RESTRICTED WALKS IN REGULAR TREES

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Abstract. Let $\mathcal{T}$ be the Cayley graph of a finitely generated free group F. Given two vertices in $\mathcal{T}$ consider all the walks of a given length between these vertices that at a certain time must follow a number of predetermined steps. We give formulas for the number of such walks by expressing the problem in terms of equations in F and solving the corresponding equations.

1. Introduction

Let $\mathcal{T}$ be an infinite regular tree and n a positive integer. Fix two vertices x and y in $\mathcal{T}$. By a *walk* or a *path* between x and y we mean any finite sequence of edges that connect x and y in which backtrackings are allowed. There are many formulas in literature which give the number of walks of length n between x and y, such as recurrence formulas, generating functions, Green functions, and others. Here we consider walks of length n between x and y which at a certain time follow a number of predetermined steps.

This work was motivated by the following question of Tatiana Smirnova-Nagnibeda, in relation to finding the spectral radius of a given surface group. Let F_2 be the free group on generators a and b, K a field of characteristic 0, $T = a^{-1} + a + b^{-1} + b$ an element in the group algebra $K[F_2]$ and $c = [a, b] = aba^{-1}b^{-1}$. What is the projection, for any m, and for any m-tuple of integers $(k_1, ..., k_m)$, of $Tc^{k_1}Tc^{k_2}...Tc^{k_m}$ onto the group algebra of the subgroup generated by c? Alternately, this can be formulated as a question in the free group F_2. Given an m-tuple of integers $(k_1, ..., k_m)$, how many of the words of type $x_1c^{k_1}x_2c^{k_2}...x_mc^{k_m}$ with $x_i \in \{a^{\pm 1}, b^{\pm 1}\}$, turn out to be a power of c? In turn, this question can be translated into counting certain paths in the Cayley graph of F_2, since each word in F_2 corresponds uniquely to a walk in the Cayley graph of F_2, the infinite regular tree of degree four. In the rest of the paper we will use the formulation of the

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question in terms of the free group or in terms of walks in regular trees interchangeably.

We answer this question in the case $(k_1, \ldots, k_m) = (0, \ldots, 0, k_i, 0, \ldots, 0)$, $k_i \neq 0$, not only for the free group on two generators, but on any number of generators, by counting (see Section 5) the number of solutions $(x_1, x_2, \ldots, x_m)$ of the equation $x_1 \ldots x_i c^k x_{i+1} \ldots x_m = c^l$. This equation is a particular instance of an equation of type $WX = YU$ in a free group, where W and U are given fixed words. The study of equations in free groups is a fully-developed area, with Makanin [3] and Razborov [4] having provided an algorithm that finds the solutions to equations that have solutions, and Diekert, Gutierez and Hagenah [2] having considered solutions to equations with rational constraints. While $WX = YU$ clearly has infinitely many solutions (X, Y) in a free group and does not require the complicated machinery developed by Makanin-Razborov, when we put restrictions on the lengths of X and Y, finding the number of solutions becomes delicate. We treat the equation $WX = YU$ in Section 4. Section 3 contains results about a type of restricted words or paths which will be used in later sections, but is also of independent interest.

2. Background and Example

Let us fix a set $X = \{a_1, a_2, \ldots, a_r\}$, where r is a positive integer, and let X^{-1} be a set of formal inverses for the elements of X, that is, $X^{-1} = \{a_1^{-1}, \ldots, a_r^{-1}\}$. Let $X^{\pm} = X \cup X^{-1}$. Elements of X will be called *generators* and elements of $X^{\pm}$ will be called *letters*. For $x \in X$ set $(x^{-1})^{-1} = x$. A finite string of letters is called a *word*. We define the inverse of a word $U = x_1 \cdots x_n$ to be $U^{-1} = x_n^{-1} \cdots x_1^{-1}$. The length of U will be denoted by $|U|$. For a word W, a string of consecutive letters in W forms a *subword* of W. A word W is *reduced* if it contains no subword of the form xx^{-1} with x in $X \cup X^{-1}$. We will denote the free group on generators $a_1, \ldots, a_r$ by F_r. The elements of F_r are the reduced words in letters $a_1^{\pm 1}, \ldots, a_r^{\pm 1}$. Reduced words correspond to paths without backtracking in the Cayley graph of F_r, while *unreduced words* or simply "words" correspond to arbitrary paths in the Cayley graph.

Let a and b be the generators of F_2, K a field of characteristic 0, $T = a^{-1} + a + b^{-1} + b$ an element in the group algebra $K[F_2]$ and $c = [a, b] = aba^{-1}b^{-1}$. Let us consider the easiest case of the projection computation we mentioned in the Introduction. In the case in which $k_i = 0$ for all i, one simply counts how many words of length n in $a^{\pm 1}$ and $b^{\pm 1}$ are powers of the commutator $c = [a, b]$. This is a special case of the following computation. Let x and y be fixed points in the Cayley graph of F_r, let $l = d(x, y)$ be the distance between x and y, and let $V_l^r(n)$ be the number of paths of length

n between x and y. If $r = 2$, then the projection of $Tc^{k_1}Tc^{k_2}...Tc^{k_n}$ with $k_1 = k_2 = \ldots = k_n = 0$ is

$$\cdots + V_8^2(n)c^{-2} + V_4^2(n)c^{-1} + V_0^2(n)c^0 + V_4^2(n)c + V_8^2(n)c^2 + \ldots$$

In other words, among all the elements of length n in F_2 we get $V_0^2(n)$ of them equal to the identity, $V_4^2(n)$ equal to the commutator c, and so on. We will use the same additive notation to count the number of words in F_2 equal to commutators. Note that if $n-l$ is an odd integer, then $V_l^2(n) = 0$. Formulas for $V_l^r(n)$ have been known for a long time and are often used in the context of random walks on graphs [5, 1]. After computing the values of $V_l^2(n)$ we get, that among all the words of length 4 in F_2, there are $c^{-1} + 28c^0 + c^1$ commutators. Among all the words of length 6 in F_2, we get $16c^{-1} + 232c^0 + 16c^1$ since the generating function for $V_0^2(n)$ is $\frac{3}{1+\sqrt{4-3x^2}}$, with $V_0^2(0) = 1$, $V_0^2(2) = 4$, $V_0^2(4) = 28$, $V_0^2(6) = 232$, $V_0^2(8) = 2092$.

Bartholdi gives a generalization to the computation of $V_l^r(n)$ in [1], where he considers more general graphs, not just regular trees. His computations of the number of walks also involve the number of backtrackings or 'bumps' in the walks. One standard tool for studying random walks on graphs or groups is the *Green function*.

Definition. Let $\mathcal{G}$ be a graph with $x, y \in \mathcal{G}$ and let $p^{(n)}(x, y)$ be the probability that the walker who started at point x will be at point y at the n-th step. Then the associated Green function is

$$G(x, y|z) = \sum_{n=0}^{\infty} p^{(n)}(x, y) z^n,$$

where $z \in \mathbb{C}$.

For a regular infinite tree of degree M the Green function is [5]

$$G(x,y|z) = \frac{2(M-1)}{M-2+\sqrt{M^2-4(M-1)z^2}} \left(\frac{M-\sqrt{M^2-4(M-1)z^2}}{2(M-1)z} \right)^{d(x,y)}$$

Thus the generating function for $V_l^r(n)$ is $G(x, y|2rz)$, where $d(x, y) = l$ is fixed.

3. Restricted words

In this section we count the number of a type of reduced words that will appear in our later results.

Let $|A|$ denote the cardinality of the set A, and let $A^{-1} = \{a^{-1} : a \in A\}$.

Proposition 1. *Let $a_1, \dots, a_r$ be the generators of F_r, A and B be subsets of $\{a_1^{\pm 1}, \dots, a_r^{\pm 1}\}$. The number of elements of length n in F_r that do not start with a letter in A and do not end with a letter in B is equal to*

$$\phi'_n(A,B) = \frac{(2r-|A|)(2r-|B|)(2r-1)^{n-1} + \delta r + (-1)^n(|A||B| - \sigma r)}{2r},$$

where $\delta = |A \cap B| - |A^{-1} \cap B|$, $\sigma = |A \cap B| + |A^{-1} \cap B|$.

Proof.

Let $\chi_A(x)$ be the characteristic function for A, i.e. $\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$, let $A^+ = A \cap \{a_1, \dots, a_n\}$ and $A^- = A \cap \{a_1^{-1}, \dots, a_n^{-1}\}$. Furthermore, let $\alpha_{i,n}$ be the number of reduced words of length $n > 0$ that do not start with a letter in A, but end in a_i, and let $\bar{\alpha}_{i,n}$ be the number of reduced words of length n that do not start with a letter in A, but end in a_i^{-1}. Then we have

$$\alpha_{1,n} + \bar{\alpha}_{1,n} + \cdots + \alpha_{r,n} + \bar{\alpha}_{r,n} = (2r - |A|)(2r-1)^{n-1}, \tag{1}$$

and $\alpha_{i,1} = 1 - \chi_A(a_i)$, $\bar{\alpha}_{i,1} = 1 - \chi_A(a_i^{-1})$.

The following recursion relations hold

$$\alpha_{i,n+1} = (\alpha_{1,n} + \bar{\alpha}_{1,n} + \cdots + \alpha_{r,n} + \bar{\alpha}_{r,n}) - \bar{\alpha}_{i,n},$$

$$\bar{\alpha}_{i,n+1} = (\alpha_{1,n} + \bar{\alpha}_{1,n} + \cdots + \alpha_{r,n} + \bar{\alpha}_{r,n}) - \alpha_{i,n},$$

where $i \geq 1$.

This implies $\alpha_{i,n} - \bar{\alpha}_{i,n} = \chi_A(a_i^{-1}) - \chi_A(a_i)$ for all n and i. Now fix i. Then for any j with $1 \leq j \leq r$, when we subtract the recursion relation for $\alpha_{j,n+1}$, from the recursion relation for $\alpha_{i,n+1}$, we get $\alpha_{i,n+1} - \alpha_{j,n+1} = \bar{\alpha}_{j,n} - \bar{\alpha}_{i,n} = \alpha_{j,n} - \alpha_{i,n} + \chi_A(a_j) - \chi_A(a_j^{-1}) + \chi_A(a_i^{-1}) - \chi_A(a_i)$. Let $e_{j,n} = \alpha_{i,n} - \alpha_{j,n}$ and $\bar{e}_{j,n} = \alpha_{i,n} - \bar{\alpha}_{j,n}$. Then $e_{j,1} = \chi_A(a_j) - \chi_A(a_i)$ and it is easy to see that $e_{j,2k} = \chi_A(a_i^{-1}) - \chi_A(a_j^{-1})$, $e_{j,2k+1} = \chi_A(a_j) - \chi_A(a_i)$, and $\bar{e}_{j,n} = e_{j,n} + \chi_A(a_j^{-1}) - \chi_A(a_j)$. Equation (1) can now be written as

$$\begin{aligned} & (\alpha_{i,n} - e_{1,n}) + (\alpha_{i,n} - e_{1,n} + \chi_A(a_1) - \chi_A(a_1^{-1})) + \dots \\ + \quad & (\alpha_{i,n} - e_{r,n}) + (\alpha_{i,n} - e_{r,n} + \chi_A(a_r) - \chi_A(a_r^{-1})) = (2r - |A|)(2r-1)^{n-1}. \end{aligned}$$

This gives $\alpha_{i,n} = \frac{(2r-|A|)(2r-1)^{n-1}+2\sum_j e_{j,n}-|A^+|+|A^-|}{2r}$, where the sum runs from 1 to r. Thus

$$\begin{aligned}
\alpha_{i,2k} &= \frac{(2r-|A|)(2r-1)^{2k-1}+2r\chi_A(a_i^{-1})-|A|}{2r},\\
\alpha_{i,2k+1} &= \frac{(2r-|A|)(2r-1)^{2k}-2r\chi_A(a_i)+|A|}{2r},\\
\bar{\alpha}_{i,2k} &= \frac{(2r-|A|)(2r-1)^{2k-1}+2r\chi_A(a_i)-|A|}{2r},\\
\bar{\alpha}_{i,2k+1} &= \frac{(2r-|A|)(2r-1)^{2k}-2r\chi_A(a_i^{-1})+|A|}{2r}.
\end{aligned}$$

Now, the number of reduced words of length n that do not start with a letter in A and do not end with a letter in B is equal to

$$\begin{aligned}
&(\alpha_{1,n}+\bar{\alpha}_{1,n}+\cdots+\alpha_{r,n}+\bar{\alpha}_{r,n}) - \sum_{j:\chi_B(a_j)=1}\alpha_{j,n} - \sum_{j:\chi_B(a_j^{-1})=1}\bar{\alpha}_{j,n}\\
&= (2r-|A|)(2r-1)^{n-1} - \frac{|B|}{2r}((2r-|A|)(2r-1)^{n-1}-(-1)^n|A|) \qquad (2)\\
&\quad -(-1)^n\left(\sum_{j:\chi_B(a_j)=1}\chi_A(a_j^{-(-1)^n}) + \sum_{j:\chi_B(a_j^{-1})=1}\chi_A(a_j^{(-1)^n})\right).
\end{aligned}$$

If n is even, then we have

$$\sum_{j:\chi_B(a_j)=1}\chi_A(a_j^{-1}) + \sum_{j:\chi_B(a_j^{-1})=1}\chi_A(a_j) = |A^{-1}\cap B|$$

If n is odd, then we have

$$\sum_{j:\chi_B(a_j)=1}\chi_A(a_j) + \sum_{j:\chi_B(a_j^{-1})=1}\chi_A(a_j^{-1}) = |A\cap B|$$

By simplifying (2), one easily obtains that the number of these reduced words is

$$\begin{cases}
\frac{(2r-|A|)(2r-|B|)(2r-1)^{n-1}+|A||B|}{2r} - |A^{-1}\cap B| & \text{if } n \text{ even},\\
\frac{(2r-|A|)(2r-|B|)(2r-1)^{n-1}-|A||B|}{2r} + |A\cap B| & \text{if } n \text{ odd}.
\end{cases}$$

The desired formula follows now by averaging the two expressions, then adding and subtracting the deviation to and from the average for even and odd n, respectively. □

A more natural quantity to count is the number of reduced words that start with a letter from a given set and end with a letter from another set.

By applying the De Morgan formulas for set identities to Proposition 1 we obtain the following

Corollary 1. *Let A and B be subsets of $\{a_1^{\pm 1}, \ldots, a_r^{\pm 1}\}$. The number of elements of length n in F_r that start with a letter in A and end with a letter in B is equal to*

$$\phi_n(A,B) = \frac{|A||B|(2r-1)^{n-1} + \delta r + (-1)^n(|A||B| - 2r(|A|+|B|) + \sigma r)}{2r},$$

where $\delta = |A^{-1} \cup B| - |A \cup B|$, $\sigma = |A \cup B| + |A^{-1} \cup B|$.

4. Main results

In this section we count the number of solutions (X, Y) of the equation

$$WX = YU, \tag{3}$$

in the free group F_r, for fixed elements W and U, and fixed lengths of X and Y. The number of solutions varies widely, depending on the lengths of W and U with respect to the lengths of X and Y. We will need the following.

Definition. : (i) Let $(W)_i$ be the i-th letter in the word W, where $1 \leq i \leq |W|$, with the convention that $(W)_0 = (W)_{|W|+1} = e$, where e is the empty word.

: (ii) Define $(W)_i^j$ to be the subword of W which starts with the i-th letter of W and ends with the j-th letter of W and the convention that $(W)_i^j = e$ if $j < i$.

: (iii) Let $\gamma_{W,U}(i, n, j)$ be the *correlation function* of two words W, U. Whenever W and U are fixed we will use $\gamma(i, n, j)$ instead of $\gamma_{W,U}(i, n, j)$. The correlation function identifies whether W and U have a common maximal subword s of length exactly n, followed by j letters in W, and preceded by i letters in U. More precisely, when $n > 0$

$$\gamma(i,n,j) = \begin{cases} 1 & \text{if } (W)_{|W|-n-j+1}^{|W|-j} = (U)_{i+1}^{i+n}, \\ & (W)_{|W|-n-j} \neq (U)_i, (W)_{|W|-j+1} \neq (U)_{i+n+1} \\ 0 & \text{else.} \end{cases}$$

If $n = 0$, $$\gamma(i,0,j) = \begin{cases} 1 & \text{if } (W)_{|W|-j}(U)_{i+1} \neq e, \\ & (W)_{|W|-j} \neq (U)_i, (W)_{|W|-j+1} \neq (U)_{i+1} \\ 0 & \text{else.} \end{cases}$$

Example. Let $W = abc$ and $U = bcd$ be words in the free group on four letters. Then $\gamma(0, 2, 0) = 1$, but $\gamma(1, 1, 0) = \gamma(0, 1, 1) = 0$. In all three cases the overlap between W and U is bc, a subword of both W and U. Since bc doesn't have length 1 it follows that $\gamma(1, 1, 0) = \gamma(0, 1, 1) = 0$.

Definition. Let $L_{W,U}(N, M)$ be the number of solutions of the equation $WX = YU$, where X and Y are reduced words of length N and M, respectively.

It can be seen at once that

$$L_{U,W}(N, M) = L_{W,U}(N, M), \tag{4}$$

$$L_{W,U}(N, M) = L_{W^{-1},U^{-1}}(M, N) \tag{5}$$

$$L_{W,U}(N, M) = 0 \quad \text{whenever} \quad |U| + |W| + N + M \quad \text{is odd} \tag{6}$$

In the following propositions we adopt the convention that if e, the identity element of F_r, is in some set A, then $A = A \setminus \{e\}$.

Proposition 2. *The number $L_{W,U}(N, M)$ of solutions of $WX = YU$, where X and Y are reduced words of length N and M, respectively, is given below. Let $d = \frac{N-M+|W|-|U|}{2}$ and $n = \frac{N+M-|W|-|U|}{2}$,*

: *(i) If $N + M < \big||U| - |W|\big|$ or $n \notin \mathbb{Z}$ then $L_{W,U}(N, M) = 0$,*

: *(ii) If $\big||U| - |W|\big| \le N + M \le |W| + |U|$, $n \in \mathbb{Z}$,*

$$L_{W,U}(N, M) = \begin{cases} \displaystyle\sum_{i=0}^{\min(|U|,M)} \gamma(i, -n, i + d) & \textit{if } d \ge 0 \\ \displaystyle\sum_{i=0}^{\min(|W|,N)} \gamma(i - d, -n, i) & \textit{if } d < 0 \end{cases}$$

: *(iii) If $N + M > |W| + |U|$, $n \in \mathbb{Z}$, then*

$$L_{W,U}(N, M) = \sum_{i=\max(0,-d)}^{\min(|U|,|W|-d)} \phi'_n(A_i, B_i)$$

where $A_i = \{((W)_{|W|-d-i})^{-1}, (W)_{|W|-d-i+1}\}$,
$B_i = \{(U)_i, ((U)_{i+1})^{-1}\}$.

Proof. **:** (i) If $|W| - |X| > |U| + |Y|$, then the length of the reduced word equal to WX is strictly longer than the length of the reduced word equal to YU, so there is no solution. Similarly, if $|W| + |X| < |U| - |Y|$ there is no solution.

: (ii) For a solution (X, Y), $|X| = N$, $|Y| = M$, the length of the resulting reduced words on both sides of the equation is $N + |W| - 2|w| = M + |U| - 2|u|$, and it is easy to see that U and W must have a common subword s of exactly $|W| - M + |u| - |w| = |U| - N + |w| - |u|$ letters. From the equation above it follows that $2|w| - 2|u| = N - M + |W| - |U| = 2d$, thus $|s| = \frac{|W|+|U|-N-M}{2} = -n$.

: (iii) The equation $WX = YU$ can be rewritten as

$$W'\bar{w}wX' = Y'u\bar{u}U',$$

where w, u, X', Y' are reduced words with w and u maximal such that $W = W'\bar{w}$, $X = wX'$, $Y = Y'u$ and $U = \bar{u}U'$. From this equation we have

$$2|w| = |W| - |U| + N - M + 2|u|. \tag{7}$$

Since $N + M > |W| + |U|$, the suffix of X' is $U' = uU$ and the prefix of Y' is $W' = Ww$. Thus we can write every solution (X, Y) as $(wX''uU, WwY''u) = (wX''U', W'Y''u)$, where $X'' = Y''$ is any reduced word of length $n = \frac{M+N-|U|-|W|}{2}$ which does not begin with the inverse of the last letter of W' or w, nor end with the inverse of the first letter of U' or u, since X and Y are reduced words. Notice that the inverses of the last letter of W' and w are $(W)^{-1}_{|W|-|w|}$ and $(W)_{|W|-|w|+1}$, respectively, and the inverses of the first letter of u and U' are $(U)_{|u|}$ and $(U)^{-1}_{|u|+1}$, respectively. Note that the length of X'' is constant, regardless of the length of u and w.

The following diagram better exemplifies the equalities between the words.

$$\begin{array}{|c|c|}\hline W & \quad X \quad \\ \hline \quad Y \quad & U \\ \hline \end{array} = \begin{array}{|c|c|}\hline W'\bar{w} & \quad wX' \quad \\ \hline \quad Y'u \quad & \bar{u}U' \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline Ww & X'' & U' \\ \hline W' & Y'' & uU \\ \hline \end{array}$$

Let $d = \frac{|W|-|U|+N-M}{2}$, then it follows from (7) that $|W| - |w| = |W| - d - |u|$. For every (possibly empty) word u such that u^{-1} is a prefix of U, let $A_{|u|} = \{(W)^{-1}_{|W|-d-|u|}, (W)_{|W|-d-|u|+1}\}$ and $B_{|u|} = \{(U)_{|u|}, (U)^{-1}_{|u|+1}\}$.

Thus for a fixed u, and $n = |X''| = \frac{M+N-|U|-|W|}{2}$, the number of choices for X'' is $\phi'_n(A_{|u|}, B_{|u|})$.

To obtain the total number of solutions, we consider the cases $d \geq 0$ and $d < 0$ separately.

- $d \geq 0$
  It follows from equation (7) that the smallest length of $|w|$ for which there can be a solution is $|w| = \frac{|W|-|U|+N-M}{2}$ in which case we have $|u| = 0$. Thus $|u|$ ranges from 0 to $\min(|U|, M-n)$, while $|w|$ ranges from $\frac{|W|-|U|+N-M}{2}$ to $\frac{|W|+|U|+N-M}{2} = N - n$ (if $|U| < M - n$) or $\frac{|W|-|U|+N+M-2n}{2} = |W|$ (else).
- $d < 0$

It follows from equation (7) that the smallest length of $|u|$ for which there can be a solution is $|u| = \frac{|U|-|W|+M-N}{2}$ in which case $|w| = 0$. Thus $|w|$ ranges from 0 to $\min(|W|, N-n)$, while $|u|$ ranges from $\frac{|U|-|W|+M-N}{2}$ to $\frac{|U|+|W|+M-N}{2} = M-n$ (if $|W| < N-n$) or $\frac{|U|-|W|+M+N-2n}{2} = |U|$ (else).

In both cases, the formula follows immediately.

□

Note that Proposition 2 not only counts the number of solutions to an equation of the form (3) but the proof also sketches a strategy for computing the actual solutions of that equation. We can now use Proposition 2 to give a formula for the number of restricted walks in regular trees.

Theorem 1. *Let U be a fixed element in F_r, $\mathcal{T}$ the Cayley graph of F_r, and P a fixed point in $\mathcal{T}$. Let W be the element in F_r describing the path from the origin to P. Then the number of paths in $\mathcal{T}$, from the origin to the point P, of length $M+|U|+N$ which after M steps follow the path outlined by U and proceed with N arbitrary steps is equal to*

$$\sum_{n=0}^{N}\sum_{m=0}^{M} L_{W,U}(n,m)V_n^r(N)V_m^r(M).$$

Proof. The Theorem follows easily from Proposition 2, because for every reduced word R of length ρ, there are exactly $V_\rho^r(l)$ words of length l which are equal to R. □

5. The commutator case

For ease of notation, let $a = a_1$, $b = a_2$ and $c = [a,b] = aba^{-1}b^{-1}$. Here we consider the projection computation in the case when $k_i = 0$ for all except one value of i. Let us fix integers k and l. Then we want to find the number of solutions of the equation:

$$x_1 \dots x_i c^k x_{i+1} \dots x_m = c^l, \tag{8}$$

where $x_i \in \{a_1^{\pm 1}, \dots, a_r^{\pm 1}\}$. We count the number of solutions by first rearranging the equation as

$$c^k X = Y c^l, \tag{9}$$

where $X = x_{i+1} \dots x_m$ and $Y = (x_i \dots x_1)^{-1}$.

Let $L_{k,l}(N,M)$ be the number of solutions of the equation 9, where X and Y are reduced words of length N and M, respectively. We compute $L_{k,l}(N,M)$ by specializing our results from the previous section to the case when W and U are commutators. Clearly $L_{k,l}(N,M) = L_{l,k}(N,M)$, $L_{k,l}(N,M) = L_{-k,-l}(M,N)$, $L_{k,l}(N,M) = 0$ whenever $N+M \equiv 1 \pmod 2$, and $L_{k,l}(N,M) = L_{k,l}(M,N)$. When $N+M$ is smaller or equal to the combined length of the commutators, then we have the following number of solutions.

Proposition 3. *Let k and l be positive integers.*

: *(i) If $N+M < \big||4k|-|4l|\big|$ then $L_{k,l}(N,M) = L_{-k,l}(N,M) = 0$.*

: *(ii) For $|4k-4l| \le N+M < 4k+4l$*

$$L_{k,l}(N,M) = \begin{cases} 2 & \text{if } 4|N,\ k=l \text{ and } N=M \\ 1 & \text{if } 4|N,\ |4k-4l| = N+M \text{ or } |4k-4l| = |N-M| \\ 0 & \text{else} \end{cases}$$

$$L_{-k,l}(N,M) = \begin{cases} \min(4l,M) & \text{if } 4k+4l = M+N+2 \\ & \text{and } M \equiv 1 \pmod 4 \\ 0 & \text{else} \end{cases}$$

: *(iii) For $N+M = 4k+4l,\ M \le N$*

$$L_{k,l}(N,M) = \begin{cases} \left\lceil \frac{\min(4l,M)+1}{2} \right\rceil & \text{if } M \text{ odd} \\ 2\left\lceil \frac{\min(4l,M)}{4} \right\rceil & \text{if } M \text{ even and } M \neq 4k \\ 2\left\lceil \frac{\min(4l,M)}{4} \right\rceil + 1 & \text{if } M = 4k \end{cases}$$

$$L_{-k,l}(N,M) = \begin{cases} \min(4l,M) & \text{if } M \equiv 2 \pmod 4 \\ 0 & \text{else} \end{cases}$$

Proof. Let $n' = \frac{4k+4l-N-M}{2}$. (Here $n' = -n$, where n is as defined in Proposition 2.)

: (i) follows immediately from Proposition 2 (i).

: (ii) We can assume without loss of generality that $k \ge l$ and $N \ge M$. Here we determine $\sum_{i=0}^{\min(4l,M)} \gamma(i,n',i+d)$ for $n' > 0$. Notice that $d \ge 0$, since $d = \frac{N-M+4k-4l}{2}$. Two commutators cannot have a common maximal subword of a certain length in their interiors, but only at the end or beginning of a commutator. In other words, γ will be non-zero only when some of the following are satisfied: $i=0$, $i+d=0$, $n'=4l$.

Let us first assume that $n' = 4l$, which is equivalent to $4k-4l = N+M$. Then we must have $i=0$, so in this case $L_{k,l}(N,M) = 1$ because only $\gamma(0,n',d)$ is nonzero. Now let's assume $n' < 4l$. If $d=0$, which is equivalent to $k=l$ and $N=M$. Then we can have $i=0$ and $i=4l-n'$, so in this case $L_{k,l}(N,M) = 2$. If $d > 0$

then $i = M$, which is equivalent to $4k - 4l = N - M$, and so we get $L_{k,l}(N, M) = 1$.

Since n', d and i must be multiples of 4 in order to have solutions, we get that M and N must also be multiples of 4. The formula now follows.

If we consider $c^{-k}X = Yc^l$, then we must consider $\gamma'(i, n', i + d) := \gamma_{c^{-k},c^l}(i, n', i + d)$. Notice that $\gamma'(i, n', i + d) \neq 0$ if and only if $n' = 1$, which is equivalent to $4k + 4l = M + N + 2$. Let the common maximal subword s be the letter x. If x is the $(i + 1)$-st letter in c^l, then x^{-1} appears in a position $4k - j - 1$ in c^{-4k} where $i \equiv j \pmod 4$. Since x is two positions before or after x^{-1} in a commutator, we get that we must have $i \equiv i+d+2 \pmod 4$, which is equivalent to $d \equiv 2 \pmod 4$. Since $n' = 2$, after a few identities we get $M \equiv 1 \pmod 4$, and in this case i can have the entire range, and so the number of solutions is $\min(4l, M)$.

: (iii) Here we determine $\displaystyle\sum_{i=0}^{\min(4l,M)} \gamma(i, n, i + d)$ for $n = 0$, which is equivalent to $N + M = 4k + 4l$. Then $\gamma(i, 0, i + d)$ is nonzero if and only if $(c^k)_{4k-i-d}(c^l)_{i+1} \neq e$, $(c^k)_{4k-i-d} \neq (c^l)_i$ and $(c^k)_{4k-i-d+1} \neq (c^l)_{i+1}$. Notice that the last two conditions are equivalent. Now remove the last $i + d$ letters from c^k and the first i letters from c^l, and concatenate the resulting words to get a word of lengths $4k + 4l - 2i - d$. In order to satisfy the two conditions for γ to be nonzero we must have that $2i + d$ is $\mathcal{M}_4 + 2$ or $\mathcal{M}_4 + 3$, where $\mathcal{M}_4$ is a multiple of 4. Since $d \equiv -M \pmod 4$, we need to count for how many i between 0 and $\min(4l, M)$ the congruence $2i - M \equiv 2, 3 \pmod 4$ holds. By considering the cases when M is even or odd, we get the desired formula.

If $d = 0$ we get $M = 4k$, and we get an extra solution to the ones counted above.

As in (ii), let $\gamma'(i, 0, i + d) := \gamma_{c^{-k},c^l}(i, 0, i + d)$. In this case $\gamma'(i, 0, i + d)$ is nonzero if and only if $(c^{-k})_{4k-i-d}(c^l)_{i+1} \neq e$, $(c^{-k})_{4k-i-d} \neq (c^l)_i$ and $(c^{-k})_{4k-i-d+1} \neq (c^l)_{i+1}$. It is easy to see that these three conditions are satisfied if and only if the $(c^{-k})_{4k-i-d}^{4k-i-d+1} = xy$ and $(c^l)_i^{i+1} = yx$. This can happen, similar to case (ii), only when $d \equiv 2 \pmod 4$. Since $n' = 2$, we get that $M \equiv 2 \pmod 4$, and the entire range for i.

$\square$

The number of solutions to equation 9, when the combined length of the solutions is greater than the combined length of the commutators, is given by the following two propositions.

Proposition 4. *The number of solutions of* $c^k X = Y c^l$*, where* X *and* Y *are words in* F_r *of length* N *and* M*, respectively,* $k, l > 0$*,* $N + M > 4k + 4l$*, is given by the following formulas. Let* $d = 2k - 2l + \frac{N-M}{2}$*,* $n = \frac{N+M}{2} - 2l - 2k$*,* $R = \min(4l, 4k - d) - \max(0, -d)$*.*

(i) If $d \neq 0$ *and* $4l \neq 4k - d$*, then*

$$L_{k,l}(N,M) = \frac{(2r-1)^{n-1}(2r-2)(R(r-1)+r) + R(-1)^n(2-r)}{r} + \begin{cases} \operatorname{sgn}(d)(-1)^{\frac{d+1}{2}} & d \text{ odd}, |N-M| > |4k-4l| \\ 0 & \text{else.} \end{cases}$$

(ii) If $d = 0$ *or* $4l = 4k - d$*, then* $L_{k,l}(N,M) =$

$$\begin{cases} (R-1)\frac{(2r-2)^2(2r-1)^{n-1}+(-1)^n(4-2r)}{2r} + \frac{(2r-1)^{n+1}+(-1)^n}{r} + 1, & \text{if } d = 0,\ k = l \\ (R-1)\frac{(2r-2)^2(2r-1)^{n-1}+(-1)^n(4-2r)}{2r} + \frac{(4r-3)(2r-1)^n+(-1)^n(3-r)}{2r} + \frac{1}{2}, & \text{else.} \end{cases}$$

Proof. We begin by computing the sets A_i and B_i in Proposition 2. If $k, l > 0$, then for $\max(0, -d) \leq i \leq \min(4l, 4k - d)$

$$A_i = \begin{cases} \{a\} & i = 4k - d \\ \{b\} & i = -d \\ \{a, b\} & i \notin \{-d, 4k-d\}, i \equiv \quad -d \pmod 4 \\ \{a, b^{-1}\} & i \notin \{-d, 4k-d\}, i \equiv 1 - d \pmod 4 \\ \{a^{-1}, b^{-1}\} & i \notin \{-d, 4k-d\}, i \equiv 2 - d \pmod 4 \\ \{a^{-1}, b\} & i \notin \{-d, 4k-d\}, i \equiv 3 - d \pmod 4 \end{cases} \tag{10}$$

and

$$B_i = \begin{cases} \{a^{-1}\} & i = 0 \\ \{b^{-1}\} & i = 4l \\ \{a^{-1}, b^{-1}\} & i \notin \{0, 4l\}, i \equiv 0 \pmod 4 \\ \{a, b^{-1}\} & i \notin \{0, 4l\}, i \equiv 1 \pmod 4 \\ \{a, b\} & i \notin \{0, 4l\}, i \equiv 2 \pmod 4 \\ \{a^{-1}, b\} & i \notin \{0, 4l\}, i \equiv 3 \pmod 4 \end{cases} \tag{11}$$

We will call the cases where $i \in \{0, 4l, -d, 4k - d\}$ *borderline cases* and refer to all the other cases as the *general case*. Comparison with Proposition 2 shows that there are two borderline cases, namely $i = \max(0, -d)$ and $i = \min(4l, 4k - d)$, while the remaining $R - 1$ cases form the general case.

General case. In order to compute the sum

$$\sum_{i=\max(0,-d)+1}^{\min(4l,4k-d)-1} \phi'_n(A_i,B_i), \tag{12}$$

we will first compute the (δ,σ) pairs in Proposition 1 from (A_i,B_i) then sum up the corresponding values of the ϕ'-function. The following table can be easily obtained from the definition of δ and σ in Proposition 1. It shows the values of (δ,σ) for all possible values of i,d modulo 4.

(mod 4)	$i\equiv 0$	$i\equiv 1$	$i\equiv 2$	$i\equiv 3$
$d\equiv 0$	$(-2,2)$	$(2,2)$	$(-2,2)$	$(2,2)$
$d\equiv 1$	$(0,2)$	$(0,2)$	$(0,2)$	$(0,2)$
$d\equiv 2$	$(2,2)$	$(-2,2)$	$(2,2)$	$(-2,2)$
$d\equiv 3$	$(0,2)$	$(0,2)$	$(0,2)$	$(0,2)$

Table 1. (δ,σ) for all possible values of i,d mod 4

Thus the sum in (12) is given by

- $d\equiv 1,3 \pmod 4$

$$(R-1)\frac{(2r-2)^2(2r-1)^{n-1}+(-1)^n(4-2r)}{2r}$$

- $d\equiv 0,2 \pmod 4$

If $d\equiv 0 \pmod 4$ then $R-1=(\min(4l,4k-d)-\max(0,-d)-1)\equiv 3 \pmod 4$ and i ranges from $i\equiv 1 \pmod 4$ to $i\equiv 3 \pmod 4$, thus we have $\frac{1}{2}(R-2)+1$ values of the ϕ'-function with $(\delta,\sigma)=(2,2)$ and $\frac{1}{2}(R-2)$ values with $(\delta,\sigma)=(-2,2)$.

If $d\equiv 2 \pmod 4$ then $(R-1)$ is odd, thus we can pair the values of the ϕ' function for $(2,2)$ and $(-2,2)$ in all but one case. Since i ranges from $i\equiv 1$ or $3 \pmod 4$ to $i\equiv 1$ or $3 \pmod 4$ we have a ϕ' value for $(-2,2)$ left over in all four cases.

Hence the sum is equal to

$$(R-1)\frac{(2r-2)^2(2r-1)^{n-1}+(-1)^n(4-2r)}{2r}+(-1)^{d/2}.$$

Borderline cases. Let $i_0=\max(0,-d)$, $i_1=\min(4l,4k-d)$. We need to compute

$$\phi'_n(A_{i_0},B_{i_0})+\phi'_n(A_{i_1},B_{i_1}).$$

We are going to compute the (δ,σ) pairs and corresponding values of the ϕ'-function from Proposition 1 for all possible values of i_0 and i_1. The results follow from simple inspection of equations 10 and 11. It turns out that for certain values of i_0 and i_1 we have $\phi'_n(A_{i_0}, B_{i_0}) = \phi'_n(A_{i_1}, B_{i_1})$, thus we will list these cases under a common item and let $i \in \{i_0, i_1\}$.

- If $d > 0$ and $4l > 4k - d$, then $(\delta, \sigma) = \begin{cases} (-1,1) & d \equiv 0,1 \pmod 4 \\ (1,1) & d \equiv 2,3 \pmod 4 \end{cases}$

$$\phi'_n(A_i, B_i) = \begin{cases} \frac{(2r-2)(2r-1)^n - r + (-1)^n(2-r)}{2r} & d \equiv 0,1 \pmod 4 \\ \frac{(2r-2)(2r-1)^n + r + (-1)^n(2-r)}{2r}, & d \equiv 2,3 \pmod 4 \end{cases}$$

- If $d < 0$ and $4l < 4k - d$, then $(\delta, \sigma) = \begin{cases} (-1,1) & d \equiv 0,3 \pmod 4 \\ (1,1) & d \equiv 1,2 \pmod 4 \end{cases}$

$$\phi'_n(A_i, B_i) = \begin{cases} \frac{(2r-2)(2r-1)^n - r + (-1)^n(2-r)}{2r} & d \equiv 0,3 \pmod 4 \\ \frac{(2r-2)(2r-1)^n + r + (-1)^n(2-r)}{2r}, & d \equiv 1,2 \pmod 4 \end{cases}$$

- If $d = 0$ and $4l = 4k - d$, then $(\delta, \sigma) = (0, 0)$

$$\phi'_n(A_i, B_i) = \frac{(2r-1)^{n+1} + (-1)^n}{2r}.$$

Putting it all together. Adding up the solutions in the general and borderline cases, we obtain

- $d \equiv 0 \pmod 4$
  If $d = 0$ and $k = l$ then
  $(R-1)\frac{(2r-2)^2(2r-1)^{n-1} + (-1)^n(4-2r)}{2r} + 1 + 2\frac{(2r-1)^{n+1} + (-1)^n}{2r}$,
  If ($d = 0$ and $k \neq l$) or ($d \neq 0$, $4l = 4k - d$) then
  $(R-1)\frac{(2r-2)^2(2r-1)^{n-1} + (-1)^n(4-2r)}{2r} + 1 + \frac{(2r-1)^{n+1} + (-1)^n}{2r} + \frac{(2r-2)(2r-1)^n - r + (-1)^n(2-r)}{2r}$
  If $d \neq 0$, $4l \neq 4k - d$
  $(R-1)\frac{(2r-2)^2(2r-1)^{n-1} + (-1)^n(4-2r)}{2r} + 1 + 2\frac{(2r-2)(2r-1)^n - r + (-1)^n(2-r)}{2r}$.
- $d \equiv 1 \pmod 4$
  If $d > 0$ and $4l > 4k - d$
  $(R-1)\frac{(2r-2)^2(2r-1)^{n-1} + (-1)^n(4-2r)}{2r} + 2\frac{(2r-2)(2r-1)^n - r + (-1)^n(2-r)}{2r}$
  If ($d > 0$ and $4l < 4k - d$) or ($d < 0$ and $4l > 4k - d$) then
  $(R-1)\frac{(2r-2)^2(2r-1)^{n-1} + (-1)^n(4-2r)}{2r}$
  $+ 2\frac{(2r-2)(2r-1)^n + (-1)^n(2-r)}{2r}$
  If $d < 0$ and $4l < 4k - d$ then
  $(R-1)\frac{(2r-2)^2(2r-1)^{n-1} + (-1)^n(4-2r)}{2r} + 2\frac{(2r-2)(2r-1)^n + r + (-1)^n(2-r)}{2r}$
- $d \equiv 2 \pmod 4$ This is the simplest case. For all possible values of d, l, and k, we obtain
  $(R-1)\frac{(2r-2)^2(2r-1)^{n-1} + (-1)^n(4-2r)}{2r} - 1 + 2\frac{(2r-2)(2r-1)^n + r + (-1)^n(2-r)}{2r}$

- $d \equiv 3 \pmod 4$
  If $d > 0$ and $4l > 4k - d$ then
  $(R-1)\frac{(2r-2)^2(2r-1)^{n-1}+(-1)^n(4-2r)}{2r} + 2\frac{(2r-2)(2r-1)^n+r+(-1)^n(2-r)}{2r}$
  If ($d > 0$ and $4l < 4k - d$) or ($d < 0$ and $4l > 4k - d$) then
  $(R-1)\frac{(2r-2)^2(2r-1)^{n-1}+(-1)^n(4-2r)}{2r} + 2\frac{(2r-2)(2r-1)^n+(-1)^n(2-r)}{2r}$
  If $d < 0$ and $4l < 4k - d$ then
  $(R-1)\frac{(2r-2)^2(2r-1)^{n-1}+(-1)^n(4-2r)}{2r} + 2\frac{(2r-2)(2r-1)^n-r+(-1)^n(2-r)}{2r}$

The proposition now follows after consolidating equal cases and simplifying terms. □

Since $L_{k,l}(N,M) = L_{-k,-l}(N,M)$, the number of solutions of equation 9 when $k, l < 0$ is equal to $L_{|k|,|l|}(N,M)$. Thus it remains to compute the number of solutions when $kl < 0$. With minimal changes to the proof of Proposition 4, we obtain

Proposition 5. *The number of solutions of $c^k X = Y c^{-l}$, where X and Y are words in F_r of length N and M, respectively, $k, l > 0$, $N+M > 4k+4l$, is given by the following formulas. Let $d = 2k-2l+\frac{N-M}{2}$, $n = \frac{N+M}{2}-2l-2k$, $R = \min(4l, 4k-d) - \max(0, -d)$.*

(i) If $d \neq 0$ and $4l \neq 4k - d$, then

$$L_{k,-l}(N,M) = \frac{(2r-1)^{n-1}(2r-2)(R(r-1)+r) + R(-1)^n(2-r)}{r} - \begin{cases} (-1)^{d/2}R & d \text{ even} \\ \operatorname{sgn}(d)(-1)^{\frac{d+1}{2}} & d \text{ odd}, |N-M| > |4k-4l| \\ 0 & \text{else.} \end{cases}$$

(ii) If $d = 0$ or $4l = 4k - d$, then $L_{k,-l}(N,M) =$

$$\begin{cases} (R-1)\frac{(2r-2)^2(2r-1)^{n-1}+(-1)^n(4-2r)}{2r} + \frac{(2r-1)^{n+1}+(-1)^n(1-r)}{r} - R, & \text{if } d=0, k=l \\ (R-1)\frac{(2r-2)^2(2r-1)^{n-1}+(-1)^n(4-2r)}{2r} + \frac{(4r-3)(2r-1)^n+(-1)^n(3-2r)}{2r} - R, & \text{else.} \end{cases}$$

The propositions in this sections now allow us to compute the number of solutions of equation 8 in the following theorem, which is a special case of Theorem 1.

Theorem 2. *Let k, l be nonzero integers, and N, M be positive integers, $x_1, \ldots, x_{N+M}, a, b \in \{a_1^{\pm 1}, \ldots, a_r^{\pm 1}\}$ with a and b fixed such that $a, a^{-1} \neq b$, and let $c = [a, b]$. Then the number of solutions $(x_1, x_2, \ldots, x_{N+M})$ of the equation $x_1 \ldots x_N c^k x_{N+1} \ldots x_{N+M} = c^l$ is given by*

$$\sum_{n=0}^{N} \sum_{m=0}^{M} L_{k,l}(n,m) V_n^r(N) V_m^r(M).$$

We finish this section by computing the following expressions for the $L_{k,l}(N,M)$ function in the case $r=2$ using Propositions 4 and 5. Let $d = 2|k|-2|l|+\frac{N-M}{2}$, $n=\frac{N+M}{2}-2|l|-2|k|$, $R=\min(4|l|,4|k|-d)-\max(0,-d)$. Proposition 3 directly computes $L_{k,l}(N,M)$ for the case $N+M \le 4|k|+4|l|$. If $d \neq 0$ and $4|l| \neq 4|k|-d$, then $L_{k,l}(N,M) =$

$$3^{n-1}(R+2)+\operatorname{sgn}(kl)\begin{cases}(-1)^{d/2}R & d \text{ even}, kl<0\\ \operatorname{sgn}(d)(-1)^{\frac{d+1}{2}} & d \text{ odd}, |N-M| > \big|4|k|-4|l|\big|\\ 0 & \text{else.}\end{cases}$$

If $d=0$ or $4l=4k-d$, then $L_{k,l}(N,M)=$

$$\begin{cases}(R-1)3^{n-1}+\frac{3^{n+1}+(-1)^n}{2}+1 & d=0 \text{ and } k=l\\ (R-1)3^{n-1}+\frac{3^{n+1}-(-1)^n}{2}-R & d=0 \text{ and } k=-l\\ (R-1)3^{n-1}+\frac{5\cdot 3^n+(-1)^n}{4}+\frac{1}{2} & (d\neq 0 \text{ or } k\neq l) \text{ and } kl>0\\ (R-1)3^{n-1}+\frac{5\cdot 3^n-(-1)^n}{4}-R & \text{else.}\end{cases}$$

6. Conclusion

We have counted the number of bounded length solutions to two variable equations of the form $WX = YU$, which is equivalent to counting the number of restricted walks that lie in a given ball of an infinite regular tree of even degree.

While we have tackled only the simplest case of the general question posed in the Introduction, the methods used can be generalized to obtain formulas for more complicated cases. The expressions for the $L_{k,l}(N,M)$ function in the case $r=2$ indicate that writing out a formula for a more general case will be very tedious.

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