A ROTATION METHOD WHICH GIVES LINEAR L^p-ESTIMATES FOR POWERS OF THE AHLFORS-BEURLING OPERATOR

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ABSTRACT. In [12] the Ahlfors-Beurling operator T was represented as an average of two-dimensional martingale transforms. The same result can be proven for powers T^n. Motivated by [18], we deduce from here that $\|T^n\|_p$ are bounded from above by Cnp^*, $p^* = \max\left(p, \frac{p}{p-1}\right)$. We further improve this estimate to obtain optimal behaviour of the L^p norms in question.

Résumé

L'opérateur d'Ahlfors–Beurling T a une répresentation comme un moyenne des transformations de martingale sur le plan, voir [12]. Le même résultat existe pout T^n. On en déduit (motivés par [18]) que les normes $\|T^n\|_p$ soient bornés par Cnp^*, $p^* = \max\left(p, \frac{p}{p-1}\right)$. On peut améliorer ce résultat et obtenir la meilleur borne pour ces normes.

1. Introduction

In the article [18] Iwaniec and Martin study singular integrals that appear in regularity theory of nonlinear PDE in arbitrary dimensions. For example, they compute the L^p-norms of scalar Riesz transforms on $\mathbb{R}^n$, thus extending a well-known result of Pichorides [23]. One of the key features of their work is that they succeed in reducing the estimates of vector-valued operators on $\mathbb{R}^n$ (such as combinations of Riesz transforms, complex Riesz transforms, certain differential operators, etc.) to those of scalar valued operators on $\mathbb{C}$. There the crucial rôle is played by the Ahlfors-Beurling

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operator T, which is defined as

$$Tf(z) = -\frac{1}{\pi}\,\text{p.v.}\int_{\mathbb{C}} \frac{f(\zeta)}{(z-\zeta)^2}\,d\zeta\,,$$

its square root $\mathbf{H}_{\mathbb{C}}$ and their powers. From Vekua [25, I, §9] it emerges that

$$\mathbf{H}_{\mathbb{C}} f(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{f(\zeta)}{(\zeta - z)|\zeta - z|}\,d\zeta\,.$$

The m-th iterate of $\mathbf{H}_{\mathbb{C}}$ is the convolution operator $\mathbf{H}_{\mathbb{C}}^m$ with the kernel

$$\frac{i^{|m|}|m|}{2\pi}\left(\frac{|z|}{z}\right)^m \frac{1}{|z|^2}\,. \tag{1}$$

Here $m \in \mathbb{Z}\backslash\{0\}$. As said above, $\mathbf{H}_{\mathbb{C}}^2 = T$. Throughout the article they extensively work with $\mathbf{H}_{\mathbb{C}}^m$; most of their estimates are expressed in terms of the norm of $\mathbf{H}_{\mathbb{C}}^m$ on $L^p(\mathbb{C})$, which they denote by $H_p(m)$. However, no estimate on $H_p(m)$ itself is given. Explicitly, on p. 28 bottom they write: "(...) *As far as we are aware this is the first time estimates of the norms of such singular integral operators have been attained and in particular the reduction to the estimates of norms of two dimensional operators seems new. We should point however that the p-norms $H_p(m)$ of the m^{th} iterate of the complex Hilbert transform are as yet unknown*".

In a subsequent paper by Iwaniec and Sbordone [19] it was noticed that for odd m one can resort to the method of rotations, developed in the 1956 paper [9] of Calderón and Zygmund, see also [14, Section 4.3] or [24, Chapter II], which returns

$$H_p(2n-1) \leqslant \frac{\pi}{2}(2n-1)\cot\frac{\pi}{2p^*} \leqslant Cnp^*, \qquad \forall n \in \mathbb{N}$$

with an absolute constant C. Here

$$p^* = \max\left\{p, \frac{p}{p-1}\right\}.$$

On the other hand, the case of even m does not enjoy such a linear estimate. Obviously

$$H_p(2n) \leqslant H_p(2n-1)H_p(1) \leqslant Cn{p^*}^2\,. \tag{2}$$

The slight difference lies in the fact that the kernel of $\mathbf{H}_{\mathbb{C}}^{2n} = T^n$ is even. Calderón and Zygmund [9] derive a method for operators with even kernels as well, but that method returns the same quadratic estimate in p as (2), namely

$$\|T^n\|_p = H_p(2n) \leqslant Cn{p^*}^2\,.$$

The main goal of this note is to present another method of rotation, which works very well exactly for even kernels. We apply it to $T^n = \mathbf{H}_{\mathbb{C}}^{2n}$ which gives us a linear estimate on $H_p(2n)$.

Theorem 1. *There is an absolute constant $C > 0$ such that for all $n \in \mathbb{N}$ and $1 < p < \infty$,*

$$\|T^n\|_p \leqslant Cnp^* . \tag{3}$$

Our proof of Theorem 1 consists of two main parts. One is a generalization of Burkholder's sharp inequality for martingale transforms on the line. He proved that the L^p norm of any such operator does not exceed $p^* - 1$. We construct analogue operators on the plane (while retaining the same name) for which we show that the Burkholder's theorem is still valid.

The second ingredient of the proof is representation of T as an average of these planar martingale transforms, which was the principal result of [12]. Here we revisit the technique used there in order to obtain such a representation for arbitrary T^n. Actually, we come up with many different underlying Haar systems, called $\mathcal{H}_b$, which enables us to optimize the arising constants, and consequently we yield linearity of estimates simultaneously in n and p.

Let us state this result.

Theorem 2. *Choose $b > 0$. For every $n \in \mathbb{Z}$ we have*

$$T^n = C_b(n) \cdot T'$$

where $C_b(n) > 0$ and T' is a result of an averaging process involving martingale transforms on translated, dilated and rotated $\mathcal{H}_b$.

Hence we can estimate $H_p(m)$ for even m, since in that case the kernels of $\mathbf{H}^m_{\mathbb{C}}$ are symmetric and our averaging method works. Besides, all $\mathbf{H}^m_{\mathbb{C}}$ have the "right" order of homogeneity (i.e. -2). We cannot replace T^n in the above theorem by $\mathbf{H}^m_{\mathbb{C}}$ with odd m. For a result regarding representation of operators with odd kernels as averages of simpler ones see [22].

One can even sharpen (3) for a fixed p and get

$$\|T^n\|_p \leqslant C(n \log n)^{1-2/p^*} p^* . \tag{4}$$

This can be extracted from interpolation between (3) and the case of $p = 2$ for which $\|T^n\|_2 = 1$.

The estimate (4) is still not quite optimal. Namely, after we had already obtained (3) and (4), we learned of a theorem [10], [16] which enables us to push (4) to its limits. Since we also obtain sharp estimates from below, this yields the optimal behaviour of the norms $H_p(2n)$, as is described in the following statement.

Theorem 3. *There are absolute constants $C_1, C_2 > 0$ such that for all $n \in \mathbb{N}$ and $1 < p < \infty$,*

$$C_1 n^{1-2/p^*} p^* \leqslant \|T^n\|_p \leqslant C_2 n^{1-2/p^*} p^* . \tag{5}$$

One can take $C_1 = e^{-1}$.

To demonstrate the right inequality in (5) we will use the above-mentioned powerful result due to Christ, Rubio de Francia [10] and Hofmann [16] which regards weak type estimates for operators with even kernels. As such (since $L^{1,\infty}$ is not a locally convex space) it cannot be proven by any rotation method, and the proof is an example of hard analysis. Of course (5) implies (3).

However, there may be some advantages of our proof of (3). It is conceptually interesting and technically very simple. Another small advantage is that it gives certain control on the constant in (3):

Theorem 4. *There is* $N_0 \in \mathbb{N}$ *such that for all* $n \geqslant N_0$ *and* $1 < p < \infty$,

$$\|T^n\|_p \leqslant 2.72\, n(p^* - 1)\,. \tag{6}$$

It is interesting to compare this estimate to the estimate of $\kappa_n(p)$ in Lemma 3 below and with the Conjecture that we put after Lemma 3.

It seems to be quite difficult to derive the numerical value of $\|T^n\|_{L^1\to L^{1,\infty}}$ in [10], [16]. Estimate (6) hints that these norms are bounded by 2.72.

Constant C in (3) can represent a considerable interest. For example, for $n = 1$ there is since 1982 a well-known conjecture of T. Iwaniec [17] that

$$\|T\|_p = p^* - 1\,. \tag{7}$$

This would have very interesting analytic and geometric implications for the theory of quasiconformal mappings (see discussions in [17], [2], [21]). In particular, the main result of [1] would immediately follow from (7). Although the conjecture is yet unconfirmed, it is known that the growth of norms is indeed linear. In [13] the estimate

$$\|T\|_p \leqslant \sqrt{2}(p-1)\Big(\frac{1}{2\pi}\int_0^{2\pi} |\cos\vartheta|^p\, d\vartheta\Big)^{-1/p}, \quad p \geqslant 2\,,$$

was given. Very recently a better estimate $\|T\|_p \leqslant \sqrt{2p(p-1)}$ was obtained in [4]. Both [13] and [4] under interpolation improve to $(2-\varepsilon)(p^*-1)$. Of course [4] gives a larger ε, namely 0.425. For large p both [13] and [4] return $\lessapprox \sqrt{2}(p-1)$.

We conclude the presentation with a conjecture about exact L^p norms of T^n. Its special case $n = 1$ is the aforementioned conjecture of Iwaniec (7).

A bit of hydrodynamics. Sparked by some questions raised in [15], we consider L^p estimates of the family of Fourier multipliers on $\mathbb{C}$ given by symbols $e^{in\cos\varphi}$, $n \in \mathbb{Z}$. They are actually powers of the Fourier multiplier with symbol $e^{i\cos\varphi}$. At first sight this family represents just a small deviation from $\{T^n\,; n \in \mathbb{N}\}$, but in fact we cannot obtain for it the analogue of Theorem 2. We prove upper estimates for its L^p norms. As to the lower estimate we can obtain only a slightly weaker result.

Theorem 5. *Denote* $m_n(re^{i\varphi}) = e^{in\cos\varphi}$ *and let* S_n *be the Fourier multiplier with symbol* m_n, *that is,* $\widehat{S_n f} = m_n \widehat{f}$. *Then*

$$\|S_n\|_p \leqslant C n^{1-2/p*}(p^* - 1)\,.$$

for all $n \in \mathbb{Z}$ *and all* $1 < p < \infty$.

Proof. It is known [24, III.3.5] that each S_n can be equivalently described as a principal value convolution operator with a kernel that is homogeneous of degree -2. That is,

$$S_n f(z) = \text{p.v.} \int_{\mathbb{C}} \frac{\Omega_n(\zeta)}{|\zeta|^2} f(z - \zeta)\, d\zeta\,,$$

where Ω_n is a smooth function on the sphere with mean zero. A standard formula then gives

$$m_n(re^{i\vartheta}) = \int_0^{2\pi} \Omega_n(e^{i\varphi}) \left(\log \frac{1}{|\cos(\varphi - \vartheta)|} - i\,\frac{\pi}{2}\,\operatorname{sign}\cos(\varphi - \vartheta) \right) d\varphi\,.$$

In short,

$$m_n = \Omega_n * \Gamma\,,$$

where

$$\Gamma(e^{i\vartheta}) = \log \frac{1}{|\cos\vartheta|} - i\,\frac{\pi}{2}\,\operatorname{sign}\cos\vartheta\,.$$

If we apply the same formula to the powers $\mathbf{H}_{\mathbb{C}}^k$, we can extract from (1) the Fourier coefficients of Γ:

$$\widehat{\Gamma}(k) = \frac{2\pi}{i^{|k|}|k|}\,.$$

Thus

$$\widehat{m}_n(k) = \frac{2\pi}{i^{|k|}} \cdot \frac{\widehat{\Omega}_n(k)}{|k|}\,, \qquad k \neq 0\,. \tag{8}$$

Therefore

$$\|\Omega_n\|_2 = 2\pi \|m_n'\|_2 = 2\pi\, n \|e^{in\cos\varphi} \sin\varphi\|_2 = C\, n\,.$$

By the same reasoning as in the proof of the upper estimate of Theorem 3, see page 18, $\|S_n\|_p$ is bounded by $Cn^{2/p-1}/(p-1)$ on L^p for $1 < p < 2$. When $p > 2$ use duality:

$$\langle S_n f, g\rangle = \langle \widehat{S_n f}, \hat{g}\rangle = \langle m_n \hat{f}, \hat{g}\rangle = \langle \hat{f}, \overline{m}_n \hat{g}\rangle = \langle \hat{f}, m_{-n}\hat{g}\rangle = \langle f, S_{-n} g\rangle\,.$$

This completes the proof. □

Theorem 6. *Assuming the notation from above, there is for every $\delta > 0$ a constant C_δ such that*

$$\|S_n\|_p \geqslant C_\delta n^{1-\frac{2}{p^*}-\delta}(p^*-1)\,,$$

for all $n \in \mathbb{Z}$ and all $1 < p < \infty$.

Proof. Consider $p \in (1,2)$. Notice that $e^{in\cos\varphi}\sin\varphi$ is an odd function. Consider function ω_n given by its Fourier coefficients almost in the same way as in (8)

$$\widehat{m}_n(k) = 2\pi \cdot \frac{\widehat{\omega}_n(k)}{|k|}\,, \qquad k \neq 0\,.$$

This formula shows that

$$\omega_n = c\,nH(e^{in\cos\varphi}\sin\varphi)\,,$$

where H stands for the Hilbert transform on the circle. Function $e^{in\cos\varphi}\sin\varphi$ has the property

$$oscillation_I(e^{in\cos\varphi}\sin\varphi) \leqslant \frac{\pi}{4}\,,$$

if $|I| \leqslant \frac{1}{10n}$. This is obvious as its derivative is bounded by n uniformly. But then the derivative of $H(e^{in\cos\varphi}\sin\varphi)$ is bounded by n in BMO, and hence in any $L^r(\mathbb{T})$. Therefore,

$$oscillation_I(H(e^{in\cos\varphi}\sin\varphi)) \leqslant \frac{\pi}{4}\,,$$

if $|I| \leqslant \frac{1}{C_\varepsilon n^{1+\varepsilon}}$.

Now we need to have the same type of estimate for $K_n = \frac{\Omega_n(\zeta - z)}{|\zeta - z|^2}$, so we need the estimate for oscillation of Ω_n, not for ω_n. The factor $i^{|k|}$ is not a big problem as we just split $m_n|_{\mathbb{T}}$ to four functions as follows:

$$m_{nl}(z) = \frac{m_n(z)}{z^l}\,, \qquad l = 1,2,3,4\,,$$

average them to get

$$M_{nl}(z) = \frac{1}{4}(m_{nl}(z) + m_{nl}(iz) + m_{nl}(i^2z) + m_{nl}(i^3z))$$

and consider $m_n^l(z) = z^l M_{nl}(z)$ for $|z| = 1$. That is,

$$m_n^l(z) = \frac{1}{4}\Big(m_n(z) + \frac{m_n(iz)}{i^l} + \frac{m_n(i^2z)}{i^{2l}} + \frac{m_n(i^3z)}{i^{3l}}\Big)\,.$$

Then we consider Ω_n^l given by (8), but with $m_n^l(z)$ instead of m_n. All $m_n^l(e^{i\vartheta})$ have derivatives bounded by n. This is obvious by construction. So we can repeat our considerations verbatim for each Ω_n^l. Then oscillation for them will be estimated. But Ω_n is just $\Omega_n^1 + \Omega_n^2 + \Omega_n^3 + \Omega_n^4$.

Now choose a test function $\phi(\zeta)$ to be 0 outside the disc $D(0, \frac{1}{C_\varepsilon n^{1+\varepsilon}})$ and equal to $n^{\frac{2+2\varepsilon}{p}}$ inside this disc. Its norm in L^p is 1. If we act on it by kernel $K = \frac{\Omega(\zeta-z)}{|\zeta-z|^2}$, $|z| > 1$, then this oscillation condition allows us to write

$$\Big|\int K_n(\zeta-z)\phi(\zeta)\,dA(\zeta)\Big| \geqslant c\,n\|\phi\|_1 = C\,n^{1+(2+2\varepsilon)(\frac{1}{p}-1)} = C\,n^{\frac{2}{p}-1}n^{-2\varepsilon(1-\frac{1}{p})}\,.$$

This is a pointwise estimate.

Integrating p-th power over $\{1 < |z| < 2\}$ and taking power $1/p$ we get the estimate of the theorem with $\delta = 2\varepsilon(1-\frac{1}{p})$.

□

Remark. We do not know how to get rid of δ.

Notice that the same estimate holds for the same multiplier in $\mathbb{R}^3$.

Theorem 7. *Let φ be a polar angle in one of hyperplanes in $\mathbb{R}^3$. Denote $m_n(x,y,z) = e^{in\cos\varphi}$ and let S_n be the Fourier multiplier with symbol m_n, that is, $\widehat{S_n f} = m_n\widehat{f}$. Then*

$$\|S_n\|_p \geqslant C_\delta n^{1-\frac{2}{p^*}-\delta}(p^*-1)\,,\ \forall\delta > 0\,,$$

for all $n \in \mathbb{Z}$ and all $1 < p < \infty$.

The proof is pretty much obvious as we can use a "slice" test function.

Remark. We do not know the sharpness or estimate from above. It can be quite not sharp, and behavior in $\mathbb{R}^3$ may generate higher powers of n. But our previous theorems give a partial answer for the questions posed in hydrodynamical paper [15].

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2. Sharp estimates for martingale transforms

Let us start by recalling some definitions. We call the family of intervals $\mathcal{L} := \{[m\,2^n, (m+1)\,2^n)\,; m, n \in \mathbb{Z}\}$ the standard *dyadic lattice*. Each interval $I \subset \mathbb{R}$ gives rise to its *Haar function* h_I, defined by

$$h_I := |I|^{-1/2}(\chi_{I_+} - \chi_{I_-})\,,$$

where I_- and I_+ denote the left and the right half of the interval I respectively, and χ_E stands for the characteristic function of the set E, as usual.

Denote by $\mathcal{L}(I)$ the set of all dyadic subintervals of the interval I, including I itself. For any $p \in (1,\infty)$ and any interval I, the set $\{h_J\,;J\in\mathcal{L}(I)\}$ forms a *basis* of the space $L^p(I)$. By that we shall mean that for $f\in L^p(I)$,

$$f-\langle f\rangle_I\chi_I=\lim_{n\to\infty}\sum_{\substack{J\in\mathcal{L}(I)\\ |J|>2^{-n}|I|}}\langle f,h_J\rangle h_J\,,$$

the limit existing in the L^p-sense and $\langle f\rangle_I$ standing for the average of function f over I. A similar statement is valid for arbitrary intervals, of course.

Now we are able to define the operator T_σ by

$$T_\sigma f:=\sum_{J\in\mathcal{L}}\sigma_J\langle f,h_J\rangle\,h_J\,,$$

where $\sigma:\mathcal{L}\to S^1$ is arbitrary. Such operators are called *martingale transforms.* Note that if f is a test function, the terms $\langle f,h_J\rangle$ are nonzero only for J contained in the support of f.

2.1. **Two dimensional case.** It was proven by Burkholder [5], [6], [8] that

$$\sup_\sigma\|T_\sigma\|_{B(L^p)}=p^*-1\,. \tag{9}$$

We would like to extend this result to the martingale transforms on $\mathbb{R}^2$. For that purpose we should start with the construction of the Haar system on the plane. We repeat the definitions from [12].

The term *dyadic lattice* and the symbol $\mathcal{L}$ will now stand for the collection of all squares of the form $I\times J\subset\mathbb{R}^2$, where I and J are dyadic intervals of the same length. To each such square $Q=I\times J$ we will assign three Haar functions:

$$\begin{aligned}
h^1_Q(s,t)&=\chi_I(s)h_J(t)|I|^{-1/2}\\
h^2_Q(s,t)&=h_I(s)\chi_J(t)|J|^{-1/2}\\
h^3_Q(s,t)&=h_I(s)h_J(t)\,.
\end{aligned}$$

Symbolically,

$$h^1_Q\equiv\begin{array}{|c|}\hline \quad + \quad \\ \hline \quad - \quad \\ \hline\end{array}\qquad h^2_Q\equiv\begin{array}{|c|c|}\hline - & + \\ \hline\end{array}\qquad h^3_Q\equiv\begin{array}{|c|c|}\hline - & + \\ \hline + & - \\ \hline\end{array}$$

As previously, one can verify that the set $\{h^i_Q\,;Q\in\mathcal{L},\ i=1,2,3\}$ constitutes a basis of $L^p(\mathbb{R}^2)$. In order to distinguish it from the subsequent Haar systems, we will call it $\mathcal{H}_{orig}$. Now the *two-dimensional martingale transform* becomes the operator

$$T_\sigma f:=\sum_{Q\in\mathcal{L}}\sum_{i=1}^3\sigma^i_Q\langle f,h^i_Q\rangle\,h^i_Q\,,$$

where, as before, $\sigma^i : \mathcal{L} \to S^1$.

2.2. **Modified system.** It turns out there is a subtle reason for which we are not able to reprove (9) for these operators. See [11] where it was explained in detail. An elegant way of solving this problem was suggested to us by Guy David. He proposed associating to each square Q a different set of Haar functions:

$$h_Q^0 := h_Q^1 , \quad h_Q^+ := \frac{1}{\sqrt{2}}\,(h_Q^2 + h_Q^3) , \quad h_Q^- := \frac{1}{\sqrt{2}}\,(h_Q^2 - h_Q^3) . \tag{10}$$

Symbolically,

$$h_Q^0 \equiv \begin{array}{|cc|}\hline \multicolumn{2}{|c|}{+} \\ \hline \multicolumn{2}{|c|}{-} \\ \hline \end{array} \qquad h_Q^+ \equiv \begin{array}{|c|c|}\hline - & + \\ \hline \multicolumn{2}{|c|}{\ } \\ \hline \end{array} \qquad h_Q^- \equiv \begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{\ } \\ \hline - & + \\ \hline \end{array}$$

Let us denote the system $\{h_Q^* \,; * \in \{0,+,-\},\ Q \in \mathcal{L}\}$ by $\mathcal{H}_{new}$. In that case the associated martingale transforms do admit the desired estimates, as Theorem 8 below shows. In order to prove it we apply the following lemma of Burkholder [7], which is very useful due to its generality and sharpness. We present it here for the convenience of the reader.

Lemma 1. *Let $(\mathfrak{W}, \mathcal{F}, P)$ be a probability space, $\{\mathcal{F}_n \,; n \in \mathbb{N}\}$ a filtration in $\mathcal{F}$ and H a separable Hilbert space. Furthermore, let $(X_n, \mathcal{F}_n, P)$ and $(Y_n, \mathcal{F}_n, P)$ be H-valued martingales satisfying*

$$\|Y_0(\omega)\|_H \le \|X_0(\omega)\|_H \ \text{and}\ \|Y_n(\omega) - Y_{n-1}(\omega)\|_H \leqslant \|X_n(\omega) - X_{n-1}(\omega)\|_H \tag{11}$$

for all $n \in \mathbb{N}$ and almost every $\omega \in \mathfrak{W}$. Then for any $p \in (1, \infty)$

$$\|Y_n\|_p \leqslant (p^* - 1)\|X_n\|_p .$$

The constant $p^* - 1$ is sharp.

The property (11) is called *differential subordination.*

Theorem 8. *For any $Q \in \mathcal{L}$ and $* \in \{0,+,-\}$ let σ_Q^* be arbitrary unimodular complex numbers. Define the operator*

$$T_\sigma f := \sum_{Q \in \mathcal{L}} \left[\sigma_Q^0 \langle f, h_Q^0 \rangle h_Q^0 + \sigma_Q^+ \langle f, h_Q^+ \rangle h_Q^+ + \sigma_Q^- \langle f, h_Q^- \rangle h_Q^- \right] .$$

Then $\|T_\sigma\|_p \leqslant p^* - 1$. This estimate is sharp.

Proof. Take a test function f, supported in some $\Omega \in \mathcal{L}$, and define

$$X_{2n} := \sum_{\substack{Q\in\mathcal{L}(\Omega)\\ |Q|>4^{-n}}} \Big[\langle f, h_Q^0\rangle h_Q^0 + \langle f, h_Q^+\rangle h_Q^+ + \langle f, h_Q^-\rangle h_Q^-\Big]$$

$$X_{2n+1} := X_{2n} + \sum_{\substack{Q\in\mathcal{L}(\Omega)\\ |Q|=4^{-n}}} \langle f, h_Q^0\rangle h_Q^0$$

and

$$Y_{2n} := \sum_{\substack{Q\in\mathcal{L}(\Omega)\\ |Q|>4^{-n}}} \Big[\sigma_Q^0\langle f, h_Q^0\rangle h_Q^0 + \sigma_Q^+\langle f, h_Q^+\rangle h_Q^+ + \sigma_Q^-\langle f, h_Q^-\rangle h_Q^-\Big]$$

$$Y_{2n+1} := Y_{2n} + \sum_{\substack{Q\in\mathcal{L}(\Omega)\\ |Q|=4^{-n}}} \sigma_Q^0\langle f, h_Q^0\rangle h_Q^0\,.$$

Let $\mathcal{F}_m$ be the $\sigma-$algebra, generated by X_m. Explicitly, $\mathcal{F}_{2n}$ is generated by all dyadic squares of size 4^{-n}, while $\mathcal{F}_{2n+1}$ is generated by their upper and lower halves. Each $\mathcal{F}_{m+1}$ is properly contained in $\mathcal{F}_m$, hence $(X_m, \mathcal{F}_m, dx)$ and $(Y_m, \mathcal{F}_m, dx)$ are martingales. Moreover, it is clear that they satisfy the differential subordination:

$$|(X_{m+1} - X_m)(\omega)| = |(Y_{m+1} - Y_m)(\omega)| \qquad \forall\omega \in \mathbb{C}\,.$$

We can apply Lemma 1 and get that $\|Y_m\|_p \leqslant (p^* - 1)\|X_m\|_p$ for every $m \in \mathbb{N}$. Now use that $\lim_{m\to\infty}\|X_m\|_p = \|f\|_p$ and $\lim_{m\to\infty}\|Y_m\|_p = \|T_\sigma f\|_p$. □

3. The averaging. Proof of Theorem 2

The special case $n = 1$ of Theorem 2 first appeared in [12]. There it was proven for $\mathcal{H}_{orig}$. The general case is not significantly different. Still, we have to review the most important steps of the proof, since understanding how the constants $C(n)$ are obtained will be crucial for proving Theorem 1. We summarize the proof as it appeared in [12].

Instead of a dyadic lattice let us for a moment consider a unit *grid* $\mathcal{G}$ of squares. This is a family of squares $I \times J$, where I and J are dyadic intervals of unit length. Furthermore, for $t \in \mathbb{R}^2$ define $\mathcal{G}_t := \mathcal{G} + t$, i.e. the grid of unit squares such that one of them contains point t as one of its vertices.

Introduce

$$\mathbb{P}_t f := \sum_{Q\in\mathcal{G}_t} \langle f, h_Q^0\rangle h_Q^0\,.$$

The family $\Omega := \{\mathcal{G}_t\,; t \in \mathbb{R}^2\}$ of all unit grids naturally corresponds to the torus $\mathbb{R}^2/\mathbb{Z}^2$, which is of course in one-to-one correspondence with the

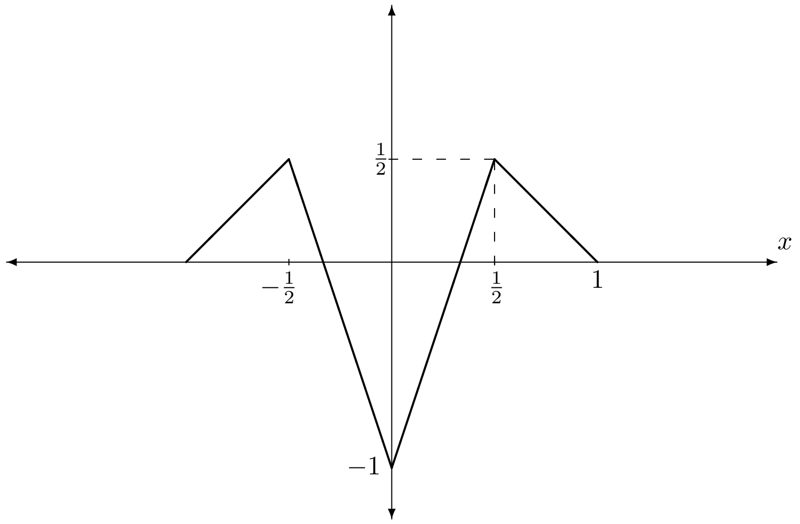

FIGURE 1. Graph of α

square $[0,1)^2$. Thus we are able to regard Ω as a probability space where the probability measure equals the Lebesgue measure on $[0,1)^2$.

Now this leads to the "mathematical expectation" of the "random variable" $\mathbb{P}$. Symbolically,

$$\mathbb{E}\mathbb{P}\, f = \int_\Omega \mathbb{P}_t f\, dt\,.$$

Since $\mathbb{E}\mathbb{P}$ is a result of integrating over a certain probability space, it makes sense to call this process the *averaging*. The structure of this operator is revealed in the following proposition. Readers interested in details of the proof should consult [12].

Proposition 1. *Assuming the notation as above, the operator $\mathbb{E}\mathbb{P}$ is a convolution operator with the kernel $F(x,y) = -\beta(x)\alpha(y)$, where*

$$\alpha = h_0 * h_0 \quad \textit{and} \quad \beta = \chi_0 * \chi_0.$$

Here χ_0 and h_0 stand (respectively) for the characteristic and Haar function of the interval $[-1/2,1/2)$. Inserting h_Q^2 in $\mathbb{P}_t$ instead of $h_Q^1 = h_Q^0$ yields $-\alpha(x)\beta(y)$, while h_Q^3 would produce $\alpha(x)\alpha(y)$.

Graphs of functions α and β are shown as Figures 1 and 2, respectively.

Instead of the unit grid we may consider a grid of squares with sides of an arbitrary length $\rho > 0$. Denote such a grid by $\mathcal{G}_t^\rho$ if $t \in \mathbb{R}^2$ is a vertex of

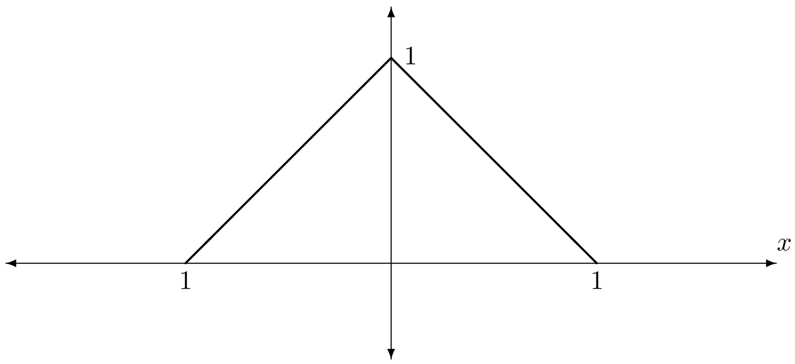

FIGURE 2. Graph of β

one of its members. Henceforth we will call ρ the *size* of the grid and t its *reference point.* We obtain another family of operators, defined by

$$\mathbb{P}_t^\rho f := \sum_{Q\in\mathcal{G}_t^\rho} \langle f, h_Q^0\rangle h_Q^0\,.$$

Applying Proposition 1 or modifying its proof, we can show the following.

Proposition 2. *Choose $\rho > 0$. Then averaging operators $\mathbb{P}_t^\rho$ returns a convolution operator with the kernel*

$$F^\rho(x,y) := \frac{1}{\rho^2}\,F\left(\frac{x}{\rho},\frac{y}{\rho}\right)\,.$$

Thus we have found the kernel of the operator, resulting from averaging over *all* grids of a *fixed* size. Our next step will be to average over all *sizes.* Let us explain what we mean by that.

Take $r > 0$. A *lattice of calibre* r is said to be a family of intervals (squares), obtained from the standard dyadic lattice $\mathcal{L}$ by dilating it by a factor r and translating by an arbitrary vector t. In other words, such a lattice (call it $\mathcal{L}_t^r$) is the union of *grids* of sizes $r\cdot 2^n\,, n\in\mathbb{Z}$, having t as their reference point.

We introduce kernels

$$k^r := \sum_{n=-\infty}^{\infty} F^{r\cdot 2^n}\,.$$

By Proposition 2,

$$k^r = \frac{1}{r^2}\sum_{n=-\infty}^{\infty}\frac{1}{4^n}\,F\Big(\frac{\cdot}{2^n r}\Big)$$

where the sum converges absolutely and uniformly on the complement of any ball centered at the origin.

The fact that k^r* is a sum of operators, obtained by averaging over grids of size $r \cdot 2^n$, hints at k^r* itself being an average, this time over *unions* of these grids, i.e. lattices of calibre r. While it is not clear what could be a probability space corresponding to all lattices of a fixed calibre, we define the above-said average as a limit of averages of truncated lattices. Then the statement makes sense and holds [12]. Virtually the same proof establishes the lemma which follows below.

For $M \in \mathbb{Z}$ let the M-th partial sum of the series k^r be

$$k^r_M := \sum_{n=-\infty}^{M} F^{r \cdot 2^n} .$$

Lemma 2. *Function k^r_M defines a bounded convolution operator on L^p. The limit $k^r* := \lim_{M\to\infty} k^r_M*$ exists in the strong sense and also gives rise to a bounded operator on L^p.*

Next step is to average over dilations, in other words, over all calibres r. It is clear that the set of all possible calibres naturally corresponds to the interval $[1, 2)$. For our purpose, the most appropriate measure on this interval turns out to be dr/r. This makes all other possible choices of intervals, e.g. $[2^n, 2^{n+1})$, have the same measure $(\log 2)$.

Averaging operators k^r*, i.e. integrating k^r with respect to the normalized measure dr/r, gives us a convolution operator once again. Call its kernel k. Then a quick computation shows

$$k(\zeta) = \frac{1}{\log 2} \int_0^\infty F^s(\zeta)\, \frac{ds}{s}$$

for $\zeta \in \mathbb{C}\backslash\{0\}$. By applying Proposition 2 we get

$$k(\zeta) = \frac{1}{\log 2} \int_0^\infty F(r\zeta)\, r\, dr\,. \tag{12}$$

Note that for $r > 0$,

$$k(re^{i\varphi}) = \frac{k(e^{i\varphi})}{r^2} \tag{13}$$

Because of this it suffices to know the behaviour of k on S^1.

Finally, we are going to perform averaging over rotations. Choose $\psi \in [0, 2\pi)$. Operators will be the same as before, just that the grids and lattices will consist of squares, rotated by the angle ψ counterclockwise with respect to the standard position. Let $U_\psi : \mathbb{C} \to \mathbb{C}$ be defined by $U_\psi(\zeta) := \zeta e^{-i\psi}$. Then the convolution kernel of the operator K_ψ, which corresponds to the average over rotated lattices, is equal to $k_\psi := k \circ U_\psi$. The operator itself satisfies the similarity relation $K_\psi = S_\psi^{-1} K_0 S_\psi$, where $S_\psi f = f \circ U_{-\psi}$.

Now let us fix $n \in \mathbb{N}$ and define a (weighted) average of operators K_ψ, which we denote by T'.

$$\begin{aligned}(T'f)(z) &:= \frac{1}{2\pi}\int_0^{2\pi}(K_\psi f)(z)\, e^{-2in\psi}\, d\psi \\ &= \frac{1}{2\pi}\int_0^{2\pi}(k_\psi * f)(z)\, e^{-2in\psi}\, d\psi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}} k(\zeta e^{-i\psi}) f(z-\zeta)\, dA(\zeta)\, e^{-2in\psi}\, d\psi\,.\end{aligned}$$

Using the observation (13) we continue as

$$\begin{aligned}(T'f)(z) &= \frac{1}{2\pi}\int_0^{2\pi}\int_{\mathbb{C}} \frac{k(e^{i(\arg\zeta-\psi)})}{|\zeta|^2} f(z-\zeta)\, dA(\zeta)\, e^{-2in\psi}\, d\psi \\ &= \frac{(-1)^n n}{\pi}\int_{\mathbb{C}}\frac{f(z-\zeta)}{|\zeta|^2}\,\frac{(-1)^n}{2n}\int_0^{2\pi} k(e^{i(\arg\zeta-\psi)})\, e^{-2in\psi}\, d\psi\, dA(\zeta) \\ &= \frac{1}{C(n)}\cdot T^n f(z)\,,\end{aligned}$$

for

$$\int_0^{2\pi} k(e^{i(\arg\zeta-\psi)})\, e^{-2in\psi}\, d\psi = e^{-2in\arg\zeta}\int_0^{2\pi} k(e^{i\varphi})\, e^{2in\varphi}\, d\varphi\,.$$

Thus we proved Theorem 2. $\square$

As the most important by-product of the proof we obtained an exact expression for the constant $C(n)$. Namely, from (12) and the calculations above it follows that

$$\begin{aligned}\frac{1}{C(n)} &= \frac{(-1)^n}{2n}\int_0^{2\pi} k(e^{i\varphi})\, e^{2in\varphi}\, d\varphi \\ &= \frac{(-1)^n}{2n\log 2}\int_0^{2\pi}\int_0^{\infty} F(re^{i\varphi})\, r\, dr\, e^{2in\varphi}\, d\varphi\,,\end{aligned}$$

therefore

$$\frac{1}{C(n)} = \frac{(-1)^n}{2n\log 2}\int_{\mathbb{R}}\int_{\mathbb{R}} F(x,y)\left(\frac{x+iy}{x-iy}\right)^n dx\, dy\,. \tag{14}$$

3.1. **Optimizing coefficients.** We want to make $C(n)$ as small as possible. Let us start by considering $\mathcal{H}_{new}$ which was defined by (10). Choose complex numbers σ^0, σ^+ and σ^- with modulus one. Our aim is to examine operators of the type

$$\mathcal{P}_t f := \sum_{Q\in\mathcal{G}_t}\Big[\sigma_0\langle f, h_Q^0\rangle h_Q^0 + \sigma_+\langle f, h_Q^+\rangle h_Q^+ + \sigma_-\langle f, h_Q^-\rangle h_Q^-\Big]\,. \tag{15}$$

The coefficients σ^0, σ^+, σ^- are chosen not to depend on squares Q, for otherwise we might already get in trouble when trying to run the first averaging process – the one over translations.

It is convenient to write the summands in terms of the functions from $\mathcal{H}_{orig}$, since for them the kernels resulting after the averaging were already computed. By using the identities (10) we get

$$\sigma_0 H_Q^1 f + \frac{\sigma_+ + \sigma_-}{2}\,(H_Q^2 + H_Q^3)f + \frac{\sigma_+ - \sigma_-}{2}\,(\langle f, h_Q^2\rangle h_Q^3 + \langle f, h_Q^3\rangle h_Q^2)$$

where $H_Q^j f = \langle f, h_Q^j\rangle h_Q^j$, $j = 1, 2, 3$.

Let us average operators $\mathcal{P}_t$ over grids. A proof, analogous to that of Proposition 1, shows that the sum of mixed terms in parentheses on the right becomes zero. Thus, by Proposition 1, the kernel we get is

$$F(x, y) = -\sigma_0\beta(x)\alpha(y) + \frac{\sigma_+ + \sigma_-}{2}\,(-\alpha(x)\beta(y) + \alpha(x)\alpha(y))\,.$$

We can assume that $\sigma_0 = 1$, for we are only interested in the maximum of the absolute value of the integral in (14). Next, α and β are even functions, while the imaginary part of $(\frac{x+iy}{x-iy})^n$ is odd in both x and y. Thus the integral of $-\alpha(x)\beta(y) + \alpha(x)\alpha(y)$ with the weight $(\frac{x+iy}{x-iy})^n$ will be real, so the maximum will be obtained when $\sigma_+ = \sigma_- = 1$ or $\sigma_+ = \sigma_- = -1$. The first choice would mean that we are eventually averaging the identity operators, so it has to be discarded. This is how we obtained the best coefficients σ in the case of $\mathcal{H}_{new}$. Hence from now on we will deal with

$$F(x, y) = \alpha(x)\beta(y) - \beta(x)\alpha(y) - \alpha(x)\alpha(y)\,. \tag{16}$$

3.2. **Rectangles.** We can perform the same averaging process for more general Haar systems. The reason to do that is that we want to refine yet further the behaviour of optimal $C(n)$. One way of introducing parameters of generality is to consider functions supported on general rectangles rather than squares.

Let us start with a rectangle whose horizontal and vertical side have lengths 1 and b, respectively. Here $b > 0$ can be arbitrary. We cover $\mathbb{R}^2$ by a grid of such rectangles and form a corresponding dyadic lattice. To each of its members we assign, as always so far, three Haar functions. It is obvious how the analogues of h_Q^0, h_Q^+ and h_Q^- (for which we retain the same name), should look like. The set of all Haar functions h_Q^0, h_Q^+ and h_Q^-, where Q runs over the dyadic lattice described above, will be denoted by $\mathcal{H}_b$. In particular, $\mathcal{H}_{new} = \mathcal{H}_1$. This construction was clearly made to fit the proof of Theorem 8, in other words,

the corresponding martingale transforms also admit L^p − norms not exceeding $p^* - 1$.

So let us define $\mathcal{P}_t$ as in (15), just that this time the grid $\mathcal{G}_t$ consists of rectangles of sizes 1 and b. A similar consideration as before shows that the choice $\sigma^0 = 1$, $\sigma^+ = \sigma^- = -1$ is optimal. In this setting, too, we can use Theorem 2, i.e. we can represent each T^n as an average of martingale transforms, arising from $\mathcal{P}_t$. The rôle of F is now assumed by kernel F_b, given by

$$F_b(x,y) = \frac{1}{b}\, F\Big(x, \frac{y}{b}\Big)\,.$$

Formula (12) is generalized in the same way:

$$k_b(x,y) = \frac{1}{\log 2}\int_0^\infty F_b(rx,ry)\, r\, dr = \frac{1}{b\log 2}\int_0^\infty F\Big(rx, r\frac{y}{b}\Big)\, r\, dr\,,$$

therefore

$$k_b(x,y) = \frac{1}{b} k\Big(x, \frac{y}{b}\Big)\,, \tag{17}$$

while (14) now takes the form

$$\begin{aligned}\frac{1}{C_b(n)} &= \frac{(-1)^n}{2n\log 2}\int_{\mathbb{R}}\int_{\mathbb{R}} F_b(x,y)\left(\frac{x+iy}{x-iy}\right)^n dx\, dy\\ &= \frac{(-1)^n}{2n\log 2}\int_{\mathbb{R}}\int_{\mathbb{R}} F(x,y)\left(\frac{x+i\,by}{x-i\,by}\right)^n dx\, dy\,.\end{aligned}$$

Remark. We can also perform the same process for arbitrary parallelograms, but that seems not to affect our final estimates.

4. Proof of Theorems 1 and 4

In order to estimate $C_b(n)$, note (compare with page 14) that the integral in (14) is a constant multiple of a Fourier coefficient of k_b, viewed as a function from $C(S^1)$. More precisely,

$$C_b(n) = \frac{(-1)^n n}{\pi \hat{k}_b(-2n)}\,. \tag{18}$$

For the purpose of showing that the optimal growth of $\inf_b |C_b(n)|$ is at most linear in n, we need to bound the Fourier coefficients of kernels k_b from below. We can do that thanks to the fact that regardless of n we have an abundant supply of kernels (corresponding to many different rectangles).

Proposition 3. *Under the above notation, there exists an absolute constant $C > 0$ such that for all $n \in \mathbb{N}$,*

$$\inf_{b>0} |C_b(n)| \leqslant Cn\,.$$

Proof. For k_b as in (17),

$$\begin{aligned}\widehat{k_b}(-2n) &= \frac{1}{2\pi}\int_0^{2\pi} k_b(e^{i\varphi})\, e^{2in\varphi}\, d\varphi \\ &= \frac{1}{\pi}\int_0^{\pi} k_b(e^{i\varphi})\, e^{2in\varphi}\, d\varphi\,.\end{aligned}$$

In the last line we used evenness of k, which follows from the same property being valid for F.

We are thankful to Fulvio Ricci for the following cute idea. Let us choose $b = 1/n$. Then (17) implies

$$\begin{aligned}\widehat{k}_{1/n}(-2n) &= \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} k_{1/n}(e^{i\varphi})\, e^{2in\varphi}\, d\varphi \\ &= \frac{n}{\pi}\int_{-\pi/2}^{\pi/2} k(\cos\varphi, n\sin\varphi)\, e^{2in\varphi}\, d\varphi \\ &= \frac{1}{\pi}\int_{\mathbb{R}} k(\cos(\vartheta/n), n\sin(\vartheta/n))\, e^{2i\vartheta}\, \chi_{(-n\pi/2,\, n\pi/2)}(\vartheta)\, d\vartheta\,.\end{aligned}$$

At this point we would like to apply the dominated convergence theorem. Denote the integrand above by $\Psi_n(\vartheta)$.

Since k is continuous on $\mathbb{C}\backslash\{0\}$ and homogeneous of degree -2, we get

$$|\Psi_n(\vartheta)| \leqslant \|k\|_{L^\infty(S^1)}\, \frac{\chi_{(-n\pi/2,\, n\pi/2)}(\vartheta)}{\cos^2(\vartheta/n) + n^2\sin^2(\vartheta/n)}\,.$$

It follows that, for some absolute constants C_1, C_2 and all $\vartheta\in\mathbb{R}$,

$$|\Psi_n(\vartheta)| \leqslant \frac{\|k\|_{L^\infty(S^1)}}{C_1 + C_2\vartheta^2}\,.$$

The function on the right belongs to $L^1(\mathbb{R})$, therefore we may bring the limit as $n\to\infty$ inside the last integral to conclude that

$$\lim_{n\to\infty} \widehat{k}_{1/n}(-2n) = \frac{1}{\pi}\int_{\mathbb{R}} k(1,\vartheta)\, e^{2i\vartheta}\, d\vartheta\,.$$

Same reasoning shows that for arbitrary $\lambda > 0$,

$$\lim_{n\to\infty} \widehat{k}_{\lambda/n}(-2n) = \frac{1}{\pi}\int_{\mathbb{R}} k(1,\vartheta)\, e^{2i\lambda\vartheta}\, d\vartheta\,. \tag{19}$$

The integral on the right can be viewed, up to a constant, as the Fourier transform of the function $k(1,\cdot)$ calculated at the point -2λ. Since this function is not identically zero, the integral cannot vanish for all λ. Hence there is $n_0\in\mathbb{N}$ and $\varepsilon > 0$ such that for every $n \geqslant n_0$,

$$\sup_{b>0} |\widehat{k}_b(-2n)| > \varepsilon\,. \tag{20}$$

Since the same inequality (possibly for different ε) is true also for indices up to n_0, we saw that (20) is valid for *all* $n \in \mathbb{N}$. This proves Theorem 1. □

Proof of Theorem 4. Let as take a closer look at the integral in (19). By recalling (12) we can write it as

$$\frac{1}{\log 2} \int_{\mathbb{R}} \int_0^\infty F(r, r\vartheta)\, e^{2i\lambda\vartheta}\, r\, dr\, d\vartheta\,.$$

Next we use that the function F is even in the second variable and supported on $[-1, 1]^2$ to get

$$\frac{2}{\log 2} \int_0^1 \int_0^1 F(x, y)\, \cos \frac{2\lambda y}{x}\, dx\, dy\,.$$

As F is a concrete function (16) one can estimate this integral. Numerical evaluations show that the expression's inverse attains the smallest value of approximately 2.716. This means (compare with (18) on p. 16) that for large n and for all p simultaneously

$$\|T^n\|_p \ \leqslant 2.72\, n(p^* - 1)\,.$$

□

5. Proof of Theorem 3

First we are going to prove the right half of (5). For that purpose we apply a theorem concerning weak boundedness of singular integrals with rough kernels. It was proven independently by Christ, Rubio de Francia [10] and Hofmann [16]. The formulation in [16] is explicit about behaviour of the estimates. We present it for the sake of the reader's convenience.

Let $\Omega \in L^q(S^1)$ *for some* $q > 1$*, such that* $\int_{S^1} \Omega = 0$*. If* Ω *is also homogeneous of degree* 0*, then*

$$Tf(z) = \text{p.v.} \int_{\mathbb{R}^2} f(z - \zeta) \frac{\Omega(\zeta)}{|\zeta|^2}\, d\zeta$$

defines an operator which is of weak type $(1, 1)$*. Its bound depends linearly on* $\|\Omega\|_q$*.*

Denote by $\Omega_n(z)/|z|^2$ the kernel of T^n. By (1) we know that

$$\Omega_n(z) = \frac{(-1)^n n}{\pi} \left(\frac{\bar{z}}{z}\right)^n\,.$$

The theorem quoted above implies that each T^n is of weak type $(1, 1)$ with a constant that can be bounded from above by Cn, where $C > 0$ is absolute.

Now a combination of real and complex interpolation between weak (1,1) and strong (2,2) –recall that each T^n is an isometry on L^2– yields

$$\|T^n\|_p \leqslant \frac{Cn^{2/p-1}}{p-1}$$

for all $n \in \mathbb{N}$ and $1 < p < 2$, and the result for $p > 2$ follows by duality.

5.1. **Lower estimates.** The operators T^n can be characterised by the property $\partial^n f = T^n(\bar{\partial}^n f)$, where f belongs to a suitable Sobolev space. This can be used to obtain lower estimates of $\|T^n\|_p$. When $n = 1$ it is well known that lower bounds are provided by radial stretch functions. Therefore they are the most natural candidate for extremals for arbitrary n. In that case we extend this example as follows. Take $z \in \mathbb{C}$ and define

$$f_{n,\alpha}(z) = \begin{cases} z^n |z|^{-2\alpha} & ; \quad |z| \leqslant 1 \\ \bar{z}^{-n} & ; \quad |z| \geqslant 1 \end{cases} .$$

Then, for $p \in (1, \infty)$,

$$\lim_{\alpha \to 1/p} \frac{\|\partial^n f_{n,\alpha}\|_p}{\|\bar{\partial}^n f_{n,\alpha}\|_p} = \kappa_n(p)\,,$$

where

$$\kappa_n(p) = \prod_{k=0}^{n-1} \frac{k - 1/p + 1}{k + 1/p}\,.$$

From now until the end of the section we will assume that $p \geqslant 2$. We can afford that because of duality.

Let us list few simple observations regarding this product.

- Every $\kappa_n(p)$ contains the factor $\kappa_1(p) = p - 1$. Each of the factors in $\kappa_n(p)$ is an increasing function of p.
- $$\lim_{p \to \infty} \frac{\kappa_n(p)}{p-1} = n$$
- $$\|T^n\|_p \geqslant \kappa_n(p) \geqslant p - 1 \geqslant \kappa_n(2) = 1\,.$$

More information about $\kappa_n(p)$ is provided by the following elementary result.

Lemma 3. *For all $n \in \mathbb{N}$ and $1 < p < \infty$,*

$$e^{-1} n^{1-2/p^*}(p^* - 1) \leqslant \kappa_n(p) \leqslant e\, n^{1-2/p^*}(p^* - 1)\,.$$

Proof. Denote

$$\varepsilon = \varepsilon(p) = \frac{1}{2} - \frac{1}{p}\,.$$

Define

$$\gamma_n(p) = \frac{\kappa_n(p)}{n^{1-2/p^*}(p^*-1)} = \frac{1}{n^{2\varepsilon}} \prod_{k=1}^{n-1} \frac{k+1/2+\varepsilon}{k+1/2-\varepsilon}\,.$$

First we are going to estimate $\gamma_n(p)$ by an absolute constant from above. Since

$$\frac{k+1/2+\varepsilon}{k+1/2-\varepsilon} = 1 + \frac{2\varepsilon}{k+1/2-\varepsilon} \leqslant 1 + \frac{2\varepsilon}{k}\,,$$

we see that

$$\gamma_n(p) \leqslant \frac{1}{n^\delta} \prod_{k=1}^{n-1} \left(1 + \frac{\delta}{k}\right)\,,$$

where $\delta = 2\varepsilon$. Recall that $p \geqslant 2$, which implies $0 \leqslant \delta < 1$. Hence

$$\begin{aligned} \log \gamma_n(p) &\leqslant \sum_{k=1}^{n-1} \log\left(1 + \frac{\delta}{k}\right) - \delta \log n \\ &\leqslant \delta \left(\sum_{k=2}^{n} \frac{1}{k} - \log n + 1 - \frac{1}{n} \right) \\ &\leqslant \delta \left(1 - \frac{1}{n}\right) \end{aligned}$$

and so $\gamma_n(p) < e$ for every $n \in \mathbb{N}$ and $p \in [2, \infty)$.

To estimate $\gamma_n(p)$ from below, repeat the above reasoning for $1/\gamma_n(p)$. □

Since $\|T^n\|_p \geqslant \kappa_n(p)$, this also settles Theorem 3. □

Based on these findings, we acutally think that the norms of T^n could be described by the following statement.

Conjecture.

$$\|T^n\|_p = \kappa_n(p)\,.$$

References

[1] K. Astala: *Area distortion of quasiconformal mappings*, Acta Math. **173** (1994), no. 1, 37–60.

[2] A. Baernstein II, S. J. Montgomery-Smith: *Some conjectures about integral means of ∂f and $\overline{\partial} f$*, Complex analysis and differential equations (Uppsala, 1997), Acta Univ. Upsaliensis Skr. Uppsala Univ. C Organ. Hist. **64**, Uppsala Univ., Uppsala, (1999), 92–109.

[3] R. Bañuelos, P. Méndez-Hernández: *Space-time Brownian motion and the Beurling-Ahlfors transform*, Indiana Univ. Math. J. **52** (2003), no. 4, 981–990.

[4] R. Bañuelos, P. Janakiraman: *L^p−bounds for the Beurling–Ahlfors transform*, preprint, 2005.

[5] D. L. Burkholder: *Boundary value problems and sharp inequalities for martingale transforms*, Ann. Prob. **12**, no. 3 (1984), 647–702.

[6] D. L. Burkholder: *A proof of Pełczyński's conjecture for the Haar system*, Studia Math. **91** (1988), no. 1, 79–83.

[7] D. L. Burkholder: *Sharp inequalities for martingales and stochastic integrals*, Colloque Paul Lévy sur les Processus Stochastiques (Palaiseau, 1987), Astérisque **157-158** (1988), 75–94.

[8] D. L. Burkholder: *Explorations in martingale theory and its applications*, Ecole d'Eté de Probabilités de Saint-Flour XIX – 1989, Lecture Notes in Math. **1464**, Springer, Berlin (1991), 1–66.

[9] A. P. Calderón, A. Zygmund: *On singular integrals*, Amer. J. Math. **78** (1956), 289–309.

[10] M. Christ, J. L. Rubio de Francia: *Weak type (1,1) bounds for rough operators, II*, Invent. math. **93** (1988), 225–237.

[11] O. Dragičević, S. Petermichl, A. Volberg: *Sharp estimates of martingale transforms in higher dimensions and applications to the Ahlfors-Beurling operator*, ArXiv ...

[12] O. Dragičević, A. Volberg: *Sharp estimate of the Ahlfors–Beurling operator via averaging martingale transforms*, Michigan Math. J. **51** (2003), no. 2, 415–435.

[13] O. Dragičević, A. Volberg: *Bellman function, Littlewood-Paley estimates and asymptotics for the Ahlfors-Beurling operator in $L^p(\mathbb{C})$*, Indiana Univ. Math. J. **54**, no. 4 (2005), 971–995.

[14] J. Duoandikoetxea: *Fourier Analysis*, translated and revised by D. Cruz-Uribe, American Mathematical Society, 2001.

[15] Y. Giga, K. Inui, A. Mahalov, S. Matsui: *Navier-Stokes equations in a rotating frame in $\mathbb{R}^3$ with initial data nondecreasing at infinity*, preprint, 2005.

[16] S. Hofmann: *Weak $(1,1)$ boundedness of singular integrals with nonsmooth kernel*, Proc. Amer. Math. Soc. **103** (1982), no. 1, 260–264.

[17] T. Iwaniec: *Extremal inequalities in Sobolev spaces and quasiconformal mappings*, Z. Anal. Anwendungen **1** (1982), 1–16.

[18] T. Iwaniec, G. Martin: *Riesz transforms and related singular integrals*, J. reine angew. Math. **473** (1996), 25–57.

[19] T. Iwaniec, C. Sbordone: *Riesz transforms and elliptic PDEs with VMO coefficients*, J. Anal. Math. **74** (1998), 183–212.

[20] O. Lehto, K. I. Virtanen: *Quasiconformal Mappings in the Plane*, Springer-Verlag, 1973.

[21] F. Nazarov, A. Volberg: *Heat extension of the Beurling operator and estimates for its norm* St. Petersburg Math. J. **15** (2004), no. 4, 142–158.

[22] S. Petermichl, S. Treil, A. Volberg: *Why the Riesz transforms are averages of the dyadic shifts?*, Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations (El Escorial, 2000), Publ. Mat. (2002), Vol. Extra, 209–228.

[23] S. K. Pichorides: *On the best values of the constants in the theorems of M. Riesz, Zygmund and Kolmogorov*, Studia Math. **44** (1972), 165–179.

[24] E. M. Stein: *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[25] I. N. Vekua: *Verallgemeinerte analytische Funktionen*, Akademie-Verlag Berlin, 1963.

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