More generally for any map $A \to B$ of kernel I, an element of B is said integral over A if it is integral over A/I. As any monic polynomial of $(A/I)[X]$ can be lifted in a monic polynomial of $A[X]$, it is equivalent to say that $b \in B$ is integral over A if it exists $P \in A[X]$ monic such that $P(b) = 0$. $A \subset B$ is said *integrally closed* if any element integral over A is in A. The set of integrally closed monomorphism of rings is noted $IntClo$. The following proposition is [Mat, thm. 9.1].

Proposition 48. *For any monomorphism of rings $A \subset B$, the subset C of elements integral over A in B is a ring, and $C \subset B$ is integrally closed.*

This constructs a factorisation system on monomorphisms of rings, with the right class being $IntClo$. To have a factorisation for every morphism, we use the $(Surj, Mono)$ factorisation. A map $A \to B$ of kernel I is called *integrally surjective* if every element of B is integral over A. The set of integrally surjective maps is note $IntSurj$. The archetypal example of a integrally surjective map is a integral extension $A \to (A/I)[x]/P(x)$ for some ideal I and some monic polynomial P.

Proposition 49. *$IntSurj$ and $IntClo$ are the left and right classes of a unique factorisation system.*

As $IntClo \subset Mono$ and $Surj \subset IntSurj$, the $(IntSurj, IntClo)$ factorisation system compares to the $(Surj, Mono)$ as $(Loc, Cons)$ and $(indEt, Hens)$ compared in §3.3.6: they define a triple factorisation system

$$A \xrightarrow{Surj} C \xrightarrow{Mono \ \& \ IntSurj} D \xrightarrow{IntClo} B$$

where $A \to C \to B$ is the $(Surj, Mono)$ factorisation and $A \to D \to B$ the $(IntSurj, IntClo)$ factorisation.

Proposition 50. *The $(IntSurj, IntClo)$ factorisation system is left generated by the set of maps $A \to (A/I)[x]/P(x)$ where A is of finite presentation, I some finitely generated ideal of A and P a monic polynomial.*

Proof. First, it is clear by definition that such a map $A \to (A/I)[x]/P(x)$ is in $IntSurj$. Then, as a factorisation system is entirely determine by one of the left or right classes, it is sufficient to prove that the class of maps right orthogonal to $A \to (A/I)[x]/P(x)$ is $IntClo$. For a map $B \to C$, a lifting for the square

$$\begin{array}{ccc} \mathbb{Z}[x] & \longrightarrow & B \\ \downarrow & \nearrow & \downarrow \\ \mathbb{Z} & \longrightarrow & C \, , \end{array}$$

exists iff the kernel of B is reduced to 0, *i.e.* that $B \to C$ is a monomorphism. Now for a square (with P monic and $B \to C$ a monomorphism)

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & \nearrow & \downarrow {\scriptstyle mono} \\ A[x]/P(x) & \longrightarrow & C \end{array}$$

the image of x in C is an element integral over B and any such can be defined by such a square. The existence of a lift states that any element integral over B is image of an element in B, *i.e.* that B is integrally closed in C. □

The following lemma justifies the name chosen for this topology.

Lemma 3.14. *A finitely presented map $A \to B$ is integrally surjective map iff it is proper.*

Proof. An integrally surjective map $A \to B$ of kernel I decomposed in a quotient followed by an integral extension $A \to C = A/I \to B$. Quotient are always proper and so are integral extensions when they are finitely presented (as they are finite morphisms), which is the case here by cancellation. □

As a consequence of theorem 2.5 (the compatibility will be proven below), general integrally surjective maps are inductive limits of proper map of finite presentation. Despite this coincidence, we have chosen to keep the more sophisticated name 'integrally surjective' as it reflects better our practical manipulations of rings.

3.6.1. *Finiteness context and points.* The finiteness context $\mathcal{C}^f = (CRings^f)^o$ is still taken to be the opposite of subcategory of $CRings$ of finitely presented maps. Using 3.5.1, it remains to show that an integral extension $A \to B$ is a ind-object in finitely presented integral extension can be seen by writing B as the colimit of the filtered diagram of its sub-A-algebras generated by a finite number of elements. The distinguished class of covering families will be constructed in §3.6.3.

Proposition 51. *A ring is a point of the $(IntSurj, IntClo)$ factorisation system iff it is an algebraically closed field.*

Proof. A ring A corresponds to a point iff any finitely presented integrally surjective map $A \to B$ admits a section. From prop. 50, it is necessary and sufficient to prove this only for maps $A \to B$ where $B = A/I$ for some finitely generated ideal I or $B = A[x]/P(x)$ and some monic or zero polynomial P. Prop. 40 says that existence of retraction for quotients $A \to A/I$ implies that A is a field. A field A is now a point iff every monic polynomial has a root in A. But with coefficients in a field every polynomial is proportional to a monic one and A is a point iff every polynomial has a root in A. $\square$

Proposition 52. *The set of points of a ring A is in bijection with the set of prime ideals of A.*

Proof. As $Surj \subset IntSurj$, $pt_{Prop}(A) \subset pt_{Dom}(A)$. The inverse inclusion is a consequence of the existence of an algebraic closure for every field. $\square$

3.6.2. *Covering families and local objects.* A family $\{A \to A_i\}$ of integrally surjective finitely presented maps is a point covering family iff any map $A \to \overline{k}$ to a residual algebraically closed field factors through some $A \to A_i$. This is equivalent to the fact that any map $A \to k$ to a residue field of A lift through one of the $A \to A_i$ after an algebraic extension of k.

Proposition 53. *Pointed local objects are integrally closed domain which fraction field is algebraically closed.*

Proof. Let K be an algebraically closed field, and $A \to K$ an integrally closed map. We need only to show that the fraction field $K(A) = A[(A^*)^{-1}]$ of A is algebraically closed. But the stability by localisation of integral closure implies that $K(A) \to K[(A^*)^{-1}] \simeq K$ is again integrally closed. $\square$

In analogy with strict henselian local rings, such rings will be called *strict integrally closed domains*.

Proposition 54. *Local objects are integrally closed domain which fraction field is algebraically closed.*

Proof. Let A be a local object. As it must be a local object for the $(Surj, Loc)$ factorisation system, it must be an integral domain. Now we have to prove

that the map $A \to K(A) = A[(A^*)^{-1}]$ is integrally closed. As is it already a monomorphism it is sufficient to prove that it has the unique right lifting property with respect to maps $A \to A[x]/P(x)$ where P is monic. We are going to use the same argument as for the local objects of etale topology. Given such a map $A \to A[x]/P(x)$ lifting the fraction field of A, it can be completed in a $(IntSurj, IntClo)$-covering family by adjoining $A \to (A/p)^{int}$ for prime ideals different from 0. Then the hypothesis on A gives a retraction of one the map of the cover which can only be $A \to A[x]/P(x)$. This gives a lifting square

$$\begin{array}{ccc} A & === & A \\ {\scriptstyle IntSurj}\downarrow & \nearrow & \downarrow {\scriptstyle IntClo} \\ A[x]/P(x) & \longrightarrow & K(A) \end{array}$$

so the lift is unique because $(IntSurj, IntClo)$ is a unique lifting system.

To prove that the fraction field $K(A)$ is algebraically closed, we are going to prove that any algebraic extension $K(A) \to K(A)[x]/P(x)$ where P is irreducible in $K(A)[X]$ has a retraction. The composite $A \to K(A) \to K(A)[x]/P(x)$ factors as $A \to A' \to K$ where A' is the integral closure of A in K, this map $A \to A'$ is a $(IntSurj, IntClo)$-covering family (or can be completed as such in the same way as before) and thus admits a retraction, which gives the wanted retraction for $K(A)$. □

3.6.3. *Distinguished covering families.* In order to apply theorem 2.5 we need to show that the condition of being a strict integrally closed domain can be tested using only finitely presented point covering families. We are going to copy the situation of §3.3.4.

A point covering family $\{B \to B_i, i\}$ of an A-algebra B is said *distinguished* if all the $B \to B_i$ are maps of finite presentation of A-algebras and satisfy one of the following two conditions

a. it is a $(Mono^o, Surj^o)$ point covering family,

b. or it consists of single integral extension (such map will be called an *integral covering map*).

Lemma 3.15. *Any finitely presented integrally surjective map $B \to C$ between finitely presented A-algebras can be factored into a finitely presented quotient followed by a finitely presented integral covering map.*

Proof. We use the $(Surj, Mono)$ factorisation on $B \to C$ to obtain a quotient D/I of B with I the kernel of $B \to C$. I is finitely generated so $D \to B$ is finitely presented and so is $D \to C$ by cancellation.

We have to prove that $D \to C$ is an integral covering map. C is generated by some finite set of elements c_i zero of some monic polynomials of $B[X]$. If K is algebraically closed and $D \to K$ is a point, $K \to C \otimes_D K$ is an algebraic extension generated by the image of the c_i (because the relations are monic, they

are still non trivial and $C \otimes_D K$ is not empty). So as K is algebraically closed there exists a retraction, proving that any point of D lift though $D \to C$. $\qquad\square$

Proposition 55. *An A-algebra B is a strictly integrally closed ring iff it lifts through any distinguished covering families.*

Proof. The necessary condition is clear by characterisation of local objects as strict integrally closed rings. Reciprocally, the lifting condition with respect to finitely presented $(Mono^o, Surj^o)$ point covering families says that B is a integral domain (lemma. 43).

If $K(B)^{alg}$ is an algebraic closure of the fraction field of B, we are going to prove that $B \to K(B)^{alg}$ is integrally closed. It has to have the left lifting property with respect to finitely presented integrally surjective map $C \to D$ between finitely presented A-algebras, we can transform this problem into a lifting through an integral covering map.

$$
\begin{array}{ccc}
C/I & \longrightarrow & B/IB \\
\nearrow \; \downarrow \text{\scriptsize int.cov.map} & & \quad\quad \nwarrow u \\
C \longrightarrow & B & \simeq \\
\downarrow \quad \searrow \ell & & \downarrow \\
D/ID & & \\
\searrow \quad \nearrow & & \\
D & \longrightarrow & K(B)^{alg}
\end{array}
$$

where I is the kernel of $C \to D$. The map u exists and is an isomorphism as B/IB is a quotient of B still containing the generic point. And the lift ℓ exists by property of B. $\qquad\square$

3.6.4. *Spectra and moduli interpretation.* $Prop = (CRings^o = (IntClo^o, IntSurj^o),$ $(CRings^f)^o)$ is a compatible factorisation context, we can apply theorem 2.5 to prove the following.

Proposition 56. $SPEC_{Prop}(A)$ *classifies A-algebras that are strict integrally closed domains and $Spec_{Prop}(A)$ classifies integrally surjective A-algebras that are strict integrally closed domains. In particular those algebras can have automorphisms and neither of the two spectra is spatial.*

The two notions of points agree.

Proposition 57. *For $A \in CRings$, the set of points of $Spec_{Prop}(A)$ is in bijection with $pt_{Prop}(A)$.*

Proof. We need to prove that the set of points of $Spec_{Prop}(A)$ is in bijection with that of prime ideals of A. We proceed as in prop. 36. Given a prime

ideal and the associated integral domain quotient $A \to A/p$, we consider $A/p \to K(A/p) \to K(A/p)^{alg}$ where $K(A/p)^{alg}$ is an algebraic closure of the fraction field $K(A/p)$. The $(IntSurj, IntClo)$ factorisation of this maps defines an object $(A/p)^{sint}$ which is a point of $Spec_{Prop}(A)$. $(A/p)^{sint}$ is called the *strict integral closure* of A at p. The map $A/p \to K(A/p)^{alg}$ is injective and so is $A/p \to (A/p)^{sint}$ which implies that p is the kernel of $A \to (A/p)^{sint}$. We have constructed an injective map from prime ideals to points of $Spec_{Prop}(A)$; we prove now the surjectivity. For $A \to B$ a point of $Spec_{Prop}(A)$, B being an integral domain, the kernel of $A \to B$ is a prime ideal. With the notation of before, we have a diagram

$$
\begin{array}{ccccc}
A & \xrightarrow{\ IntSurj\ } & B & \xrightarrow{\ IntClo\ } & K(B) \\
{\scriptstyle Surj}\downarrow & \nearrow{\scriptstyle Mono} & & & \uparrow{\scriptstyle IntClo} \\
A/p & \xrightarrow[IntSurj]{} (A/p)^{sint} & \xrightarrow[IntClo]{} & K(A/p)^{alg}
\end{array}
$$

presenting $A \to (A/p)' \to K(B)$ as another factorisation of $A \to B \to K(B)$, so $B \simeq (A/p)^{sint}$. $\qquad\square$

3.7. **Proper Nisnevich topology.** Integral domains, integrally closed domains and strict integrally closed domains behave like local rings, henselian local rings and strictly henselian local rings, so it is tempting to define a Nisnevich localisation of the $(IntSurj, IntClo)$ setting so that local object are non strict integrally closed domains.

We consider the class $\mathcal{F}$ of fields and the associated Nisnevich forcing of the previous setting. A $(IntSurj, IntClo)$ point covering family $\{A \to A_i, i\}$ of A is $\mathcal{F}$-localising iff for any map $A \to K$ to a field, there exists an i and a factorisation of $A \to K$ through $A \to A_i$. In particular, $(Mono^o, Surj^o)$ point covering families are $\mathcal{F}$-localising. The Nisnevich context $NSurj := (CRings^o = (IntClo^o, IntSurj^o), (CRing^f)^o, \mathcal{F})$ will be called the *proper Nisnevich context*.

Proposition 58. *A ring is in the saturation of $\mathcal{F}$ iff it is an integrally closed domain.*

Proof. Let A be an integrally closed domain, *i.e.* a integral domain such that the map $A \to K(A)$ to the fraction field is integrally closed, and $A \to A_i$ a $\mathcal{F}$-localising point covering family. By definition of such a family there exists an i and a factorisation $A \to A_i \to K(A)$ of $A \to K(A)$. This forces $A \to A_i$ to be an integral extension and, as A is integrally closed, there exists a retraction. The reciprocal part has already been proven in the proof of prop. 54. $\qquad\square$

The following lemma is a consequence of $Surj \subset Intsurj$ and of the definition of Nisnevich covering families.

Lemma 3.16. *$(Mono^o, Surj^o)$ point covering families are proper Nisnevich covering families.*

3.7.1. *Distinguished covering families.* The finiteness context $\mathcal{C}^f = (CRings^f)^o$ is taken to be the opposite of subcategory of $CRings$ of finitely presented maps. Compatibility conditions have been checked in §3.3.4, we need only to construct a class of distinguished covering families sufficient to detect integrally closed rings.

A proper Nisnevich point covering family $\{B \to B_i, i\}$ of an A-algebra B is said *distinguished* if it is of finite presentation over A, *i.e.* there exist $A \to B' \to B$ where $A \to B'$ is of finite presentation and all $B \to B_i$ are pushout of some maps $B' \to B_i'$ between algebra of finite presentation, and satisfies one of the following two conditions

a. it is a $(Mono^o, Surj^o)$ point covering family,

b. or the family is reduced to two elements $B' \to B_0'$ and $B' \to B_1'$ where $B_0' = B/b$ for some $b \in B$ and B_1' is an integrally extension of B' such that $B'[b^{-1}] \to B_1'[b^{-1}]$ admit a retraction.

Geometrically (for the Zariski topology), this last condition says that the covering family is distinguished if it contains a finitely presented Zariski closed set Z and cover its complement by some integral extension that has a section over the complement of Z.

Proposition 59. *A A-algebra B is an integrally closed domain iff it lifts through any distinguished proper Nisnevich covering families.*

Proof. We need to prove only the sufficient part. Lifting through finitely presented $(Mono^o, Surj^o)$ point covering families says that B is an integral domain (lemma. 3.10), we need then to show that, if $K(B)$ is it fraction field of B, the map $B \to K(B)$ is integrally closed, *i.e.* has the left lifting property with respect integrally surjective map between finitely presented A-algebras of the type $C \to (C/I)[x]/P(x)$ for some finitely presented I and some monic polynomial P, we can use the same trick as in prop. 55 and suppose $I = 0$. We are going to complete $C \to D$ into a distinguished proper Nisnevich covering family. In a diagram

$$
\begin{array}{ccc}
C & \longrightarrow & B \\
\downarrow & \nearrow^{\ell} & \downarrow \\
C[x]/P(x) & \longrightarrow & K(B)
\end{array}
$$

we can always assume C to be an integral domain by quotienting by the kernel of $C \to K(B)$, so x can be describe in $K(C)$ as some fraction a/b so $C[b^{-1}] \to C[b^{-1}][x]/P(x)$ has a section. This will be the distinguished localisation of the covering family, we complete it in a cover with $C \to C/b$. Now by hypothesis $C \to B$ will factor one of the two maps of the cover, and it cannot be $C \to C/b$ as the map $K(C) \to K(B)$ send b to an invertible element. $\square$

3.7.2. *Spectra and moduli interpretation.* $PNis=(CRings^o=(IntClo^o, IntSurj^o)$, $(CRings^f)^o, \mathcal{F})$ is a compatible Nisnevich context so we can apply theorem 2.5.

Proposition 60. *For a ring A, points of $SPEC_{PNis}(A)$ are A-algebras that are integrally closed domains and points of $Spec_{PNis}(A)$ are integral extension of quotients of A that are integrally closed domains.*

As in §3.4.3, the small Proper Nisnevich spectrum of A have in general more points than the set of prime ideals of A. Also, a prime ideal p of A still define two points of $Spec_{PNis}(A)$, the first one is the point of $Spec_{Prop}(A)$ associated to p and the second on is the integrally closed domain obtained by the $(IntSurj, IntClo)$ factorisation of the residue map $A \to \kappa(p)$.

3.8. **Remarks on the previous settings.** It is folkloric that etale and proper maps look alike, but the structure behind this duality is still undefined. We do not formalize this structure here, but we think our approach using factorisation systems should help and we group here a few remarks in this spirit.

Etale-Proper duality. We would like to sketch here a parallel between the six previous studied contexts. Recall that $\mathcal{F}$ is the subcategory of $CRings$ generated by fields, and that $f\mathcal{F}$ that generated by fat fields (§3.2.2). The finiteness context being understood as $CRings^f$, the parallel is the following.

	Etale context	Proper context
Primary factorisation system	$(indEt, Hens)$	$(IntSurj, IntClo)$
Secondary factorisation system	$(Loc, Cons)$	$(Surj, Mono)$
Nisnevich context	$((Loc, Hens), f\mathcal{F})$	$((Surj, Mono), \mathcal{F})$

Where the 'secondary factorisation system' is obtained from the primary one by looking only at those maps in the left class that are epimorphisms in $CRings$: localisations are those ind-etale maps that are epimorphisms and surjections are those integrally surjective maps that are epimorphisms. The secondary factorisation context can be thought as a way to extract open embbedings from etale maps. Also both Nisnevich localising classes are exactly the points of the secondary factorisation context. We are not sure how much these remarks are meaningful, but they do sketch a general structure. Thinking as $\mathcal{C}$ as $CRings^o$, one can define canonically from a factorisation context $(\mathcal{C} = (A, B), \mathcal{C}^f)$ a secondary factorisation context as $(\mathcal{C} = (^\perp(\mathcal{B} \cap Mono), \mathcal{B} \cap Mono), \mathcal{C}^f)$ and a Nisnevich context $(\mathcal{C} = (A, B), \mathcal{C}^f, \mathcal{P}t_{(\mathcal{B} \cap Mono)^f})(\mathcal{C})$.

Points and local objects. We recall the comparison between the points and local objects for the different contexts.

	Etale context	Proper context
Secondary points	fat fields	fields
Primary points	fat separably closed fields	algebraically closed fields
Secondary local objects	local rings	integral domain
Primary local objects	strict Henselian local rings	strict integrally closed domains
Nisnevich local objects	Henselian local rings	integrally closed domains

It is remarkable that for the four factorisation systems the set of points of a ring A is always the set of prime ideals of A and that it always coincide with the set of points of the associated spectra (*i.e.* every local object is pointed). Also for every prime ideal $p \subset A$ there exists always a (essentially unique) distinguished map $A \to \kappa(p)^!$ where $\kappa(p)^!$ is the residue field or some extension of it at p, such that the local object at p can be constructed by factorising $A \to \kappa(p)^!$ for the underlying factorisation system.

Other dual notions.

	Etale context	Proper context
secondary generators	$\mathbb{Z}[x] \to \mathbb{Z}[x, x^{-1}]$ $(\mathbb{G}_m \to \mathbb{A}^1)$	$\mathbb{Z}[x] \to \mathbb{Z}$ $(\{0\} \to \mathbb{A}^1)$
locality condition	$x + y$ invertible $\Rightarrow$ x or y invertible $(x + y \in \mathbb{G}_m \Rightarrow$ x or $y \in \mathbb{G}_m)$	$xy = 0 \Rightarrow$ $(x = 0$ or $y = 0)$
completion	henselisation	normalisation

Normalisation of a noetherian ring A: if p_i are the minimal prime of A and $\kappa(p_i)$ the associated residue fields, NA is the middle object of the $(IntSurj, Intclo)$ factorisation of $A \to \prod_i \kappa(p_i)$. It is always a product of the normalisation NA_i of the A/p_i, indeed the idempotents associated with $\prod_i \kappa(p_i)$ are elements integral over $\mathbb{Z}$ so they belong to NA.

Henselisation of a semilocal ring A: if m_i are the maximal prime of A and $\kappa(m_i)$ the associated residue fields, HA is the middle object of the $(indEt, Hens)$

factorisation of $A \to \prod_i \kappa(m_i)$. As A is the product of its localisations A_{m_i}, HA is the product of the henselisation HA_i of the A_{m_i}.

Dual lifting properties. The duality between the Etale and Proper contexts can be also thought as follow. Having in mind that points of a local ring are all generisation of the closed point, and that points of a integral domain are all specialisation of the generic point, the dual lifting properties for etale and proper maps are dual in the same sense than a category and its opposite. Another illustration of this is the fact that the poset of points of Zariski and Domain spectra are opposite categories. All this recall Grothendieck's smooth and proper functors [Ma1] for which a functor $F\colon C \to D$ is smooth iff its opposite $F\colon C^o \to D^o$ is proper. It is stated in [Ma1] that this property of functors has no analog in algebraic geometry, but these dual topologies could be a hint toward a more precise analogy. However the classes of smooth and proper functors are not know (yet?) to be part of factorisation systems so a link with our theory is not obvious.

The example to follow (§3.9.3) of left and right fibrations of category also has a flavour of the same kind of duality, but the situation is clearer in this setting as the opposition of categories exchange the two dual factorisation systems. Is there an operation of the same kind exchanging the etale and proper factorisation systems?

3.9. **Other examples.** This section sketches the results of the study of some common unique factorisation systems. Proofs are left to the reader.

3.9.1. ($Epi, Mono$) *topology in a topos.* We investigate the (Epi,Mono) factorisation system of maps of a topos $\mathcal{T}$, the finiteness context is taken to be the whole of $\mathcal{T}$.

An object $P \neq \emptyset \in \mathcal{T}$ is a point iff any monomorphism $U \to P$ ($U \neq \emptyset$) admits a section. This forces $U \to P$ to be an isomorphism: points are objects without any proper subobject. These objects are called *atoms* of the topos [Jo2, C.3.5.7]. Maps between atoms are always epimorphisms and all quotients of atoms are atoms. Points of an object X are called atomic subobjects of X, any two atomic subobjects are either equal or disjoint in X. Any morphism $A \to X$ with A an atom factors through a unique atomic subobject of X, so the set of points of X is that of its atomic subobjects. The family of all atomic subobjects of X is the finest point covering of X, so local objects coincide with points and $Spec_{atom}(X)$ is the topos of presheaves over the set of atomic subobjets of X.

We are going to illustrate this in the topos $BG = G\text{-}Sets$ classifying G-torsors for some discrete group G. Objects of BG are sets with a right action of G and can be thought as particular groupoids, a map is a monomorphism if, viewed as a map of groupoids, it is fully faithful. Points of ($Epi, Mono$) system of BG are sets with a transitive action of G. The category of all points is then the orbit category of G and the set of points of $X \in BG$ is simply the set of orbits of the

action of G. A point covering family is a family of monomorphisms surjective on orbits, or view through the associated groupoids, a family of fully faithfull maps globally essentially surjective. The family of all orbits of a given X is the finest point covering family of X, and $Spec_{atom}(X)$ is equivalent to the topos of presheaves on the set of orbits of X.

3.9.2. $(Epi, Mono)$ *topology in an abelian category and discrete projective spaces.* Any abelian category $\mathcal{C}$ has an $(Epi, Mono)$ unique factorisation system, its initial object 0 is also final and so not strict but this is not important. The finiteness context is taken to be the whole of $\mathcal{C}$.

Points are non zero objects without any proper subobject, *i.e.* simple objects. Any map to M from a simple object is either 0 or a monomorphism, the set of points of M is then the set of simple or null subobjects of M. The family of all simple subobject of M is the finest point covering family of M, so all local objects are points and the small spectrum $Spec_{Epi}(M)$ is the topos of presheaves on the poset of simple or null subobjects of M. All simple subobjects correspond to closed points and 0 to a generic point.

If $\mathcal{C}$ is the category of vector spaces over some field k, $Spec_{Mono}(M)$ is a sort of discrete projective space for M, with an extra generic point. Forgetting about this generic point, a map $M \to N$ can be though as inducing a partially defined transformation (it is not defined on the kernel of $M \to N$) between the associated projective spaces.

The big spectrum $SPEC_{Mono}(0)$ is the category of presheaves over the category of simple objects of $\mathcal{C}$. And the structure sheaf map $Spec(M) \to SPEC(0)$ send a simple subobject of M to its underlying simple object.

As $\mathcal{C}$ has both finite limits and finite colimits, this system is easily dualisable in $(\mathcal{C}^o = (Epi, Mono) = (Mono^o, Epi^o)$. Points of $\mathcal{C}^o$ are objects without any proper quotient, which are again simple objects; the set of points of an object M is that of simple or null quotients of M and $Spec_{Epi}(M)$ is a the "dual" projective space of $Spec_{Mono}(M)$ still with an extra point, which is this time the only to be closed.

3.9.3. *Discrete fibrations of categories.* We are going to study two unique factorisation systems on the category CAT of small categories, the reference for all the results is [Joy].

Let $[n]$ be the ordinal with $n + 1$ elements $0 < \cdots < n$ viewed as a category. $[0]$ is the punctual category. The two functors $[0] \to [1]$ will be called $\underline{0}$ and $\underline{1}$. In CAT, the unique factorisation system $(Fin, DRFib)$ is defined as left generated by $\underline{1}$, Fin is called the class of *final functors*, and $DRFib$ the class of *discrete right fibrations*. There is a dual system $(Ini, DLFib)$ left generated by $\underline{0}$, Ini is called the class of *initial functors*, and $DLFib$ the class of *discrete left fibrations*. It is easy to see that $C \to D \in LFib$ iff $C^o \to D^o \in RFib$.

We are only going to detail the factorisations in a special case: if $c\colon [0] \to C$ is an object of a category C, the $(Fin, DRFib)$ factorisation of c is $[0] \to C_{/c} \to C$ and the $(Ini, DLFib)$ factorisation of c is $[0] \to {}_c\backslash C \to C$. We want say much of the left classes only that in the previous factorisation $[0] \to C_{/c}$ points the final object of $C_{/c}$ and $[0] \to {}_c\backslash C$ the initial object of ${}_c\backslash C$. As for the right classes, it can be shown that any $D \to C \in DRFib$ is associated a presheaf $F\colon C^o \to \mathcal{S}$ such that D is isomorphic to $C_{/F}$ and that any $D \to C \in DRFib$ is associated a functor $F\colon C \to \mathcal{S}$ such that D is isomorphic to ${}_F\backslash C$. From this we can deduced that the categories $DRFib_{/C}$ and $DLFib_{/C}$ are respectfully equivalent to the category $\widehat{C}$ of contravariant functors $C \to \mathcal{S}$ and to that $\check{C}$ of covariant functors $C \to \mathcal{S}$.

We are now going to study the $(Fin, DRFib)$ system, the associated factorisation topology will be called the *right topology*. A point is a non empty category P such that any any discrete right fibration $C_{/F} \to C$ has a section. Using the Yoneda embedding in $\widehat{C}$, this condition says every presheaf on C has a global section. Such categories can be highly non trivial (Δ is an example) and the set of points of category is difficult to described, but fortunately the point covering families are simple to understand. Certainly $[0]$ is a point, and so a point covering family of C has to be globally surjective on the objects of C. This condition is also sufficient: indeed if $P \to C$ is a point of C, it will lift through a covering family $U_i \to C$ iff one of the fiber product $U_i \times_C P$ is not empty, but if $U_i \to C$ is assumed surjective on the points, it cannot happen that all fiber products are empty.

A local object is a category such that any epimorphic family of presheaves contains a presheaf with a global section. In particular any category with a terminal object is a local object (as proven already by the factorisation $c\colon [0] \to C_{/c} \to C$). We don't know if all local object are of this type, neither if they are all pointed.

A discrete right fibration $C_{/F} \to C$ is surjective on the objects iff $F(c) \neq \emptyset$ for all $c \in C$ iff $F \to \in \widehat{C}$ is an epimorphism. In the same way a family $U_i \to C$ of discrete right fibrations is globally surjective on objects iff it is globally epimorphic in $\widehat{C}$. The small site of C is $\widehat{C}$ and the previous remark show that the topology is the canonical one, so $Spec_{Right}(C)$ is the topos $\widehat{C}$. Its category of points is that of pro-objects of C.

$SPEC_{Right}(C)$ is the topos of presheaves over $CAT_{/C}$. Every object $c \in C$ define a point of $\widehat{C}$, the associated local object is $C_{/c}$ and the structural map is $C_{/c} \to C$. Using a topological vocabulary, one can say that $C_{/c}$ is the *right localisation* of C at c.

For the $(Ini, DLFib)$ system the same reasonning leads a topology called the *left topology* and to $Spec_{Left}(C)$ being the topos $\check{C}$.

Analogy with the etale-proper duality. The pair of $(Fin, LFib)$ and $(Ini, RFib)$ looks dual in the same sense that $(Loc, Cons)$ and $(Surj, Mono)$ or $(IndEt, Hens)$ and $(IntSurj, IntClo)$ are in $CRings$. $(Fin, LFib)$ is left generated by $1\colon [0] \to [1]$ and $(Ini, RFib)$ is left generated by $0\colon [0] \to [1]$, thinking of $[1] = 0 \to 1$ as a specialisation morphism, 0 is then generic point and 1 the closed point. With this vocabulary a discrete right fibration lift any generisation of any object that is lifted and so behave as an open map, and a discrete left fibration lift any speciali-sation of any object that is lifted and so behave like a closed map. This situation is to compare with the facts that Zariski open embeddings lift any generisation of any point that is lifted and that closed embeddings lift any specialisation of any point that is lifted.

Also, the generators $\mathbb{G}_m \to \mathbb{A}^1$ and $\{0\} \to \mathbb{A}^1$ of the $(Loc, Cons)$ and $(Surj, Mono)$ systems on $CRings$, which also are a generic point and a closed point. However, seen geometrically in $CRings^o$ the generators are this time in the right class.

Moreover in this case, $\widehat{C}$ and $\check{C}$ have a duality pairing given by the coend:

$$\begin{aligned} \widehat{C} \times \check{C} &\longrightarrow \mathcal{S} \\ (F, G) &\longmapsto \int^C F \times G \end{aligned}$$

This pairing is moreover "exact" in the sense that the natural map $\widehat{C} \to CAT(\check{C}, \mathcal{S})$ is an equivalence on the subcategory of functors commuting with all limits and $\check{C} \to CAT(\widehat{C}, \mathcal{S})$ is an equivalence on the subcategory of functors commuting with all colimits.

Is this a feature of the same duality? Does a similar pairing exist for spectra of rings?

Locality properties between the two systems. Those two system have also some compatibility conditions together. The left class of a factorisation system is not in general stable by base change but Fin and Ini are stable by base change along $DLFib$ and $DRFib$ respectively. This has an interesting consequence as a map $C \to D$ can be characterized to be final iff its pull-back along every $_{d\backslash}D \to D$ for some $d \in D$ is final

$$\begin{array}{ccc} {}_{d\backslash}C & \longrightarrow & C \\ \downarrow & & \downarrow \\ {}_{d\backslash}D & \longrightarrow & D. \end{array}$$

Now this can be read using a topological langage: $_{d\backslash}C$ is the localisation of C at d in $\widehat{D}$ and being a final maps is a local property for the Right topology. Dually of course, being initial is a local property for the Left topology. Also, these topologies can be used to interpret Quillen's theorem A and many definitions of

[Ma1] as proving locality properties of some classes of functors with respect to the left or right topology.

Groupoids. Restricted to the category of groupoids, $DRFib$ and $DLFib$ coincide and define the class of *coverings functors* (discrete fibrations) and In and Fin coincide too and define the class of *connected functors*. In fact both factorisation systems restrict to the categrory of groupoids and define a factorisation system compatible with weak equivalence such that, when groupoids are taken as models for homotopy 1-types, it induces the 0-th Postnikov system of §3.9.5.

3.9.4. *A dual topological realisation for simplicial sets.* Let Δ be the category of finite (non empty) ordinals and order preserving maps. Writing $[n] := (0 < 1 < \cdots < n)$ for the $(n+1)^{th}$ ordinal, a map $u\colon [n] \to [m] \in \Delta$ decomposes into $[n] \to [p] \to [m]$ where $[n] \to [p]$ is a *surjection* and $[p] \to [m]$ a *monomorphism*. This factorisation system is left generated by the single map $[1] \to [0]$.

   The category $SSets = \widehat{\Delta}$ of presheaves on Δ is the category of simplicial sets, objects of Δ view in $SSet$ will be noted $\Delta[n]$ and called simplices.

Lemma 3.17. *If $\mathcal{C}$ is a full subcategory of a cocomplete category $\mathcal{D}$, any unique factorisation system $\mathcal{C} = (\mathcal{A}, \mathcal{B})$ left generated by compact objects extend to a unique unique factorisation system $\mathcal{D} = (\mathcal{A}', \mathcal{B}')$ such that $\mathcal{A} = A' \cap C$ and $\mathcal{A} = A' \cap C$.*

Proof. Let G be a set of left generators, so $\mathcal{B} = G^\perp$ and $\mathcal{A} =^\perp \mathcal{B}$ in $\mathcal{C}$. We define now $\mathcal{B}' := G^\perp$ and $\mathcal{A}' :=^\perp \mathcal{B}'$ in $\mathcal{D}$. It is clear that $\mathcal{C} \cap \mathcal{B}' = \mathcal{B}$ and so we have also $\mathcal{C} \cap \mathcal{A}' = \mathcal{A}$. Now the set of generators G can always be completed to satisfies assumptions of prop. 3 so we only have to prove that the factorisation in $\mathcal{D}$ of a map in $\mathcal{C}$ coincide with the factorisation in $\mathcal{C}$, but this is obvious by unicity of the factorisation. $\qquad\square$

Corollary 5. *The unique factorisation system $(Surj, Mono)$ on Δ can be extended to $SSet$ in a system noted $(Deg, NDeg)$.*

   A map in Deg will be called *degenerated* and a map in $NDeg$ *non degenerated*.

Proposition 61. *$NDeg$ is the class of maps of simplicial sets $u\colon Y \to X$ sending non degenerate simplices of Y to non degenerate simplices of X. In particular, a map $\Delta[n] \to X$ is in $NDeg$ iff it is a non degenerate simplex of X.*

Proof. First we claim that a particular case of the factorisation is the one of the Eilenberg-Zilber lemma saying that a map $\Delta[n] \to X \in SSet$ factors through a unique $\Delta[n']$ where $n' \le n$ so that the map $\Delta[n] \to \Delta[n']$ is a surjection and $\Delta[n'] \to X$ is a non degenerate simplex. So the simplex $\Delta[n] \to X$ is degenerated iff $n' < n$. Using this factorisation on the top and bottom arrows, we can develop

any lifting square in

$$\begin{array}{ccc}
\Delta[n] \xrightarrow{\text{surj.}} \Delta[n'] \xrightarrow{\text{non deg.}} Y \\
\text{surj.} \downarrow \qquad \downarrow \qquad \downarrow \\
\Delta[m] \xrightarrow[\text{surj.}]{} \Delta[m'] \xrightarrow[\text{non deg.}]{} X
\end{array}$$

where $\Delta[n'] \to \Delta[m']$ is a surjection by cancellation. The map $Y \to X$ is orthogonal to surjection of simplices iff the map $\Delta[n'] \to \Delta[m']$ is an isomorphism. But this condition says exactly that a non degenerated simplex of Y is send to a non degenerated simplex of X. □

Raw spectrum. The finiteness context is taken to be the whole $SSet$.

Proposition 62. *The only point is $\Delta[0]$.*

Proof. It is easy to see that $\Delta[0]$ is a point. Reciprocally, a simplicial set X is a point if $Y \to X \in NDeg$ every it admit a section. Applied to $\Delta[0] \to X$ this forces X to be $\Delta[0]$. □

The set of points of an object X is exactly the set of vertices X. A family of maps $U_i \to X \in NDeg$ is a point covering family iff it is surjective on vertices. For any simplicial set X, the family of maps $\Delta[0] \to X$ is the finest cover of X. As a consequence, the only local simplex is $\Delta[0]$ (and of course every local object is pointed local).

Proposition 63. $Spec_{NDeg}(X) \simeq \mathcal{S}^{X_0}$.

Proof. For any $U \to X$, the nerve of the covering by simplices of U is constant si a presheaf $F\colon NDeg^o_{/X} \longrightarrow \mathcal{S}$ is a sheaf for the factorisation topology iff $F(u\colon U \to X) = \prod_{x \in U_0} F(u(x))$. □

Simplectic Nisnevich Spectrum. To make this setting a bit more interesting, we are going to make a Nisnevich localisation along the category Δ of simplices. Covering families of the Nisnevich context $\Delta Nis := (SSets = ((Deg, NDeg), SSets, \Delta)$ are families of maps $U_i \to X \in NDeg$ lifting not only vertices but any simplex of X.

Lemma 3.18. *The family of all maps $\Delta[n] \to X \in NDeg$ for all n, is a Nisnevich covering family of X.*

Proof. We need to prove that any $\Delta[m] \to X$ factors through one of the $\Delta[n] \to X \in NDeg$, but this Eilenberg-Zilber lemma. □

Corollary 6. *Local objects of the Nisnevich context ΔNis are simplices.*

Proof. By definition of the context, simplices are local. Reciprocally by lemma 3.18 it is enough to use the family of all $\Delta[n] \to X \in NDeg$. Let $d\colon \Delta[n] \to X$ be a map of the family having a section s, s is in $NDeg$ and so is sd. But the only non degenerate endomorphism of $\delta[n]$ is the identity, so d is an isomorphism. □

As a consequence, the set of points of the Nisnevich spectrum $Spec_{\Delta,NDeg}(X)$ is the set of maps $\Delta[n] \to X \in NDeg$, *i.e.* the set of non degenerate simplices of X.

Proposition 64. *Let $P(n)$ be the poset of faces of $\Delta[n]$. $Spec_{\Delta Nis}(\Delta[n])$ is the topos of presheaves over $P(n)$. In particular this is a spatial topos whose poset of points is $P(n)$.*

Proof. For the first assertion, we just need to prove that the topology is trivial, but any cover of $\Delta[m]$ admits a copy of $\Delta[m]$ so the identity is the finest cover. The category of points is $Pro(P(n))$ which turns out to be equivalent to $P(n)$. This is a consequence of the fact that any functor $f\colon I \to P(n)$ where I is a filtered category factors through a category J with a terminal object (hence every pro-object will be representable). To see this it is enough to consider I to be a poset, and a poset is filtered iff for any two objects i and j, there exists an object k and two arrows $k \to i$ and $k \to j$. If $f\colon I \to P(n)$ is a filtered diagram, $f(i), f(j)$ and $f(k)$ are faces of $\Delta[n]$ and if $f(i)$ is a vertex then necessarily $f(k) = f(i)$ and $f(i)$ is a vertex of $f(j)$. This implies that there can be at most one vertex of $\Delta[n]$ in the image of f and this vertex is a terminal element for the image poset of f, proving our assertion. If no vertices are in the image of f, there can be at most a single edge in the image of f which is then the terminal element of the image poset. If no edges are in the image of f, one has to continue the same argument with higher dimensional faces. □

Corollary 7. *$Spec_{\Delta Nis}(X)$ is topological space such that any non-degenerate $\Delta[n] \to X$ is an open embedding.*

Remark. The small Nisnevich spectra of a simplicial set X can be thought as a geometric realisation of X as it is a spatial object that does not see the degenerate part of X. This geometric realisation is such that any vertex of X is open in $Spec_{\Delta Nis}(X)$ and as show the computation of $Spec_{\Delta Nis}(\Delta[n])$, it can be thought as a cellular complex dual of the usual geometric realisation (use for example in the theory of Poincaré duality).

This "duality" raises the question of the existence of another factorisation system on $\widehat{\Delta}$ for which the small spectra of a simplicial set would be (a combinatorial form of) the usual geometric realisation. Unfortunately, for this realisation, the only open of a n-simplex would be the cell of dimension n but such a cell without its boundary is not a simplicial object. In fact, ordinary geometric realisation

being constructed by glueing along closed subsets, they are not local for the topology of the realisation and toposic techniques do not seem relevant here.

3.9.5. *Postnikov factorisation systems in Hot.* We present here a situation that is not an example our setting but an example of a natural generalisation to $(\infty, 1)$-categories and homotopically unique factorisation systems.

If C is an $(\infty, 1)$-category, a homotopically unique factorisation system is still the data of two classes (A, B) factoring all maps but the axiom of unicity is replaced by the higher analog: the ∞-groupoid of maps between two factorisation has to be contractible. We claim that our constructions of the small and big spectra generalize, but they are now ∞-toposes [Re, HAG1, Lu].

We are going to sketch the study of the Postnikov factorisation systems in the case of the $(\infty, 1)$-category $\mathcal{H}$ of homotopy types. But it could be any $(\infty, 1)$-topos. Fix $n \in \mathbb{N} \cup \{\infty\}$. A type X is said to be *n-truncated* if all its homotopy invariants of rank $> n$ are trivial. A map $X \to Y$ is said to be *n-truncated* if all its homotopy fibers are n-truncated. The class of n-truncated maps is noted *n-Trunc*. A map $X \to Y$ is said to be *n-connected* if it has the left lifting property with respect to n-truncated maps. The class of n-connected maps is noted *n-Con*.

Proposition 65. *n-Con and n-Trunc are respectively the left and right class of a unique factorisation system on Hot.*

Proof. This just a reformulation of the relative theory of Postnikov towers: any map $X \to Y$ factors as $X \to P_n X \to Y$ where $P_n X \to Y$ is n-truncated and is a homotopical terminal object for the category of all factorisation $X \to Z \to Y$ where Z is n-truncated. $\qquad\square$

$P \in Hot$ is a point iff every n-truncated map $Y \to P$, with $Y \neq \emptyset$ admits a section. This is equivalent to P being n-connected (having no homotopy invariants of rank $\leq n$). Then one can show that a family $U_i \to X$ of n-truncated map is a cover iff it is surjective on connected components. The small site of X is the $(\infty, 1)$-category $\Pi_n(X)$ (in fact an $(n, 1)$-category) of all n-truncated maps over X, this is an n-topos in the sense of [Lu, 6.4] and the factorisation topology coincide with the canonical topology, so the small spectrum of X is the n-topos $\Pi_n(X)$. Higher Galois theory says that n-truncated coverings of X depend in fact only of the nth stage $P_n(X)$ of the Postnikov tower of $X \to *$, *i.e.* $\Pi_n(X)$ is the topos of representations of the n-groupoid $P_n(X)$.

In the case of the category $\mathcal{H}$ of homotopy types, the case $n = \infty$ is trivial as the small spectrum of X is the ∞-topos $\mathcal{H}_{/X}$. But in the general case of $\mathcal{T}$ a non t-complete ∞-topos [Re, HAG1, Lu], the case $n = \infty$ of the small spectrum gives the *t-completion* of $\mathcal{T}_{/X}$.

Introducing some finiteness contexts can also give some known features. Any kind of condition on the homotopy invariants of the homotopy fibers of maps of $\mathcal{H}$

will create a finiteness context (that may not be compatible). For example if we looked at maps whose homotopy fibers have finitely many non trivial homotopy invariants and if those invariants have an underlying finite set, the associated small spectra are the toposes of representations of the profinite completions of the n-groupoids $P_n(X)$. If we look at maps whose homotopy fibers have homotopy invariants that are p-groups, the associated small spectra should be the toposes of representations of the p-completions of the n-groupoids $P_n(X)$.

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