LARGE DEVIATION FOR BSDE WITH SUBDIFFERENTIAL OPERATOR

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Abstract. In this paper we prove that the solution of a backward stochastic differential equation, which involves a subdifferential operator and associated to a family of reflecting diffusion processes, converges to the solution of a deterministic backward equation and satisfies a large deviation principle.

1. Introduction

Let $X^{s,x,\varepsilon}$ be the diffusion process that is the unique solution of the stochastic differential equation

$$(1)\quad X_t^{s,x,\varepsilon} = x + \int_s^t b(X_r^{s,x,\varepsilon})dr + \sqrt{\varepsilon}\int_s^t \sigma(X_r^{s,x,\varepsilon})dB_r, \quad 0 \le s \le t \le T,$$

where $b : I\!R^d \longrightarrow I\!R^d$ is a uniformly Lipschitz continuous function, all the elements of the diffusion matrix σ are bounded, uniformly lipschitz continuous functions, and B is a standard Brownian motion in $I\!R^d$. The existence and uniqueness of the strong solution $X^{s,x,\varepsilon}$ of (1) is standard (see, for example, see Dembo and Zeitouni [4]). It is known, thanks to the works of Freidlin and Wentzell [7], that $X^{s,x,\varepsilon}$ converges in probability, as ε goes to 0, to the solution $\chi^{s,x}$ of the following deterministic equation

$$(2)\qquad \chi_t^{s,x} = x + \int_s^t b(\chi_r^{s,x})dr, \quad 0 \le s \le t \le T,$$

and satisfies a large deviation principle. This result has been generalized recently by Rainero [15] to the case of backward stochastic differential equation (BSDE for short), related to the family of diffusion processes $\{X^{s,x,\varepsilon}\}$,

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of the form

$$(3)\quad Y_t^{s,x,\varepsilon} = h(X_T^{s,x,\varepsilon}) + \int_t^T f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon})dr - \int_t^T Z_r^{s,x,\varepsilon} dB_r, \quad 0 \le s \le t \le T,$$

where h and f are a given functions and satisfy some appropriate assumptions. The author has proved that the solution $(Y^{s,\varepsilon}, Z^{s,\varepsilon})$ of equation (3) converges, as ε goes to 0, to $(Y^{s,x}, 0)$ solution of the following backward deterministic equation

$$Y_t^{s,x} = h(\chi_T^{s,x}) + \int_t^T f(r, \chi_r^{s,x}, Y_r^{s,x}, 0)dr, \quad 0 \le s \le t \le T,$$

and satisfies a large deviation principle.

Backward stochastic differential equations of type (3) have been first introduced by Pardoux and Peng [12]. A solution for such equation is a couple of adapted processes (Y, Z) with values in $I\!R^k \times I\!R^{k\times d}$ which mainly satisfies equation (3). The aim of Pardoux and Peng was to give a probabilistic interpretation of a solution of second order quasi-linear partial differential equation. Since then, those equations have been intensively investigated due to their connections with financial mathematics, optimal control and stochastic game, non-linear PDEs and homogenization (see, for example, [5, 6, 8, 9, 14, 12, 2, 3, 10, 1] and the references therein).

In this paper, we are interested to the system of forward-backward stochastic differential equations

$$(4)\quad \begin{cases} X_t^{s,x,\varepsilon} = \\ = x + \int_s^t b(X_r^{s,x,\varepsilon})dr + \sqrt{\varepsilon}\int_s^t \sigma(X_r^{s,x,\varepsilon})dB_r + \rho_t^{s,x,\varepsilon} - \rho_s^{s,x,\varepsilon}, 0 \le s \le 1 \le T, \\ \rho_t^{s,x,\varepsilon} = \int_0^t \nabla\psi(X_r^{s,x,\varepsilon})d|\rho^{s,x,\varepsilon}|_r, \ |\rho^{s,x,\varepsilon}|_t = \int_0^t \mathbf{1}_{\{X_r^{s,x,\varepsilon}\in\partial\Theta\}}d|\rho^{s,x,\varepsilon}|_r, \end{cases}$$

$$(5)\quad \begin{cases} Y_t^{s,x,\varepsilon} = \\ = h(X_T^{s,x,\varepsilon}) + \int_t^T f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon})dr - \int_t^T Z_r^{s,x,\varepsilon}dB_r - \int_t^T U_r^{s,x,\varepsilon}dr \\ (Y_t^{s,x,\varepsilon}, U_t^{s,x,\varepsilon}) \in \partial\Pi, \text{ and } I\!E\int_0^T \Pi(Y_r^{s,x,\varepsilon})dr < +\infty, \end{cases}$$

where ψ is $\mathcal{C}_b^2(I\!R^d)$ function and ρ is a bounded variation process such that $\rho_0 = 0$, Π is a proper lower semicontinuous convex function and $\partial\Pi$ is a subdifferntial operator. Equations of type (5) have been introduced by Pardoux and Rascanu [13]. A solution of such equations is a triple of process (Y, Z, U) with values in $I\!R^k \times I\!R^{k\times d} \times I\!R^k$ and satisfies equation (5). Our aim is to prove that the solution $(X^{s,x,\varepsilon},\rho^{s,x,\varepsilon},Y^{s,x,\varepsilon},Z^{s,x,\varepsilon},U^{s,x,\varepsilon})$ of system (4)-(5) converges, as ε goes to 0, to the solution $(\chi^{s,x},\rho^{s,x},Y^{s,x},Z^{s,x},U^{s,x})$ of the following system of forward-backward deterministic equation

$$\begin{cases} \chi_t^{s,x} = x + \int_s^t b(\chi_r^{s,x})dr + \rho_t^{s,x} - \rho_s^{s,x} \\ \rho_t^{s,x} = \int_0^t \nabla\psi(\chi_r^{s,x})d|\rho^{s,x}|_r,\ |\rho^{s,x}|_t = \int_0^t \mathbf{1}_{\{\chi_r^{s,x}\in\partial\Theta\}}d|\rho^{s,x}|_r \\ \\ Y_t^{s,x} = h(\chi_T^{s,x}) + \int_t^T f(r,\chi_r^{s,x},Y_r^{s,x},0)dr - \int_t^T U_r^{s,x}dr \\ (Y_t^{s,x},U_t^{s,x}) \in \partial\Pi, \text{ and } I\!\!E\int_0^T \Pi(Y_r^{s,x})dr < +\infty, \end{cases}$$

and that $Y^{s,x,\varepsilon}$ satisfies a large deviation principle. Our paper is, in fact, a generalization of the two works cited before.

2. Assumptions and problem formulation

Let $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\leq 1}))$ be a stochastic basis such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, $\mathcal{F}_{t+} = \bigcap_{\epsilon>0}\mathcal{F}_{t+\epsilon} = \mathcal{F}_t$, $\forall t\leq 1$, and suppose that the filtration is generated by a d-dimensional Brownian motion $(B_t)_{t\leq 1}$.
On the other hand, let

• Θ be an open connected bounded subset of $I\!\!R^d$, which is such that for a function $\psi\in\mathcal{C}_b^2(I\!\!R^d)$, $\Theta=\{\psi>0\}$, $\partial\Theta=\{\psi=0\}$, and $|\nabla\psi(x)|=1$, $x\in\partial\Theta$. Note that at any boundary point $x\in\partial\Theta$, $\nabla\psi(x)$ is a unit normal vector to the boundary, pointing towards the interior of Θ. The above assumptions imply that there exists a constant $\delta>0$ such that for all $x\in\partial\Theta$, $x'\in\overline{\Theta}$

$$2\langle x'-x,\nabla\psi(x)\rangle + \delta|x-x'|^2 \geq 0. \tag{6}$$

• $b:\overline{\Theta}\longrightarrow I\!\!R^d$, $\sigma:\overline{\Theta}\longrightarrow I\!\!R^{d\times d}$ be functions such that :

(A1) There exists a constant $C>0$ such that

$$|b(x)|+|\sigma(x)|\leq C, \forall x\in\overline{\Theta}$$

$$|b(x)-b(x^{'})|+|\sigma(x)-\sigma(x^{'})|\leq C|x-x^{'}|, \forall x,x'\in\overline{\Theta}.$$

(A2) The matrix $a=\sigma\sigma^*$ is uniformly elliptic, that is, there exists a constant $\gamma>0$ such that

$$a(x)\geq\gamma|x|^2, \forall x\in\overline{\Theta}.$$

• $h\in\mathcal{C}(\overline{\Theta};I\!\!R^k)$, $f\in\mathcal{C}([0,1]\times\overline{\Theta}\times I\!\!R^k\times I\!\!R^{k\times d};I\!\!R^k)$ be functions satisfying the following assumptions :

(A3) There exist constants $\alpha \in I\!R,\ K > 0,\ c > 0,\ \mu > 0$ such that

$$\begin{array}{l} (i)\, \forall t, \forall x, \forall y, \forall (z, z'), \\ \qquad \mid f(t,x,y,z) - f(t,x',y,z') \mid \leq \mu(\mid z - z' \mid + \mid x - x' \mid) \\ (ii)\, \forall t, \forall x, \forall z, \forall (y, y'), \\ \qquad \langle y - y',\ f(t,x,y,z) - f(t,x,y',z) \rangle \leq \alpha \mid y - y' \mid^2 \\ (iii)\, \forall x, \forall x',\ |h(x) - h(x')| \leq c|x - x'|, \\ (vi)\, \forall t, \forall x, \forall y, \forall z, \quad \mid f(t,x,y,z) \mid \leq K(1 + \mid y \mid + \mid z \mid) \\ (v)\, \forall x, \quad |h(x)| \leq K(1 + |x|). \end{array}$$

• $\Pi : I\!R^k \rightarrow] - \infty, +\infty]$, be a proper lower semicontinuous convex function such that
(A4) There exists a constant $C > 0$ such that

$$\Pi(h(x)) \leq C(1 + |x|), \quad \forall x \in \overline{\Theta},$$

$$\Pi(y) \geq \Pi(0) = 0,\ \forall y \in I\!R^k.$$

We need also the following notations :
• $\mathcal{C}[0,T]$ denotes the space of continuous functions $\Phi : [0,T] \longrightarrow I\!R^d$ such that $f(0) \in \overline{\Theta}$.
• $\overline{\mathcal{C}}[0,T]$ denotes the space of continuous functions $\Psi : [0,T] \longrightarrow \overline{\Theta}$.
• $\mathcal{V}[0,T]$ denotes the space of functions $\rho : [0,T] \longrightarrow I\!R^d$ with bounded variation and $\rho(0) = 0$.
For $\rho \in \mathcal{V}[0,T]$, $|\rho|_t$ denotes the total variation of ρ in the interval $[0,t]$.
Consider the system of forward-backward stochastic differential equations

(7)
$$\begin{cases} X_t^{s,x,\varepsilon} = \\ = x + \int_s^t b(X_r^{s,x,\varepsilon})dr + \sqrt{\varepsilon} \int_s^t \sigma(X_r^{s,x,\varepsilon})dB_r + \rho_t^{s,x,\varepsilon} - \rho_s^{s,x,\varepsilon}, 0 \leq s \leq t \leq T, \\ \rho_t^{s,x,\varepsilon} = \int_0^t \nabla\psi(X_r^{s,x,\varepsilon})d|\rho^{s,x,\varepsilon}|_r,\ |\rho^{s,x,\varepsilon}|_t = \int_0^t \mathbf{1}_{\{X_r^{s,x,\varepsilon} \in \partial\Theta\}} d|\rho^{s,x,\varepsilon}|_r \end{cases}$$

(8)
$$\begin{cases} Y_t^{s,x,\varepsilon} = \\ = h(X_T^{s,x,\varepsilon}) + \int_t^T f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon})dr - \int_t^T Z_r^{s,x,\varepsilon} dB_r - \int_t^T U_r^{s,x,\varepsilon} dr \\ (Y_t^{s,x,\varepsilon}, U_t^{s,x,\varepsilon}) \in \partial\Pi, \text{ and } I\!E \int_0^T \Pi(Y_r^{s,x,\varepsilon})dr < +\infty, \end{cases}$$

where

$$\partial\Pi(u) = \{u^* \in I\!R^k : \langle u^*, v - u \rangle + \Pi(u) \leq \Pi(v), \forall v \in I\!R^k\}$$

Note that the subdifferential operator $\partial\Pi : I\!R^k \longrightarrow 2^{I\!R^k}$ is a maximal monotone operator, that is

$$\langle u' - v', u - v\rangle \geq 0. \quad \forall (u, u'), (v, v') \in \partial\Pi.$$

The existence and uniqueness of the strong solution $X^{s,x,\varepsilon}$, under assumption **(A1)**, for equation (7) is standard (see, for example, Lions and Sznitman [11] or Saisho [16]). It follows also from the result of Pardoux and Rascanu [13] that, under assumptions **(A3)** and **(A4)**, there exists a unique triple $(Y^{s,x,\varepsilon}, Z^{s,x,\varepsilon}, U^{s,x,\varepsilon})$ for equation (8).
The objective of this work is to prove that the solution of forward-backward stochastic differential equation (7)-(8) converges and satisfies a large deviation principle.

For the sake of simplicity, we put, in general, $s = 0$. Of course, the results hold true for all $s \in [0, T]$. We denote then by $X^{x,\varepsilon} := X^{0,x,\varepsilon}$, $Y^{0,x,\varepsilon} := Y^{x,\varepsilon}$,...

3. Large deviation principle and convergence of the solution of the forward equation

Before giving a large deviation principle for the reflecting diffusion process $X^{s,x,\varepsilon}$, we recall the following

Definition 3.1. *The family of processes $(X_t, 0 \leq t \leq T)$ which depends on a parameter ε is said to satisfy a large deviation principle with a rate function $S(\Psi)$ if the following condition hold for every Borel set $A \subseteq \mathcal{C}[0, T]$*

$$1. \quad \limsup_{\varepsilon \to 0} \varepsilon \ln(P(X^\varepsilon \in A)) \leq \inf_{\Psi \in \overline{A}} S(\Psi)$$

$$2. \quad \liminf_{\varepsilon \to 0} \varepsilon \ln(P(X^\varepsilon \in A)) \geq - \inf_{\Psi \in \overset{\circ}{A}} S(\Psi),$$

where $\overline{A}$ is the closure of A and $\mathring{A}$ the interior of A.

Let $\Phi \in \mathcal{C}[0, T]$, $\Psi \in \overline{\mathcal{C}}[0, T]$, $\rho \in \mathcal{V}[0, T]$ such that

$$(9) \qquad \begin{cases} \Psi(t) = \Phi(t) + \rho(t), \\ \rho_t = \int_0^t \nabla\psi(\Psi_r) d|\rho|_r, \ |\rho|_t = \displaystyle\int_0^t \mathbf{1}_{\{\Psi(r) \in \partial\Theta\}} d|\rho|_r \end{cases}$$

For Φ and Ψ defined as above, let $\Psi = F(\Phi)$. It is known from Lions and Sznitman [11] or Saisho [16] that F is continuous. We have the following theorem

Theorem 3.1. *The process $X^{x,\varepsilon}$ given by equation (7) satisfies a large deviation principle with rate function $S(\Psi)$ defined by*

$$S(\Psi) = \frac{1}{2} \inf_{\Phi \in F^{-1}(\Psi)} \int_0^T (\dot{\Phi}(s) - b(\Psi(s)))^* a^{-1}(\Psi(s))(\dot{\Phi}(s) - b(\Psi(s)))ds,$$

with the fact that $S(\Psi) = \infty$ if $F^{-1}(\psi) = \emptyset$ or Φ is not absolutely continuous.

Proof. The result follows by using the contraction principle (see Dembo and Zeitouni [4]) and a large deviation principle for diffusion processes (see Stroock [18] or [4], see also Sheu [17] for other assumptions on Θ).

Remark 3.1. *The function $S(\Psi)$ has the following properties*
1. $S(\Psi)$ is lower semi-continuous in Ψ.
2. If $S(\Psi) < \infty$, then there exists $\Phi \in \mathcal{C}[0,T]$ such that $F(\Phi) = \Psi$ and

$$S(\Psi) = \frac{1}{2} \int_0^T (\dot{\Phi}(s) - b(\Psi(s)))^* a^{-1}(\Psi(s))(\dot{\Phi}(s) - b(\Psi(s)))ds.$$

Let $(\chi^{s,x}, \rho^{s,x})$ be the solution of the following deterministic equation

$$\begin{cases} \chi_t^{s,x} = x + \displaystyle\int_s^t b(\chi_r^{s,x})dr + \rho_t^{s,x} - \rho_s^{s,x} \\ \rho_t^{s,x} = \int_0^t \nabla\psi(\chi_r^{s,x})d|\rho^{s,x}|_r, \ |\rho^{s,x}|_t = \displaystyle\int_0^t \mathbf{1}_{\{\chi_r^{s,x} \in \partial\Theta\}} d|\rho^{s,x}|_r. \end{cases}$$

We get the following

Lemma 3.1. *For all $\varepsilon \in]0,1]$, there exists a constant $C > 0$, independent of x and ε, such that*

$$\mathbb{E} \sup_{0 \le t \le T} |X_t^{x,\varepsilon} - \chi_t^x|^2 \le C\varepsilon. \tag{10}$$

Proof. Applying Itô's formula to

$$e^{-\delta(\psi(X_t^{x,\varepsilon}) + \psi(\chi_t^x))}|X_t^{x,\varepsilon} - \chi_t^x|^2,$$

where δ is given by the inequality (6), we get a.s for all $t \in [0,T]$

(11)
$$\begin{aligned}
&e^{-\delta(\psi(X_t^{x,\varepsilon})+\psi(\chi_t^x))}|X_t^{x,\varepsilon}-\chi_t^x|^2\\
&=2\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}\big[\langle X_r^{x,\varepsilon}-\chi_r^x,\sqrt{\varepsilon}\sigma(X_r^{x,\varepsilon})dB_r\rangle+\\
&+\langle X_r^{x,\varepsilon}-\chi_r^x, b(X_r^{x,\varepsilon})-b(\chi_r^x)\rangle dr\big]\\
&+2\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}\big[\langle X_r^{x,\varepsilon}-\chi_r^x,\nabla\psi(X_r^{x,\varepsilon})\rangle d|\rho^{x,\varepsilon}|_r-\\
&-\langle X_r^{x,\varepsilon}-\chi_r^x,\nabla\psi(\chi_r^x)\rangle d|\rho^x|_r\big]\\
&+\varepsilon\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}|\sigma(X_r^{x,\varepsilon})|^2dr\\
&-\delta\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}|X_r^{x,\varepsilon}-\chi_r^x|^2\bigg((\nabla\psi(X_r^{x,\varepsilon}))^*\sqrt{\varepsilon}\sigma(X_r^{x,\varepsilon})+\\
&+\tfrac{\varepsilon}{2}tr(D^2\psi(X_r^{x,\varepsilon})\sigma\sigma^*(X_r^{x,\varepsilon})+\langle\nabla\psi(X_r^{x,\varepsilon}),b(X_r^{x,\varepsilon})\rangle+\\
&+\langle\nabla\psi(\chi_r^x),b(\chi_r^x)\rangle\bigg)dr-\delta\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}|X_r^{x,\varepsilon}-\\
&-\chi_r^x|^2\bigg(|\nabla\psi(X_r^{x,\varepsilon})|^2d|\rho^{x,\varepsilon}|_r+|\nabla\psi(X_r^x)|^2d|\rho^x|_r\bigg)\\
&+\tfrac{\varepsilon\delta^2}{2}\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}|X_r^{x,\varepsilon}-\chi_r^x|^2|(\sigma(X_r^{x,\varepsilon}))^*\nabla\psi(X_r^{x,\varepsilon})|^2dr\\
&-2\varepsilon\delta\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}\langle X_r^{x,\varepsilon}-\chi_r^x,\sigma(X_r^{x,\varepsilon})\rangle(\sigma(X_r^{x,\varepsilon})^*\nabla\psi(X_r^{x,\varepsilon})dr.
\end{aligned}$$

Since $|\nabla\phi|=1$ for all $x\in\partial\Theta$, by inequality (6) we have

$$\begin{aligned}
&2\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}\langle X_r^{x,\varepsilon}-\chi_r^x,\nabla\psi(X_r^{x,\varepsilon})\rangle d|\rho^{x,\varepsilon}|_r\\
&-\delta\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}|X_r^{x,\varepsilon}-\chi_r^x|^2|\nabla\psi(X_r^{x,\varepsilon})|^2d|\rho^{x,\varepsilon}|_r\le 0,
\end{aligned}\tag{12}$$

and

$$\begin{aligned}
&-2\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}\langle X_r^{x,\varepsilon}-\chi_r^x,\nabla\psi(X_r^x)\rangle d|\rho^x|_r\\
&-\delta\int_0^t e^{-\delta(\psi(X_r^{x,\varepsilon})+\psi(\chi_r^x))}|X_r^{x,\varepsilon}-\chi_r^x|^2|\nabla\psi(X_r^x)|^2d|\rho^x|_r\le 0.
\end{aligned}\tag{13}$$

The result is then a consequence of the boundeness of b, σ, ψ, $\nabla\psi$, $D^2\psi$, inequalities (12)–(13) and Burkholder-Davis-Gundy inequality.

Remark 3.2. *As a consequence of Lemma 3.1, the solution of the reflecting diffusion process $X^{x,\varepsilon}$ converges to the deterministic path χ^x in L^2.*

4. Convergence and large deviation principle for the solution of the backward equation

Let $(\chi^{(s,x)}, \rho^{s,x}, Y^{(s,x)}, 0, U^{(s,x)})$ be the solution of the following deterministic equations

$$\begin{cases} \chi_t^{s,x} = x + \int_s^t b(\chi_r^{s,x})dr + \rho_t^{s,x} - \rho_s^{s,x} \\ \rho_t^{s,x} = \int_0^t \nabla\psi(\chi_r^{s,x})d|\rho^{s,x}|_r, \ |\rho^{s,x}|_t = \int_0^t \mathbf{1}_{\{\chi_r^{s,x}\in\partial\Theta\}}d|\rho^{s,x}|_r \end{cases}$$

$$\begin{cases} Y_t^{s,x} = h(\chi_T^{s,x}) + \int_t^T f(r, \chi_r^{s,x}, Y_r^{s,x}, 0)dr - \int_t^T U_r^{s,x}dr \\ (Y_t^{s,x}, U_t^{s,x}) \in \partial\Pi, \text{ and } I\!\!E\int_0^T \Pi(Y_r^{s,x})dr < +\infty. \end{cases} \tag{14}$$

We have the following theorem

Theorem 4.1. $\forall\varepsilon\in]0,1]$, *there exists a constant* $C > 0$, *independent of* s, x *and* ε, *such that*

$$\begin{aligned} &I\!\!E[\sup_{s\le t\le T}|Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2 + \int_s^T |Z_r^{s,x,\varepsilon}|^2dr] \\ &\le C[I\!\!E(X_T^{s,x,\varepsilon} - \chi_T^{s,x}|^2) + I\!\!E\int_s^T |X_r^{s,x,\varepsilon} - \chi_r^{s,x}|^2dr]. \end{aligned} \tag{15}$$

Proof. Applying Itô's formula to $|Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2$, we get

$$\begin{aligned} &I\!\!E|Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2 + I\!\!E\int_s^T|Z_r^{s,x,\varepsilon}|^2dr + 2I\!\!E\int_s^T\langle Y_r^{s,x,\varepsilon} - Y_r^{s,x}, U_r^{s,x,\varepsilon} - U_r^{s,x}\rangle dr \\ &\le I\!\!E(h(X_T^{s,x,\varepsilon}) - h(\chi_T^{s,x}|^2) \\ &+2I\!\!E\int_s^T\langle Y_r^{s,x,\varepsilon} - Y_r^{s,x}, f(r, X_r^{s,x,\varepsilon}, Y_r^{s,x,\varepsilon}, Z_r^{s,x,\varepsilon}) - f(r, \chi_r^{s,x}, Y_r^{s,x}, 0)\rangle dr. \end{aligned}$$

But $\langle Y_r^{s,x,\varepsilon} - Y_r^{s,x}, U_r^{s,x,\varepsilon} - U_r^{s,x}\rangle \ge 0$, $dP\times dr$ a.e and f satisfies conditions $\mathbf{A3}(i)-(ii)$, then

$$\begin{aligned} &I\!\!E|Y_t^{s,x,\varepsilon} - Y_t^{s,x}|^2 + I\!\!E\int_s^T|Z_r^{s,x,\varepsilon}|^2dr \\ &\le I\!\!E(h(X_T^{s,x,\varepsilon}) - h(\chi_T^{s,x}|^2) + 2\alpha I\!\!E\int_s^T|Y_r^{s,x,\varepsilon} - Y_r^{s,x}|^2dr \\ &+2\mu I\!\!E\int_s^T|Y_r^{s,x,\varepsilon} - Y_r^{s,x}||X_r^{s,x,\varepsilon} - \chi_r^{s,x}|dr + 2\mu I\!\!E\int_s^T|Y_r^{s,x,\varepsilon} - Y_r^{s,x}||Z_r^{s,x,\varepsilon}|dr \end{aligned}$$

Henceforth

$$\begin{aligned}&\mathbb{E}[|Y_t^{s,x,\varepsilon}-Y_t^{s,x}|^2+\int_s^T|Z_r^{s,x,\varepsilon}|^2dr]\\&\leq C[\mathbb{E}(X_T^{s,x,\varepsilon}-\chi_T^{s,x}|^2)+\mathbb{E}\int_s^T|X_r^{s,x,\varepsilon}-\chi_r^{s,x}|^2dr],\end{aligned}$$

where C is a positive constant. The result then follows from Burkholder-Davis-Gundy inequality.

Remark 4.1. *As a consequence of Theorem 4.1 and Lemma 3.1, we get*

$$\mathbb{E}[\sup_{s\leq t\leq T}|Y_t^{s,x,\varepsilon}-Y_t^{s,x}|^2+\int_s^T|Z_r^{s,x,\varepsilon}|^2dr]\leq C\varepsilon, \tag{16}$$

where C is a positive constant and then the solution of the BSDE (8) converges to the deterministic solution of the equation (14).

We now consider the BSDE in the case $k=1$. We want to prove that the process $Y^{s,x,\varepsilon}$ satisfies a large deviation principle. For that reason, we recall the link between Variational Inequality (VI, for short) and BSDE. For all $\varepsilon\geq 0$, we consider the following VI

$$\left\{\begin{array}{l}\frac{\partial u^\varepsilon}{\partial t}(t,x)+\mathcal{L}^{x,\varepsilon}u^\varepsilon(t,x)+\\+f\left(t,x,u^\varepsilon(t,x),\left((\nabla u^\varepsilon)^*\sqrt{\varepsilon}\sigma\right)(t,x)\right)\in\partial\Pi\left(u^\varepsilon(t,x)\right),\\\qquad\qquad t\in\left]0,T\right[,\ x\in\Theta\\\frac{\partial u^\varepsilon}{\partial n}(t,x)\in\partial\Pi\left(u^\varepsilon(t,x)\right),\ x\in\partial\Theta\\u^\varepsilon(T,x)=h(x),\ \ x\in\overline{\Theta},\end{array}\right. \tag{17}$$

where $\mathcal{L}^{x,\varepsilon}$ is the second order partial differential operator

$$\mathcal{L}^{x,\varepsilon}:=\frac{\varepsilon}{2}\sum_{i,j=1}^d(\sigma\sigma^*)_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^d b_i\frac{\partial}{\partial x_i},$$

and at point $x\in\partial\Theta$

$$\frac{\partial}{\partial n}:=\sum_{i=1}^d\frac{\partial\psi}{\partial x_i}(x)\frac{\partial}{\partial x_i},$$

then we have, for each $(t,x)\in[0,T]\times\overline{\Theta}$,

$$u^\varepsilon(t,x)=Y_t^{t,x,\varepsilon}, \tag{18}$$

both in the sense that any classical solution of the VI (17) is equal to $Y_t^{t,x,\varepsilon}$, and $Y_t^{t,x,\varepsilon}$ is, in the case where all coefficients are continuous, a viscosity solution of the VI (17) (see Pardoux and Rascanu [13]). Moreover, we have also that

$$Y_t^{s,x,\varepsilon}=u^\varepsilon(t,X_t^{s,x,\varepsilon}).$$

Let $s \in [0,T]$ and $\varepsilon \geq 0$, we define the following applications :

$$F^\varepsilon(\Psi) := [t \to u^\varepsilon(t,\Psi_t)],\ t \in [s,T],\ \ \Psi \in \overline{\mathcal{C}}[s,T] \text{ satisfying equation (9).}$$

Hence $Y_t^{s,x,\varepsilon} = F^\varepsilon(X^{s,x,\varepsilon})(t)$, for all $t \in [0,T]$, and $Y^{s,x,\varepsilon} = F^\varepsilon(X^{s,x,\varepsilon})$. For $\varepsilon = 0$, u and F stand for u^0 and F^0. We have the following theorem

Theorem 4.2. *$Y^{x,\varepsilon}$ satisfies a large deviation principle with a rate function*

$$S'(\Psi') = \inf\{S(\Psi) | \Psi'_t = F(\Psi)(t) = u(t,\Psi_t),\ \forall t \in [0,T]\}.$$

Proof. In order to apply the contraction principle, we need to prove that F^ε, $\varepsilon \geq 0$ are continuous and $\{F^\varepsilon\}$ converges uniformly to F on every compact of $\overline{\mathcal{C}}[0,T]$. Since u^ε is continuous, it is not hard to prove that F^ε is also continuous. The uniform convergence of $\{F^\varepsilon\}$ is a consequence of Remark 4.1.

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