MONOPOLES IN ARBITRARY DIMENSION

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Abstract. A self-contained study of monopole configurations of pure Yang-Mills theories and a discussion of their charges is carried out in the language of principal bundles. A n-dimensional monopole over the sphere $\mathbb{S}^n$ is a particular type of principal connection on a principal bundle over a symmetric space K/H which is K-invariant, where $K = SO(n+1)$ and $H = SO(n)$. It is shown that principal bundles over symmetric spaces admit a unique K-invariant principal connection called canonical, which also satisfy Yang-Mills equations. The geometrical framework enables us to describe their associated field strengths in purely algebraic terms and compute the charge of relevant (Yang-type) monopoles avoiding the use of coordinates. Besides, two corrections on known results are performed in this paper. First, it is proven that the Yang monopole should be considered a connection invariant by $Spin(5)$ instead of by $SO(5)$, as Yang did in his original article [Y78]. Second, unlike the way suggested in [GT06], we give the correct characteristic class to be used to calculate the charge of the monopoles introduced by Gibbons and Townsend.

1. Introduction

Monopoles in gauge theories have deserved a lot of attention since Dirac introduced his magnetic monopole [D31], mainly due to the fact that monopoles carry an intrinsically associated charge which only takes discrete values, something that could easily explain the observable quantization of the charge in electromagnetic theory. Recall that the Dirac monopole can be seen as a static singular solution on $\mathbb{R}^3$ of a field theory with gauge group $U(1)$. In practice, monopoles have never been observed, and their existence is only justified from a theoretical point of view in order to build a bridge between classical and quantum field theories. After Dirac and the explosion of the popularity of gauge theories, there have been other attempts to generalize the concept of monopole to different (non-abelian) gauge groups in higher dimensions. Among them, [Y78] is one of the most celebrated generalizations.

One of the most remarkable aspects of monopoles is that their charge is related to the topological properties of the underlying space and strongly depends on the way the gauge potential is attached to it. In other words, monopoles cannot be understood *at a local level* but their properties need to be described form a *global point of view*. In particular, unless additional boundary conditions are

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required, there cannot exist monopoles in the Euclidean space $\mathbb{R}^n$, $n \in \mathbb{N}$, but, on the contrary, monopoles exhibit a singularity at the origin $0 \in \mathbb{R}^n$, where the charge is supposed to be. Therefore, $0 \in \mathbb{R}^n$ needs to be removed. On the other hand, it is widely known that the theory of principal bundles provides the most satisfactory framework to study and develop gauge theories from a geometrical (global) point of view (see [DV80], [B81], and [EGH80]). Although the reader is supposed to be familiar with the geometrical framework of gauge theories, we are going to recall in this paper the main features of principal bundles for the sake of a clearer exposition.

If we restrict to pure Yang-Mills theories, the framework of principal bundles over $\mathbb{R}^n \backslash \{0\}$ seems to be the main mathematical tool to tackle monopoles. However, the classification theory of principal bundles over paracompact manifolds ([M56a] and [M56b]) requires in general a rather sophisticated topological machinery that we would like to avoid as much as possible. Since $\mathbb{R}^n \backslash \{0\}$ is homotopic to $\mathbb{S}^{n-1}$, we can study principal bundles either over $\mathbb{R}^n \backslash \{0\}$ or $\mathbb{S}^{n-1}$ indistinguishably as far as the global properties of monopoles is concerned; for a given gauge group G, principal bundles over $\mathbb{R}^n \backslash \{0\}$ and $\mathbb{S}^{n-1}$ are homomorphic and their structure can be recovered from one to the other. Remember that two principal bundles are called homomorphic is there exits a smooth map between them equivariant with respect to the actions of the gauge group. The key point is that $\mathbb{S}^n$ is a homogeneous space; for example, $\mathbb{S}^n \cong SO(n+1)/\,SO(n)$, where $SO(n)$ denotes the special orthogonal group. Since such spaces and their associated structures have been extensively studied, a huge geometrical machinery is consequently available to deal with them.

Using a geometrical language, gauge potentials and field strengths in gauge field theories are described in terms of principal connections on principal bundles and their curvature, respectively. On the other hand, the Chern-Weil homomorphism provides a mechanism to associate to the curvature some *de Rham* cohomology classes $H^{2k}(\mathbb{S}^n)$ of even order, known as characteristic classes. Roughly speaking, the Chern-Weil homomorphism allows us to remove the dependence of the field strength on the *gauge indices* (or the *color*, in a physics language), which should not appear in any observable physical quantity. In this context, a monopole configuration on $\mathbb{S}^n$ is a principal bundle $\pi\colon P \to \mathbb{S}^n$ with a principal connection such that:

(i) There exists a characteristic class in $H^n(\mathbb{S}^n)$ whose integral over $\mathbb{S}^n$ is different from zero. This means that we can associate a non-vanishing charge to the monopole. As we will discuss in Section 5, there is no general consensus on which topological invariant should represent the charge of a monopole and some authors chose others. Observe that n needs to be even in order to $n/2$ be an integer. That is, there will be no monopoles in even (spatial) dimensions.

(ii) The principal connection is $SO(n+1)$ invariant. This property is usually referred to as ***spherical symmetry*** of the monopole in the literature. In particular, it implies that we need to be able to define an action of the group of rotations of $\mathbb{R}^{n+1}$ on our principal bundle so that the principal connection is invariant with respect to it. This is *not* always possible, as it actually happens for the Yang's monopole, despite the explicit reference to the $SO(5)$ invariance Yang did in [Y78]. We will see that, in the Yang case, spherical symmetry needs to be implemented through an action of $Spin(5)$ instead of $SO(5)$, contrary to what was usually thought.

It is customary in gauge theories to give monopole configurations locally on coordinate patches and then to impose some compatibility conditions where these patches overlap. The use of coordinates is sometimes unavoidable in computations, but it is often very tedious. Fortunately, there are many features that can be seen intrinsically. The purpose of our paper is to convey to the physics community some of the global tools from differential geometry perfectly tailored to study monopoles. The main contributions of this paper are the following:

1. We explicitly show that there exists a bijective correspondence between principle bundles over the Euclidean space $\mathbb{R}^{2n+1}\backslash\{0\}$ and principal bundles over the sphere $\mathbb{S}^{2n}$ and their principal connections are Yang-Mills if and only if they are Yang-Mills on the latter.

2. We will see that on $\mathbb{S}^{2n}$, seen as a symmetric space, only the so-called canonical connections are $SO(2n+1)$-invariant. Moreover, it is proved (see Proposition 11) that they automatically satisfy the Yang-Mills equations.

3. Despite the widely spread idea that the Yang monopole on $\mathbb{S}^4$ is $SO(5)$ invariant, it is shown that the concept of spherical symmetry needs to be implemented by its universal covering group $Spin(5)$. This is because there does not exist any principal bundle with structural group $SU(2)$ admitting a (left) $SO(5)$ action. When describing the monopole on $\mathbb{S}^4$ by means of local sections as Yang did, $Spin(5)$ acts through $SO(5)$, which explains why such a confusion arises.

4. We make precise some of the results about monopole configurations found in the literature. Explicitly, in Section 6 we discuss that the charge of the monopoles over $\mathbb{S}^{2n}$ with gauge group $SO(2n)$, $n > 2$, recently introduced in ([GT06]) can only be implemented through the so-called Euler class. Although, broadly speaking, the main ideas behind Gibbons and Townsend $SO(2n)$-monopoles do not differ too much from ours, the way they introduce the field strength and the charge of the monopole is imprecise and leads them to assert wrong statements. We fix this point by clarifying the way to define properly these concepts in geometrical terms.

5. We give a depiction of monopoles on homogeneous symmetric spaces only in algebraic terms (Section 4). More concretely, if $\pi\colon P \to K/H$ is a principal bundle over a symmetric Lie space related to a monopole with gauge group G, K and $H \subset K$ two Lie groups, then a monopole is completely described in terms of the Lie algebras $\mathfrak{k}$, $\mathfrak{h}$, and $\mathfrak{g}$. This simplifies a lot the amount of manipulations needed to compute any relevant quantity associated to monopoles (no local coordinates are needed) and, what is more important, allows us to go from a geometrical framework to an algebraic one which, in practice, makes quantities computable. For example, we show in Section 4 and 5 that field strengths and Chern classes can be easily computed for monopoles without much effort.

6. We clarify the structure of monopole configurations from a geometrical point of view. This means that our approach is global as we try to emphasize the intrinsic nature of the structures involved in such configurations and, consequently, avoid using local coordinates. As we said, this approach seems to be suitable since the properties of monopoles are topological.

7. We gather some results on principal bundles over homogeneous spaces which have appeared since the late 1950's and make them available to physicists interested in monopoles. Although they are widely known among geometers, there still exists surprisingly some confusion in the community about the precise meaning of some concepts such as spherically symmetric potentials, for instance, or the relationship between the charge of a monopole and the topological invariants of a principal bundle expressed by the Chern-Weil homomorphism.

The paper is structured as follows: in Section 2, we recall on the one hand the main geometric tools of principal bundles emphasizing their importance in gauge theories and, on the other, we proof that principal bundles over $\mathbb{S}^{2n}$ and $\mathbb{R}^{2n+1}\backslash\{0\}$ can be recovered ones from the others. In Section 3, we introduce homogeneous principal bundles $P_\lambda \to K/H$ over homogeneous spaces. These bundles, which admit a left action by the Lie group K, are the geometric background for monopole configurations. We characterize the principal connections (gauge potentials) $\omega \in \Omega^1(P;\mathfrak{g})$ which are invariant by K and show that, when K/H is a symmetric space, there exists a unique connection with these properties. We present in Section 4 an explicit procedure to give the spherically symmetric field strengths Ω^ω associated to monopole configurations in terms of the Lie algebras of the groups involved. This procedure is implemented in some examples. In Section 5, we recall the Chern-Weil homomorphism, a mechanism to associate some *de Rham* cohomology classes of $\mathbb{S}^{2n}$ to the field strength Ω^ω of $P_\lambda \to \mathbb{S}^{2n}$. We also show how to define the charge of a monopole from these classes using the algebraic description of Ω^ω given in Section 4. Finally, in Section 6, we apply the tools developed throughout the paper to revise the classical examples by Dirac and Yang, and the more recent monopoles introduced by Gibbons and Townsend ([GT06]).

Notation: All manifolds M in this paper will be of class C^{∞}. The set of smooth vector fields on M will be denoted by $\mathfrak{X}(M)$ and the set of differential forms by $\Omega(M)$. If M and N are two manifolds, the tangent map of a smooth function $F\colon M \to N$ at a point $m \in M$ between the tangent spaces T_mM and $T_{F(m)}N$ of M and N at $m \in N$ and $F(m)$ respectively will be denoted by T_mF. The symbol d will be reserved for the exterior differential $d\colon \Omega(M) \to \Omega(M)$. If V is a real vector space, $\Lambda(V) = \oplus_{k\geq 0}\Lambda^k(V)$ will be the space of multilinear alternating maps from V to $\mathbb{R}$. On the other hand, S_n will denote the symmetric group of order $n \in \mathbb{N}$ and $|\sigma| = \pm 1$ the parity of a permutation $\sigma \in S_n$. The wedge product of two forms $\alpha \in \Omega^k(M)$ and $\beta \in \Omega^l(M)$ is defined as

$$
\begin{aligned}
&(\alpha \wedge \beta)(X_1, \ldots, X_{k+l}) \\
&\qquad = \frac{1}{k!l!} \sum_{\sigma \in S_{k+l}} (-1)^{|\sigma|} \alpha\left(X_{\sigma(1)}, \ldots, X_{\sigma(k)}\right) \beta\left(X_{\sigma(k+1)}, \ldots, X_{\sigma(k+l)}\right),
\end{aligned}
$$

$\{X_1, \ldots, X_{k+l}\} \subset \mathfrak{X}(M)$, and the differential $d\alpha$ satisfies

$$
\begin{aligned}
d\alpha(X_1, \ldots, X_{k+1}) = &\sum_{i=1}^{k} (-1)^{i+1} \alpha\left(X_1, \ldots, \widehat{X_i}, \ldots, X_{k+1}\right) + \\
&\sum_{i<j} (-1)^{i+j} \alpha\left([X_i, X_j], X_1, \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_{k+1}\right).
\end{aligned}
$$

It is worth noticing that, in the literature, some authors sometimes use different factors in these expressions.

2. Geometric preliminaries

We recalled in the introduction that principal bundles over $\mathbb{R}^{2n+1}\backslash\{0\}$ and $\mathbb{S}^{2n}$ are in a bijective correspondence. In this section, we are going to give more details about how this bijection works. The idea is to use it in subsequent sections to switch from $\mathbb{R}^{2n+1}\backslash\{0\}$ to $\mathbb{S}^{2n}$ and take advantage of the geometric tools available when $\mathbb{S}^{2n}$ is considered as a homogeneous space. Moreover, we want to see that, if a principal connection on $\mathbb{S}^{2n}$ satisfies the Yang-Mills equations, so does the corresponding induced connection on $\mathbb{R}^{2n+1}\backslash\{0\}$. The rest of this section is devoted to recalling the basics of gauge theories such as principal connections (Subsection 2.2) and the Hodge operator (Subsection 2.3). After introducing Yang-Mills connections, we will conclude the section by seeing that a principal connection is Yang Mills on $\mathbb{S}^{2n}$ if and only if it is Yang-Mills on the corresponding bundle over $\mathbb{R}^{2n+1}\backslash\{0\}$ (Proposition 1).

2.1. Correspondence between principal bundles over $\mathbb{R}^{2n+1}\backslash\{0\}$ and $\mathbb{S}^{2n}$.

Let $\pi\colon P \to M$ be a principal bundle with structural group G over a manifold M and right action $R\colon G \times P \to P$. Let $f\colon N \to M$ a smooth function from a manifold N to M. The pull-back of π by f is a fiber bundle over N defined as

$$f^*(P) = \{(p,x) \in P \times N \mid \pi(p) = f(x)\}$$
$$\overline{\pi} : f^*(P) \to N, \ \ \overline{\pi}((p,x)) = x.$$

With the natural right action $(p,x) \cdot g = (R_g(p), x)$, $g \in G$, inherited from $\pi \colon P \to M$, it is easy to verify that $\overline{\pi} \colon f^*(P) \to N$ is indeed a principal bundle. An important result is that, if $f, h \colon N \to M$ are two homotopic smooth maps, then the pull-backs $f^*(P)$ and $h^*(P)$ are isomorphic (see [I89, page 121],[M01]), that is, there exists a map $F \colon f^*(P) \to h^*(P)$ over the identity on N such that $F(z \cdot g) = F(z) \cdot g$ for any $z \in f^*(P)$. This rather simple result allows us to explicitly draw the bijection between principal bundles over $\mathbb{R}^{2n+1} \backslash \{0\}$ and $\mathbb{S}^{2n}$, respectively. Indeed, let $\pi \colon P \to \mathbb{R}^{2n+1} \backslash \{0\}$ be a principal bundle and let $P|_{\mathbb{S}^{2n}}$ be the restriction of P to $\mathbb{S}^{2n}$, which coincides with the pull-back of P by the inclusion of the sphere into $\mathbb{R}^{2n+1} \backslash \{0\}$ ([I89, page 120]). On the other hand, the map

$$\begin{array}{rcl} f : \mathbb{R}^{2n+1} \backslash \{0\} & \longrightarrow & \mathbb{S}^{2n} \subset \mathbb{R}^{2n+1} \backslash \{0\} \\ x & \longmapsto & \frac{x}{\|x\|}, \end{array} \tag{2.1}$$

is homotopic to the identity $\mathrm{Id} \colon \mathbb{R}^{2n+1} \backslash \{0\} \to \mathbb{R}^{2n+1} \backslash \{0\}$, where $\|x\| = \sqrt{\sum_{i=1}^{2n+1} (x^i)^2}$ denotes the Euclidean norm. Therefore, the principal bundles $f^*(P)$ and P are isomorphic. But clearly $f^*(P) = f^*(P|_{\mathbb{S}^{2n}})$. So we conclude that principal bundle structures on $\mathbb{S}^{2n}$ are induced by restriction from those on $\mathbb{R}^{2n+1} \backslash \{0\}$ and, conversely, that principal bundles over $\mathbb{S}^{2n}$ induce principal bundles over $\mathbb{R}^{2n+1} \backslash \{0\}$ by means of (2.1), both procedures being commutative. As a consequence, we can study monopole configurations on the sphere $\mathbb{S}^{2n}$ and then pull them back onto $\mathbb{R}^{2n+1} \backslash \{0\}$ using the projection (2.1). Before that, we will continue recalling more geometric ingredients of gauge theories; concepts that are quite common for physicists in the context of Riemannian geometry but less known in more general principal bundle framework.

2.2. **Principal connections.** Let $\pi \colon P \to M$ be a principal bundle with structural group G. A ***principal connection*** $\omega \in \Omega^1(P; \mathfrak{g})$ is a one form on P with values in the Lie algebra $\mathfrak{g}$ of G such that

$$R_g^* \omega = \mathrm{Ad}_{g^{-1}} \omega, \tag{2.2a}$$

$$\omega(p) \left(\left. \frac{d}{dt} \right|_{t=0} R_{\exp(t\eta)}(p) \right) = \eta \tag{2.2b}$$

for any $g \in G$, $p \in P$, and $\eta \in \mathfrak{g}$. In this expression Ad denotes the adjoint representation of G on $\mathfrak{g}$ and $\exp \colon \mathfrak{g} \to G$ the usual exponential map. We will denote the vector field $\left. \frac{d}{dt} \right|_{t=0} R_{\exp(t\eta)}$ simply by η_P, $\eta \in \mathfrak{g}$. Any principal connection $\omega \in \Omega^1(P; \mathfrak{g})$ defines the ***horizontal space*** $\mathrm{Hor}_p = \ker \omega$ at any $p \in P$ such that $T_pP = \mathrm{Hor}_p \oplus \mathrm{Ver}_p$, where $\mathrm{Ver}_p \subset T_pP$ is the ***vertical space*** $\mathrm{Ver}_p = \ker T_p\pi$. An arbitrary form is called ***horizontal*** if it vanishes when contracted with vector fields in the vertical space.

Given a $\mathfrak{g}$ valued r-form $\varphi \in \Omega^r(P, \mathfrak{g})$ on a principal bundle $\pi\colon P \to M$ and a principal connection $\omega \in \Omega^1(P, \mathfrak{g})$, the ***covariant derivative*** $D^\omega\varphi$ of φ is defined at any point $p \in P$ as $D^\omega\varphi(p) := d\varphi|_{H_p}$. That is, we calculate the standard exterior differential of φ and then we restrict it to the horizontal space. In particular, the ***curvature*** of the connection is $\Omega^\omega := D^\omega\omega$. When regarded as a potential, we will usually refer to the curvature as the ***field strength***. It is customary to find the curvature in the literature written as $\Omega^\omega = d\omega + \frac{1}{2}[\omega, \omega]$. This is the so-called ***structural equation***. If $\varphi \in \Omega^r(P, \mathfrak{g})$ and $\psi \in \Omega^k(P, \mathfrak{g})$, the bracket $[\cdot, \cdot]$ is defined as

$$
\begin{aligned}
&[\varphi, \psi](X_1, \ldots, X_{r+k}) \\
&\qquad = \frac{1}{r!k!} \sum_{\sigma \in S_{r+k}} (-1)^{|\sigma|} \left[\varphi\left(X_{\sigma(1)}, \ldots, X_{\sigma(r)}\right), \psi\left(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+k)}\right)\right]_{\mathfrak{g}}.
\end{aligned}
$$

In this equation, the bracket $[\cdot, \cdot]_{\mathfrak{g}}$ is that of the Lie algebra, $X_1, \ldots, X_{r+k} \subset \mathfrak{X}(P)$ are arbitrary vector fields on P, and S_{r+k} denotes the permutation group of $r+k$ elements.

2.3. **The Hodge operator and Yang-Mills connections.** Given a principal bundle $\pi\colon P \to M$ with structural Lie group G, the ***adjoint bundle*** $\mathrm{Ad}(P)$ is the associated bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$. That is, the space of equivalent classes of $P \times \mathfrak{g}$ under the equivalence relation $(p, \xi) \sim (R_g(p), \mathrm{Ad}_{g^{-1}}\xi)$, $p \in P$, $\xi \in \mathfrak{g}$, and $g \in G$. It is a rather standard result in differential geometry (see [M07, Theorem 19.14]) that the space $\Omega_{equiv}(P; \mathfrak{g})^{\mathrm{Hor}}$ of horizontal $\mathfrak{g}$-valued forms on P which are G-equivariant by (2.2a) can be identified with the space $\Omega(M; \mathrm{Ad}(P))$ of $\mathrm{Ad}(P)$-valued differential forms on the base manifold M. This identification works as follows: having a (principal) connection $\omega \in \Omega^1(P; \mathfrak{g})$ amounts to having a splitting of the exact short sequence

$$
0 \longrightarrow V_p \longrightarrow T_pP \overset{\curvearrowleft\Gamma_p}{\longrightarrow} T_{\pi(p)}M
$$

at any point $p \in P$ such that $X = \Gamma_p(T_p\pi(X)) \in \mathrm{Hor}_p$ for any $X \in T_pP$. Thus, we naturally associate to any $\varphi \in \Omega^r(P; \mathfrak{g})^{\mathrm{Hor}}$ the $\mathrm{Ad}(P)$-valued form $\widetilde{\varphi} \in \Omega^r(M; \mathrm{Ad}(P))$ such that

$$
\widetilde{\varphi}(m)(Y_1, \ldots, Y_r) = [p, \varphi(p)(\Gamma_p(Y_1), \ldots, \Gamma_p(Y_r))]^\sim \tag{2.3}
$$

for any $Y_1, \ldots, Y_r \in \mathfrak{X}(M)$. In (2.3), $p \in \pi^{-1}(m)$, and the bracket $[\cdot, \cdot]^\sim$ denotes the equivalent class of a point $(p, \xi) \in P \times \mathfrak{g}$ into $P \times_{\mathrm{Ad}} \mathfrak{g}$. It is not difficult to check that (2.3) does not depend on the choice of the fiber point $p \in \pi^{-1}(m)$.

Suppose now that M is a n-dimensional Riemann manifold with Riemannian volume form $\mu \in \Omega^n(M)$ and we have a Ad-invariant metric $\mathbf{h}$ on $\mathfrak{g}$. For example, $\mathbf{h}$ could be taken to be (minus) the Killing-Cartan form if G was a semi-simple compact Lie group. Recall that the inverse of the Riemann metric on M can be used to define a $C^\infty(M)$-bilinear pairing

$$\langle \cdot, \cdot \rangle_M : \Omega^q(M) \times \Omega^q(M) \longrightarrow C^\infty(M), \ q \in \mathbb{N},$$

([B81, Chapter 0]). On the other hand, $\mathbf{h}$ induces a metric on the fibers of the vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g} \to M$ in a standard way. We keep on denoting this metric by $\mathbf{h}$. Both $\langle \cdot, \cdot \rangle_M$ and $\mathbf{h}$ applied together define a $C^\infty(M)$-bilinear product

$$\langle \cdot, \cdot \rangle : \Omega^q(M; \mathrm{Ad}(P)) \times \Omega^q(M; \mathrm{Ad}(P)) \longrightarrow C^\infty(M).$$

Additionally, the induced metric $\mathbf{h}$ allows us to define a wedge pairing

$$\wedge : \Omega^r(M; \mathrm{Ad}(P)) \times \Omega^q(M; \mathrm{Ad}(P)) \longrightarrow \Omega^{r+q}(M)$$

via the equality

$$(\varphi \wedge \psi)(m)(Y_1, \ldots, Y_{r+q}) := \frac{1}{r!q!} \sum_{\sigma \in S_{r+q}} (-1)^{|\sigma|} \mathbf{h}_m \left(\varphi\left(Y_{\sigma(1)}, \ldots, Y_{\sigma(r)}\right), \varphi\left(Y_{\sigma(r+1)}, \ldots, Y_{\sigma(r+q)}\right)\right)$$

for any $\varphi \in \Omega^r(M; \mathrm{Ad}(P))$, $\psi \in \Omega^q(M; \mathrm{Ad}(P))$, and any $Y_1, \ldots, Y_{r+q} \in \mathfrak{X}(M)$. More importantly, there is a natural operator called the ***Hodge operator***

$$* : \Omega^r(M; \mathrm{Ad}(P)) \longrightarrow \Omega^{n-r}(M; \mathrm{Ad}(P))$$

characterized by the relation

$$\theta \wedge *\varphi = \langle \theta, \varphi \rangle \mu \in \Omega^n(M)$$

for any $\varphi \in \Omega^r(M; \mathrm{Ad}(P))$ and any $\theta \in \Omega^{n-r}(M; \mathrm{Ad}(P))$. The Hodge operator defines the inner product

$$(\theta, \varphi) := \int_M \theta \wedge *\varphi = \int_M \langle \theta, \varphi \rangle \mu$$

provided this integral exists. Finally, given $\omega \in \Omega^1_{equiv}(P; \mathfrak{g})$, the ***covariant codifferential*** δ_ω is defined by

$$\delta_\omega \varphi = -(-1)^{n(r+1)} * \circ D^\omega \circ *\varphi \in \Omega^{r-1}_{equiv}(P; \mathfrak{g})^{\mathrm{Hor}}, \ \varphi \in \Omega^r_{equiv}(P; \mathfrak{g})^{\mathrm{Hor}},$$

where we have used the identification $\Omega(M; \mathrm{Ad}(P)) = \Omega_{equiv}(P; \mathfrak{g})^{\mathrm{Hor}}$ in order to apply the Hodge operator to a $\mathfrak{g}$-valued horizontal form on P.

In a pure Yang-Mills theory, the ***Yang-Mills functional*** YM associates to any principal connection $\omega \in \Omega^1(P; \mathfrak{g})$ the real number

$$YM(\omega) := (\Omega^\omega, \Omega^\omega) = \int_M \Omega^\omega \wedge *\Omega^\omega.$$

Roughly speaking, the Yang-Mills functional gives a measure of the total curvature of the principal connection ω. Critical points of the functional, the so called ***Yang-Mills connections***, are the most important for physical purposes because their corresponding field strengths model physical interactions in gauge theories. A classical result shows that $\omega \in \Omega^1(P;\mathfrak{g})$ is a Yang-Mills connection if and only if

$$\delta_\omega \Omega^\omega = 0 \tag{2.4}$$

(see [B81, Theorem 5.2.3] for a modification of (2.4) in the presence of currents).

Now, suppose that $\omega \in \Omega^1(P;\mathfrak{g})$ is a Yang-Mills connection of some bundle $\pi\colon P \to \mathbb{S}^{2n}$. We have already argued that the map (2.1) can be used to define principal bundle structures on $\mathbb{R}^{2n+1}\backslash\{0\}$ from those on $\mathbb{S}^{2n}$. Let $F\colon f^*(P) \to P$ be the bundle homomorphism from the pull-back of π by $f\colon \mathbb{R}^{2n+1}\backslash\{0\} \to \mathbb{S}^{2n}$ given in Equation (2.1). The next proposition, whose proof can be found in the Appendix, shows that the principal connection $F^*(\omega) \in \Omega^1(f^*(P);\mathfrak{g})$ on $f^*(P)$ is also Yang-Mills.

Proposition 1. *Let $\pi\colon P \to \mathbb{S}^{2n}$ be a principal bundle with structural group G and let $\omega \in \Omega^1(P;\mathfrak{g})$ be a principal connection. Let $f\colon \mathbb{R}^{2n+1}\backslash\{0\} \to \mathbb{S}^{2n}$ be as in (2.1) and $F^*\colon f^*(P) \to P$ the corresponding principal bundle homomorphism. Then*

$$\delta^{F^*(\omega)}\Omega^{F^*(\omega)} = -\frac{1}{\overline{\pi}^*(r^2)}F^*(\delta^\omega\Omega^\omega),$$

where $r \in C^\infty(\mathbb{R}^{2n+1}\backslash\{0\})$ is the radius function $r(x) = \|x\|$, $x \in \mathbb{R}^{2n+1}\backslash\{0\}$. In particular, ω is a Yang-Mills connection if and only if $F^*(\omega)$ is a Yang-Mills connection.

3. Principal bundles over homogeneous spaces

The aim of this section is to introduce the main geometrical ingredients to study gauge theories over homogeneous spaces. Since we are interested in gauge theories over the n-dimensional sphere $\mathbb{S}^n = SO(n+1)/SO(n)$. However, among all the possible principal bundle structures over $\mathbb{S}^n$, we need to characterize those admitting a (left) $SO(n+1)$-action in order to talk properly about spherically symmetric quantities. This will be done in the first subsection. We will see that these principal bundles can be labelled by a Lie group homomorphism $\lambda\colon SO(n) \to G$ from the isotropy group to the gauge group. Moreover, they can be understood as homogeneous spaces themselves, a perspective that will be extremely fruitful. At the end we will give a characterization of the four more relevant examples of our study, the principal bundles which will correspond to Dirac, Yang, and Gibbons-Townsend monopoles. Once we have learnt how to build such principal bundles, we will characterize in Subsection 3.2 the principal connections (gauge potentials) which are invariant by the rotations group in terms of linear maps $W\colon \mathfrak{so}(n) \to \mathfrak{g}$ satisfying some compatibility conditions. Finally, in Subsection 3.3, we introduce symmetric spaces, a particular subclass of homogeneous spaces whose Lie algebra can be suitably decomposed. For example, the sphere $\mathbb{S}^n$ is a symmetric space. Over them, we will show that there exists a unique $SO(n+1)$-invariant principal connection; that is, a monopole potential one-form. This means that requiring a principal bundle to admit a $SO(n+1)$-action equals to having an essentially unique spherically symmetric configuration on it.

3.1. **Homogeneous principal bundles.** Let K and G be two Lie groups and $H \subset K$ a closed subgroup. A ***homogeneous principal bundle*** $\pi\colon P \to K/H$ with structural group G is a principal bundle over a homogeneous space K/H together with a left K-action on P by automorphisms which projects to the left multiplication of K on the base manifold K/H. According to [HSV80] and [W58], homogeneous principal bundles $\pi\colon P \to K/H$ with structural group G are (modulo isomorphisms) in one-to-one correspondence with group homomorphisms $\lambda\colon H \to G$ (modulo conjugation) so that $\pi\colon P \to K/H$ is isomorphic to the ***associated bundle*** $P_\lambda := K \times_H G$; that is, the space of orbits of the right action

$$\begin{array}{rcl} \Psi_\lambda : (K \times G) \times H & \longrightarrow & K \times G \\ ((k,g),h) & \longmapsto & (kh, \lambda(h)^{-1}g)\,. \end{array} \tag{3.1}$$

Denoting the elements p of P_λ as equivalent classes, $p = [k,g]^\sim$ such that $k \in K$ and $g \in G$, the projection π is simply given by $[k,g]^\sim \longmapsto kH \in K/H$. If $p \in \pi^{-1}(o)$ is some point in the equivalence class $o \in K/H$ of $e \in K$, the homomorphism $\lambda\colon H \to G$ can be understood by the relation

$$h \cdot p = p \cdot \lambda(h),\ h \in H,$$

where the dot $\cdot$ denotes the left action of K or the right action of G on P_λ respectively. We encourage the reader to check with [M07] for a brief review on the basic facts about associated bundles.

Furthermore, P_λ can be also seen as the homogeneous space $(K \times G)/\widetilde{H}$, where $\widetilde{H}$ is the closed subgroup $\widetilde{H} = \{(h, \lambda(h)) \mid h \in H\} \subset K \times G$, clearly isomorphic to H: that is why the principal bundles P_λ are called homogeneous. The isomorphism works as follows:

$$\begin{array}{rcl} \Upsilon : (K \times G)/\widetilde{H} & \longrightarrow & P_\lambda \\ \overline{(k,g)} & \longmapsto & [k, g^{-1}]^\sim, \end{array} \tag{3.2}$$

where $\overline{(k,g)}$ and $[k,g^{-1}]^\sim$ denote the equivalent class of $(k,g) \in K \times G$ in $(K \times G)/\widetilde{H}$ and P_λ respectively.

Finally, we fix some notation for later convenience. The *left* action $L_{P_\lambda} : K \times P_\lambda \to P_\lambda$ and a *right* action $R_\lambda\colon G \times P_\lambda \to P_\lambda$ that we have on a homogeneous principal bundle are respectively given by

$$(L_\lambda)_{\bar{k}}\left([k,g]^\sim\right) = [\bar{k}k, g]^\sim \text{ and } (R_\lambda)_{\bar{g}}\left([k,g]^\sim\right) = [k, g\bar{g}]^\sim, \tag{3.3}$$

where $g, \bar{g} \in G$, $k, \bar{k} \in K$.

Remark 2. In the general classification theory of bundles, two principal bundles with the same base manifold and the same structural group are called equivalent if there exists a homomorphism between them which projects onto the identity map on the basis. When the base manifold is the n-dimensional sphere $\mathbb{S}^n$, such equivalence classes are in bijection with the elements of the homotopy group $\pi_{n-1}(G)$ provided the gauge group G is connected (see [S51]). Take for example $n = 3$ and $G = SO(3)$. Since $\pi_2(SO(3)) = 0$, we know that, essentially, there exists a unique principal bundle over $\mathbb{S}^3$ with structural group $SO(3)$. Namely, $\pi\colon SO(4) \to \mathbb{S}^3 = SO(4)/SO(3)$. Therefore, $\pi\colon SO(4) \to \mathbb{S}^3$ is trivializable and $SO(4)$ is diffeomorphic to $SO(3) \times \mathbb{S}^3$. However, they are not isomorphic as Lie groups (see Proposition 14). On the other hand, there exist at least two homomorphisms $\lambda\colon SO(3) \to G = SO(3)$ which are not conjugated: the trivial homomorphism $\lambda(h) = e \in SO(3)$ for any $h \in SO(3)$, and the identity homomorphism, $\lambda = \mathrm{Id}$. So, according to what we have said so far, there exist two different principal bundles over $\mathbb{S}^3$ with gauge group $SO(3)$ admitting a left action of $SO(4)$. Is this a contradiction? The answer is no. Everything relies on the notion of *equivalence* of principal bundles we use. In general, when we forget about the $SO(4)$-left action, there always exists a fiber preserving diffeomorphism between two any principal bundles over $\mathbb{S}^3$. But my notion of equivalence changes when $SO(4)$ acts upon our principal bundles in the way we stated. Then, the previous diffeomorphism needs to be also equivariant with respect to the two $SO(4)$ actions, a requirement that prevents some bundles from being equivalent. In other words, we can define at least two different $SO(4)$-left actions on the unique principal bundle over $\mathbb{S}^3$ with gauge group $SO(3)$ in a non-equivalent way.

Remark 3. The theory of ***equivariant principal bundles*** tries to describe those principal bundles $\pi\colon P \to M$ with structural Lie group G such that both P and M are left acted upon another Lie group K such that the projection π is K-equivariant and the actions of K and G commute. This is a much more general framework that reduces to ours when $M = K/H$ is a homogeneous space, where $H \subseteq K$ is a closed Lie subgroup. Under some general assumptions and in particular for the case $\mathbb{S}^n = SO(n+1)/SO(n)$, $n \geq 3$, it can be checked that the number of isomorphic principal bundles $\pi\colon P \to M$ with structural group G over a left K-manifold M is finite provided that G is compact and the isotropy groups K_m are semi-simple, $m \in M$ ([HH03, Corollary 8.6]). In particular, the number of principal bundles over $\mathbb{S}^n$, $n \geq 3$, with structural group G compact admitting a $SO(n+1)$-left action is finite.

Examples 4. Let $\mathcal{R}(n, G)$ be the set of smooth homomorphisms from $SO(n)$ to G modulo conjugation by elements of G. We will describe $\mathcal{R}(n, G)$ for some values of $n \in \mathbb{N}$ and some Lie groups G that will allow us to study later on some of the monopole configurations found in the literature (see [HH03] and references therein).

- **(i)** $n = 2$ and $G = U(1)$. Given that $SO(2) = U(1)$, the set of homomorphisms $\mathcal{R}(2, U(1))$ is $\lambda\colon U(1) \to U(1)$ modulo conjugation. It is well known that such a set can be labelled by $\mathbb{Z}$, the set of integers. Regarding

$U(1) = \{e^{iz} : z \in [0, 2\pi)\}$, we can chose the homomorphisms

$$\begin{array}{rcl} \lambda_m : U(1) & \longrightarrow & U(1) \\ e^{iz} & \longmapsto & (e^{iz})^m = e^{izm} \end{array}, \quad m \in \mathbb{Z},$$

as representatives of the equivalent classes of $\mathcal{R}(2, U(1))$.

(ii) $n = 4$ and $G = SO(3)$. The ***algebra of quaternions*** $\mathbb{H}$ is usually defined abstractly as a 4-dimensional real vector space with a multiplication $(x, y) \mapsto xy$, $x, y \in \mathbb{H}$, which satisfies the usual associative and distributive laws and with a distinguished basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ satisfying the following commutation relations

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$$
$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k},\ \mathbf{jk} = -\mathbf{kj} = \mathbf{i},\ \mathbf{ki} = -\mathbf{ik} = \mathbf{j}.$$

The ***modulus*** of a quaternion $x = x_0\mathbf{1} + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is $|x| = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{1/2}$. The set of unit quaternions $S^3 := \{x \in \mathbb{H} \mid |x| = 1\}$ is isomorphic to $SU(2)$ and homeomorphic to the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ ([N97, Theorem 1.1.4]). Moreover, $S^3 \times S^3$ is the universal covering group of $SO(4)$ ([M07, Example 4.32]) so that

$$SO(4) \cong \left(S^3 \times S^3\right) / \{(\mathbf{1}, \mathbf{1}), (-\mathbf{1}, -\mathbf{1})\}.$$

On the other hand, $S^3 = SU(2)$ is the universal covering group of $SO(3) \cong S^3/\{\pm\mathbf{1}\}$. The set $\mathcal{R}(4, SO(3))$ contains three elements: the trivial homomorphism and those induced from the projections $S^3 \times S^3 \to S^3$ given by $\sigma_1(x, y) = x$ and $\sigma_2(x, y) = y$.

(iii) $n = 4$ and $G = SO(4)$. Using the identification

$$SO(4) \cong \left(\mathbb{S}^3 \times \mathbb{S}^3\right) / \{(\mathbf{1}, \mathbf{1}), (-\mathbf{1}, -\mathbf{1})\}$$

as in **(ii)**, the set $\mathcal{R}(4, SO(3))$ contains five elements: the trivial homomorphism, the identity $\mathrm{Id}\colon SO(4) \to SO(4)$, which give rise to the principal bundle $SO(5) \to SO(5)/SO(4)$, and three homomorphisms induced by the maps $\sigma_3, \sigma_4, \delta\colon \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$ given by $\sigma_3(x, y) = (x, x)$, $\sigma_4(x, y) = (y, y)$, and $\delta(x, y) = (y, x)$.

(iv) $n = 2k \geq 6$ and $G = SO(2k)$, $k \in \mathbb{N}$. The set $\mathcal{R}(2k, SO(2k))$ contains three elements: the trivial homomorphism, the identity $\mathrm{Id}\colon SO(2k) \to SO(2k)$, whose associated principal bundle is $SO(2k+1) \to SO(2k+1)/SO(2k)$, and the conjugation δ by the diagonal matrix $(-1, \ldots, -1, 1)$. Observe that $\delta \in O(2k)$ but $\delta \notin SO(2k)$. ■

3.2. **Invariant principal connections.** Let $\pi\colon P \to K/H$ be a homogeneous principal bundle as in the previous subsection. We say that a principal connection ω is ***K-invariant*** if $(L_\lambda)_k^*\omega = \omega$ for any $k \in K$. One can prove that, if $\mathfrak{k}$, $\mathfrak{h}$, and $\mathfrak{g}$ denote the Lie algebra of K, H, and the gauge group G respectively, K-invariant principal connections on $\pi\colon P_\lambda \to K/H$ are in one-to-one correspondence with linear maps $W\colon \mathfrak{k} \to \mathfrak{g}$ such that

(i) $W(\xi) = T_e\lambda\,(\xi)$ for any $\xi \in \mathfrak{h}$,
(ii) $W\,(\mathrm{Ad}_h\,\xi) = \mathrm{Ad}_{\lambda(h)}\,(W(\xi))$ for any $\xi \in \mathfrak{k}$ and any $h \in H$.

(see [W58], [KN69a]). From now on, we are going to refer to these linear maps W as ***Wang maps***. Given a Wang map $W\colon \mathfrak{k} \to \mathfrak{g}$, the principal connection $\omega \in \Omega^1(P_\lambda; \mathfrak{g})$ is given by

$$\omega_{p_o}(\xi_{P_\lambda}) = W(\xi) \tag{3.4}$$

where $\xi \in \mathfrak{k}$, o denotes the equivalent class of $e \in K$ in K/H, $p_o \in \pi^{-1}(o)$ is any arbitrary point on the fiber of $o \in K/H$, and ξ_{P_λ} is the vector field induced on P_λ by the K-action ([KN69a, Theorem 11.5]). Observe that, since ω is K-invariant and G acts transitively on the fibers of P_λ, (3.4) and (2.2a) characterizes ω completely.

One of the most important examples of homogeneous spaces are those called reductive. Recall that a homogeneous space K/H is called ***reductive*** if the Lie algebra $\mathfrak{k}$ can be written as $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ and $\mathrm{Ad}_h\,(\mathfrak{m}) \subseteq \mathfrak{m}$. For a reductive homogeneous space K/H, the linear map $\mathbf{W}\colon \mathfrak{k} \to \mathfrak{g}$ defined as $\mathbf{W}|_{\mathfrak{h}} = T_e\lambda$ and $\mathbf{W}|_{\mathfrak{m}} = 0$ is called the ***canonical connection***.

Example 5. It can be shown that the principal H-bundle $K \to K/H$ admits a K-invariant connection if and only if K/H is reductive ([KN69a, Theorem 11.1]). The canonical connection $\boldsymbol{\omega} \in \Omega^1\,(K, \mathfrak{h})$ on $K \to K/H$ is given by the $\mathfrak{h}$-valued part of the ***Maurer-Cartan form*** ω_{MC} which is defined by $\omega_{MC}(k)\,(\xi_K(k)) = \xi \in \mathfrak{k}$, $k \in K$. That is, $\boldsymbol{\omega}(k)(\xi_K(k)) = \mathrm{proj}_{\mathfrak{h}}(\xi)$. ■

Principal connections can be used to induce connections on associated bundles (see [M07, 19.8] and subsequent sections for a general approach to this subject). The details of this mechanism and the proof of the following proposition, that we include here for the sake of a more complete exposition, are postponed to the Appendix A.2. The proposition claims that the principal connection of P_λ is induced from that of $K \to K/H$ whenever K/H is reductive. Although it can be found in the literature, its proof is frequently omitted, so we decided to prove it ourselves explicitly.

Proposition 6. *Let K/H be reductive. Then, the canonical connection on P_λ is induced from the canonical connection of $K \to K/H$.*

3.3. **Symmetric spaces.** We are now going to describe invariant connections over a particular class of homogeneous spaces: symmetric spaces. Symmetric spaces are usually presented in the context of Riemannian geometry. Most of the content of this subsection is extracted from [KN69b], which the reader is encourage to check with. We will see that, over a symmetric space K/H, the canonical connection is the unique principal connection which is K-invariant. Since the sphere $\mathbb{S}^n$ is a symmetric space, it means that there will exist a unique monopole configuration on any homogeneous principal bundle over $\mathbb{S}^n$.

Let M be a n-dimensional Riemann manifold with an ***affine connection*** ∇, that is, a connection in the frame bundle. Let $U \subseteq M$ be an open neighborhood, $x \in U$ a fixed point, and $X_x \in T_xM$. Denote by $\exp(X_x)$ the value of the geodesic $\gamma(t)$ at time $t = 1$ which satisfies $\gamma(0) = x$, $\dot{\gamma}(0) = X_x$. This value exists for X_x in a suitable small neighborhood of $0 \in T_xM$. A diffeomorphism $\varphi\colon M \to M$ is called an ***affine transformation*** if it is a diffeomorphism and $T\varphi\colon TM \to TM$ maps each parallel vector field along a curve $\tau\colon (-\varepsilon, \varepsilon) \to M$, $\varepsilon > 0$, into a parallel vector field along the curve $\varphi(\tau)$. A ***symmetry*** s_x at a point $x \in U$ is a diffeomorphism of U onto itself which sends $\exp(X_x)$ into $\exp(-X_x)$. Observe that a symmetry s_x is involutive: $s_x \circ s_x = \mathrm{Id}$. If there exists an affine transformation s_x for any $x \in M$, then M is said to be ***affine locally symmetric***. M is said ***affine symmetric*** if the symmetry s_x can be extended to a global affine transformation of M for any $x \in M$.

The group of affine transformations of an affine symmetric manifold M is a Lie group which acts transitively on it ([KN69b, Chapter XI, Theorem 1.4]). If K denotes the identity component of such group, then $M = K/H$, where H denotes the subgroup of those affine transformations in K leaving a point $o \in M$ fixed ([H78, Chapter IV, Theorem 3.3]). Taking this remark into account, we say that a triple (K, H, σ) is a ***symmetric space*** if K, H are Lie groups, $H \subset K$, $\sigma\colon K \to K$ is an involutive automorphism, and $K_\sigma^e \subseteq H \subseteq K_\sigma$. Here K_σ denotes the set of elements of K which are invariant by σ and K_σ^e the identity component of K_σ. In the case of an affine symmetric manifold M, the automorphism σ is given by $\sigma(k) = s_o \circ k \circ s_o^{-1}$ where s_o is a symmetry at o. On the contrary, each symmetry s_x can be recovered from σ as $s_x = k \circ s_o \circ k^{-1}$, $x \in M$. In general, s_o is defined to be the involutive diffeomorphism of K/H onto itself induced by the automorphism σ.

Example 7. The n-dimensional sphere $\mathbb{S}^n$ is a symmetric space. Indeed, if $K = SO(n+1)$ and $o = (1, 0, \overset{n-2)}{\ldots}, 0) \in \mathbb{S}^n \subset \mathbb{R}^{n+1}$, then

$$H = \begin{pmatrix} 1 & 0 \\ 0 & SO(n) \end{pmatrix} \cong SO(n)$$

and $\mathbb{S}^n = SO(n+1)/SO(n)$. ■

In terms of the Lie algebras $\mathfrak{k}$ and $\mathfrak{h}$ of K and H, respectively, a symmetric space (K, H, σ) is described as follows. To start with, we see from the involutivity of σ that $T_e\sigma\colon \mathfrak{k} \to \mathfrak{k}$ has eigenvalues $+1$ and -1. Then, the Lie algebra $\mathfrak{k}$ can be written as $\mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h}$ is the eigenspace associated to the eigenvalue 1 and $\mathfrak{m}$ is the eigenspace associated to -1. Moreover,

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$$

and $\mathrm{Ad}_H(\mathfrak{m}) \subset \mathfrak{m}$ ([KN69b, Chapter XI, Proposition 2.1 and 2.2]). That is, symmetric spaces are reductive.

When the gauge group G is a subgroup of $GL(n;\mathbb{R})$, homogeneous principal bundles $P_\lambda \to K/H$ can be regarded as subbundles of the frame bundle. This is the case in our examples. Then, any K-invariant principal connection on P_λ (i.e., a Wang map $W\colon \mathfrak{k} \to \mathfrak{g} \subset \mathfrak{gl}(n;\mathbb{R})$) can be consequently considered as K-invariant affine connection (i.e., a Wang map $W\colon \mathfrak{k} \to \mathfrak{gl}(n;\mathbb{R})$). The next theorem is the most important as far as characterizing invariant affine connections on symmetric spaces is concerned.

Theorem 8 ([KN69b, Theorem 3.1 and 3.3])**.** *Let (K, H, σ) be a symmetric space. The canonical connection is the only affine connection on K/H which is invariant by the symmetries s_x of M, $x \in M$. Furthermore, a K-invariant (indefinite) Riemannian metric on K/H, if there exists any, induces the canonical connection on M.*

The previous theorem is important for the following reason. We defined monopoles as those configurations invariant by $SO(n)$ because elements of $SO(n)$ are physically relevant symmetries of our base space-time $\mathbb{R}^n\backslash\{0\}$. However, in more general models, there may not exist any natural action of $SO(n)$ onto the base manifold M, which is supposed to be a Riemann manifold according to General Relativity. In this case, the group of symmetries s_x seems to be the natural candidate to replace $SO(n)$ in the definition of spherical symmetry. In other words, we should require monopoles to be invariant by the symmetries s_x, $x \in M$, instead of by $SO(n)$.

Nevertheless, we are interested so far in connections which are invariant not by the symmetries but by the action of K. As Laquer shows in [L92], except for very concrete cases, the canonical connection is the unique affine connection on a symmetric space (K, H, σ) which is K-invariant. Therefore, the unique connection available to construct monopoles.

Theorem 9 ([L92, Theorem 2.1])**.** *Let K be a simple Lie group and (K, H, σ) a symmetric space. The set of K-invariant affine connections on K/H consists of just the canonical connection in all cases except for the following:*

$$\begin{array}{cl} SU(n)/SO(n) & n \geq 3, \\ SU(2n)/SP(n) & n \geq 3, \\ E_6/F_4. & \end{array} \tag{3.5}$$

Each of these spaces has a one-dimensional family of invariant affine connections.

4. The algebraic setting

In this section we are going to describe algebraically the space $\Omega_{equiv}(P_\lambda;\mathfrak{g})^K$ of $\mathfrak{g}$-valued forms which are G-equivariant in the sense of (2.2a) and K-invariant by the left action L_λ (3.3). The field strength Ω^ω will be then a multilinear map from $\mathfrak{k}$ to $\mathfrak{g}$ easily expressed in terms of the corresponding Wang map. Carrying out such identification is quite simple. Since two arbitrary points in P_λ

are always linked by the composition of the actions of K and G on P_λ, any $\alpha \in \Omega_{equiv}(P_\lambda;\mathfrak{g})^K$ is fully characterized by its values on a fixed point $p \in P_\lambda$. Suppose that $p \in \pi^{-1}(o)$ is $p = [e,e]^\sim$ as in the proof of Proposition 6. Since the isomorphism (3.2) allows us to identify T_pP_λ with $(\mathfrak{k}\times\mathfrak{g})/\widetilde{\mathfrak{h}}$, it seems reasonable to express $\Omega_{equiv}(P_\lambda;\mathfrak{g})^K$ as a suitable set of forms defined on $\mathfrak{k}\times\mathfrak{g}$ satisfying some restrictions. We will particularize in Subsection 4.2 the canonical field strengths of the homogeneous principal bundles introduced in Examples 4, which will correspond to the field strengths of Dirac, Yang, and Gibbons-Townsend monopoles. Moreover, we will also prove that they satisfy the Yang-Mills connections (Proposition 11) and, therefore, give rise to monopole configurations indeed.

First of all, observe that $\Omega_{equiv}(P_\lambda;\mathfrak{g})^K$ coincides with the space of $\mathfrak{g}$-valued forms forms on P_λ $(K\times G)$-equivariant with respect to the left $(K\times G)$-actions

$$\begin{array}{rcl}\Psi:(K\times G)\times P_\lambda & \longrightarrow & P_\lambda \\ ((k,g),[k_2,g_2]^\sim) & \longmapsto & [kk_2,g_2g^{-1}]^\sim\end{array} \tag{4.1}$$

and

$$\begin{array}{rcl}\rho:(K\times G)\times\mathfrak{g} & \longrightarrow & \mathfrak{g} \\ ((k,g),\xi) & \longmapsto & \mathrm{Ad}_g\,\xi.\end{array} \tag{4.2}$$

That is, $\varphi \in \Omega_{equiv}(P_\lambda;\mathfrak{g})^K$ if and only if $\Psi^*_{(k,g)}(\varphi) = \rho_{(k,g)}\circ\varphi$ for any $(k,g)\in K\times G$. On the other hand, if

$$\begin{array}{rcl}\widetilde{\Psi}:(K\times G)\times(K\times G)/\widetilde{H} & \longrightarrow & (K\times G)/\widetilde{H} \\ \left((k,g),\overline{(k_2,g_2)}\right) & \longmapsto & \overline{(kk_2,gg_2)}\end{array} \tag{4.3}$$

is the natural left action of $K\times G$ on the quotient space $(K\times G)/\widetilde{H}$, the isomorphism $\Upsilon\colon (K\times G)/\widetilde{H}\to P_\lambda$ introduced in (3.2) is such that the following diagram commutes

$$\begin{array}{ccc} P_\lambda & \overset{\Psi_{(k,g)}}{\longrightarrow} & P_\lambda \\ \Upsilon\uparrow & \# & \uparrow\Upsilon \\ (K\times G)/\widetilde{H} & \underset{\widetilde{\Psi}_{(k,g)}}{\longrightarrow} & (K\times G)/\widetilde{H}\end{array}$$

for any $(k,g)\in K\times G$. Therefore, $\varphi\in\Omega_{equiv}(P_\lambda;\mathfrak{g})^K$ if and only if

$$\widetilde{\Psi}^*_{(k,g)}(\Upsilon^*\varphi) = \rho_{(k,g)}\circ\Upsilon^*(\varphi) = \mathrm{Ad}_g\circ\Upsilon^*(\varphi).$$

In this situation, provided that $K\times G$ is connected, one of the consequences of [CE48, Theorem 13.1] is that the space $\Omega_{equiv}((K\times G)/\widetilde{H};\mathfrak{g})$ of $\mathfrak{g}$-valued forms on $(K\times G)/\widetilde{H}$ which are $(K\times G)$-invariant with respect to the actions (4.3) and (4.2) is isomorphic to the graded differential algebra $\Lambda_{\widetilde{\mathfrak{h}}}(\mathfrak{k}\times\mathfrak{g};\mathfrak{g})$, the space of $\mathfrak{g}$-valued chains on $\mathfrak{k}\times\mathfrak{g}$ such that

(i) vanish on $\widetilde{\mathfrak{h}} = \{\xi\in\mathfrak{h}\mid(\xi,T_e\lambda(\xi))\in\mathfrak{k}\times\mathfrak{g}\}$ and

(ii) if $\varphi \in \Lambda^n(\mathfrak{k}\times\mathfrak{g};\mathfrak{g})$, $z, z_1, \ldots, z_n \in \mathfrak{k}\times\mathfrak{g}$, $z = (\xi,\eta)$, $z_i = (\xi_i,\eta_i)$ with $\xi,\xi_i \in \mathfrak{k}$ and $\eta,\eta_i \in \mathfrak{g}$ for any $i = 1,\ldots,n$, then

$$[T_e\lambda(\xi), \varphi(z_1,\ldots,z_n)] = \sum_{i=1}^{n} \varphi(z_1,\ldots,[z,z_i],\ldots,z_n), \tag{4.4}$$

where $[z,z_i] = ([\xi,\xi_i], [T_e\lambda(\xi),\eta_i]) \in \mathfrak{k}\times\mathfrak{g}$.

Let

$$\Phi : \Omega_{equiv}((K\times G)/\widetilde{H};\mathfrak{g}) \cong \Lambda_{\widetilde{\mathfrak{h}}}(\mathfrak{k}\times\mathfrak{g};\mathfrak{g})$$

be the isomorphism between $\Omega_{equiv}((K\times G)/\widetilde{H};\mathfrak{g})$ and $\Lambda_{\widetilde{\mathfrak{h}}}(\mathfrak{k}\times\mathfrak{g};\mathfrak{g})$. For example, Φ sends a principal connection $\omega \in \Omega^1_{equiv}(P_\lambda;\mathfrak{g})$ associated to a Wang map $W\colon \mathfrak{k}\to\mathfrak{g}$ to the one chain $\widetilde{W}\colon \mathfrak{k}\times\mathfrak{g}\to\mathfrak{g}$ given by $\widetilde{W}(\xi,\eta) = W(\xi)-\eta$, $\xi\in\mathfrak{k}$, $\eta\in\mathfrak{g}$. We define the ***horizontal projector*** $\mathrm{Hor}_{\widetilde{W}}\colon \mathfrak{k}\times\mathfrak{g}\to\mathfrak{k}\times\mathfrak{g}$ as $\mathrm{Hor}_{\widetilde{W}}(\xi,\eta) = (\xi, W(\xi))$ and the ***vertical projector*** $\mathrm{Ver}_{\widetilde{W}}\colon \mathfrak{k}\times\mathfrak{g}\to\mathfrak{k}\times\mathfrak{g}$ $\mathrm{Ver}_{\widetilde{W}}(\xi,\eta) = (0, \eta - W(\xi))$. We made the dependence on the Wang map W explicit in order to distinguish these vertical and horizontal projectors from those associated to TP_λ and ω. In this context, the ***exterior differential*** operator $\mathbf{d}\colon \Lambda^n(\mathfrak{k}\times\mathfrak{g};\mathfrak{g}) \to \Lambda^{n+1}(\mathfrak{k}\times\mathfrak{g};\mathfrak{g})$ is defined as

$$\begin{aligned}\mathbf{d}\varphi(z_1,\ldots,z_{n+1}) = &\sum_{i=1}^{n+1}(-1)^{i-1}[\eta_i, \varphi(z_1,\ldots,\widehat{z}_i,\ldots,z_{n+1})]\\ &+\sum_{i<j}(-1)^{i+j}\varphi(([\xi_i,\xi_j],[\eta_i,\eta_j]), z_1,\ldots,\widehat{z}_i,\ldots,\widehat{z}_j,\ldots,z_{n+1}),\end{aligned}$$

where $z_i = (\xi_i,\eta_i)$ with $\xi_i\in\mathfrak{k}$ and $\eta_i\in\mathfrak{g}$ for any $i = 1,\ldots,n+1$. In the same way that we introduced the covariant derivative D^ω on $\Omega(P_\lambda;\mathfrak{g})$ from a principal connection $\omega\in\Omega^1(P_\lambda;\mathfrak{g})$, we consider the ***exterior covariant derivative*** $D^{\widetilde{W}} := \mathbf{d}\circ\mathrm{Hor}_{\widetilde{W}}$ which satisfies

$$D^{\widetilde{W}}\circ\Phi = \Phi\circ D^\omega$$

([T08, Proposition 2]). In particular, the field strength $\Phi\circ\Omega^\omega$ equals $\Omega^{\widetilde{W}} := D^{\widetilde{W}}\circ\widetilde{W} = \mathbf{d}\widetilde{W} + \frac{1}{2}[\widetilde{W},\widetilde{W}]$ and

$$\Omega^{\widetilde{W}}(z_1,z_2) = [W(\xi_1),W(\xi_2)] - W([\xi_1,\xi_2]),$$

where $z_i = (\xi_i,\eta_i)\in\mathfrak{k}\times\mathfrak{g}$, $i = 1,2$.

The field strength Ω^ω is a K-invariant ***horizontal form***, that is, it vanishes when contracted with any vector field taking values on the vertical space. It can also be checked that the image of horizontals forms $\Omega_{equiv}(P_\lambda;\mathfrak{g})^{\mathrm{Hor}}$ under Φ are those chains in $\Lambda_{\widetilde{\mathfrak{h}}}(\mathfrak{k}\times\mathfrak{g};\mathfrak{g})$ which only depend on elements in the horizontal space $\mathrm{Hor}_{\widetilde{W}}(\mathfrak{k}\times\mathfrak{g})$. Suppose that K/H is a symmetric space such that $\mathfrak{k} = \mathfrak{h}\oplus\mathfrak{m}$, $[\mathfrak{h},\mathfrak{h}]\subseteq\mathfrak{h}$, $[\mathfrak{h},\mathfrak{m}]\subseteq\mathfrak{m}$, and $[\mathfrak{m},\mathfrak{m}]\subseteq\mathfrak{h}$. Observe that $\mathrm{Hor}_{\widetilde{W}}(\xi,\eta) = (\xi, T_e\lambda(\xi))\in\widetilde{\mathfrak{h}}$ if $\xi\in\mathfrak{h}$. Then, since the chains in $\Lambda_{\widetilde{\mathfrak{h}}}(\mathfrak{k}\times\mathfrak{g};\mathfrak{g})$ vanish on $\widetilde{\mathfrak{h}}$, we can therefore identify $\Lambda_{\widetilde{\mathfrak{h}}}(\mathfrak{k}\times\mathfrak{g};\mathfrak{g})$ with the space $\Lambda_{\mathfrak{h}}(\mathfrak{m};\mathfrak{g})$ of $\mathfrak{g}$-valued chains on $\mathfrak{m}$ such that, if $\varphi\in\Lambda^r_{\mathfrak{h}}(\mathfrak{m};\mathfrak{g})$,

$$[T_e\lambda(\xi),\varphi(v_1,\ldots,v_r)]=\sum_{i=1}^{n}\varphi(v_1,\ldots,[\xi,v_i],\ldots,v_n)$$

where $\xi\in\mathfrak{h}$ and $\{v_1,\ldots,v_r\}\subset\mathfrak{m}$ (see (4.4)).

Example 10. If K/H is reductive, then $\mathfrak{k}=\mathfrak{h}\oplus\mathfrak{m}$ and $\mathrm{Ad}_h(\mathfrak{m})\subseteq\mathfrak{m}$ for any $h\in H$. The field strength $\Omega^{\widetilde{\mathbf{W}}}$ associated to the Wang map (canonical connection)

$$\mathbf{W}(\xi)=\begin{cases}T_e\lambda(\xi)\text{ if }\xi\in\mathfrak{h}\\0\text{ if }\xi\in\mathfrak{m}.\end{cases}$$

is given by $\Omega^{\widetilde{\mathbf{W}}}(v_1,v_2)=-T_e\lambda\left(\mathrm{proj}_{\mathfrak{h}}([v_1,v_2])\right)$, $v_1,v_2\in\mathfrak{m}$. ■

4.1. **Yang-Mills equations on symmetric spaces.** We are going to show that the curvature associated to the canonical connection on a symmetric space satisfies the Yang-Mills equations (Proposition 11). Thus, let $M=K/H$ be a homogeneous symmetric space, $\mathfrak{k}=\mathfrak{h}\oplus\mathfrak{m}$, and let P_λ be a homogeneous principal bundle given by the Lie group homomorphism $\lambda\colon H\to G$. The left K-action L_{P_λ} (3.3) on P_λ induces a natural K-action on $\Omega_{equiv}(P_\lambda;\mathfrak{g})^{\mathrm{Hor}}$ by means of the pull-backs $(L_{P_\lambda})^*_k$, $k\in K$, and hence on $\Omega(M;\mathrm{Ad}(P_\lambda))$ by the identification $\Omega(M;\mathrm{Ad}(P_\lambda))=\Omega_{equiv}(P_\lambda;\mathfrak{g})^{\mathrm{Hor}}$. If the Riemann metric on K/H and its associated volume form are K-invariant, so is the product $\langle\cdot,\cdot\rangle$ and the Hodge operator $*$ commutes with the K-action. That is,

$$*(k\cdot\varphi)=k\cdot(*\varphi)$$

for any $k\in K$ and any $\varphi\in\Omega(K/H;\mathrm{Ad}(P_\lambda))$ (see [T08, Subsection 2.6]). Consequently, $*$ preserves the space of K-invariant forms $\Omega(K/H;\mathrm{Ad}(P_\lambda))^K$. Since $P_\lambda\cong(K\times G)/\widetilde{H}$ and $\Phi:\Omega_{equiv}((K\times G)/\widetilde{H};\mathfrak{g})^{\mathrm{Hor}}\cong\Lambda_{\mathfrak{h}}(\mathfrak{m};\mathfrak{g})$, this implies that the Hodge operator $*$ can be carried to $\Lambda_{\mathfrak{h}}(\mathfrak{m};\mathfrak{g})$ simply imposing that

$$\Phi\circ*=*\circ\Phi,\tag{4.5}$$

where we also denote the new Hodge operator in the right hand side of (4.5) by $*$. Additionally, the covariant codifferential δ^ω is K-invariant as well for any $\omega\in\Omega^1_{equiv}(P_\lambda;\mathfrak{g})$ and, since both the Hodge operator and the covariant derivative commute with Φ, the operator

$$\begin{array}{rcl}\delta_{\widetilde{W}}:\Lambda_{\mathfrak{h}}(\mathfrak{m};\mathfrak{g}) & \longrightarrow & \Lambda_{\mathfrak{h}}(\mathfrak{m};\mathfrak{g})\\ \varphi & \longmapsto & -(-1)^{n(|\varphi|+1)}*\circ D^{\widetilde{W}}\circ*\varphi,\end{array}$$

where $\widetilde{W}=\Phi(\omega)$, is such that

$$\Phi\circ\delta^\omega=\delta_{\widetilde{W}}\circ\Phi.$$

Then, $\widetilde{W} = \Phi(\omega)$ is a Yang-Mills connection if and only if

$$\delta_{\widetilde{W}} \Omega^{\widetilde{W}} = 0. \tag{4.6}$$

In the following paragraphs, we are going to introduce some notation and carry out a few computations that will be useful later when working out some examples. In particular, we will justify (4.6) explicitly for the canonical connection on symmetric spaces and will explicitly exhibit the field strength of Example 10 for the homogeneous principal bundles given in Examples 4.

Recall that being $M = K/H$ symmetric, the Lie algebra $\mathfrak{k}$ can be decomposed as $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{m}$ such that $[\mathfrak{h}, \mathfrak{h}] \subseteq \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] \subseteq \mathfrak{m}$, and $[\mathfrak{m}, \mathfrak{m}] \subseteq \mathfrak{h}$. Let $n = \dim(K/H) = \dim(\mathfrak{m})$. Let $\{\xi_1, \ldots, \xi_{\dim(\mathfrak{h})}\}$ be a basis of $\mathfrak{h}$ and $\{\upsilon_1, \ldots, \upsilon_n\}$ be a basis of $\mathfrak{m}$. The commutation relations between the elements of the basis of $\mathfrak{h}$ and $\mathfrak{m}$ can be written as

$$[\xi_\alpha, \xi_\beta] = \sum_{\gamma=1}^{\dim(\mathfrak{h})} c^{\gamma}{}_{\alpha\beta} \xi_\gamma, \ [\xi_\alpha, \upsilon_i] = \sum_{j=1}^{n} d^{j}{}_{\alpha i} \upsilon_j, \ [\upsilon_i, \upsilon_j] = \sum_{\alpha=1}^{\dim(\mathfrak{h})} e^{\alpha}{}_{ij} \xi_\alpha.$$

On the other hand, let $\{\eta_1, \ldots, \eta_{\dim(\mathfrak{g})}\}$ be a basis of $\mathfrak{g}$, the Lie algebra of the structural group of a homogeneous principal bundle $\pi\colon P_\lambda \to K/H$ and suppose that

$$[\eta_a, \eta_b] = \sum_{c=1}^{\dim(\mathfrak{g})} r^{c}{}_{ab} \eta_c.$$

The dual basis associated to $\{\xi_1, \ldots, \xi_{\dim(\mathfrak{h})}\}$, $\{\upsilon_1, \ldots, \upsilon_n\}$, and $\{\eta_1, \ldots, \eta_{\dim(\mathfrak{g})}\}$ will be denoted with the same greek letters with upper indices, that is, $\{\xi^1, \ldots, \xi^{\dim(\mathfrak{h})}\}$, $\{\upsilon^1, \ldots, \upsilon^n\}$, and $\{\eta^1, \ldots, \eta^{\dim(\mathfrak{g})}\}$ respectively. The field strength $\Omega^{\widetilde{\mathbf{W}}}$ associated to the canonical connection (see Example 10) can be written as

$$\Omega^{\widetilde{\mathbf{W}}}_\lambda\left(\upsilon^i, \upsilon^j\right) = -T_e\lambda\left(\sum_{\alpha=1}^{\dim(\mathfrak{h})} e^{\alpha}{}_{ij} \xi_\alpha\right) = -\sum_{\alpha=1}^{\dim(\mathfrak{h})} \sum_{a=1}^{\dim(\mathfrak{g})} e^{\alpha}{}_{ij} \lambda^a_\alpha \eta_a, \tag{4.7}$$

$\upsilon^i, \upsilon^j \in \mathfrak{m}$, where $(\lambda^a_\alpha)^{a=1,\ldots,\dim(\mathfrak{g})}_{\alpha=1,\ldots,\dim(\mathfrak{h})}$ denotes the matrix of $T_e\lambda$ in the basis $\{\xi_1, \ldots, \xi_{\dim(\mathfrak{h})}\}$ and $\{\eta_1, \ldots, \eta_{\dim(\mathfrak{g})}\}$. In (4.7) we have made the dependence of $\Omega^{\widetilde{\mathbf{W}}}_\lambda$ with the homomorphism λ explicit.

K-invariant metrics on K/H are in one-to-one correspondence with $\mathrm{Ad}(H)$-invariant scalar products on $\mathfrak{m}$ ([KN69b, Chapter X Proposition 3.1]). Similarly, K-invariant volume forms on K/H correspond to $\mathrm{Ad}(H)$-invariant volume forms on $\mathfrak{m}$. So let $\mathbf{h}_\mathfrak{m}$ be the scalar product on $\mathfrak{m}$ inducing our Riemann structure on K/H and let μ be its corresponding volume element (we are not going to differentiate between the volume element on K/H and $\mathfrak{m}$). The metric $\mathbf{h}_\mathfrak{m}$ yields the musical isomorphism

$$\begin{array}{rcl} \flat:\mathfrak{m} & \longrightarrow & \mathfrak{m}^* \\ \upsilon & \longmapsto & \mathbf{h}_{\mathfrak{m}}(\upsilon,\cdot), \end{array}$$

whose inverse will be denoted by $\#\colon \mathfrak{m}^* \to \mathfrak{m}$. The musical isomorphisms will be used to lower and raise indices as it is customary in physics. For example, if $\{\varphi_i^a\}_{i=1,\ldots,n}^{a=1,\ldots,\dim(\mathfrak{g})}$ are the components of the $\mathfrak{g}$-valued one form $\varphi \in \Lambda^1(\mathfrak{m};\mathfrak{g})$, $\varphi = \sum_{i=1}^n \sum_{a=1}^{\dim(\mathfrak{g})} \varphi^a{}_i \upsilon^i \otimes \eta_a$, then $\{\varphi^{ai}\}_{i=1,\ldots,n}^{a=1,\ldots,\dim(\mathfrak{g})}$ will be the components of $\varphi^\# \in \Lambda^1(\mathfrak{m}^*;\mathfrak{g})$, $\varphi^\# = \sum_{i=1}^n \sum_{a=1}^{\dim(\mathfrak{g})} \varphi^{ai}\upsilon_i \otimes \eta_a$. That is, $\varphi^{ai} = \sum_{j=1}^n h^{ij}\varphi^a{}_j$, where $(h^{ij})_{i,j=1,\ldots,n}$ is the inverse matrix of $(h_{ij})_{i,j=1,\ldots,n}$, $h_{ij} = \mathbf{h}_{\mathfrak{m}}(\upsilon_i,\upsilon_j)$. It is worth noticing that, in principle, the elements of the dual basis $\{\upsilon^1,\ldots,\upsilon^n\}$ do not correspond to $\{\mathbf{h}_{\mathfrak{m}}(\upsilon_1,\cdot),\ldots,\mathbf{h}_{\mathfrak{m}}(\upsilon_n,\cdot)\}$. In other words, υ^i needs not be $\mathbf{h}_{\mathfrak{m}}(\upsilon_i,\cdot)$, $i=1,\ldots,n$. In order to solve this situation and avoid a confusing notation, we may suppose that $\{\upsilon^1,\ldots,\upsilon^n\}$ is an orthonormal basis with respect to $\mathbf{h}_{\mathfrak{m}}$. Then, $(h_{ij})_{i,j=1,\ldots,n}$ equals the identity matrix.

Finally, let $\varphi \in \Lambda_{\mathfrak{h}}(\mathfrak{m};\mathfrak{g})$ be expressed in the form

$$\varphi = \sum_{a=1}^{\dim(\mathfrak{g})} \sum_{i_1,\ldots,i_r}^{n} \varphi^a{}_{i_1\ldots i_r}(\upsilon^{i_1}\wedge\ldots\wedge\upsilon^{i_r})\otimes\eta_a.$$

It is shown in [T08] that

$$(*\varphi)^b{}_{j_1\ldots j_{n-r}} = \frac{1}{r!}\,|\mu|^{1/2} \sum_{i_1,\ldots,i_r=1}^{n} \varphi^{bi_1\ldots i_r}\epsilon_{i_1\ldots i_r j_1\ldots j_{n-r}}$$

where $|\mu| = \det(\mu) \neq 0$, ϵ is the completely antisymmetric Levi-Civita symbol, and the indices $i_1,\ldots,i_r$ have been raised with $\#$. Moreover, if W is the Wang map associated to the principal connection $\omega \in \Omega^1_{equiv}(P_\lambda;\mathfrak{g})$ then, for any $\varphi \in \Lambda_{\mathfrak{h}}(\mathfrak{m};\mathfrak{g})$,

$$\left(\delta_{\widetilde{W}}\varphi\right)(\zeta_1,\ldots,\zeta_r) = -\sum_{i=1}^{n}\left[(W|_{\mathfrak{m}})^\#(\upsilon^i),\varphi(\upsilon_i,\zeta_1,\ldots,\zeta_r)\right], \tag{4.8}$$

$\zeta_1,\ldots,\zeta_r \in \mathfrak{m}$ ([T08, Example 2.13]).

Proposition 11. *The canonical connection on a symmetric space is a Yang-Mills connection.*

Proof. The canonical connection satisfies $\mathbf{W}|_{\mathfrak{m}} = 0$. We see from (4.8) that $\delta_{\widetilde{\mathbf{W}}} = 0$. Consequently $\delta_{\widetilde{\mathbf{W}}}\Omega^{\widetilde{\mathbf{W}}} = 0$ and $\boldsymbol{\omega} = \Phi^{-1}(\widetilde{\mathbf{W}})$ is Yang-Mills. □

4.2. **Examples: invariant field strengths on the sphere.** We want to compute in this subsection the curvature associated to the canonical connection for the principal bundles described in Examples 4. Recall that they were principal bundles over the sphere $\mathbb{S}^n$ for some values of $n \in \mathbb{N}$ and several gauge groups G.

These curvatures will be useful later on in order to calculate the charge of the monopole for some of the classical examples found in the literature (Section 6).

Let $\mathfrak{k} = \mathfrak{so}(n+1) = \mathfrak{h} \oplus \mathfrak{m}$, where $\mathfrak{h} = \mathfrak{so}(n)$ are the $(n+1) \times (n+1)$ matrices of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \ B \text{ skew-symmetric of degree } n,$$

and $\mathfrak{m}$ is the subspace of all matrices of the form

$$\begin{pmatrix} 0 & -v^\top \\ v & 0 \end{pmatrix}, \tag{4.9}$$

where v is a (column) vector in $\mathbb{R}^n$. Let $\{\xi_{\alpha,\beta}\}_{\alpha>\beta}$, $\alpha, \beta \in \{1, \dots, n\}$, be the basis of $\mathfrak{so}(n)$ such that $\xi_{\alpha,\beta}$ is the matrix whose entries are 1 in the position (α, β), -1 in the position (β, α), and 0 elsewhere. Observe that, for the sake of a clearer notation, we label the basis of $\mathfrak{h}$ with two indices instead of a single one. Let $\{v_1, \dots, v_n\}$ be the canonical basis of $\mathbb{R}^n$, $v_i = (0, \overset{i-1)}{\dots}, 1, 0, \dots, 0)$, which is also a basis of $\mathfrak{m}$ using the correspondence given by (4.9). Then,

$$[v_i, v_j] = v_i v_j - v_j v_i = \xi_{j,i}, \ i < j,$$

Therefore, $[v_i, v_j] = \sum_{\alpha,\beta} e^{\alpha\beta}_{ij} \xi_{\alpha,\beta}$ implies $e^{\alpha\beta}_{ij} = 1$ if $\alpha = j$ and $\beta = i$ and 0 otherwise. In order to be coherent with our notation, we set $\xi_{i,j} = -\xi_{j,i}$ whenever $i > j$. Thus

$$\Omega^{\widetilde{\mathbf{W}}}_{\lambda}(v_i, v_j) = -\sum_{a=1}^{\dim(\mathfrak{g})} \lambda^a_{ji} \eta_a. \tag{4.10}$$

Let us particularize the field strength (4.10) for those gauge groups G given in Examples 4.

Examples 12.

(i) $n = 2$ and $G = U(1)$. Here $\mathfrak{g} = \mathfrak{h} = \mathfrak{u}(1)$ are both isomorphic to $i\mathbb{R}$, so $\dim(\mathfrak{g}) = \dim(\mathfrak{h}) = 1$. On the other hand, $\mathfrak{m} = \mathbb{R}^2$. Let $\lambda_m \colon U(1) \to U(1)$ given by $\lambda_m(\mathrm{e}^{iz}) = \mathrm{e}^{izm}$. Then $T_e\lambda_m \colon i\mathbb{R} \to i\mathbb{R}$ equals multiplying by m and $\Omega^{\widetilde{\mathbf{W}}}_{\lambda_m}(v_1, v_2) = -im \in i\mathbb{R} \cong \mathfrak{u}(1)$.

(ii) $n = 4$ and $G = SO(3)$. As we have already seen, the set $\mathcal{R}(4, SO(3))$ contains three elements. The trivial homomorphism $\lambda_{trivial} \colon SO(4) \to SO(3)$ sends any $h \in SO(4)$ to $e = \mathrm{Id} \in SO(3)$, $T_e\lambda_{trivial} = 0$, and consequently the corresponding fields strength $\Omega^{\widetilde{\mathbf{W}}}_{\lambda_{trivial}} = 0$ vanishes identically.

Let $\lambda_l \colon SO(4) \to SO(3)$, $l = 1, 2$, be the homomorphism induced by $\sigma_l \colon S^3 \times S^3 \to S^3$ respectively such that $\sigma_1(x, y) = x$ and $\sigma_2(x, y) = y$. One can prove that $\mathfrak{so}(4) = \mathfrak{so}(3)^{(1)} \times \mathfrak{so}(3)^{(2)}$, where $\mathfrak{so}(3)^{(l)}$ is the subalgebra spanned by $\{A^l, B^l, C^l\}$, $l = 1, 2$, such that $A^l = -\xi_{2,1} + (-1)^l \xi_{4,3}$, $B^l = -\xi_{3,2} + (-1)^l \xi_{4,1}$, and $C^l = -\xi_{3,1} + (-1)^{l+1} \xi_{4,2}$ (see [I81, Section 3]. The different sign in our expressions is due to a different choice of the basis $\{\xi_{\alpha,\beta}\}_{\alpha>\beta}$ of $\mathfrak{so}(4)$). Our initial basis can be written in terms of $\{A^l, B^l, C^l\}$, $l=1,2$, as

$$\begin{aligned} \xi_{2,1} &= -\frac{1}{2}\left(A^1+A^2\right),\ \xi_{3,1} = -\frac{1}{2}\left(C^1+C^2\right), \\ \xi_{3,2} &= -\frac{1}{2}\left(B^1+B^2\right),\ \xi_{4,1} = -\frac{1}{2}\left(B^1-B^2\right), \\ \xi_{4,2} &= \frac{1}{2}\left(C^1-C^2\right),\ \xi_{4,3} = -\frac{1}{2}\left(A^1-A^2\right). \end{aligned} \tag{4.11}$$

Both $\{A^1, B^1, C^1\}$ and $\{A^2, B^2, C^2\}$ can be regarded as basis of $\mathfrak{so}(3)$. The field strengths $\Omega^{\widetilde{\mathbf{W}}}_{\lambda_l}\colon \mathbb{R}^4\times\mathbb{R}^4\to\mathfrak{so}\,(3)$ satisfy

$$\begin{aligned} \Omega^{\widetilde{\mathbf{W}}}_{\lambda_l}(\upsilon_1,\upsilon_2) &= \frac{1}{2}A^l,\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_l}(\upsilon_1,\upsilon_3) = \frac{1}{2}C^l, \\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_l}(\upsilon_1,\upsilon_4) &= \frac{(-1)^{l+1}}{2}B^l,\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_l}(\upsilon_2,\upsilon_3) = \frac{1}{2}B^l, \\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_l}(\upsilon_2,\upsilon_4) &= \frac{(-1)^{l}}{2}C^l,\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_l}(\upsilon_3,\upsilon_4) = \frac{(-1)^{l+1}}{2}A^l, \end{aligned}$$

$l=1,2$.

(iii) $n=4$ and $G=SO(4)$. In this example, $\mathcal{R}\,(4, SO(4))$ contains 5 elements. The trivial homomorphism has associated a zero field strength. The identity $\lambda_{\mathrm{Id}}\colon SO(4)\to SO(4)$ has tangent map $T_e\lambda_{\mathrm{Id}}\colon \mathfrak{so}(4)\to\mathfrak{so}(4)$ such that $T_e\lambda_{\mathrm{Id}} = \mathrm{Id}|_{\mathfrak{so}(4)}$. Thus, $\Omega^{\widetilde{\mathbf{W}}}_{\lambda_{\mathrm{Id}}}(\upsilon_i,\upsilon_j) = -\xi_{ji}$, $\upsilon_i,\upsilon_j\in\mathbb{R}^4$, $i<j$.

Let $\lambda_i\colon SO(4)\to SO(4)$, $i=3,4$, be the homomorphism induced by $\sigma_i\colon S^3\times S^3\to S^3\times S^3$ respectively such that $\sigma_3(x,y)=(x,x)$ and $\sigma_4(x,y)=(y,y)$. Let $\{A^1,B^1,C^1,A^2,B^2,C^2\}$ be the basis of $\mathfrak{so}(4)$ introduced in **(ii)**. We are going to consider $\mathfrak{so}(4)$ as $\mathfrak{so}(3)\times\mathfrak{so}(3)$ and both $\{A^1,B^1,C^1\}$ and $\{A^2,B^2,C^2\}$ indistinguishably as bases of $\mathfrak{so}(3)$. Then, using (4.11),

$$\begin{aligned} \Omega^{\widetilde{\mathbf{W}}}_{\lambda_i}(\upsilon_1,\upsilon_2) &= \frac{1}{2}(A^i,A^i),\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_i}(\upsilon_1,\upsilon_3) = \frac{1}{2}(C^i,C^i), \\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_i}(\upsilon_1,\upsilon_4) &= \frac{(-1)^{i+1}}{2}(B^i,B^i),\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_i}(\upsilon_2,\upsilon_3) = \frac{1}{2}(B^i,B^i), \\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_i}(\upsilon_2,\upsilon_4) &= \frac{(-1)^{i}}{2}(C^i,C^i),\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_i}(\upsilon_3,\upsilon_4) = \frac{(-1)^{i+1}}{2}(A^i,A^i), \end{aligned}$$

$i = 1, 2$. Finally, let $\lambda_\delta \colon SO(4) \to SO(4)$ be the homomorphism induced by $\delta \colon S^3 \times S^3 \to S^3 \times S^3$, $\delta(x, y) = (y, x)$. In this case,

$$\Omega^{\widetilde{\mathbf{W}}}_{\lambda_\delta}(v_1, v_2) = \frac{1}{2}(A^2, A^1),\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_\delta}(v_1, v_3) = \frac{1}{2}(C^2, C^1),$$
$$\Omega^{\widetilde{\mathbf{W}}}_{\lambda_\delta}(v_1, v_4) = \frac{1}{2}(-B^2, B^1),\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_\delta}(v_2, v_3) = \frac{1}{2}(B^2, B^1),$$
$$\Omega^{\widetilde{\mathbf{W}}}_{\lambda_\delta}(v_2, v_4) = \frac{1}{2}(C^2, -C^1),\ \Omega^{\widetilde{\mathbf{W}}}_{\lambda_\delta}(v_3, v_4) = \frac{1}{2}(-A^2, A^1).$$

(iv) $n = 2k \geq 6$ and $G = SO(2k)$, $k \in \mathbb{N}$. As in the item **(ii)**, $\mathcal{R}(2k, SO(2k))$ contains three elements. The trivial homomorphism and the identity $\lambda_{\mathrm{Id}} \colon SO(2k) \to SO(2k)$ are similar to **(iii)**. The other element in $\mathcal{R}(2k, SO(2k))$ is the conjugation $\delta \colon SO(2k) \to SO(2k)$ by the diagonal matrix with entries $(-1, \dots, -1, 1)$. The tangent map $T_e\delta \colon \mathfrak{so}(2k) \to \mathfrak{so}(2k)$ acts on the basis $\{\xi_{\alpha,\beta}\}_{\alpha > \beta}$, $\alpha, \beta \in \{1, \dots, 2k\}$, as follows

$$T_e\delta(\xi_{\alpha,\beta}) = \begin{cases} \xi_{\alpha,\beta} \text{ if } \alpha \neq 2k \\ -\xi_{\alpha,\beta} \text{ if } \alpha = 2k. \end{cases}$$

Thus, $\Omega^{\widetilde{\mathbf{W}}}_{\delta}(v_i, v_j) = -\xi_{j,i}$ if $i < j$ and $j \neq 2k$ and $\Omega^{\widetilde{\mathbf{W}}}_{\delta}(v_i, v_{2k}) = \xi_{2k,i}$. ■

5. The Chern-Weil homomorphism. Characteristic classes

This section aims at recalling the concept of characteristic class and how the Chern-Weil homomorphism works. Roughly speaking, given a principal bundle $\pi \colon P \to M$, the Chern-Weil homomorphism associates an even differential form on M to the curvature Ω^ω. In order to do that, a symmetric Ad-invariant polynomial on $\mathfrak{g}$ is required so that the dependence of Ω^ω on the gauge indices can be removed. The most remarkable point is that the differential form on M defines a *de Rham* cohomology class which is independent of the principal connection $\omega \in \Omega^1(P; \mathfrak{g})$ under consideration. When its degree matches the dimension of M, the integral of such form over M defines a topological quantity that is interpreted as the charge of the configuration described by $\pi \colon P \to M$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let $S^k(\mathfrak{g}^*)$ be the set of maps $f \colon \mathfrak{g} \times \overset{k)}{\dots} \times \mathfrak{g} \to \mathbb{R}$ (or $\mathbb{C}$) which are multilinear and symmetric. That is, $f\left(\eta_{\sigma(1)}, \dots, \eta_{\sigma(k)}\right) = f(\eta_1, \dots, \eta_k)$ for any permutation $\sigma \in S_k$ of k elements. Let $S(\mathfrak{g}^*)^G = \oplus_{k \geq 0} S^k(\mathfrak{g}^*)^G$ be the symmetric algebra of multilinear functions on $\mathfrak{g}$ which are Ad-invariant. Explicitly, $f \in S^k(\mathfrak{g}^*)^G$ if $f \in S^k(\mathfrak{g}^*)$ and

$$f(\mathrm{Ad}_g(\eta_1), \dots, \mathrm{Ad}_g(\eta_k)) = f(\eta_1, \dots, \eta_k)$$

for any $g \in G$, any $\eta_1, \dots, \eta_k \in \mathfrak{g}$. For later convenience, we remark that the algebra $S(\mathfrak{g}^*)^G$ is isomorphic to the algebra $P(\mathfrak{g}^*)^G = \oplus_{k \geq 0} P^k(\mathfrak{g}^*)^G$ of Ad-invariant homogeneous polynomials on $\mathfrak{g}$ ([N00, Section 6.2]). This isomorphism works through *the polarization formula.* Indeed, if $f \in P^k(\mathfrak{g}^*)^G$ is an homogeneous polynomial of degree k, we define $\mathrm{Sym}(f) \in S^k(\mathfrak{g}^*)^G$ as

$$\mathrm{Sym}(f)(\eta_1, \ldots, \eta_k) = \frac{1}{k!} \sum_{i=0}^{k-1} (-1)^i \sum_{j_r \neq j_s} f\left(\eta_{j_1} + \ldots + \eta_{j_{k-i}}\right). \tag{5.1}$$

For example, if $k = 3$, then

$$\begin{aligned} \mathrm{Sym}(f)(\eta_1, \eta_2, \eta_3) = \frac{1}{6} [& f(\eta + \eta_2 + \eta_3) - f(\eta_1 + \eta_2) - f(\eta_1 + \eta_3) - f(\eta_2 + \eta_3) \\ & + f(\eta_1) + f(\eta_2) + f(\eta_3)]. \end{aligned}$$

Let $\pi \colon P \to M$ be a principal fiber bundle with structural Lie group G. Let $\omega \in \Omega^1_{equiv}(P; \mathfrak{g})$ be a principal connection and let $\Omega^\omega \in \Omega^2_{equiv}(P; \mathfrak{g})^{\mathrm{Hor}}$ its curvature. If $f \in S^k(\mathfrak{g}^*)^G$, then the $2k$-form

$$\begin{aligned} & \bar{f}(\Omega^\omega)(p)(X_1, \ldots, X_{2k}) \\ = & \frac{1}{2^k} \sum_{\sigma \in S_{2k}} (-1)^{|\sigma|} f\left(\Omega^\omega(p)(X_{\sigma(1)}, X_{\sigma(2)}), \ldots, \Omega^\omega(p)(X_{\sigma(2k-1)}, X_{\sigma(2k)})\right) \end{aligned}$$

is G-invariant and horizontal. Therefore, there exists a uniquely defined $2k$-form $cw(f, P, \omega) \in \Omega^{2k}(M)$ such that

$$\pi^*(cw(f, P, \omega)) = \bar{f}(\Omega^\omega).$$

The form $cw(f, P, \omega)$ is called the ***Chern-Weil form*** of f. What is more important, $cw(f, P, \omega)$ is closed, so there is a well defined *de Rham* cohomology class $[cw(f, P, \omega)] \in H^{2k}(M)$ called the ***characteristic class*** of the invariant polynomial f ([B81, Theorem 10.4.3], [M07, Theorem 20.3]), which is independent of the particular choice of $\omega \in \Omega^1_{equiv}(P; \mathfrak{g})$ ([B81, Theorem 10.4.11]). That is, it only depends on the fiber bundle structure of P. It is worth noticing that the proof of this fact uses that the polynomial f is *symmetric*. For example, the characteristic classes of a trivial principal bundle all vanish. In addition, the mapping

$$\begin{array}{rcl} Cw_P : S(\mathfrak{g}^*)^G & \longrightarrow & H^*(M) \\ f & \longmapsto & [cw(f, P, \omega)] \end{array}$$

is a homomorphism of commutative algebras, known as the ***Chern-Weil homomorphism***. If two principal bundles P and P' over M are isomorphic, they give rise to the same Chern-Weil homomorphism ([B81, Theorem 10.4.8]).

We are going to assume from now on that our Lie group G is contained in $GL(m, \mathbb{R})$ for some $m \in \mathbb{N}$. Roughly speaking, G may be thought as a classical matrix Lie group. For such groups, the adjoint action of G on $\mathfrak{g}$ has a simple expression. That is,

$$\mathrm{Ad}_g(\xi) = g\xi g^{-1}, \; g \in G, \; \xi \in \mathfrak{g},$$

where $g\xi g^{-1}$ is a product of matrices. For a matrix $A \in \mathfrak{gl}(m,\mathbb{R})$, the ***characteristic coefficient*** $c_k^m(A)$ are implicitly given by the equation

$$\det\left(t\,\mathrm{Id} + \frac{i}{2\pi}A\right) = \sum_{k=0}^{m} t^{m-k} c_k^m(A),\ t \in \mathbb{R}.$$

The characteristic coefficients are homogeneous polynomials of degree k which are Ad-invariant. Furthermore, they satisfy the recursive formula ([M07, Lemma 20.9])

$$c_k^m(A) = \frac{1}{k}\sum_{j=0}^{k-1}(-1)^{k-j-1}\left(\frac{i}{2\pi}\right)^{k-j} c_j^m(A)\,\mathrm{trace}(A^{k-j}),\ A \in \mathfrak{gl}(m,\mathbb{R}). \tag{5.2}$$

For example, it is easy to show from (5.2) that

$$c_0^m(A) = 1,\ c_1(A) = \frac{i}{2\pi}\,\mathrm{trace}\,A,\ c_2^m(A) = -\frac{1}{8\pi^2}\left[(\mathrm{trace}\,A)^2 - \mathrm{trace}\left(A^2\right)\right]$$
$$c_3^m(A) = -\frac{i}{48\pi^3}\left[(\mathrm{trace}\,A)^3 - 3\,\mathrm{trace}\left(A^2\right)\mathrm{trace}\,A + 2\,\mathrm{trace}\left(A^3\right)\right].$$

The k-th ***Chern class*** is defined as

$$c_k(P) := Cw_P(\mathrm{Sym}(c_k^m)) \in H^{2k}(M).$$

Among other characteristic classes, we choose the Chern classes because, despite the presence of the imaginary unit $i \in \mathbb{C}$ in their definition, they are actually real cohomology classes provided that G is a subgroup of the unitary group $U(m)$ as in our examples ([M07, 20.13]). If we write Ω^ω as a matrix valued two form $\left((\Omega^\omega)^i_j\right)_{i,j=1,\ldots,m}$, then

$$\pi^*\left(cw\left(\mathrm{Sym}(c_k^m), P, \omega\right)\right)$$
$$= \frac{(-1)^k}{(2\pi i)^k\,k!}\sum_{i_1<\ldots<i_k}\sum_{\sigma\in S_k}(-1)^{|\sigma|}(\Omega^\omega)^{i_1}_{\sigma(i_1)}\wedge\ldots\wedge(\Omega^\omega)^{i_k}_{\sigma(i_k)} \tag{5.3}$$

([KN69b, page 309]).

Example 13. The characteristic coefficients are usually algebraically independent and generate the algebra of polynomial functions on $\mathfrak{g}$ invariant by Ad_G, at least for some of the classical matrix groups such as $U(m)$ ([KN69b, Chapter XII]). However, we are going to deal in this example with $G = SO(m)$ whose Lie algebra $\mathfrak{g}$ is the algebra of skew-symmetric matrices or order $m \in \mathbb{N}$. The characteristic coefficients c_k^m are then equal to zero if k is odd, as it can be inductively checked from (5.2). Moreover, if $m = 2q+1$ is odd, then $\{c_2^m, \ldots, c_{2q}^m\}$ are indeed algebraically independent and generate $P\left(\mathfrak{so}(m)^*\right)^{SO(m)}$ ([KN69b, Chapter XII Theorem 2.7]). If $m = 2q$ is even, however, there exists a polynomial function Pf (unique up to a sign) such that $c_{2q}^m = (-1)^q(2\pi)^{-2q}\,\mathrm{Pf}^2$ and the functions $\{c_2^m, \ldots, c_{2(q-1)}^m, \mathrm{Pf}\}$ are algebraically independent and generate

$P(\mathfrak{so}(m)^*)^{SO(m)}$. The polynomial Pf is called ***the Pfaffian*** and, up to a factor, equals the square root of the determinant of a matrix. If the matrix $A \in \mathfrak{so}(m)$ is written as $A = (A^i_j)_{i,j=1,\dots,2q}$, then

$$\mathrm{Pf}\,(A) = \frac{1}{2^q q!} \sum_{\eta \in S_{2q}} (-1)^{|\eta|} A^{\eta(1)}_{\eta(2)} \cdots A^{\eta(2q-1)}_{\eta(2q)}.$$

The ***Euler class*** $\chi(P)$ is defined as $\frac{1}{\pi^q} Cw_P\,(\mathrm{Pf})$. If the curvature $\Omega^\omega \in \Omega^2_{equiv}(P;\mathfrak{g})^{\mathrm{Hor}}$ of some principal connection $\omega \in \Omega^1_{equiv}(P;\mathfrak{g})$ on $\pi\colon P \to M$ is written as a matrix valued two form $\left((\Omega^\omega)^i_j\right)_{i,j=1,\dots,2q}$, then

$$\pi^*\left(cw\,(\mathrm{Sym}(\mathrm{Pf}), P, \omega)\right) = \frac{1}{2^q \pi^q q!} \sum_{\eta \in S_{2q}} (-1)^{|\eta|} (\Omega^\omega)^{\eta(1)}_{\eta(2)} \wedge \cdots \wedge (\Omega^\omega)^{\eta(2q-1)}_{\eta(2q)} \quad (5.4)$$

([KN69b, Chapter XII Theorem 5.1]). ■

Finally, we are going to introduce the charge of a monopole. So let $\pi\colon P_\lambda \to \mathbb{S}^n$ be a homogeneous principal bundle over the n-dimensional sphere. The sphere equals the symmetric space $SO(n+1)/SO(n)$. It is a Riemann manifold with the Riemannian structure inherited from $\mathbb{R}^{n+1}$. We know by Theorem 9 that the canonical connection is the unique which is invariant by $SO(n+1)$. Let $\boldsymbol{\omega} \in \Omega^1_{equiv}(P;\mathfrak{g})$ denote the canonical connection and $\Omega^{\boldsymbol{\omega}} \in \Omega^2_{equiv}(P;\mathfrak{g})^{\mathrm{Hor}}$ its curvature, which is $SO(n+1)$-invariant by the left translations L_λ (Eq. (3.3)). Then $cw\,(\mathrm{Sym}(f), P_\lambda, \boldsymbol{\omega}) \in \Omega^{2k}(\mathbb{S}^n)$, $f \in P^k(\mathfrak{g}^*)^G$, is also invariant by the natural left $SO(n+1)$-action we have on $\mathbb{S}^n$. Suppose that $n = 2q$ is even. In that case, $cw(\mathrm{Sym}(f), P_\lambda, \boldsymbol{\omega})$, $f \in P^q(\mathfrak{g}^*)^G$, is proportional to the volume element μ of $\mathbb{S}^n$ induced from the standard metric, i.e.,

$$cw(\mathrm{Sym}(f), P_\lambda, \boldsymbol{\omega}) = \mathbf{d}\mu$$

for some function $\mathbf{d} \in C^\infty(\mathbb{S}^n)$. Since μ is also $SO(n+1)$ invariant, so is $\mathbf{d} \in C^\infty(\mathbb{S}^n)$. But the only functions on $\mathbb{S}^n$ which are invariant by the special orthogonal group are the constants, so $\mathbf{d} \in \mathbb{R}$. Regarding $\mathbb{S}^n$ as an imbedded submanifold of $\mathbb{R}^{n+1}$, we may consider $\mathbf{d}$ as a function of the radius. It is worth observing that, once $f \in P^q(\mathfrak{g}^*)^G$ is given, $\mathbf{d}$ can be easily computed from the expression of $\Omega^{\boldsymbol{\omega}}$ given in (4.10) (see examples in Section 6). The quantity

$$Q := \int_{\mathbb{S}^n} cw\,(\mathrm{Sym}(f), P_\lambda, \boldsymbol{\omega}) = \mathbf{d}\,\mathrm{vol}(\mathbb{S}^n) \quad (5.5)$$

will be called the ***charge of the monopole***. Up to a factor, it can be interpreted as the flow of the field strength $\Omega^{\boldsymbol{\omega}}$ trough the surface of the sphere $\mathbb{S}^n$. However, in order to match the order of $\Omega^{\boldsymbol{\omega}}$ with the dimension of $\mathbb{S}^n$ we need some characteristic class $cw\,(\mathrm{Sym}(f), P_\lambda, \boldsymbol{\omega})$, for example the Chern class (of suitable order). It is worth noticing that the charge of the monopole does not depend on the fact that we have worked with the canonical connection $\boldsymbol{\omega}$ because, as we already said, the Chern-Weil homomorphism does not depend on $\boldsymbol{\omega}$. In other words, it is a topological invariant. Obviously, the charge depends strongly on the choice of the invariant polynomial $f \in P^q(\mathfrak{g}^*)^G$ or, equivalently, on the characteristic class $cw\left(\mathrm{Sym}(f), P_\lambda, \boldsymbol{\omega}\right)$ and, for some $f \in P^q(\mathfrak{g}^*)^G$, *it could be zero even for non-trivial bundles.* As we will discuss in the examples, we will define the charge integrating on $\mathbb{S}^n$ either the Chern class $c_q(P)$, $n = 2q$, or the Euler class $\chi(P)$ in order to label all the non-isomorphic principal bundles over $\mathbb{S}^n$ with a different value of their charge. These two classes are, up to a constant factor, essentially the unique characteristic classes we can use to define the charge in most classical matrix Lie groups.

6. Examples

6.1. **The Dirac monopole.** The first one in introducing the concept of monopole was Dirac in the context of electromagnetic field theory [D31]. Dirac showed that there exist static singular solutions of the Maxwell equations on $\mathbb{R}^3\backslash\{0\}$ with a pointwise magnetic source placed at the origin $0 \in \mathbb{R}^3$. In order to be gauge invariant, the magnetic charge needed to be an integer in appropriate units. Since there is no evidence of the existence of such magnetic charge (despite the efforts carried out to find it since then), Dirac monopoles might have seemed useless at first sight. Nevertheless, and more importantly, the fact that the magnetic charge can only take discrete values implies in turn that the electric charge needs do so, as we experimentally observe. In other words, both the magnetic and the electric charge are *quantized.* Thus the relevance of such magnetic monopoles.

A free electromagnetic field is a Yang-Mills theory with gauge group $U(1)$. In particular, Dirac's monopoles are described as principal bundles over $\mathbb{S}^2$ (that is, principal bundles over $\mathbb{R}^3\backslash\{0\}$) with structural group $U(1)$. Since we require the potential vector field, and its corresponding field strength, to be $SO(3)$-invariant, such principal bundles $\pi\colon P_{\lambda_m} \to \mathbb{S}^2$ are in one-to-one correspondence with the homomorphisms $\lambda_m\colon U(1) \to U(1)$, $m \in \mathbb{Z}$, introduced in Examples 4 **(i)**. The $SO(3)$-invariant field strength $\Omega^{\boldsymbol{\omega}}$ is built from the canonical connection and computed at $o \in SO(3)/U(1) \cong \mathbb{S}^2$ in Subsection 4.2 **(i)**. Recall that $\Omega^{\widetilde{\mathbf{W}}}_{\lambda_m}(v_1, v_2) = -im$, where $\{v_1, v_2\}$ is the canonical basis of $\mathbb{R}^2 = T_o\mathbb{S}^2$. The charge associated to these configurations is given by integrating the first Chern class $[\frac{i}{2\pi}\,\mathrm{trace}\,\Omega^{\boldsymbol{\omega}}] \in H^2(\mathbb{S}^2)$ over $\mathbb{S}^2$. As we already pointed out, $\frac{i}{2\pi}\,\mathrm{trace}\,\Omega^{\boldsymbol{\omega}} = \mathbf{d}\mu$ for some constant $\mathbf{d} \in \mathbb{R}$ and where μ is the volume 2-form of $\mathbb{S}^2$. The constant $\mathbf{d}$ can be calculated as follows,

$$\mathbf{d} = \frac{i}{2\pi}\,\mathrm{trace}\left(\Omega^{\boldsymbol{\omega}}(o)\,(v_1, v_2)\right) = \frac{i}{2\pi}\,\mathrm{trace}\left(\Omega^{\widetilde{\mathbf{W}}}_{\lambda_m}(v_1, v_2)\right) = \frac{m}{2\pi},$$

and the charge

$$Q_m := \frac{i}{2\pi}\int_{\mathbb{S}^2}\mathrm{trace}\,\Omega^{\boldsymbol{\omega}} = \frac{m}{2\pi}\int_{\mathbb{S}^2}\mu = 2m.$$

6.2. **The Yang monopole.** Yang monopoles are non-trivial solutions of Yang-Mills theories on $\mathbb{R}^4\backslash\{0\}$ (equivalently on $\mathbb{S}^4$) with gauge group $G = SU(2)$. Unlike the general approach throughout this paper, where we considered the sphere as a quotient of orthogonal groups, we are now going to regard $\mathbb{S}^4$ as a quotient of spin groups, i.e., $\mathbb{S}^4 = Spin(5)/Spin(4)$. That is, we are going to describe principal bundles $\pi\colon P \to \mathbb{S}^4$ with gauge group $SU(2)$ and a left $Spin(5)$ action projecting onto the $Spin(5)$ action on $Spin(5)/Spin(4)$, which obviously coincides with the standard $SO(5)$ action on $\mathbb{S}^4$. In his paper [Y78], Yang describes monopole configurations on $\mathbb{S}^4$ which are invariant by the standard action of $SO(5)$. In our opinion, his description is imprecise and he should have talked about $Spin(5)$ invariant monopoles. Indeed, as the next proposition shows, there does not exist any non-trivial principal bundle over $\mathbb{S}^4$ with gauge group $SU(2)$ supporting a $SO(5)$ left action. However, since Yang worked with potentials and field strengths on $\mathbb{S}^4$ using local sections, he did not realize that his $SO(5)$ action actually came from a $Spin(5)$ action on the whole bundle.

Proposition 14. *The unique homomorphism of Lie groups $\lambda\colon SO(4) \to SU(2)$ from $SO(4)$ to $SU(2)$ is the trivial homomorphism, $\lambda(h) = e \in SU(2)$ for any $h \in SO(4)$.*

Proof. As we saw in Examples 4 **(ii)**, $Spin(4) = S^3 \times S^3$ and $SO(4) = (S^3 \times S^3)/\{(1,1),(-1,-1)\}$, where S^3 is the quaternionic sphere. Let $\tau\colon Spin(4) \to SO(4)$ be the covering homomorphism. On the other hand, we already argued that $SU(2) = S^3$. Recall that, modulo conjugation, the unique homomorphisms between $Spin(4)$ and $SU(2)$ are the trivial one and the projections $\sigma_l\colon S^3 \times S^3 \to S^3$, $l = 1, 2$, such that $\sigma_1(x, y) = x$ and $\sigma_2(x, y) = y$, $(x, y) \in S^3 \times S^3$.

Suppose that there exists a homomorphism $\lambda\colon SO(4) \to SU(2)$ different from the trivial one. Then, $\lambda \circ \tau\colon Spin(4) \to SU(2)$ is conjugated to σ_1 or σ_2. Assume that it is conjugated to σ_1. Therefore, there exists some $g \in SU(2)$ such that

$$\lambda_1 = g(\lambda \circ \tau) g^{-1} = g\lambda g^{-1} \circ \tau.$$

Replacing λ with $g\lambda g^{-1}$ if necessary, we may suppose that $\sigma_1 = \lambda \circ \tau$, where λ is different from the trivial homomorphism. But this is clearly a contradiction, since $\tau((x, y)) = \tau((-x, -y)) \in SO(4)$ and $\sigma_1((x, y)) \neq \sigma_1((-x, -y))$. □

In conclusion, we have two non-trivial homogeneous principal bundles $\pi_{\sigma_l}\colon P_{\sigma_l} \to \mathbb{S}^4$ associated to the homomorphisms $\sigma_l\colon Spin(4) = S^3 \times S^3 \to SU(2) = S^3$, $l = 1, 2$. It is worth noting that $\pi_{\sigma_1}\colon P_{\sigma_1} \to \mathbb{S}^4$ was already identified in [ACO83, Subsection 4.3] as the principal bundle behind the BPST instanton.

We want to compute the charge Q (Eq. (5.5)) of $\pi_{\sigma_l}\colon P_{\sigma_l} \to \mathbb{S}^4$, $l = 1, 2$, by means of the second Chern class. According to [N00], two principal bundles over $\mathbb{S}^4$ and gauge group $SU(2)$ are isomorphic if and only if they have the same Chern number. Therefore, the charge provided by the second Chern class seems a good topological invariant to differentiate the two non-trivial monopole configurations.

The field strengths associated to the $Spin(5)$-invariant canonical connections of $\pi_{\sigma_l}\colon P_{\sigma_l} \to \mathbb{S}^4$, $l=1,2$, are given in Subsection 4.2 **(ii)**. Indeed, $\Omega^{\widetilde{\mathbf{W}}}_{\lambda_l}$, $l=1,2$, in Subsection 4.2 **(ii)** are the curvatures associated to the homomorphisms $\lambda_l\colon SO(4) \to SO(3)$ which, in turn, are induced from $\sigma_l\colon Spin(4) \to SU(2)$. Since in order to compute de curvatures of the canonical connections we only need the tangent maps $T_e\lambda_l\colon \mathfrak{so}(4) \to \mathfrak{so}(3)$ and the Lie algebras $\mathfrak{spin}(4)$ and $\mathfrak{su}(2)$ coincide with $\mathfrak{so}(4)$ and $\mathfrak{so}(3)$ respectively, $\Omega^{\widetilde{\mathbf{W}}}_{\lambda_l}\colon \mathbb{R}^4 \times \mathbb{R}^4 \to \mathfrak{su}(2)$ are the $Spin(5)$-invariant curvatures evaluated at $p=[e,e]^\sim \in P_{\sigma_l}$. However, observe that these field strengths take values in two subalgebras of $\mathfrak{so}(4)$, those generated by the matrices $\{A^l, B^l, C^l\}$, $l=1,2$, which are isomorphic to $\mathfrak{su}(2)$. We need to implement these isomorphisms explicitly since, in order to compute the second Chern class using (5.3), $\mathfrak{su}(2)$ must be regarded as Lie algebra of complex matrices contained in $\mathfrak{u}(m)$ for some $m \in \mathbb{N}$. The easiest solution is to establish the correspondence

$$A^l \mapsto \frac{i}{2}\sigma^1 = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}, \quad B^l \mapsto \frac{i}{2}\sigma^2 = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \quad C^l \mapsto -\frac{i}{2}\sigma^3 = \begin{pmatrix} -i/2 & 0 \\ 0 & i/2 \end{pmatrix},$$

$l=1,2$, where $\{\sigma^1,\sigma^2,\sigma^3\}$ are the Pauli matrices.

The second Chern classes $cw\,(S(c_2^2), P_{\sigma_l}, \omega)$, $l=1,2$, are proportional to the canonical volume element μ of $\mathbb{S}^4$, $cw\,(S(c_2^2), P_{\sigma_l}, \omega) = \mathbf{d}\mu$. The constant of proportionality $\mathbf{d}$ can be obtained as

$$\mathbf{d} = cw\left(S(c_2^2), P_{\sigma_l}, \omega\right)(\upsilon_1, \upsilon_2, \upsilon_3, \upsilon_4),$$

where $\{\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4\}$ is the canonical orthonormal basis of $\mathbb{R}^4 \cong T_o\mathbb{S}^4$. Identifying $\mathbb{R}^4 = \mathfrak{m} \subset \mathfrak{so}(4)$ with the horizontal space Hor_p at $p=[e,e]^\sim \in P_{\sigma_l}$, $\mathbf{d}$ can be computed inserting the explicit expressions for $\Omega^{\widetilde{\mathbf{W}}}_{\lambda_l}$ in (5.3). We prefer, however, giving directly the charge $Q = \mathbf{d}\,\mathrm{vol}\,(\mathbb{S}^4)$, which equals $-\frac{1}{8}$ for the homogeneous bundle $\pi_{\sigma_1}\colon P_{\sigma_1} \to \mathbb{S}^4$ and $\frac{1}{8}$ for $\pi_{\sigma_2}\colon P_{\sigma_2} \to \mathbb{S}^4$. These results are in complete agreement with [Y78] and we therefore omit explicit computations. If Yang gave the Chern number -1 and 1 respectively to these bundles was because, in his definition of the second Chern class, he chose a coefficient 8 times greater than ours. Since two principal $SU(2)$-bundles over $\mathbb{S}^4$ are isomorphic if and only if they have the same Chern number ([N00]), the two principal bundles with non-vanishing charge we obtained are isomorphic to Yang's.

6.3. **Gibbons and Townsend monopoles.** In his paper [GT06], Gibbons and Townsend study monopole configurations over the sphere $\mathbb{S}^{2q}$, with gauge group $SO(2q)$, $q \geq 2$. However, they only deal with the principal bundle $SO(2q+1) \to SO(2q+1)/SO(2q)$, which corresponds to the homogeneous principal bundle $P_{\lambda_{\mathrm{Id}}}$ given by the identity homomorphism $\lambda_{\mathrm{Id}}\colon SO(2q) \to SO(2q)$ ([GT06, Section 4]), and exhibit the corresponding $SO(2q+1)$-invariant (canonical) connection. In addition, they define the charge of the monopole as the integral over $\mathbb{S}^{2q}$ of the $2q$-form

$$\mathrm{trace}\left(\Omega^{\boldsymbol{\omega}} \wedge \overset{q)}{\ldots} \wedge \Omega^{\boldsymbol{\omega}}\right), \tag{6.1}$$

where Ω^ω is the field strength associated to the canonical principal connection. Up to a constant factor, this characteristic class coincides with the Chern class for the case $k=2$. In our opinion, some authors choose (6.1) to define the charge (see for instance [M08]) because it is a straightforward generalization of the integrand trace $(\Omega^\omega \wedge \Omega^\omega)$ used by Yang to compute the charge of his monopole. In [Y78], Yang points out that he deliberately chooses the second Chern class. Nevertheless, it is not clear to which $\mathrm{Ad}_{SO(2q)}$-invariant polynomial $f \in P(\mathfrak{so}(2q)^*)^{SO(2q)}$, if any, corresponds the $2q$-form (6.1). Moreover, it is claimed in [GT06], but no proof is provided, that the field strength Ω^ω can be written in a suitable basis of $\mathfrak{so}(2q)$ such that

$$\int_{\mathbb{S}^{2q}} \operatorname{trace}\left(\Omega^{\boldsymbol{\omega}} \wedge \overset{q)}{\cdots} \wedge \Omega^{\boldsymbol{\omega}}\right) \neq 0. \tag{6.2}$$

In our opinion, this result is not correct. The argument against (6.2) works as follows: since $\operatorname{trace}(\Omega^{\boldsymbol{\omega}} \wedge \overset{q)}{\cdots} \wedge \Omega^{\boldsymbol{\omega}})$ is proportional to the natural volume element μ of $\mathbb{S}^{2q}$, we only need to compute the constant of proportionality $\mathbf{d}$ in order to value (6.2). Furthermore, this computation can be carried out at any point $m \in \mathbb{S}^{2q}$ of the sphere. If $\{\upsilon_1, \ldots, \upsilon_{2q}\}$ is an orthonormal basis of $T_m\mathbb{S}^{2q} \cong \mathbb{R}^{2q}$, then

$$\mathbf{d} = \operatorname{trace}\left(\Omega^{\boldsymbol{\omega}} \wedge \overset{q)}{\cdots} \wedge \Omega^{\boldsymbol{\omega}}\right)(\upsilon_1, \ldots, \upsilon_{2q}).$$

Let $o \in \mathbb{S}^{2q}$. Since the charge is a topological invariant, we can compute it using any field strength on $SO(2q+1) \to SO(2q+1)/SO(2q)$. According to Subsection 4.2 **(iv)**, the field strength at o is given by $\Omega^{\widetilde{\mathbf{W}}}_{\lambda_{\mathrm{Id}}}(\upsilon_i, \upsilon_j) = -\xi_{j,i} = \xi_{i,j} \in \mathfrak{so}(2q)$. The matrix $\xi_{j,i}$ has entries

$$(\xi_{j,i})^\alpha_\beta = (-1)^{U(i-j)}(-1)^{U(\beta-\alpha)}\delta^\alpha_j \delta_{i\beta}, \tag{6.3}$$

where U is the Heaviside step function, $U(x)=1$ if $x>0$ and $U(x)=0$ if $x \leq 0$. Therefore

$$\begin{aligned} &\operatorname{trace}\left(\Omega^{\boldsymbol{\omega}} \wedge \overset{q)}{\cdots} \wedge \Omega^{\boldsymbol{\omega}}\right)(\upsilon_1, \ldots, \upsilon_{2q}) \\ &= \frac{1}{2^q}\operatorname{trace}\Big(\sum_{\sigma \in S_{2q}} (-1)^{|\sigma|}\,\Omega^{\widetilde{\mathbf{W}}}_{\lambda_{\mathrm{Id}}}\left(\upsilon_{\sigma(1)}, \upsilon_{\sigma(2)}\right) \cdots \Omega^{\widetilde{\mathbf{W}}}_{\lambda_{\mathrm{Id}}}\left(\upsilon_{\sigma(2q-1)}, \upsilon_{\sigma(2q)}\right)\Big) \\ &= \frac{(-1)^q}{2^q}\operatorname{trace}\Big(\sum_{\sigma \in S_{2q}} (-1)^{|\sigma|}\,\xi_{\sigma(2),\sigma(1)} \cdots \xi_{\sigma(2q),\sigma(2q-1)}\Big) \end{aligned}$$

but $\xi_{\sigma(2),\sigma(1)} \cdots \xi_{\sigma(2q),\sigma(2q-1)} = 0$ for any $\sigma \in S_{2q}$ because the matrix product $\xi_{j,i}\xi_{r,s}$ is zero if the indices (j,i) are different from (r,s). Thus, $\operatorname{trace}\left(\Omega^{\boldsymbol{\omega}} \wedge \overset{q)}{\cdots} \wedge \Omega^{\boldsymbol{\omega}}\right)=0$.

The Chern class is not useful to define the monopole charge either, since it also vanishes. The details are given in Subsection A.3 in the Appendix for the sake of a clearer exposition. Things are different as far as the Euler class is concerned.

Indeed, we also prove in Subsection A.3 that

$$Q = \int_{\mathbb{S}^{2q}} cw\left(\mathrm{Sym}(\mathrm{Pf}), P_{\lambda_{\mathrm{Id}}}, \boldsymbol{\omega}\right) = 2,$$

which is obviously the Euler-Poincaré characteristic of the sphere $\mathbb{S}^{2q}$. Since $SO(2q+1) \to \mathbb{S}^{2q}$ can be regarded as the orthogonal frame bundle, this equality is simply a restatement of one of the possible versions of the Gauss-Bonet Theorem (see [D78, page 112]).

If $q \geq 3$, then, up to isomorphism, there only exists another principal bundle structure over $\mathbb{S}^{2q}$, $\pi\colon P_\delta \to \mathbb{S}^{2q}$, that given by the homomorphism $\delta\colon SO(2q) \to SO(2q)$ introduced in Subsection 4.2 **(iv)**. In order to obtain the Euler class $\chi\left(P_\delta\right)$, one can repeat the same computations carried out in Subsection A.3 in the Appendix just replacing $\Omega^{\widetilde{\mathbf{W}}}_{\lambda_{\mathrm{Id}}}$ with $\Omega^{\widetilde{\mathbf{W}}}_{\delta}$. If we do so, it is not difficult to realize that a -1 appears in each term of (A.9) and, therefore, $\int_{\mathbb{S}^{2q}} \chi\left(P_\delta\right) = -2$. In other words, $P_{\lambda_{\mathrm{Id}}}$ and P_δ have the same charge with opposite sign. The details are left to the reader.

Appendix A

A.1. **Proof of Proposition 1.** Before proving Proposition 1, we need an auxiliary lemma:

Lemma 15. *Let $\alpha \in \Omega^k\left(\mathbb{S}^{2n}\right)$ and $f\colon \mathbb{R}^{2n+1}\backslash\{0\} \to \mathbb{S}^{2n}$ as in Equation (2.1). If $r \in C^\infty\left(\mathbb{R}^{2n+1}\backslash\{0\}\right)$ is the radius function, $r\left(x\right) = \|x\|$, then*

$$*f^*(\alpha) = r^{2(n-k)} f^*\left(*\alpha\right) \wedge dr. \tag{A.1}$$

Proof. Let $y \in \mathbb{R}^{2n+1}\backslash\{0\}$ be an arbitrary point and $z = y/\left\|y\right\| \in \mathbb{S}^{2n}$. We can take global Euclidean coordinates $(x^1, \ldots, x^{2n+1})$ on $\mathbb{R}^{2n+1}\backslash\{0\}$ such that $y = (0, \overset{2n)}{\ldots}, 0, r(y))$. Then $z = (0, \overset{2n)}{\ldots}, 0, 1) \in \mathbb{S}^{2n} \subset \mathbb{R}^{2n+1}\backslash\{0\}$. That is, z can be regarded as the north pole of the sphere $\mathbb{S}^{2n}$. The tangent space $T_y(\mathbb{R}^{2n+1}\backslash\{0\})$ can be decomposed as the direct sum

$$T_y(\mathbb{R}^{2n+1}\backslash\{0\}) = T_y S_{r(y)} \oplus W_y$$

where $T_y S_{r(y)}$ is the tangent space to the sphere $S_{r(y)}$ of radius $r(y)$ at y and W_y is its orthogonal complement, in the radial direction. The first $2n$ coordinates $(x^1, \ldots, x^{2n+1})$ we have on $\mathbb{R}^{2n+1}\backslash\{0\}$ can be used around z on $\mathbb{S}^{2n}$ by means of the local diffeomorphism

$$\left(x^1, \ldots, x^{2n}\right) \longmapsto \left(x^1, \ldots, x^{2n}, \sqrt{1 - \sum\nolimits_{i=1}^{2n} \left(x^i\right)^2}\right).$$

Observe that the vector fields $\left\{\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^{2n}}\right\}$ form an orthonormal basis at $z \in \mathbb{S}^{2n}$ and that, as a vector space, $T_y S_{r(y)}$ is isomorphic to $T_z \mathbb{S}^{2n}$. In this context, it is easy to see that

$$T_y f = \frac{1}{r(y)} \operatorname{Id} \circ \operatorname{proj}|_{T_y S_{r(y)}},$$

where $\operatorname{proj}|_{T_y S_{r(y)}} : T_y(\mathbb{R}^{2n+1} \backslash \{0\}) \to T_y S_{r(y)}$ denotes the projection onto $T_y S_{r(y)}$ and the isomorphism $T_y S_{r(y)} \cong T_z \mathbb{S}^{2n}$ has been used. Consequently, if α is locally written as $\sum_{i_1<\ldots<i_k} \alpha_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k}$ around z, it is immediate to see that

$$f^*(\alpha)(y) = \frac{1}{r^k(y)} \sum_{i_1<\ldots<i_k} \alpha_{i_1\ldots i_k}(z) \left(dx^{i_1} \wedge \ldots \wedge dx^{i_k}\right)(y).$$

On the other hand,

$$(*\alpha)(z) = \frac{1}{k!} \sum_{j_1<\ldots<j_{2n-k}} \sum_{i_1<\ldots<i_k} \alpha^{i_1\ldots i_k}(z)\, \varepsilon^{1\ldots 2n}_{i_1\ldots i_k j_1 \ldots j_{2n-k}} \left(dx^{j_1} \wedge \ldots \wedge dx^{j_{2n-k}}\right)(z),$$

$$f^*(*\alpha)(y) = \frac{1}{k! r^{2n-k}(y)} \sum_{j_1<\ldots<j_{2n-k}} \sum_{i_1<\ldots<i_k} \alpha^{i_1\ldots i_k}(z)\, \varepsilon^{1\ldots 2n}_{i_1\ldots i_k j_1 \ldots j_{2n-k}} \left(dx^{j_1} \wedge \ldots \wedge dx^{j_{2n-k}}\right)(y),$$

and $*(f^*\alpha)(y)$ equals

$$\frac{1}{k! r^k(y)} \sum_{j_1<\ldots<j_{2n+1-k}} \sum_{i_1<\ldots<i_k} \alpha^{i_1\ldots i_k}(z)\, \varepsilon^{1\ldots 2n+1}_{i_1\ldots i_k j_1 \ldots j_{2n+1-k}} \left(dx^{j_1} \wedge \ldots \wedge dx^{j_{2n+1-k}}\right)(y). \tag{A.2}$$

In these equations, $\varepsilon^{1\ldots 2n}_{i_1\ldots i_k j_1 \ldots j_{2n-k}}$ denotes the totally antisymmetric symbol and $\alpha^{i_1\ldots i_k}(z) = \alpha_{i_1\ldots i_k}(z)$ because, in the coordinates we chose, the matrix of the Euclidean metric is diagonal on both $y \in \mathbb{R}^{2n+1}\backslash\{0\}$ and $z \in \mathbb{S}^{2n}$. Now, in each non-zero term on the right hand side of (A.2), the differential dx^{2n+1} appears. We can move it to the last right position just taking into account a possible additional $(-1)^{|\sigma|}$ for a suitable permutation σ. This $(-1)^{|\sigma|}$, however, cancels with the same $(-1)^{|\sigma|}$ that comes from moving the index $j_l = 2n+1$ in $\varepsilon^{1\ldots 2n+1}_{i_1\ldots i_k j_1 \ldots j_{2n+1-k}}$ to the last right position. In this case, $\varepsilon^{1\ldots 2n+1}_{i_1\ldots i_k j_1 \ldots j_l} = \varepsilon^{1\ldots 2n}_{i_1\ldots i_k j_1 \ldots j_{2n-k}}$. Therefore, (A.2) equals

$$\begin{aligned} \frac{1}{k! r^k(y)} \left(\sum_{j_1<\ldots<j_{2n-k}} \sum_{i_1<\ldots<i_k} \alpha^{i_1\ldots i_k}(z)\, \varepsilon^{1\ldots 2n}_{i_1\ldots i_k j_1 \ldots j_{2n-k}} \left(dx^{j_1} \wedge \ldots \wedge dx^{j_{2n-k}}\right)(y) \right) \wedge dx^{2n+1}(y) \\ = r^{2(n-k)}(y) \left(f^*(*\alpha)(y)\right) \wedge dx^{2n+1}(y). \end{aligned}$$

Finally, observe that dr coincides with dx^{2n+1} at y, $dr(y) = dx^{2n+1}(y)$, so

$$*f^*(\alpha)(y) = r^{2(n-k)}(y) \left(f^*(*\alpha)(y)\right) \wedge dr(y).$$

Since the point $y \in \mathbb{R}^{2n+1}\backslash\{0\}$ we chose was completely arbitrary, we conclude that (A.1) holds globally. □

Proof of Proposition 1. Let $\varphi \in \Omega^k_{equiv}(P; \mathfrak{g})^{\mathrm{Hor}}$ and let $F^* \colon f^*(P) \to P$ be the natural bundle homomorphism from the pull-back of $\pi \colon P \to S^{2n}$ by f (see 2.1).

$F^*(\varphi)$ can be naturally seen as a form in $\Omega^k_{equiv}(f^*(P);\mathfrak{g})^{\mathrm{Hor}}$. It is not difficult to realize then from Lemma 15 that

$$*F^*(\varphi) = \overline{\pi}^*\left(r^{2(n-k)}\right) F^*(*\varphi)\ \bar{\wedge}\ \overline{\pi}^*(dr),$$

where $\overline{\pi}\colon f^*(P) \to \mathbb{R}^{2n+1}\backslash\{0\}$ and the product $\bar{\wedge}$ of two forms $\beta \in \Omega^r(f^*(P);\mathfrak{g})$ and $\alpha \in \Omega^q(f^*(P))$ must be understood through the product of an element of the vector space $\mathfrak{g}$ by a real number; that is,

$$\beta\ \bar{\wedge}\ \alpha(Y_1,\ldots,Y_{r+q}) = \frac{1}{r!q!}\sum_{\sigma\in S_{r+q}}(-1)^{|\sigma|}\underbrace{\alpha\left(Y_{\sigma(1)},\ldots,Y_{\sigma(r)}\right)}_{\in\mathbb{R}}\ \underbrace{\beta\left(Y_{\sigma(r+1)},\ldots,Y_{\sigma(r+q)}\right)}_{\in\mathfrak{g}},$$

for any $\{Y_1,\ldots,Y_{r+q}\}\subset\mathfrak{X}(f^*(P))$.

Let now $\omega\in\Omega^1_{equiv}(P;\mathfrak{g})$ be a principal connection and $F^*(\omega)\in\Omega^1_{equiv}(f^*(P);\mathfrak{g})$ the corresponding principal connection on $\overline{\pi}\colon f^*(P)\to\mathbb{R}^{2n+1}\backslash\{0\}$. Observe that

$$T_yF(\mathrm{Hor}_y) = T_yF\left(\ker\left(F^*(\omega)(y)\right)\right) = \ker\omega(F(y)) = \mathrm{Hor}_{F(y)}\subset T_{F(y)}P$$

therefore, as far as their field strengths is concerned ([M07, 17.5]),

$$\begin{aligned}\Omega^{F^*(\omega)} &= D^{F^*(\omega)}(F^*(\omega)) = d\circ F^*(\omega)|_{\mathrm{Hor}_y}\\ &= F^*(d\circ\omega)|_{\mathrm{Hor}_y} = F^*\left(d\circ\omega|_{\mathrm{Hor}_{F(y)}}\right) = F^*\left(\Omega^\omega\right).\end{aligned}$$

Then,

$$\begin{aligned}-\delta^{F^*(\omega)}\Omega^{F^*(\omega)} &= *\circ D^{F^*(\omega)}\circ *\left(\Omega^{F^*(\omega)}\right) = *\circ D^{F^*(\omega)}\circ *\left(F^*\left(\Omega^\omega\right)\right)\\ &= *\circ D^{F^*(\omega)}\left(\overline{\pi}^*\left(r^{2(n-2)}\right)F^*\left(*\Omega^\omega\right)\bar{\wedge}\,\overline{\pi}^*(dr)\right)\\ &= *\circ d\left(\overline{\pi}^*\left(r^{2(n-2)}\right)F^*\left(*\Omega^\omega\right)\ \bar{\wedge}\,\overline{\pi}^*(dr)\right)\big|_{\mathrm{Hor}}\\ &= *\left(\overline{\pi}^*\left(r^{2(n-2)}\right)F^*\left(d\circ*(\Omega^\omega)|_{\mathrm{Hor}}\right)\bar{\wedge}\,\overline{\pi}^*(dr)\right)\end{aligned}$$

where in the last line we have used that $\overline{\pi}^*(dr)$ was already a horizontal form. Thus,

$$\delta^{F^*(\omega)}\Omega^{F^*(\omega)} = -*\left(\overline{\pi}^*\left(r^{2(n-2)}\right)F^*\left(D^\omega\circ*(\Omega^\omega)\right)\bar{\wedge}\,\overline{\pi}^*(dr)\right)$$

Now, if $\alpha\in\Omega^k_{equiv}(P;\mathfrak{g})^{\mathrm{Hor}}$, then

$$*\circ*(\alpha) = (-1)^{k(m-k)}\alpha, \tag{A.3}$$

where $m=2n+1$ or $m=2n$ if the base manifold is $\mathbb{R}^{2n+1}\backslash\{0\}$ or $\mathbb{S}^{2n}$ respectively. On the other hand, by Lemma 15,

$$\begin{aligned}*\left(F^*\left(*\circ D^\omega\circ*(\Omega^\omega)\right)\right) &= \overline{\pi}^*\left(r^{2(n-1)}\right)F^*\left(*\circ*\circ D^\omega\circ*(\Omega^\omega)\right)\overline{\pi}^*(dr)\\ &= (-1)^{2n-1}\,\overline{\pi}^*\left(r^{2(n-1)}\right)F^*\left(D^\omega\circ*(\Omega^\omega)\right)\bar{\wedge}\,\overline{\pi}^*(dr).\end{aligned}\tag{A.4}$$

Taking the Hodge operator in both sides of (A.4) and using (A.3),

$$\left(F^*\left(*\circ D^\omega\circ*(\Omega^\omega)\right)\right) = -*\left(\overline{\pi}^*\left(r^{2(n-1)}\right)F^*\left(D^\omega\circ*(\Omega^\omega)\right)\bar{\wedge}\,\overline{\pi}^*(dr)\right),$$

so

$$\delta^{F^*(\omega)}\Omega^{F^*(\omega)} = \overline{\pi}^* \left(\frac{r^{2(n-2)}}{r^{2(n-1)}}\right) F^* \left(* \circ D^\omega \circ *(\Omega^\omega)\right) = -\frac{1}{\overline{\pi}^* \left(r^2\right)} F^* \left(\delta^\omega \Omega^\omega\right). \quad \square$$

A.2. **Proof of Proposition 6.** Sometimes, principal connections are more conveniently described by means of a one form $\Phi \in \Omega^1(P; VP)$ with values on the vertical bundle $VP = \cup_{p\in P} \mathrm{Ver}_p$,

$$\Phi_p(X) = T_e R_p \circ \omega_p(X).$$

In this expression $X \in \mathfrak{X}(P)$, $p \in P$, $e \in G$ denotes the unit element, and $R_p \colon G \to P$ is the right action $R_p(g) := R(g,p)$ for any $g \in G$. The principal connection Φ satisfies that $TR_g \circ \Phi = \Phi \circ TR_g$ or, equivalently, $\Phi = TR_{g^{-1}} \circ \Phi \circ TR_g$ for any $g \in G$.

In the particular case of homogeneous principal bundles $\pi \colon P_\lambda \to K/H$, principal connections $\Phi_\lambda \in \Omega^1(P_\lambda; VP_\lambda)$ can be built from principal connections $\Phi \in \Omega^1(K; VK)$ on $K \to K/H$. In order to show how this construction works, we are going to explicitly describe TP_λ. First of all, it can be proved that $T\pi \colon TK \to T(K/H)$ is again a principal bundle with structural group TH with right action,

$$\begin{array}{rcl} TR : TK \times TH & \longrightarrow & TK \\ \left((k, X_k), (h, X_h)\right) & \longmapsto & \left(kh, T_h L_k(X_h) + T_k R_h(X_k)\right), \end{array} \tag{A.5}$$

where $X_h \in T_h H$ and $X_k \in T_k K$. In addition, if $\mathrm{inv} \colon H \to H$, $\mathrm{inv}(h) := h^{-1}$ denotes the inverse map of the Lie group H, TH acts on TG by the right action

$$\begin{array}{rcl} TG \times TH & \to & TK \\ \left((g, X_g), (h, X_h)\right) & \mapsto & \left(\lambda(h)^{-1} g, T_g L_{\lambda(h)^{-1}}(X_g) + T_{\lambda(h)^{-1}} R_g \circ T_{h^{-1}}\lambda \circ T_h \,\mathrm{inv}(X_h)\right), \end{array} \tag{A.6}$$

so that the tangent space TP_λ equals the associated bundle $TK \times_{TH} TG$ ([M07, Theorem 18.18]). That is, TP_λ is the orbit space of $TK \times TG$ under the TH-action $T\Psi_\lambda$. Using the fact that $TP_\lambda = TK \times_{TH} TG$, the connection Φ_λ induced from Φ is defined by the following commutative diagram:

$$\begin{array}{ccc} TK \times TG & \overset{\Phi\times \mathrm{Id}}{\longrightarrow} & TK \times TG \\ {\scriptstyle Tq}\downarrow & & \downarrow {\scriptstyle Tq} \\ TK \times_{TH} TG & \underset{\Phi_\lambda}{\longrightarrow} & TK \times_{TH} TG = T\left(K \times_H G\right), \end{array} \tag{A.7}$$

where $q \colon K \times G \to K \times_H G$ sends each element to its corresponding equivalent class in $K \times_H G$ and Tq is its tangent map.

Proof of Proposition 6. Take $p = [e,e]^\sim \in P_\lambda$ on the fiber $\pi^{-1}(o)$ and let $\xi \in \mathfrak{k}$. Since the K-action on P_λ is simply the left action L_λ introduced in (3.3), the infinitesimal generator ξ_{P_λ} at p corresponds to the equivalent class $[\xi, 0]_p^\sim$ in

$TK \times_{TH} TG$. Observe that $[\xi, 0]_p^\sim$ denotes the orbit of $((e,\xi),(e,0)) \in TK \times TG$ under the action of TH. By (A.5) and (A.6), $[\xi, 0]_p^\sim$ is equivalent to

$$[X_h + T_e R_h(\xi), T_{h^{-1}}\lambda \circ T_h \operatorname{inv}(X_h)]^\sim_{(h,\lambda(h)^{-1})}$$

for any $X_h \in T_h H$, $h \in H$. Taking $h = e \in H$, we have

$$[\xi, 0]_p^\sim = [\eta + \xi, -T_e\lambda(\eta)]_p^\sim . \tag{A.8}$$

In (A.8), we have written $\eta \in \mathfrak{h} = T_e H$ instead of X_e and have used $T_e \operatorname{inv} = -\operatorname{Id}$.

On the other hand, $\Phi \in \Omega^1(K; VK)$ coincides with the projection $\operatorname{proj}_{\mathfrak{h}} \colon \mathfrak{k} \to \mathfrak{h}$ from $\mathfrak{k}$ to $\mathfrak{h}$ at $e \in K$. Therefore, (A.7) implies

$$(\Phi_\lambda)([e,e]^\sim)(\xi_{P_\lambda}) = (\Phi_\lambda)([e,e]^\sim)\left([\xi,0]^\sim_{(e,e)}\right) = [\operatorname{proj}_{\mathfrak{h}}(\xi), 0]^\sim_{(e,e)}.$$

By (A.8) with $\eta = -\operatorname{proj}_{\mathfrak{h}}(\xi)$, $[\operatorname{proj}_{\mathfrak{h}}(\xi), 0]^\sim_{(e,e)}$ is equivalent to

$$\left[0, T_e\lambda\left(\operatorname{proj}_{\mathfrak{h}}(\xi)\right)\right]^\sim_{(e,e)} .$$

Now, for any $\eta \in \mathfrak{g}$, $T_e(R_\lambda)_p(\eta) \in TP_\lambda$ equals $[0,\eta]_p^\sim$ in $TK \times_{TH} TG$. Hence, the principal connection $\omega_p = (T_e(R_\lambda)_p)^{-1} \circ (\Phi_\lambda)_p$ satisfies

$$\begin{aligned}
\omega(p)(\xi_{P_\lambda}) &= (T_e(R_\lambda)_p)^{-1}\left([\operatorname{proj}_{\mathfrak{h}}(\xi), 0]^\sim_{(e,e)}\right) \\
&= (T_e(R_\lambda)_p)^{-1}\left(\left[0, T_e\lambda\left(\operatorname{proj}_{\mathfrak{h}}(\xi)\right)\right]^\sim_{(e,e)}\right) \\
&= T_e\lambda\left(\operatorname{proj}_{\mathfrak{h}}(\xi)\right) = \mathbf{W}(\xi)
\end{aligned}$$

if $\mathbf{W} \colon \mathfrak{k} \to \mathfrak{g}$ is the canonical connection. □

A.3. **Characteristic classes of Gibbons-Townsend monopoles.** According to what we said in Subsection 6.3, we are going to explicitly show that the q-th Chern class of the principal bundle $SO(2q+1) \to \mathbb{S}^{2q} = SO(2q+1)/SO(2q)$ is zero. If q is odd, then the characteristic coefficient c_q^{2q} is zero and, consequently, so is the corresponding q-th Chern class. If q is even then, by (5.3),

$$\begin{aligned}
&(-1)^q (2\pi i)^q q!\, \pi^*\left(cw\left(\operatorname{Sym}(c_q^{2q}), P, \boldsymbol{\omega}\right)\right)(\upsilon_1, \ldots, \upsilon_{2q}) \\
&\quad = \sum_{i_1 < \ldots < i_q} \sum_{\eta \in S_q} (-1)^{|\eta|} (\Omega^{\boldsymbol{\omega}})^{i_1}_{\eta(i_1)} \wedge \ldots \wedge (\Omega^{\boldsymbol{\omega}})^{i_q}_{\eta(i_q)} (\upsilon_1, \ldots, \upsilon_{2q}) \\
&\quad = \frac{1}{2^q} \sum_{i_1 < \ldots < i_q} \sum_{\eta \in S_q} (-1)^{|\eta|} \sum_{\sigma \in S_{2q}} (-1)^{|\sigma|} \left(\Omega^{\widetilde{\mathbf{W}}}_{\lambda_{\mathrm{Id}}}\left(\upsilon_{\sigma(1)}, \upsilon_{\sigma(2)}\right)\right)^{i_1}_{\eta(i_1)} \cdots \left(\Omega^{\widetilde{\mathbf{W}}}_{\lambda_{\mathrm{Id}}}\left(\upsilon_{\sigma(2q-1)}, \upsilon_{\sigma(2q)}\right)\right)^{i_q}_{\eta(i_q)} \\
&\quad = \frac{1}{2^q} \sum_{i_1 < \ldots < i_q} \sum_{\eta \in S_q} (-1)^{|\eta|} \sum_{\sigma \in S_{2q}} (-1)^{|\sigma|} \left(\xi_{\sigma(1),\sigma(2)}\right)^{i_1}_{\eta(i_1)} \cdots \left(\xi_{\sigma(2q-1),\sigma(2q)}\right)^{i_q}_{\eta(i_q)} .
\end{aligned}$$

Using (6.3), $\left(\xi_{\sigma(1),\sigma(2)}\right)^{i_1}_{\eta(i_1)} \cdots \left(\xi_{\sigma(2q-1),\sigma(2q)}\right)^{i_q}_{\eta(i_q)}$ equals

$$\prod_{r \in \{1,2,\ldots,q\}} (-1)^{U(\sigma(2r)-\sigma(2r-1))} (-1)^{U(\eta(i_r)-i_r)} \delta^{i_r}_{\sigma(2r-1)} \delta_{\sigma(2r)\eta(i_r)} .$$

But $\delta^{i_r}_{\sigma(2r-1)}\delta_{\sigma(2r)\eta(i_r)}$ must be zero for some $r \in \{1,2,\ldots,q\}$ for any $\sigma \in S_{2q}$ because $\{\sigma(2r-1),\sigma(2r)\}$ cover all the indices in $\{1,2,....,2q\}$ as r ranges from 1 to q but $\{i_r,\eta(i_r)\}$ only q of them. Thus,

$$\pi^*\left(cw\left(\mathrm{Sym}(c_q^{2q}),P,\boldsymbol{\omega}\right)\right)(\upsilon_1,\ldots,\upsilon_{2q})=0$$

and the Chern class vanishes.

The same argument applied to the Euler class (5.4) shows that

$$\begin{aligned}
&2^{2q}\pi^q q!\, cw\left(\mathrm{Sym}(\mathrm{Pf}),P_{\lambda_{\mathrm{Id}}},\boldsymbol{\omega}\right)(\upsilon_1,\ldots,\upsilon_{2q})\\
&=\left(\sum_{\eta\in S_{2q}}(-1)^{|\eta|}\left(\Omega^{\boldsymbol{\omega}}\right)^{\eta(1)}_{\eta(2)}\wedge\cdots\wedge\left(\Omega^{\boldsymbol{\omega}}\right)^{\eta(2q-1)}_{\eta(2q)}\right)(\upsilon_1,\ldots,\upsilon_{2q})\\
&=\frac{1}{2^q}\sum_{\eta\in S_{2q}}(-1)^{|\eta|}\sum_{\sigma\in S_{2q}}(-1)^{|\sigma|}\left(\Omega^{\widetilde{\mathbf{W}}}_{\lambda_{\mathrm{Id}}}\left(\upsilon_{\sigma(1)},\upsilon_{\sigma(2)}\right)\right)^{\eta(1)}_{\eta(2)}\cdots\left(\Omega^{\widetilde{\mathbf{W}}}_{\lambda_{\mathrm{Id}}}\left(\upsilon_{\sigma(2q-1)},\upsilon_{\sigma(2q)}\right)\right)^{\eta(2q-1)}_{\eta(2q)}\\
&=\frac{1}{2^q}\sum_{\eta\in S_{2q}}(-1)^{|\eta|}\sum_{\sigma\in S_{2q}}(-1)^{|\sigma|}\left(\xi_{\sigma(1),\sigma(2)}\right)^{\eta(1)}_{\eta(2)}\cdots\left(\xi_{\sigma(2q-1),\sigma(2q)}\right)^{\eta(2q-1)}_{\eta(2q)}.
\end{aligned}\tag{A.9}$$

Using (6.3), (A.9) equals

$$\begin{aligned}
&\frac{1}{2^q}\sum_{\eta\in S_{2q}}(-1)^{|\eta|}\sum_{\sigma\in S_{2q}}(-1)^{|\sigma|}\prod_{i\in\{1,3,\ldots,2q-1\}}(-1)^{U(\sigma(i+1)-\sigma(i))}(-1)^{U(\eta(i+1)-\eta(i))}\delta^{\eta(i)}_{\sigma(i)}\delta_{\sigma(i+1)\eta(i+1)}\\
&\qquad=\frac{1}{2^q}\sum_{\eta\in S_{2q}}(-1)^{|\eta|}(-1)^{|\eta|}=\frac{(2q)!}{2^q}.
\end{aligned}$$

Since $\mathrm{vol}\left(\mathbb{S}^{2q}\right)=\frac{2^{2q+1}\pi^q q!}{(2q)!}$, we conclude that the charge Q of the monopole is

$$Q=\int_{\mathbb{S}^{2q}}cw\left(\mathrm{Sym}(\mathrm{Pf}),P_{\lambda_{\mathrm{Id}}},\boldsymbol{\omega}\right)=\frac{1}{2^{2q}\pi^q q!}(2q)!\,\mathrm{vol}\left(\mathbb{S}^{2q}\right)=2$$

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