REDUCTION OF PERIODIC DIFFERENCE SYSTEMS TO LINEAR OR AUTONOMOUS ONES

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Abstract. We extend Floquet theory for reducing nonlinear periodic difference systems to autonomous ones (actually linear) by using normal form theory.

1. Introduction and statement of the main results

The theory of difference equations (or recurrence relations, iterated maps), the methods used in their solutions and their wide applications have advanced beyond their adolescent stage to occupy a central position in applicable analysis. In fact, in the last few years, the proliferation of the subject is witnessed by hundred of research articles and several monographs, see for instance [1, 6].

Among all the attractive topics, the so–called "Floquet Theory" is a typical one, which in general deals with periodic linear systems. The asymptotic properties are determined by the periodic map (or monodromy operator) and in particular, by their spectra. So, if concerning periodic orbits of nonlinear systems, the local behavior of a semi–flow close to a periodic orbit is, up to the first order, determined by the derivatives of the flow map on its normal bundle. We refer the works of Coddington and Levinson [4] and Hartman [8] for the classical Floquet theory, to Hale [7] for the case of retarded functional differential equations, to Heney [9] for time periodic linear perturbations of analytic semigroups. Also see [11] for its extension for nonlinear periodic differential equations and to the book [12] for partial differential equations. The motivation of our paper is to extend Floquet

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theory to reduce the nonlinear periodic difference systems to autonomous ones by studying their normal forms.

In the context of difference equations, we consider linear homogeneous periodic non–autonomous difference systems of the form

$$x_{n+1} = A_n x_n, \tag{1}$$

where $x_n \in \mathbb{R}^d$, $n \in \mathbb{Z}$, A_n is a real $d \times d$ matrix whose entries are function of n satisfying $A_n = A_{n+m}$ for a positive integer m and is *non–degenerated*, i.e., $\det A_n \neq 0$ for all $n \in \mathbb{Z}$. As usual $\mathbb{Z}_+$ denotes the set of non–negative integers and $M = A_{m-1}A_{m-2}\cdots A_0$ denotes the *monodromy* of system (1).

Theorem 1. *There exists a sequence of non–degenerated real matrices $\{B_n\}_{n\in\mathbb{Z}}$, with $B_n = B_{n+m}$ and a constant matrix D such that by the coordinate substitution $x_n = B_n y_n$ system* (1) *is transformed into*

$$y_{n+1} = Dy_n$$

if and only if the monodromy M and the period m satisfy
(1) *m is odd;*
(2) *or M has no negative real eigenvalues;*
(3) *or the Jordan blocks of the Jordan Normal Form (for short, JNF) of M corresponding to the negative real eigenvalues appear pairwise.*

This theorem can be seen as an extension of the classical Floquet's theory to periodic difference systems. Detailed discussions can be found in [6, 1] and in section 2 of our paper. We note that the real transformed autonomous system of system (1) can be obtained with at most a $2m$–periodic transformation whatever M and m be.

Consider the m–periodic difference system

$$x_{n+1} = F_n(x_n) = A_n x_n + f_n(x_n), \tag{2}$$

where $x_n \in U$ a neighborhood of the origin in $\mathbb{R}^d$, A_n is a real $d \times d$ non–degenerated n–depending matrix satisfying $A_n = A_{n+m}$, $f_n = f_{n+m}$ and the coefficients of $f_n : U \to \mathbb{R}^n$ are C^∞ functions such that $f_n(x) = O(\|x\|^2)$ for $n = 0, \ldots, m-1$.

For any matrix A, we denote the set of its eigenvalues by $\lambda(A) = (\lambda_1, \ldots, \lambda_d) \in \mathbb{C}^d$. For simplicity, sometimes we write $\lambda(A)$ as λ without misunderstanding. The d–tuple $\lambda(A)$ is called *weakly non–resonant* if for $j = 1, \ldots, d$ and $k \in \mathbb{Z}_+^d$, $|k| = \sum_{i=1}^d k_i \geq 2$ the following conditions are satisfied

$$\lambda_j \lambda^{-k} \neq e^{\frac{2i\pi}{m}\sqrt{-1}}, \quad i = 1, 2, \ldots, m-1,$$

where $\lambda^{-k} = \lambda_1^{-k_1}\lambda_2^{-k_2}\cdots\lambda_d^{-k_d}$. The d–tuple $\lambda(A)$ are called *non–resonant* if for $j = 1, \ldots, d$ and $k \in \mathbb{Z}_+^d$, $|k| = \sum_{i=1}^d k_i \geq 2$ the following conditions are satisfied: $\lambda_j \lambda^{-k} \neq 1$. Moreover, we say $\lambda(A)$ or A is *hyperbolic* if the module of λ_j is different from 1 for $j = 1, \dots, d$.

Assume that in the following difference system

$$y_{n+1} = G_n(y_n) = D_n y_n + g_n(y_n), \tag{3}$$

G_n satisfies the same conditions as F_n in system (2). Then system (2) and (3) are said to be C^k *equivalent* if system (2) can be changed into (3) under the coordinate substitution $x_n = H_n(y_n) = B_n y_n + h_n(y_n)$, where $H_n(y)$ are C^k functions in y, $H_{n+m} = H_n$ and B_n are non-degenerated. In the following theorem, we will show that the linearization of system (2) depends greatly on the monodromy of its linear part.

Theorem 2. *Let M be the monodromy of the linear part of system* (2). *Assume $\lambda(M)$ is non–resonant, then in a neighborhood of the origin in $\mathbb{R}^d$ system* (2) *is C^∞ equivalent to its linear part.*

Now our purpose is to find proper conditions, under which system (2) can be C^∞ equivalent to a real autonomous difference system. Obviously, the first step is to change the linear part. Because of Theorem 1, here we only consider the case that $A_n = A$ is a constant real matrix in system (2).

Theorem 3. *Assume $A_n = A$ is a hyperbolic constant real matrix and $\lambda(A)$ is weakly non–resonant, then in a neighborhood of the origin in $\mathbb{R}^d$ system* (2) *is C^∞ equivalent to an autonomous difference system with linear part given by the matrix A.*

From the above theorems we can get the following result.

Corollary 4. *Let M be the monodromy of the linear part of system* (2). *Assume $\lambda(M) = (\lambda_1, \dots, \lambda_d)$ is hyperbolic and $\lambda_j > 0$ for $j = 1, \dots, d$, then in a neighborhood of the origin in $\mathbb{R}^d$ system* (2) *is C^∞ equivalent to an autonomous difference system.*

The structure of our paper is as follows. In section 2, we present the proof of Theorem 1. In section 3, using normal forms of periodic systems we prove Theorem 3, 2 and Corollary 4.

2. Proof of Theorem 1

In this section following the analogous way for the periodic differential systems we can defined the Poincaré map for the difference systems. Additionally we provide a strong lemma, which characterizes difference systems and their corresponding Poincaré maps. Together with some detailed discussions in linear algebra, we provide the proof of Theorem 1. Our arguments naturally imply the classical Floquet's theory for periodic difference systems.

Let $s \in \mathbb{Z}$ and $\phi_n^s(x_n)$ be the solution of system (2) with initial condition $\phi_n^0(x_n) = x_n$. Then we obtain

$$\phi_n^s = \begin{cases} F_{n+s-1} \circ \cdots \circ F_{n+1} \circ F_n, & s > 0; \\ \text{id}, & s = 0; \\ F_{n+s}^{-1} \circ \cdots F_{n-2}^{-1} \circ F_{n-1}^{-1}, & s < 0. \end{cases}$$

The Poincaré map of system (2) is defined as Φ: $x_0 \mapsto \phi_0^m(x_0)$. Notice that we have $\phi_{n+m}^s = \phi_n^s$. As usual we denote by C^k with $k \in \mathbb{Z}_+$, $k = \infty$ and $k = \omega$ a C^k function, a C^∞ function and an analytic function, respectively.

Lemma 5. *Let $k \in \mathbb{Z}_+ \cup \{\infty\} \cup \{\omega\}$. Two real C^k periodic difference systems of period m are C^k equivalent if and only if their Poincaré maps are C^k conjugate.*

Proof. Necessity: Denote by $\phi_n^s(x_n)$ and $\psi_n^s(x_n)$ the solutions of these two systems under consideration with initial conditions $\phi_n^0(x_n) = x_n$, $\psi_n^0(x_n) = x_n$, respectively. By the assumption of necessity, there exists a n–depending C^k coordinate transformation $y_n = H_n(x_n)$ satisfying $\psi_n^s(H_n(x_n)) = H_{n+s}(\phi_n^s(x_n))$, where $H_n = H_{n+m}$. Let $\Phi(x_0) = \phi_0^m(x_0)$ and $\Psi(x_0) = \psi_0^m(x_0)$ be their Poincaré maps respectively, then

$$\Psi \circ H_0(x_0) = \psi_0^m(H_0(x_0)) = H_m(\phi_0^m(x_0)) = H_0 \circ \Phi(x_0).$$

Sufficiency: Let ϕ_n^s, ψ_n^s, Φ and Ψ be the ones defined above. By the assumption of sufficiency, there exists a C^k diffeomorphism H such that $\Psi \circ H = H \circ \Phi$. Let $H_s = \psi_0^s \circ H \circ \phi_s^{-s}$. Then H_s is a s–depending C^k diffeomorphism with $H_0 = H$. Notice that

$$\psi_n^s(H_n(x_n)) = \psi_0^{n+s} \circ H \circ \phi_n^{-n} = H_{n+s}(\phi_n^s(x_n))$$

and

$$H_{n+m} = \psi_0^{n+m} \circ H \circ \phi_{n+m}^{-n-m} = \psi_0^n \circ \Psi \circ H \circ \Phi^{-1} \circ \phi_n^{-n} = \psi_0^n \circ H \circ \phi_n^{-n} = H_n.$$

This completes the proof. □

Lemma 6. *Let M be a nonsingular $d \times d$ matrix and $m \in \mathbb{Z}$. Then the following statements hold.*
(a) Assume that M is a complex matrix. Then there exists a complex matrix D such that $M = D^m$.
(b) Assume that M is a real matrix. Then there exists a real matrix D such that $M = D^m$ if and only if M and m satisfy
(b1) m is odd;
(b2) or M has no negative real eigenvalues;
(b3) or the Jordan blocks in the JNF of M corresponding to the negative real eigenvalues appear pairwise.

Before giving the proof of Lemma 6, we recall some useful definitions and a lemma provided in [11]. For a given matrix M, if there is a matrix B satisfying

$$M = e^B = \sum_{k=0}^{\infty} \frac{B^k}{k!},$$

we say that $B = \ln M$ is the *logarithm* of M. Similarly we call the matrix D the *m–th root* of M if $D^m = M$. By the Jordan normal form theory if M is real and non–degenerated, then there exists an invertible $d \times d$ real matrix T, such that $M = TJT^{-1}$ and

$$J = \operatorname{diag}(A, B, C) = \operatorname{diag}(A_1, \ldots, A_r, B_1, \ldots, B_s, C_1, \ldots, C_t),$$

where

$$
\begin{aligned}
A_m &= \begin{pmatrix} \lambda_m & & & \\ 1 & \lambda_m & & \\ & \ddots & \ddots & \\ & & 1 & \lambda_m \end{pmatrix}_{n_m \times n_m}; \\
B_j &= \begin{pmatrix} \mu_j & & & \\ 1 & \mu_j & & \\ & \ddots & \ddots & \\ & & 1 & \mu_j \end{pmatrix}_{p_j \times p_j}; \\
C_k &= \begin{pmatrix} D_k & & & \\ E_2 & D_k & & \\ & \ddots & \ddots & \\ & & E_2 & D_k \end{pmatrix}_{2q_k \times 2q_k}; \\
D_k &= \begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{pmatrix}, \quad E_2 = \operatorname{diag}(1, 1),
\end{aligned} \tag{4}
$$

Here $\lambda_m > 0$, $1 \le m \le r$, $\mu_j < 0$, $1 \le j \le s$, α_k, $\beta_k \in \mathbb{R}$, $\beta_k \neq 0$, $1 \le k \le t$. The next result appears in [11].

Lemma 7. *The following statements hold.*
(a) *Let M be a complex matrix. Then M has a logarithm if and only if M is non–degenerated.*
(b) *Assume that M is a non–degenerated real matrix. Then the following hold.*
($b1$) *The blocks of type A and C in the JNF of M corresponding to the positive and complex eigenvalues, always have real logarithm.*
($b2$) *The Jordan block of type B in the JNF of M corresponding to the negative real eigenvalues has a real logarithm if and only if every block appears*

pairwise, i.e., there is an even number of such blocks: $B = \operatorname{diag}(B_1, \ldots, B_{2m})$ *with* $B_{2i-1} = B_{2i}$ *for* $i = 1, \ldots, m$.

Proof of Lemma 6. (a) From Lemma 7(a), we can choose $D = e^{\ln M/m}$ straightforwardly. Thus $D^m = M$.

(b) Assume that the JNFs of M and D are $\operatorname{diag}(\widetilde{A}, \widetilde{B}, \widetilde{C})$ and diag(A,B,C), respectively. We distinguish them in the detailed form (4) also by the superscript.

First, we note that M has a real m-th root if and only if in its JNF $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{C}$ all have real m-th roots, respectively. Thus, by Lemma 7($b1$), we only consider the block $\widetilde{B}$ in the JNF of M, which means that statement ($b2$) holds.

When m is odd, notice that $-\widetilde{B}$ only has positive eigenvalues. So again, by Lemma 7($b2$), there exists a real matrix $\overline{D}$ such that $\overline{D}^m = -\widetilde{B}$, i.e., $(-\overline{D})^m = \widetilde{B}$. This proves statement ($b1$).

When m is even, the sufficiency of result ($b3$) is from Lemma 7($b2$). Now we prove the necessity. By the assumption of necessity, there exists a real matrix $\overline{D}$ such that $\overline{D}^m = \widetilde{B}$. Since the eigenvalues of A^m and B^m are positive, $\widetilde{B}$ must come from C_k^m for some $1 \le k \le t$, which corresponds to the complex eigenvalues. Notice that in (4) we have

$$D_k = \begin{pmatrix} \alpha_k & \beta_k \\ -\beta_k & \alpha_k \end{pmatrix} = e^{a_k} P_k = e^{a_k} \begin{pmatrix} \cos b_k & \sin b_k \\ -\sin b_k & \cos b_k \end{pmatrix},$$

where

$$a_k = \frac{1}{2} \ln(\alpha_k^2 + \beta_k^2), \quad b_k = \arccos\left(\frac{\alpha_k}{\sqrt{\alpha_k^2 + \beta_k^2}}\right) + 2n\pi.$$

Then denoting $\tau_k = \sqrt{\alpha_k^2 + \beta_k^2}$, we obtain

$$C_k^m =$$

$$= \begin{pmatrix} (\tau_k P_k)^m & 0 & 0 & 0 \\ \binom{m}{1} (\tau_k P_k)^{m-1} & (\tau_k P_k)^m & 0 & 0 \\ \vdots & \ddots & \ddots & \\ \binom{m}{q_k - 1} (\tau_k P_k)^{m-q_k+1} & \ldots & \binom{m}{1} (\tau_k P_k)^{m-1} & (\tau_k P_k)^m \end{pmatrix}_{2q_k \times 2q_k}.$$

Here $\binom{m}{k} = k!(m-k)!/m!$. Since the eigenvalues of C_k^m are negative, $\sin(mb_k) = 0$ and $\cos(mb_k) = -1$. Therefore, we obtain

$$C_k^m = -\tau_k^m \begin{pmatrix} E_2 & 0 & 0 & 0 \\ \binom{m}{1}(\tau_k P_k)^{-1} & E_2 & 0 & 0 \\ \vdots & \ddots & \ddots & \\ \binom{m}{q_k-1}(\tau_k P_k)^{-q_k+1} & \dots & \binom{m}{1}(\tau_k P_k)^{-1} & E_2 \end{pmatrix}_{2q_k\times 2q_k},$$

whose JNF is the matrix $\mathrm{diag}(\widetilde{B}_{\widetilde{k}}, \widetilde{B}_{\widetilde{k}})$, where

$$\widetilde{B}_{\widetilde{k}} = \begin{pmatrix} -\tau_k^m & & & \\ 1 & -\tau_k^m & & \\ & \ddots & \ddots & \\ & & 1 & -\tau_k^m \end{pmatrix}_{q_k\times q_k}.$$

So the Jordan blocks in JNF of M corresponding to the negative real eigenvalues appear pairwise. This completes the proof. □

Proof of Theorem 1. From Lemma 7, the assumptions of the theorem implies that there is a real matrix D such that $D^m = M$. By Lemma 5, we can obtain the result. □

Now assume in system (1) that $x_n \in \mathbb{C}^d$, A_n is a complex matrix and that the other conditions are fulfilled. Then we have the classical theorem.

Corollary 8 (Floquet)**.** *If we write $M = D^m$, then there exists a change of variable $x_n = S_n y_n$, with $S_{n+m} = S_n$ for all $n \in \mathbb{Z}$, such that system (1) becomes*

$$y_{n+1} = Dy_n \tag{5}$$

Proof. Using the same method than in Lemma 5, we can write $S_n = A_{n-1}\cdots A_0 D^{-n}$. It is easy to check that S_n is what we want. This completes the proof □

3. Proof of theorems 3 and 2

Denote by $\mathcal{F}(\mathbb{R})$ the set of the d–dimensional vectors whose components are formal power series of d variables with coefficients real periodic functions of $n \in \mathbb{Z}$ with period m. We say that two formal m–periodic systems

$$x_{n+1} = Ax_n + F_n(x_n)$$

and

$$y_{n+1} = Ay_n + G_n(y_n),$$

where A is a real $d \times d$ matrix, $F_n = O(\|x_n\|)$, $G_n = O(\|y_n\|^2) \in \mathcal{F}(\mathbb{R})$ are *formally equivalent* if there exists a formal m–periodic change of variables $x_n = y_n + h_n(y_n)$, $h_n(y_n) = O(\|y_n\|^2) \in \mathcal{F}(\mathbb{R})$, $h_n = h_{n+m}$, which transforms one system into the other.

Lemma 9. *Assume $\lambda(A)$ is weakly non–resonant, then system* (2) *is formally equivalent to an autonomous system $y_{n+1} = Ay_n + F(y_n)$.*

In order to prove this lemma, we need to study a linear operator on the d–dimensional vector space H_d^l, whose components are valued homogeneous polynomials of d variables of degree l with complex coefficients.

Lemma 10. *Let A be a complex $d \times d$ matrix and $\lambda(A) = (\lambda_1, \ldots, \lambda_d)$. Define a linear operator T_A on H_d^l as follows*

$$T_A h(x) := Ah(A^{-1}x),$$

for $h(x) \in H_d^l$. Then the set of eigenvalues of T_A is

$$\left\{ \lambda_j \prod_{i=1}^{d} \lambda_i^{-k_i} : k_i \in \mathbb{Z}_+, \sum_{i=1}^{d} k_i = l, j = 1, \ldots, d \right\}.$$

Proof. We separate the proof into three cases.
Case 1: We assume $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$. Choose a basis $x^k e_j$ of H_d^l, where $k = (k_1, \ldots, k_d)$, $|k| = \sum_{i=1}^d k_i = l$, $x^k = \prod_{i=1}^d x_i^{k_i}$, e_j is the unit vector having 1 in the j–th coordinate. By the definition of the operator T_A, we have

$$\begin{aligned} T_A x^k e_j &= A(A^{-1}x)^k e_j = \prod_{i=1}^{d} \lambda_i^{-k_i} A x^m e_j \\ &= \left(\lambda_j \prod_{i=1}^{d} \lambda_i^{-k_i} \right) x^m e_j = \lambda_j \lambda^{-k} x^m e_j, \end{aligned}$$

which means that $x^k e_j$ is the eigenvector of the operator T_A and that $\lambda_j \lambda^{-k}$ is the corresponding eigenvalue.

Case 2: We assume that A is diagonalizable. Let $A = PJP^{-1}$, where P is a non–degenerated complex matrix and $J = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$. If we write $x = Py$ and $h(x) = Pg(y)$, then we have

$$h(A^{-1}x) = Pg(P^{-1}A^{-1}Py),$$

which implies

$$T_A Pg(y) = APg(P^{-1}A^{-1}Py).$$

So we define another linear operator T_J on H_d^l as follows

$$T_J g(y) = Jg(J^{-1}y).$$

Let μ be an eigenvalue of the operator T_A with eigenvector $h(x) \neq 0$, i.e., $T_A h(x) = \mu h(x)$. Then

$$T_J g(y) = P^{-1} T_A h(x) = \mu P^{-1} h(x) = \mu g(y).$$

That is, μ is the eigenvalue of T_J. Moreover, since all the above reasoning are invertible, the eigenvalues of T_J and T_A are the same. Now the situation is the same as in Case 1.

Case 3. Assume A is not diagonalizable. Let $A(\varepsilon) \to A$ as $\varepsilon \to 0$ be a family of matrices depending on the parameter ε, which satisfies that $A(\varepsilon)$ is diagonalizable. Let $\lambda(\varepsilon) = (\lambda_1(\varepsilon), \ldots, \lambda_n(\varepsilon))$ be the eigenvalues of $A(\varepsilon)$, then $\lambda(\varepsilon) \to \lambda$ as $\varepsilon \to 0$. Since the entries of the matrix representation of T_A is a finite addition and multiplication of entries of A, we have also $T_{A(\varepsilon)} \to T_A$ as $\varepsilon \to 0$. Finally, by Cases 1 and 2 the proof follows. □

Proof of Lemma 9. Assume that the change of variables $x_n = y_n + h_n(y_n)$ with $h_n(y) = O(\|y\|^2) \in \mathcal{F}(\mathbb{R})$ transforms system (2) into the system $y_{n+1} = Ay_n + g_n(y_n)$. Then we obtain

$$h_{n+1}(Ay_n + g_n(y_n)) = Ah_n(y_n) + f_n(y_n + h_n(y_n)) - g_n(y_n),$$

or equivalently

$$\begin{aligned} h_{n+1}(y_n \quad &+ \quad g_n(A^{-1}y_n)) = Ah_n(A^{-1}y_n) \\ &+ \quad f_n(A^{-1}y_n + h_n(A^{-1}y_n)) - g_n(A^{-1}y_n). \end{aligned} \tag{6}$$

Using the Taylor expansions of the functions $f_n(x)$, $h_n(y)$ and $g_n(y)$ with respect to x, y and y respectively, we have

$$f_n = \sum_{l=2}^{\infty} f_{n,l}(x), \quad h_n = \sum_{l=2}^{\infty} h_{n,l}(y), \quad g_n = \sum_{l=2}^{\infty} g_{n,l}(y), \tag{7}$$

where $f_{n,l}$, $h_{n,l}$ and $g_{n,l}$ are d–dimensional vectors whose components are homogeneous polynomials of degree l for every $n \in \mathbb{Z}$. Substituting the equalities of (7) into equation (6), we solve equation (6) for $h_{n,l}$, $n \in \mathbb{Z}$, inductively by comparing the terms of degree l for $l = 2, 3, \ldots$ Setting $h_{n,1} = g_{n,1} = 0$, assuming that we have already determined the terms of degree no greater than $l-1$ for $l \geq 2$, and comparing the terms of degree l with respect to y_n, finally we get that

$$h_{n+1,l} = T_A h_{n,l} + F_{n,l} - G_{n,l}, \tag{8}$$

where T_A is the linear operator on H_d^l defined in Lemma 10, $G_{n,l}$ is the m–periodic coefficient of $g_{n,l}(A^{-1}y_n)$ with respect to y_n, $F_{n,l}$ is the coefficient vector of the term of degree l of the expression

$$f_n\left(A^{-1}y_n+\sum_{j=1}^{l-1}h_{n,j}(A^{-1}y_n)\right)-\sum_{j=1}^{l-1}h_{n+1,j}\left(y_n+\sum_{j=1}^{l-1}g_{n,j}(A^{-1}y_n)\right),$$

which is m–periodic and known already by the induction assumption.

Our purpose is to find a m–periodic solution $h_{n,l}$ of equation (8) by choosing well $G_{n,l}=G_l$ a constant independent on n. Every solution of equation (8) has the form

$$h_{n,l}=T_A^n h_{0,l}-\sum_{j=0}^{n-1}T_A^j G_l+\sum_{j=0}^{n-1}T_A^{n-j-1}F_{j,l},$$

where $T_A^0=I$ is the identity operator on H_d^l. Now we seek the solution $h_{n,l}$, which satisfies $h_{n,l}=h_{n+m,l}$ for $n\in\mathbb{Z}$. Therefore, we obtain

$$T_A^n\left((T_A^m-I)h_{0,l}-\sum_{j=0}^{m-1}T_A^jG_l+\sum_{j=0}^{m-1}T_A^{m-j-1}F_{j,l}\right)=0,$$

or equivalently

$$(T_A^m-I)h_{0,l}-\sum_{j=0}^{m-1}T_A^jG_l+\sum_{j=0}^{m-1}T_A^{m-j-1}F_{j,l}=0. \tag{9}$$

In fact, by the above equalities, we know that $h_{n,l}=h_{n+m,l}$ for $n\in\mathbb{Z}$ if and only if $h_{0,l}=h_{m,l}$ because $F_{n,l}$ is m–periodic and $\det(A)\neq 0$. Let $\widetilde{T}=\sum_{j=0}^{m-1}T_A^j$ and $\widetilde{F}=-\sum_{j=0}^{m-1}T_A^{m-j-1}F_{j,l}$, then we obtain

$$\widetilde{T}((T_A-I)h_{0,l}-G_l)=\widetilde{F},$$

Because $T_A^m-I=\widetilde{T}(T_A-I)$. If $\{\mu_i\}$ is the set of eigenvalues of T_A, then $\{\widetilde{\mu_i}\}$ is the set of eigenvalues of $\widetilde{T}$, which is given by

$$\widetilde{\mu_i}=\begin{cases} m & \text{if}\quad \mu_i=1,\\[2ex] \dfrac{\mu_i^m-1}{\mu_i-1} & \text{if}\quad \mu_i\neq 1.\end{cases}$$

Since $\lambda(A)$ is weakly non–resonant, $\widetilde{T}$ is invertible. Denote by $(T_A-I)H_d^l$ the image of the operator T_A-I on $H_d^l(\mathbb{R})$ and by R_d^l its complementary subspace: $H_d^l(\mathbb{R})=(T_A-I)H_d^l\oplus R_d^l$. Let

$$\widetilde{T}^{-1}\widetilde{F}=\widetilde{T}_1+\widetilde{T}_2,$$

where $\widetilde{T}_1\in(T_A-I)H_d^l$ and $\widetilde{T}_2\in R_d^l$. Taking $G_l=-\widetilde{T}_2$, then we obtain $(T_A-I)h_{0,l}=\widetilde{T}_1$. Thus, there exists $h_{0,l}\in H_d^l(\mathbb{R})$ satisfying equation (9);

i.e., equation (8) has a m–periodic solution choosing well G_l. This completes the proof. $\square$

Lemma 11. *Given a formal Taylor series $h(x)$, there exists a C^∞ function $\widetilde{h}(x)$ whose formal Taylor series expansion is equal to h at the origin.*

Proof. Suppose that

$$h(x) \sim \sum_{k\in\mathbb{Z}_+^d} a_k \frac{x^k}{k!}.$$

Let $\mu_k = (|a_k|+1)k!$, then the series

$$\widetilde{h}(x) = \sum_{k\in\mathbb{Z}_+^n} a_k \psi(\mu_k\|x\|)\frac{x^k}{k!}$$

defines a C^∞ function with the prescribed power series h, where ψ is the C^∞ cut–off function defined as following

$$\psi(r) = \begin{cases} 1, & 0 \le r \le 1/2; \\ 0, & r > 1. \end{cases}$$

This completes the proof. $\square$

The next result is proved in [2] and [10].

Lemma 12. *If two hyperbolic germs of C^∞ diffeomorphism are formally conjugate, then they are C^∞ conjugate.*

Proof of Theorem 2. Since $\lambda(M)$, the eigenvalues of the linear monodromy, satisfies the non–resonant conditions, then the Poincaré map is formally linearizable. See for more details [5, 3]. By the same condition, we know that M is hyperbolic. So by Lemma 12, this Poincaré map is C^∞ linearizable. Thus the corresponding periodic difference system is linearizable by Lemma 5. This completes the proof. $\square$

Proof of Theorem 3. By Lemma 7, system (2) can be formally changed into a real formal autonomous difference system

$$y_{n+1} = Ay_n + F(y_n)$$

by the formal coordinate substitution

$$x_n = y_n + h_n(y_n), \quad h_{n+m} = h_n.$$

Applying Lemma 11, there exists a C^∞ m–periodic function $\widetilde{h}_n$, $\widetilde{h}_{n+m} = \widetilde{h}_n$, satisfying $Jet^\infty_{x_n=0}\widetilde{h}_n(x_n) = Jet^\infty_{x_n=0}h_n(x_n)$ for $n \in \mathbb{Z}$, where the notation $Jet^\infty_{x_n=0}h_n(x_n)$ means the sequence of coefficients of the Taylor expansion of h_n at the origin. Now under the change of variables $x_n = y_n + \widetilde{h}_n(x_n)$, system (2) is C^∞ equivalent to

$$y_{n+1} = Ay_n + \widetilde{F}_n(y_n), \tag{10}$$

where $\widetilde{F}_n = \widetilde{F}_{n+m}$ with $Jet^\infty_{y_n=0}\widetilde{F}_n(y_n) = Jet^\infty_{y_n=0}F(y_n)$. Using again Lemma 11, we know that there exists a C^∞ function $\widetilde{f}$ such that $Jet^\infty_{y_n=0}$ $\widetilde{f}(y_n) = Jet^\infty_{y_n=0}F(y_n)$. Therefore, system (10) can be written in the new form

$$y_{n+1} = Ay_n + \widetilde{f}(y_n) + r_n(y_n), \tag{11}$$

where $r_{n+m} = r_n$ and $Jet^\infty_{y_n=0}r_n(y_n) = 0$. Denote by Φ the Poincaré map of the system

$$y_{n+1} = Ay_n + \widetilde{f}(y_n). \tag{12}$$

We specially note that this Poincaré map is a m–time map, i.e., Φ: $y_0 \mapsto y_m$. Let Ψ be the Poincaré map of system (11). Then we have $Jet^\infty_{\cdot=0}\Phi(\cdot) = Jet^\infty_{\cdot=0}\Psi(\cdot)$. Since A is hyperbolic by our assumptions, we obtain that Φ and Ψ are hyperbolic. Therefore, by Lemma 12 they are C^∞ conjugate. Finally using Lemma 5 we complete the proof. $\square$

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