L^p-SOLUTIONS TO BSDES WITH SUPER-LINEAR GROWTH COEFFICIENT. APPLICATION TO DEGENERATE SEMILINEAR PDES.

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ABSTRACT. We consider multidimensional backward stochastic differential equations (BSDEs). We prove the existence and uniqueness of solutions when the coefficient grow super-linearly, and moreover, can be neither locally Lipschitz in the variable y nor in the variable z. This is done with super-linear growth coefficient and a p-integrable terminal condition ($p > 1$). As application, we establish the existence and uniqueness of solutions to degenerate semilinear PDEs with superlinear growth generator and an $\mathbb{L}^p$-terminal data, $p > 1$. Our result cover, for instance, the case of PDEs with logarithmic nonlinearities.

1. Introduction

Let $(W_t)_{0 \leq t \leq T}$ be a r-dimensional Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. Let $(\mathcal{F}_t)_{0 \leq t \leq T}$ denote the natural filtration of (W_t) such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and ξ be an $\mathcal{F}_T$-measurable d-dimensional random variable. Let f be an $\mathbb{R}^d$-valued function defined on $[0,T] \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times r}$ such that for all $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times r}$, the map $(t,\omega) \longrightarrow f(t,\omega,y,z)$ is $\mathcal{F}_t$-progressively measurable. We consider the following BSDE,

$$(E^{(\xi,f)}) \qquad Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s \qquad 0 \leq t \leq T$$

Linear BSDEs have appeared long time ago, both as the equations for the adjoint process in stochastic control, as well as the model behind the Black and Scholes formula for the pricing and hedging of options in mathematical finance. However the first publishing paper on nonlinear BSDEs,[29], appeared only in 1990 where the existence and uniqueness of the solution under conditions including basically the Lipschitz continuity of the driver f.

In the last decade, the theory of BSDEs has found further important applications and has become a powerful tool in many field, above all financial mathematics, optimal control and stochastic game, non-linear PDEs and homogenization. The collected texts [14] give a useful introduction to the theory of BSDEs and some of their applications. See [10, 9, 11, 15, 24, 27, 28, 30] and the references therein for applications of BSDEs to PDEs, homogenization as well as in mathematical finances.

From the beginning, many authors attempted to improve the result of [29] by weakening the Lipschitz continuity of the coefficient f, see e.g [1, 2, 3, 7, 12, 13, 17, 18, 19, 21, 25, 26], or the L^2-integrability of the initial data ξ, see [15, 7]).

A third direction in the theory of BSDEs has been the developed recently by introducing the notions of of weak solutions, i.e. a solution which can be not adapted to the filtration generated by the initial driver Brownian motion. This allow to improve slightly the regularity condition on the coefficient f, see [4, 8, 24]. However, if one mimics the methodology developed to define weak solutions for forward Itô's SDE, it is important to introduce a reasonable topology on the canonical space of (Y, Z) which allows one to get reasonable compactness properties of the laws, as well as the identification of the limits. This fact is very difficult to prove in the context of BSDEs, in particular for the variable Z.

In another hand, the difficulty encountered for establishing the existence and uniqueness of strong solutions to BSDEs, with relatively weak conditions on the coefficient, stay essentially on the fact that the gradient component Z is only known implicitly, by the Itô's representation theorem, as the integrand of a Brownian stochastic integral, i.e. we know information on $(\int Z)$ but not on (Z) himself. For instance, we don't know if Z is $\mathbb{P}$-square integrable (resp. time continuous) or not . Consequently the usual localization technique by exit times could not applied naturally.

Recently in [1, 2, 3], new results on the existence and uniqueness, as well as on the stability of the solutions for multidimensional BSDEs with local assumptions (on the two variables y, z) on the coefficient are established by using a localization which is more adapted to BSDEs. However in [1, 2, 3], the terminal data remains square integrable and the conditions imposed on the coefficient f are uniform in ω and hence can not cover for example the stochastic Lipschitz condition.

The main purpose of the present paper is to extend our previous work [3] in several ways.

First, we prove existence and uniqueness of the strong solution to the BSDE (E^f) when the coefficient f can be neither locally monotone in y nor locally Lipschitz in z, moreover f may has a super linear growth in its two variables y and z. For example f can take the form $f(y,z) = -y\log|y| + h(z)\sqrt{|log|z||}$ for some fonction $h : \mathbb{R}^d \times \mathbb{R}^r \longmapsto \mathbb{R}^d$. Moreover, the assumptions which we impose on f are local not in y, z only but also in ω. This allow us to cover

some BSDEs with stochastically monotone coefficient also. We give some examples which are covered by our result and, in our knowledge, not covered by the previous works. Second, the terminal data is merely p-integrable with $p > 1$. Our conditions on the coefficient seem to be new for the classical Itô's SDEs also.
As application, we prove existence and uniqueness of the solution to certain system of semi-linear PDEs having a generator $f(s, x, u, \nabla u$ which is super-linear on u and ∇u. For example, we cover the nonlinearities of the form $f(s, x, u, \nabla u = -u \log |u|$ and/or $+$ $f(s, x, u, \nabla u) = h(\nabla u)\sqrt{|log|\nabla u||}$. Both the result as well as its proof are news. We prove, in particular, that the semi-linear system $\frac{\partial u(s,x)}{\partial s} + \mathcal{L}u(s, x) + f(s, x, u(s, x), \nabla u(s, x) = 0, \quad u(T, x) = g(x)$ has a unique solution if and only if 0 is the unique solution of the linear system $\frac{\partial u(s,x)}{\partial s} + \mathcal{L}u(s, x) = 0, \quad u(T, x) = 0$, where $\mathcal{L}$ is the second order parabolic operator associated to a given $\mathbb{R}^d$-diffusion process. This fact is completely proved here by using the BSDEs. This proof seems to be new also.

The paper is organized as follows. In section 2, we state the assumptions and the main result. In section 3 we give some examples which are (in our knowledge) not covered by the previous works on BSDEs. Section 3 is devoted to the proof of the main result. The proofs mainly consist to establishing an a priori estimate between two solutions (Y^1, Z^1), (Y^2, Z^2), with respectively the data (f^1, ξ^1) and (f^2, ξ^2), from which we deduce the existence of solutions by approximating (f, ξ) by a suitable sequence (f_n, ξ_n) and by using a suitable localization close to those of [1, 2, 3].

As in [3], this estimate is obtained by applying Itô's formula to $(|Y^1 - Y^2|^2 + \varepsilon)^{\frac{\beta}{2}}$ for some $1 < \beta < p \wedge 2$ and $\varepsilon > 0$, instead of $|Y^1 - Y^2|^2$ as is usually done. This enables us to treat BSDEs with super-linear growth coefficient in the two variables y and z. However, in contrast to [3], we don't use the L^2-weak compactness of the approximating process (Y^n, Z^n). We prove directly that (Y^n, Z^n) converges strongly in some L^q, $1 < q < 2$. We first establish the existence and uniqueness of a solution for a small time duration and then, we use the continuation procedure to extend the result to an arbitrarily prescribed time duration. The uniqueness as well as the stability of solutions are established by similar arguments. Our method makes it possible to prove both existence and uniqueness, as well as the stability of solutions by using the same computations. In section 4, apply our result to prove the existence and uniqueness of a weak L^p-solution for degenerate semi-linear PDEs with super-linear generators. Using the BSDEs, we prove in particular that the uniqueness for linear PDEs gives the uniqueness the associated semi-linear PDEs.

2. Definition, assumptions, main result and examples.

Throughout this paper, $p > 1$ is an arbitrary fixed real number and all the considered processes are $(\mathcal{F}_t)$-predictable.

2.1. **Definition.** A solution of equation $(E^{(\xi,f)})$ is an $(\mathcal{F}_t)$-adapted and $\mathbb{R}^{d+dr}$-valued process (Y, Z) such that

$$\mathbb{E}\sup_{t\leq T}|Y_t|^p + \mathbb{E}\left[\int_0^T |Z_s|^2 ds\right]^{\frac{p}{2}} + \mathbb{E}\int_0^T |f(s, Y_s, Z_s)|ds < +\infty$$

and satisfies $(E^{(\xi,f)})$.

2.2. **Assumptions.** We consider the following assumptions on (ξ, f):

$$\textbf{(H.0)} \quad \begin{cases} \text{There are } M \in \mathbb{L}^0(\Omega; \mathbb{L}^1([0,T]; \mathbb{R}_+)), \\ K \in \mathbb{L}^0(\Omega; \mathbb{L}^2([0,T]; \mathbb{R}_+)) \text{ and } \gamma \in]0, \dfrac{1 \wedge (p-1)}{2}[\\ \text{such that:} \quad \mathbb{E} \mid \xi \mid^p \, e^{\frac{p}{2}\displaystyle\int_0^T \lambda_s ds} < \infty, \\ \text{where } \lambda_s := 2M_s + \dfrac{K_s^2}{2\gamma} \end{cases}$$

(H.1) f is continuous in (y, z) for almost all (t, ω)

$$\textbf{(H.2)} \quad \begin{cases} \text{There are } \eta \text{ and } f^0 \in \mathbb{L}^0(\Omega \times [0,T]; \mathbb{R}_+) \text{ satisfying} \\ \mathbb{E}\left(\displaystyle\int_0^T e^{\displaystyle\int_0^s \lambda_r dr} \eta_s ds\right)^{\frac{p}{2}} < \infty \text{ and} \\ \mathbb{E}\left(\displaystyle\int_0^T e^{\frac{1}{2}\displaystyle\int_0^s \lambda_r dr} f_s^0 ds\right)^{p} < \infty \text{ such that:} \\ \langle y, f(t,y,z)\rangle \;\; \leq \;\; \eta_t + f_t^0|y| + M_t|y|^2 + K_t|y||z| \end{cases}$$

(H.3) $\begin{cases} \text{There are } \overline{\eta} \in \mathbb{L}^q(\Omega \times [0,T]; \mathbb{R}_+)) \text{ (for some } q > 1) \text{ and} \\ \alpha \in]1, p[, \alpha' \in]1, p \wedge 2[\text{ such that:} \\ \mid f(t,\omega,y,z) \mid \ \leq \ \overline{\eta}_t + \mid y \mid^\alpha + \mid z \mid^{\alpha'} \end{cases}$

(H.4) $\begin{cases} \text{There are } v \in \mathbb{L}^{q'}(\Omega \times [0,T]; \mathbb{R}_+)) \text{ (for some } q' > 0) \text{ and} \\ K' \in \mathbb{R}_+ \text{ such that for every } N \in \mathbb{N} \text{ and every } y, y'\, z, z' \\ \text{satisfying } \mid y \mid, \ \mid y' \mid, \mid z \mid, \mid z' \mid \leq N \\ \mathbb{1}_{v_t(\omega) \leq N} \langle y - y', f(t,\omega,y,z) - f(t,\omega,y',z') \rangle \\ \leq K' \,|y - y'|^2 \, log A_N + \sqrt{K' \log A_N}\, |y - y'|\, |z - z'| + K' \dfrac{\log A_N}{A_N} \\ \text{where } A_N \text{ is a increasing sequence and satisfies } A_N > 1, \\ \lim_{N \to \infty} A_N = \infty \text{ and } A_N \leq N^\mu \text{ for some } \mu > 0. \end{cases}$

2.3. **The main result.**

Theorem 2.1. *If **(H.0)-(H.4)** hold then $(E^{(\xi,f)})$ has a unique solution (Y,Z). Moreover we have*

$$\mathbb{E} \sup_t \mid Y_t \mid^p e^{\frac{p}{2}\int_0^t \lambda_s ds} + \mathbb{E}\left[\int_0^T e^{\int_0^s \lambda_r dr} \mid Z_s \mid^2 ds\right]^{\frac{p}{2}}$$

$$\leq C \left\{ \mathbb{E} \mid \xi \mid^p e^{\frac{p}{2}\int_0^T \lambda_s ds} + \mathbb{E}\left(\int_0^T e^{\int_0^s \lambda_r dr} \eta_s ds\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r dr} f_s^0 ds\right)^p \right\}$$

for some constant C depending only on p and γ.

We shall give some examples of BSDEs which satisfy the assumptions of Theorem 2.1. In our knowledge, these examples are not covered by the previous works in multidimensional BSDEs.

2.4. **Examples. Example 1.** Let $f(y) := -y \log |\, y \,|$ then for all $\xi \in \mathbb{L}^p(\mathcal{F}_T)$ the following BSDE has a unique solution

$$Y_t = \xi - \int_t^T Y_s \log |\, Y_s \,| ds - \int_t^T Z_s dW_s.$$

Indeed, f satisfies **(H.1)-(H.3)** since $\langle y, f(y) \rangle \leq 1$ and $\mid f(y) \mid \leq 1 + \frac{1}{\varepsilon} \mid y \mid^{1+\varepsilon}$ for all $\varepsilon > 0$. In order to verify **(H.4)**, thanks to triangular inequality, it is sufficient to treat separately the two cases: $0 \leq \mid y \mid, \mid y' \mid \leq \frac{1}{N}$ and

$\frac{1}{N} \leq \mid y \mid, \mid y' \mid \leq N$.
In the first case, since the map $x \mapsto -x \log x$ increases for $x \in]0, e^{-1}]$, we obtain for $N > e$

$$\begin{aligned} |f(y) - f(y')| &\leq |f(y)| + |f(y')| \\ &\leq 2\frac{\log N}{N} \end{aligned}$$

In the second case, the finite increments theorem applied to f in the interval $[|y|, |y'|]$ shows that

$$|f(y) - f(y')| \leq (1 + \log N) \mid y - y' \mid .$$

Hence **(H.4)** is satisfied for every $N > e$ with $v_s = 0$ and $A_N = N$.

Example 2. Let $g(y) := y \log \frac{\mid y \mid}{1 + \mid y \mid}$ and $h \in \mathcal{C}(\mathbb{R}^{dr}; \mathbb{R}_+) \bigcap \mathcal{C}^1(\mathbb{R}^{dr} - \{0\}; \mathbb{R}_+)$ be such that

$$h(z) = \begin{cases} |z|\sqrt{-\log|z|} & \text{if } |z| < 1 - \varepsilon_0 \\ |z|\sqrt{\log|z|} & \text{if } |z| > 1 + \varepsilon_0 \end{cases}$$

where $\varepsilon_0 \in]0, 1[$. Finally, we put $f(y, z) := g(y)h(z)$. Then for every $\xi \in \mathbb{L}^p(\mathcal{F}_T)$ the following BSDE has a unique solution

$$Y_t = \xi + \int_t^T f(Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

It is not difficult to see that f satisfies (H1). We shall prove that f satisfies (H2)-(H4).

(i) Since g is continuous, $g(0) = 0$ and $g(y)$ tends to 0 as $|y|$ tends to ∞, we deduce that g is bounded. Moreover, g satisfied $\langle y - y', g(y) - g(y') \rangle \leq 0$. Indeed, in one dimensional case it is not difficult to show that g is an increasing function. Since, $-\langle y, y' \rangle \log \frac{|y|}{1+|y|} \leq -|y||y'| \log \frac{|y|}{1+|y|}$ (because $\log \frac{|y|}{1+|y|} \leq 0$), we can reduce the multidimensional case to the one dimension case by developing the inner product as follows,

$$\begin{aligned}
&\langle y-y', g(y)-g(y')\rangle \leq \\
&\quad \leq |y|^2 \log\frac{|y|}{1+|y|} + |y'|^2 \log\frac{|y'|}{1+|y'|} - |y||y'|(\log\frac{|y|}{1+|y|} + \log\frac{|y'|}{1+|y'|}) \\
&\quad = (|y|-|y'|)(|y|\ \log\frac{|y|}{1+|y|} - |y'|\ \log\frac{|y'|}{1+|y'|}) \\
&\quad = \langle |y|-|y'|, g(|y|)-g(|y'|)\rangle \\
&\quad \leq 0
\end{aligned}$$

(ii) The function $h(z)$ satisfies for all $\varepsilon > 0$

$$0 \leq h(z) \leq M + \frac{1}{\sqrt{2\varepsilon}} \mid z \mid^{1+\varepsilon}, \qquad \text{where} \quad M = \sup_{|z|\leq 1+\varepsilon_0} \mid h(z) \mid$$

The last inequality follows since $\sqrt{2\varepsilon \log|z|} = \sqrt{\log|z|^{2\varepsilon}} \leq |z|^\varepsilon$ for each $\varepsilon > 0$ and $|z| > 1$. (H3) now follows directly from the previous observations (i) and (ii). (H2) is satisfied since $\langle y, f(y,z)\rangle = \langle y, g(y)\rangle h(z) \leq 0$. To verify **(H.4)** it suffices to show that for every z, z' such that $\mid z \mid, \mid z' \mid \leq N$

$$\mid h(z) - h(z') \mid \leq c\left(\sqrt{\log N} \mid z - z' \mid + \frac{\log N}{N}\right)$$

for some positive constant c and N large enough. This can be proved by considering separately the following five cases, $0 \leq \mid z \mid, \mid z' \mid \leq \frac{1}{N}$, $\frac{1}{N} \leq \mid z \mid, \mid z' \mid \leq 1-\varepsilon_0$, $1-\varepsilon_0 \leq \mid z \mid, \mid z' \mid \leq 1+\varepsilon_0$ and $1+\varepsilon_0 \leq \mid z \mid, \mid z' \mid \leq N$.

In the first case (i.e. $0 \leq \mid z \mid, \mid z' \mid \leq \frac{1}{N}$), since the map $x \mapsto x\sqrt{-\log x}$ increases for $x \in [0, \frac{1}{\sqrt{e}}]$, we obtain for $N > \sqrt{e}$, $|h(z) - h(z')| \leq |h(z)| + |h(z')| \leq 2\frac{1}{N}\sqrt{-\log\frac{1}{N}} \leq 2\frac{1}{N}\log N$.
The other cases can be proved by using the finite increments theorem.

Example 3. Let $(X_t)_{t\leq T}$ be an $(\mathcal{F}_t)-$adapted and $\mathbb{R}^k-$valued process satisfying the following forward stochastic differential equation

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

where $X_0 \in \mathbb{R}^k$ and $\sigma, b : [0,T] \times \mathbb{R}^k \to \mathbb{R}^{kr} \times \mathbb{R}^k$ are measurable functions such that $\|\sigma(s,x)\| \leq c$ and $|b(s,x)| \leq c(1+|x|)$, for some constant c.

Lemma 2.1. *There exist $\kappa > 0$ and $C > 0$ depending only on c, T and k such that*

$$\mathbb{E}\exp\left(\kappa \sup_{t\leq T} | X_t |^2\right) \leq C\exp(C \mid X_0 \mid^2).$$

Consider the following BSDE

$$Y_t = g(X_T) + \int_t^T \mid X_s \mid^{\overline{q}} Y_s - Y_s \log \mid Y_s \mid ds - \int_t^T Z_s dW_s.$$

where $\overline{q} \in]0,2[$ and g is a measurable function satisfying $\mid g(x) \mid \leq c\exp c \mid x \mid^{\overline{q}'}$, for some constants $c > 0$, $\overline{q}' \in [0,2[$.
The previous BSDE has a unique solution (Y, Z) which satisfies: for every $p > 1$ there exists a positive constant C such that

$$\mathbb{E}\sup_t \mid Y_t \mid^p + \mathbb{E}\left[\int_0^T \mid Z_s \mid^2 ds\right]^{\frac{p}{2}} \leq C\exp\left(C \mid X_0 \mid^2\right).$$

Indeed, we can show that
i) $\langle y, f(t,y)\rangle \leq 1+ \mid X_t \mid^{\overline{q}} \mid y \mid^2$
ii) Using Young inequality we obtain, for every $\epsilon > 0$ there is a constant $c_\epsilon > 0$ such, that

$$\mid f(t,y) \mid \leq c_\epsilon(1+ \mid X_t \mid^{\overline{q}c_\epsilon} + \mid y \mid^{1+\epsilon})$$

iii) f satisfies assumption **(H.4)** with $v_s = \exp \mid X_s \mid^{\overline{q}}$ and $A_N = N$.

The following example shows that our assumptions enable to treat BSDEs with stochastic monotone coefficient

Example 4. Let (ξ, f) satisfying **(H.0)-(H.3)** and

$$\textbf{(}H'\textbf{.4)} \begin{cases} \text{There are a positive process } C \text{ satisfying } \mathbb{E}\int_0^T e^{q'C_s} ds < \infty \\ \text{(for some } q' > 0\text{) and } K' \in \mathbb{R}_+ \text{ such that:} \\ \langle y - y', f(t,\omega,y,z) - f(t,\omega,y',z')\rangle \leq \\ \qquad \leq K' \mid y - y' \mid^2 \{C_t(\omega) + \mid \log \mid y - y' \mid \mid\} \\ \qquad + K' \mid y - y' \mid\mid z - z' \mid \sqrt{C_t(\omega) + \mid \log \mid z - z' \mid \mid}. \end{cases}$$

In particular we have for all z, z'

$$|f(t,\omega,y,z) - f(t,\omega,y,z')| \leq K' \mid z - z' \mid \sqrt{C_t(\omega) + \mid \log \mid z - z' \mid \mid}.$$

Then the following BSDE has a unique solution

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s.$$

To check that **(H.4)** is satisfied, it enough to show that for some constant c we have

$\langle y - y', f(t, y, z) - f(t, y', z)\rangle \leq c \log N \left(\mid y - y' \mid^2 + \frac{1}{N}\right)$

$|f(t, y, z) - f(t, y, z')| \leq c\sqrt{\log N} \left(\mid z - z' \mid + \frac{1}{N}\right)$

whenever $v_s := e^{C_s} \leq N$ and $\mid y \mid, \mid y' \mid, \mid z \mid, \mid z' \mid \leq N$.
This assertion can be proved by considering the following cases

$$\mid y - y' \mid \leq \frac{1}{2N}, \quad \frac{1}{2N} \leq \mid y - y' \mid \leq 2N.$$

and

$$\mid z - z' \mid \leq \frac{1}{2N}, \quad \frac{1}{2N} \leq \mid z - z' \mid \leq 2N.$$

Example 5. Let $(X_t)_{t\leq T}$ and ξ be as in example 3, let $F(t, x, y, z)$ such that
i) $F(t, x, .)$ is continuous
ii) $|F(t, x, y, 0)| \leq C \exp(C \mid x \mid^{\hat{q}}) + \mid y \mid^{\alpha}$, for some $\hat{q}, \alpha \in]0, 2[$ and $C > 0$,
iii) $\langle F(t, x, y, z) - F(t, x, y', z'), y - y'\rangle \leq K' \mid y - y' \mid^2 + K' \mid y - y' \mid\mid z - z' \mid$.
Let $\overline{q}, \overline{q}', \overline{q}" \geq 0$ such that $\overline{q} + \overline{q}" < 2$ and $\overline{q}' + \overline{q}" < 1$, the following BSDE has a unique solution

$$Y_t = \xi + \int_t^T |X_s|^{\overline{q}"} F(s, X_s, |X_s|^{\overline{q}} Y_s, |X_s|^{\overline{q}'} Z_s)ds - \int_t^T Z_s dW_s.$$

3. Proof of Theorem 2.1.

First, we give some a priori estimates from which we derive a stability result for BSDEs and next we use this stability result and a particular approximation of (ξ, f) to complete the proof. Here, the difficulty comes from the fact that the generator f can be neither locally Lipschitz in the variable y nor in the variable z and moreover it also may have a super-linear growth in its two variables y and z.

3.1. **Estimates for the solutions of equation** $(E^{(\xi,f)})$**.** In the first step, we give estimates for the processes Y and Z.

Proposition 3.1. *Let* $\Lambda_t := |Y_t|^2 e_t + 2\int_0^t e_s \eta_s ds + \left(\int_0^t e_s^{\frac{1}{2}} f_s^0 ds\right)^2$ *and* $e_t := \exp \int_0^t \lambda_s ds$. *Assume that* $\mathbb{E} \sup_{0\leq\ s\leq T} |Y_t|^p e_t^{\frac{p}{2}} < \infty$ *and* ***(H.2)*** *hold. Then, there exists a positive constant* $C^{(p,\gamma)}$ *such that*

$$\mathbb{E} \sup_{0\leq\ s\leq T} \Lambda_s^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T e_s |Z_s|^2 ds\right)^{\frac{p}{2}} \leq C^{(p,\gamma)} \mathbb{E}\Lambda_T^{\frac{p}{2}}.$$

To prove this proposition we need the following lemmas

Lemma 3.1. *For every* $\varepsilon > 0$, *every* $\beta > 1$ *and every positive functions* h *and* g *we obtain*

$$\int_t^T (h(s))^{\frac{\beta-1}{2}} g(s)ds \leq \varepsilon \sup_{t\leq s\leq T} \mid h(s) \mid^{\frac{\beta}{2}} + \varepsilon^{1-\beta} \left(\int_t^T g(s)ds\right)^{\beta}.$$

Proof . Let $\varepsilon > 0$ and $\beta > 1$. Using Young's inequality we get for every γ and γ' such that $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$

$$\int_t^T (h(s))^{\frac{\beta-1}{2}} g(s)ds \leq \frac{1}{\gamma} \varepsilon^{\frac{(\beta-1)\gamma}{\beta}} \sup_{t\leq s\leq T} \mid h(s) \mid^{\frac{(\beta-1)\gamma}{2}} + \frac{\varepsilon^{\frac{(1-\beta)\gamma'}{\beta}}}{\gamma'} \Big(\int_t^T g(s)ds\Big)^{\gamma'}$$

We now choose $\gamma = \frac{\beta}{\beta-1}$ and use the fact that $\gamma, \gamma' > 1$. ■

Lemma 3.2. *If* ***(H.2)*** *holds then for every* $\beta > 1 + 2\gamma$ *there exist positive constants* $C_1^{(\beta,\gamma)}, C_2^{(\beta,\gamma)}$ *such that for every* $\varepsilon > 0$,*every stopping time* $\tau \leq T$ *and every* $t \leq \tau$

$$\Lambda_t^{\frac{\beta}{2}} + \int_t^\tau \Lambda_s^{\frac{\beta-2}{2}} e_s |Z_s|^2 ds \leq$$
$$\leq \varepsilon \sup_{t\leq s\leq \tau} \Lambda_s^{\frac{\beta}{2}} + \varepsilon^{(1-\beta)} C_1^{(\beta,\gamma)} \Lambda_\tau^{\frac{\beta}{2}} - C_2^{(\beta,\gamma)} \int_t^\tau \Lambda_s^{\frac{\beta}{2}-1} e_s \langle Y_s, Z_s dW_s\rangle.$$

Proof . Without loss of generality, we suppose that η and f^0 are strictly positives.
It follows by using Itô's formula that for every $t \in [0, \tau]$,

$$|Y_t|^2 e_t + \int_t^\tau |Y_s|^2 \lambda_s e_s ds =$$
$$= e_\tau |Y_\tau|^2 + 2\int_t^\tau e_s \langle Y_s,\ f(s, Y_s, Z_s)\rangle ds - \int_t^\tau e_s \mid Z_s \mid^2 ds$$
$$-2\int_t^\tau e_s \langle Y_s,\ Z_s dW_s\rangle.$$

Again Itô's formula, applied to the process Λ, shows that

$$\Lambda_t^{\frac{\beta}{2}} + \beta\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1}\left(\frac{1}{2}|Y_s|^2\lambda_s e_s + e_s\eta_s + f_s^0 e_s^{\frac{1}{2}}\left[\int_0^s f_r^0 e_r^{\frac{1}{2}}dr\right]\right)ds$$
$$= \Lambda_\tau^{\frac{\beta}{2}} + \beta\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1}\langle e_sY_s, f(s,Y_s,Z_s)\rangle ds - \frac{\beta}{2}\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1}|Z_s|^2 e_s ds$$
$$-\beta\int_t^\tau e_s\Lambda_s^{\frac{\beta}{2}-1}\langle Y_s, Z_s dW_s\rangle - \beta(\frac{\beta}{2}-1)\int_t^\tau e_s^2\Lambda_s^{\frac{\beta}{2}-2}\sum_{j=1}^r\left(\sum_{i=1}^d Y_s^i Z_s^{i,j}\right)^2 ds$$

Observe that $\sum_{j=1}^r\left(\sum_{i=1}^d Y_s^i Z_s^{i,j})\right)^2 \leq |Y_s|^2|Z_s|^2 \leq e_s^{-1}\Lambda_s|Z_s|^2$ then use the assumption **(H.2)** to get

$$\Lambda_t^{\frac{\beta}{2}} + \frac{\beta}{2}(1-2\gamma-(2-\beta)^+)\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1}e_s|Z_s|^2 ds$$
$$\leq \Lambda_\tau^{\frac{\beta}{2}} + \beta\int_t^\tau \Lambda_s^{\frac{\beta}{2}-\frac{1}{2}}f_s^0 e_s^{\frac{1}{2}}ds - \beta\int_t^\tau \Lambda_s^{\frac{\beta}{2}-1}\langle e_sY_s, \quad Z_s dW_s\rangle.$$

It follows from Lemma 3.1 with $g(s) = f_s^0 e_s^{\frac{1}{2}}$, since $\left(\int_t^\tau f_s^0 e_s^{\frac{1}{2}}ds\right)^\beta \leq \Lambda_\tau^{\frac{\beta}{2}}$, that for every $\varepsilon > 0$

$$\int_t^\tau \Lambda_s^{\frac{\beta}{2}-\frac{1}{2}}f_s^0 e_s^{\frac{1}{2}}ds \leq \varepsilon\sup_{t\leq s\leq\tau}\Lambda_s^{\frac{\beta}{2}} + \varepsilon^{1-\beta}\Lambda_\tau^{\frac{\beta}{2}}$$

Since $\beta > 1+2\gamma$ implies $1-2\gamma-(2-\beta)^+ > 0$, Lemma 3.2 is proved. ■

Lemma 3.3. *Let **(H2)** be satisfied and assume that* $\mathbb{E}\sup_{0\leq\ s\leq T}|Y_t|^p e_t^{\frac{p}{2}} < \infty$. *Then the following assertions hold*
1) *There exists a positive constant* $C^{(p,\gamma)}$ *such that for every* $\varepsilon > 0$, *we have*

$$\mathbb{E}\int_0^T \Lambda_s^{\frac{p-2}{2}}e_s|Z_s|^2 ds \leq \varepsilon\mathbb{E}\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}} + \varepsilon^{(1-p)}C_1^{(p,\gamma)}\mathbb{E}\Lambda_T^{\frac{p}{2}}.$$

2) *There exists a positive constant* $C^{(p,\gamma)}$ *such that*

$$\mathbb{E}\left(\int_0^T e_s|Z_s|^2 ds\right)^{\frac{p}{2}} \leq C^{(p,\gamma)}\mathbb{E}\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}.$$

Proof . The first assertion follows by a standard martingale localization procedure. To prove the second assertion, we successively use Lemma 3.2

(with $\varepsilon = 1$ and $\beta = 2$), the Burkholder-Davis-Gundy inequality, the fact that $e_s|Y_s|^2 \leq \Lambda_s$ and Young's inequality to obtain

$$\begin{aligned}
&\mathbb{E}\big(\int_0^T e_s|Z_s|^2 ds\big)^{\frac{p}{2}} \leq \\
&\quad\leq C_1^{(p,\gamma)}\mathbb{E}\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}\big) + C_2^{(p,\gamma)}\mathbb{E}\big(|\int_t^T e_s\langle Y_s, Z_s dW_s\rangle|^{\frac{p}{2}}\big) \\
&\quad\leq C_1^{(p,\gamma)}\mathbb{E}\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}\big) + C_2^{(p,\gamma)}\mathbb{E}\big(|\int_0^T e_s^2|Y_s|^2|Z_s|^2 ds|^{\frac{p}{4}}\big) \\
&\quad\leq C_1^{(p,\gamma)}\mathbb{E}\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}\big) + C_2^{(p,\gamma)}\mathbb{E}\big(|\int_0^T \Lambda_s e_s|Z_s|^2 ds|^{\frac{p}{4}}\big) \\
&\quad\leq C_1^{(p,\gamma)}\mathbb{E}\big(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}\big) + C_2^{(p,\gamma)}\mathbb{E}\big[(\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{4}})(\int_0^T e_s|Z_s|^2 ds)^{\frac{p}{4}}\big] \\
&\quad\leq \big[C_1^{(p,\gamma)} + +2(C_2^{(p,\gamma)})^2\big]\mathbb{E}\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}} + \frac{1}{2}\mathbb{E}\big[(\int_0^T e_s|Z_s|^2 ds)^{\frac{p}{2}}\big] \\
&\quad\leq [2C_1^{(p,\gamma)} + 4(C_2^{(p,\gamma)})^2]\,\mathbb{E}\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}}.
\end{aligned}$$

Lemma 3.3 is proved. ■

Lemma 3.4. *Let the assumptions of Lemma 3.3 be satisfied. Then, there exists a constant $C^{(p,\gamma)}$ such that*

$$\mathbb{E}\sup_{0\leq\; s\leq T}\Lambda_s^{\frac{p}{2}} \;\leq\; C^{(p,\gamma)}\mathbb{E}\Lambda_T^{\frac{p}{2}}.$$

Proof . Lemma 3.2 and the Burkholder-Davis-Gundy inequality show that there exists a universal positive constant c such that for every $\varepsilon > 0$, $t \leq T$

$$\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}} &\leq \varepsilon\mathbb{E}\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}} + \varepsilon^{(1-p)}C_1^{(p,\gamma)}\mathbb{E}\Lambda_T^{\frac{p}{2}} \\
&\quad + cC_2^{(p,\gamma)}\mathbb{E}\left(\int_0^T \Lambda_s^{p-2}(|Y_s|^2 e_s)e_s|Z_s|^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

Young's inequality gives, for every $\varepsilon' > 0$,

$$\begin{aligned}
\mathbb{E}\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}} &\leq \varepsilon\mathbb{E}\sup_{0\leq s\leq T}\Lambda_s^{\frac{p}{2}} + \varepsilon^{(1-p)}C_1^{(p,\gamma)}\mathbb{E}\Lambda_T^{\frac{p}{2}} \\
&\quad + \varepsilon'\mathbb{E}\sup_{0\leq s\leq T}\Lambda_t^{\frac{p}{2}} + \frac{\left[cC_2^{(p,\gamma)}\right]^2}{\varepsilon'}\mathbb{E}\int_0^T \Lambda_s^{\frac{p-2}{2}} e_s|Z_s|^2 ds.
\end{aligned}$$

Applying Lemma 3.3, we find for every $\varepsilon" > 0$

$$\mathbb{E}\sup_{0\le s\le T}\Lambda_t^{\frac{p}{2}} \;\le\; (\varepsilon+\varepsilon'+\frac{\left[cC_2^{(p,\gamma)}\right]^2\varepsilon"}{\varepsilon'})\mathbb{E}\sup_{0\le s\le T}\Lambda_s^{\frac{p}{2}} + (\varepsilon^{(1-p)}C_1^{(p,\gamma)}+\frac{\left[cC_2^{(p,\gamma)}\right]^2C_1^{(p,\gamma)}(\varepsilon")^{(1-p)}}{\varepsilon'})\mathbb{E}\Lambda_T^{\frac{p}{2}}.$$

Choose suitable $\varepsilon,\varepsilon',\varepsilon"$ to conclude the proof. ■

Proof of Proposition 3.1. It follows from Lemma 3.3 and Lemma 3.4. ■

Proposition 3.2. *If **(H.3)** holds then,*

$$\mathbb{E}\int_0^T|f(s,Y_s,Z_s)|^{\hat\beta}ds \le \le 9^{p+q}(1+T)\big[1+\mathbb{E}\int_0^T\overline{\eta}_s^q ds+\mathbb{E}\sup_{0\le s\le T}|Y_s|^p+\mathbb{E}(\int_0^T|Z_s|^2ds)^{\frac{p}{2}}\big]$$

where $\hat\beta := \dfrac{2}{\alpha'}\wedge\dfrac{p}{\alpha}\wedge\dfrac{p}{\alpha'}\wedge q$.

Proof. We successively use Assumption (H.3), Young's inequality and Hölder's inequality to show that

$$\begin{aligned}
&\mathbb{E}\int_0^T|f(s,Y_s,Z_s)|^{\hat\beta}ds\\
&\le \mathbb{E}\int_0^T(\overline{\eta}_s+|Y_s|^{\alpha}+|Z_s|^{\alpha'})^{\hat\beta}ds\\
&\le 3^{\hat\beta}\mathbb{E}\int_0^T(\overline{\eta}_s^{\hat\beta}+|Y_s|^{\alpha\hat\beta}+|Z_s|^{\alpha'\hat\beta})ds\\
&\le 3^{\hat\beta}\mathbb{E}\int_0^T((1+\overline{\eta}_s)^{\hat\beta}+(1+|Y_s|)^{\alpha\hat\beta}+(1+|Z_s|)^{\alpha'\hat\beta})ds\\
&\le 3^{\hat\beta}\mathbb{E}\int_0^T((1+\overline{\eta}_s)^{q}+(1+|Y_s|)^{p}+(1+|Z_s|)^{p\wedge 2})ds\\
&\le 3^{\hat\beta}3^{p+q}\mathbb{E}\int_0^T(1+\overline{\eta}_s^{q}+|Y_s|^{p}+|Z_s|^{p\wedge 2})ds\\
&\le 3^{\hat\beta}3^{p+q}\big[T+\mathbb{E}\int_0^T\overline{\eta}_s^q ds+T\mathbb{E}\sup_{0\le s\le T}|Y_s|^p+T^{\frac{2-(p\wedge 2)}{2}}\mathbb{E}(\int_0^T|Z_s|^2ds)^{\frac{p}{2}}\big]\\
&\le 9^{p+q}(1+T)\big[1+\mathbb{E}\int_0^T\overline{\eta}_s^q ds+\mathbb{E}\sup_{0\le s\le T}|Y_s|^p+\mathbb{E}(\int_0^T|Z_s|^2ds)^{\frac{p}{2}}\big].
\end{aligned}$$

Proposition 3.2 is proved. ■

3.2. **Estimate of the difference between two solutions.**

The next proposition gives an estimate which is a key tool in the proof of existence, uniqueness and stability of solutions.

Lemma 3.5. *Let $(\xi^i, f_i)_{i=1,2}$ satisfy **(H.3)** (with the same $\overline{\eta}, \alpha$ and α') and let (Y^i, Z^i) be a solution of $(E^{(\xi^i, f_i)})$. Then, there exist $\beta = \beta(p, q, \alpha, \alpha') \in]1, p \wedge 2[$, $r = r(p, q, \alpha, \alpha', K', \mu, q') > 0$ and $a = a(p, q, \alpha, \alpha', K', \mu, q') > 0$ such that for every $u \in [0, T], u' \in [u, T \wedge (u+r)]$, $N > 0$ and every function f satisfying **(H.4)***

$$\begin{aligned}
&\mathbb{E} \sup_{u \le t \le u'} |Y_t^1 - Y_t^2|^\beta + \mathbb{E} \int_u^{u'} \frac{|Z_s^1 - Z_s^2|^2}{(1 + |Y_s^1 - Y_s^2|^2)^{1 - \frac{\beta}{2}}} ds \\
&\le \; N A_N^{1 + \frac{\beta}{2}} \left[\mathbb{E}|Y_{u'}^1 - Y_{u'}^2|^\beta + \mathbb{E} \int_0^T \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s ds \right] + \\
&\frac{1}{A_N^a} \left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E} \int_0^T \overline{\eta}_s^q ds + \mathbb{E} \int_0^T v_s^{q'} ds \right].
\end{aligned}$$

where

$$\rho_N(f_i - f)(t, \omega) := \sup_{|y|, |z| \; \le \; N} |f(t, \omega, y, z) - f_i(t, \omega, y, z)|$$

and

$$\Theta_p^i := \mathbb{E} \sup_t |Y_t^i|^p + \mathbb{E} \left(\int_0^T |Z_s^i|^2 ds \right)^{\frac{p}{2}}.$$

Proof . Let q be the number defined in assumption **(H3)** and q', K', μ those defined in assumption **(H4)**. Let $\overline{\gamma} > 0$ be such that $1 + 2\overline{\gamma} < \hat{\beta} := \frac{2}{\alpha'} \wedge \frac{p}{\alpha} \wedge \frac{p}{\alpha'} \wedge q$ and set $K" := K' + \frac{K'}{4\overline{\gamma}}$. Let $\beta \in]1 + 2\overline{\gamma}, \hat{\beta}[$ and $\nu \in]0, (1 - \frac{\beta}{\hat{\beta}})(1 \wedge q')[$. Let $r \in]0, \frac{\nu}{\mu \hat{\beta} K"} \wedge \frac{1}{2K"} \wedge 1[$.
For $N \in \mathbb{N}$, we set

$$\overline{e}_t := (A_N)^{2K"(t-u)} \quad \text{and} \quad \Delta_t := \{ \left|Y_t^1 - Y_t^2\right|^2 + (A_N)^{-1} \} \overline{e}_t.$$

Using Itô's formula, we show that for every stopping time $\tau \in [u, u']$ and every $t \in [u, \tau]$

$$\Delta_t^{\frac{\beta}{2}} + 2\log(A_N)K" \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}} ds + \frac{\beta}{2} \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \left|Z_s^1 - Z_s^2\right|^2 ds$$

$$= \Delta_\tau^{\frac{\beta}{2}} - \beta \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right) dW_s\rangle$$

$$+\beta \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \langle Y_s^1 - Y_s^2, f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)\rangle ds$$

$$-\beta(\tfrac{\beta}{2} - 1) \int_t^\tau \overline{e}_s^2 \Delta_s^{\frac{\beta}{2}-2} \sum_{j=1}^r \left(\sum_{i=1}^d (Y_{i,s}^1 - Y_{i,s}^2)(Z_{i,j,s}^1 - Z_{i,j,s}^2) \right)^2 ds$$

$$= \Delta_\tau^{\frac{\beta}{2}} - \beta \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right) dW_s\rangle + \beta I_1 - \beta(\frac{\beta}{2} - 1) I_2, \tag{3.1}$$

where

$$I_1 := \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \langle Y_s^1 - Y_s^2, f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)\rangle ds$$

and

$$I_2 := \int_t^\tau \overline{e}_s^2 \Delta_s^{\frac{\beta}{2}-2} \sum_{j=1}^r \left(\sum_{i=1}^d (Y_{i,s}^1 - Y_{i,s}^2)(Z_{i,j,s}^1 - Z_{i,j,s}^2) \right)^2 ds.$$

In order to complete the proof of Lemma 3.5 we need to estimate I_1 and I_2.

Estimate of I_1. Let $\Phi(s) := |Y_s^1| + |Y_s^2| + |Z_s^1| + |Z_s^2| + v_s$. Since $1\!\!1_{\{\Phi_s \le N\}} \le 1\!\!1_{\{v_s \le N\}}$ and f satisfies **(H4)**, then a simple computation shows that

$$\begin{aligned}
\langle Y_s^1 - Y_s^2, & f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)\rangle \\
& \le \overline{e}_s^{\frac{-1}{2}} \Delta_s^{\frac{1}{2}} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)| 1\!\!1_{\{\Phi_s > N\}} \\
& \quad + 2N[\rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s] 1\!\!1_{\{v_s \le N\}} \\
& \quad + [K" \log(A_N) \overline{e}_s^{-1} \Delta_s + \overline{\gamma} \left|Z_s^1 - Z_s^2\right|^2] 1\!\!1_{\{\Phi_s \le N\}}
\end{aligned}$$

Therefore, using Lemma 3.1 with $h_s = \Delta_s$, we get

$$\begin{aligned}
I_1 \le & \int_t^\tau \overline{e}_s^{\frac{1}{2}} \Delta_s^{\frac{\beta-1}{2}} |f_1(s,Y_s^1,Z_s^1) - f_2(s,Y_s^2,Z_s^2)| 1\!\!1_{\{\Phi_s > N\}} ds \\
& + 2N \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} [\rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s] 1\!\!1_{\{v_s \le N\}} ds \\
& + \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} [K" \log(A_N) \overline{e}_s^{-1} \Delta_s + \overline{\gamma} \left| Z_s^1 - Z_s^2 \right|^2] 1\!\!1_{\{\Phi_s \ \le \ N\}} ds \\
\le & \, \varepsilon \sup_{s \in [u,u']} \Delta_s^{\frac{\beta}{2}} \\
& + \varepsilon^{(1-\beta)} \overline{e}_{u'}^{\frac{\beta}{2}} \int_u^{u'} |f_1(s,Y_s^1,Z_s^1) - f_2(s,Y_s^2,Z_s^2)|^\beta 1\!\!1_{\{\Phi_s > N\}} ds \\
& + 2N \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} [\rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s] 1\!\!1_{\{v_s \le N\}} ds \\
& + \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} [K" \log(A_N) \overline{e}_s^{-1} \Delta_s + \overline{\gamma} \left| Z_s^1 - Z_s^2 \right|^2] 1\!\!1_{\{\Phi_s \ \le \ N\}} ds
\end{aligned}$$

Estimate of I_2. Since

$$\begin{aligned}
\sum_{j=1}^r \left(\sum_{i=1}^d (Y_{i,s}^1 - Y_{i,s}^2)(Z_{i,j,s}^1 - Z_{i,j,s}^2) \right)^2 & \le \left| Y_s^1 - Y_s^2 \right|^2 \left| Z_s^1 - Z_s^2 \right|^2 \\
& \le \overline{e}_s^{-1} \Delta_s \left| Z_s^1 - Z_s^2 \right|^2
\end{aligned}$$

then

$$I_2 \le \int_t^\tau \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \left| Z_s^1 - Z_s^2 \right|^2 ds.$$

Now, coming back to equation (3.1) and taking into account the above estimates we get for every $\varepsilon > 0$,

$$
\begin{aligned}
&\Delta_t^{\frac{\beta}{2}} + \frac{\beta}{2}(\beta - 1 - 2\overline{\gamma}) \int_t^{\tau} \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \left|Z_s^1 - Z_s^2\right|^2 ds \\
&\leq \; \overline{e}_{\tau}^{\frac{\beta}{2}} |Y_\tau^1 - Y_\tau^2|^\beta + \frac{\overline{e}_{u'}^{\frac{\beta}{2}}}{A_N^{\frac{\beta}{2}}} + \beta\varepsilon \sup_{s\in[u,u']} \Delta_s^{\frac{\beta}{2}} \\
&\quad + \beta\varepsilon^{(1-\beta)} \overline{e}_{u'}^{\frac{\beta}{2}} \int_u^{u'} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)|^\beta \mathbb{1}_{\{\Phi_s > N\}} ds \\
&\quad + 2N\beta \overline{e}_{\tau}^{\frac{\beta}{2}} A_N^{1-\frac{\beta}{2}} \int_u^{\tau} \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s \mathbb{1}_{\{v_s \;\leq\; N\}} ds \\
&\quad - \beta \int_t^{\tau} \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \langle Y_s^1 - Y_s^2, \left(Z_s^1 - Z_s^2\right) dW_s \rangle.
\end{aligned}
\tag{3.2}
$$

For a given $\hbar > 1$, let $\tau_\hbar$ be the stopping time defined by

$$
\tau_\hbar := \inf\{s \geq u, \quad |Y_s^1 - Y_s^2|^2 + \int_u^s |Z_r^1 - Z_r^2|^2 dr \geq \hbar\} \wedge u',
$$

Choosing $\tau = \tau_\hbar$, $t = u$, then passing to the expectation in equation (3.2) we obtain, when $\hbar \to \infty$,

$$
\begin{aligned}
&\frac{\beta}{2}(\beta - 1 - 2\overline{\gamma}) \mathbb{E} \int_u^{u'} \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} \left|Z_s^1 - Z_s^2\right|^2 ds \\
&\leq \; \overline{e}_{u'}^{\frac{\beta}{2}} \mathbb{E}|Y_{u'}^1 - Y_{u'}^2|^\beta + \frac{\overline{e}_{u'}^{\frac{\beta}{2}}}{A_N^{\frac{\beta}{2}}} + \beta\varepsilon \mathbb{E} \sup_{s\in[u,u']} \Delta_s^{\frac{\beta}{2}} \\
&\quad + \beta\varepsilon^{(1-\beta)} \overline{e}_{u'}^{\frac{\beta}{2}} \mathbb{E} \int_u^{u'} |f_1(s, Y_s^1, Z_s^1) - f_2(s, Y_s^2, Z_s^2)|^\beta \mathbb{1}_{\{\Phi_s > N\}} ds \\
&\quad + 2N\beta \overline{e}_{u'}^{\frac{\beta}{2}} A_N^{1-\frac{\beta}{2}} \mathbb{E} \int_u^{u'} \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s \mathbb{1}_{\{v_s \;\leq\; N\}} ds.
\end{aligned}
\tag{3.3}
$$

Return to (3.2) and use the Burkholder-Davis-Gundy inequality to get a universal constant c such that

$$\begin{aligned}
&\mathbb{E}\sup_{u\le t\le T}\Delta_t^{\frac{\beta}{2}}\\
&\le\ \overline{e}_{u'}^{\frac{\beta}{2}}\mathbb{E}|Y_{u'}^1-Y_{u'}^2|^\beta+\frac{\overline{e}_{u'}^{\frac{\beta}{2}}}{A_N^{\frac{\beta}{2}}}+\beta\varepsilon\mathbb{E}\sup_{s\in[u,u']}\Delta_s^{\frac{\beta}{2}}\\
&\quad+\beta\varepsilon^{(1-\beta)}\overline{e}_{u'}^{\frac{\beta}{2}}\mathbb{E}\int_u^{u'}|f_1(s,Y_s^1,Z_s^1)-f_2(s,Y_s^2,Z_s^2)|^\beta 1\!\!1_{\{\Phi_s>N\}}ds\\
&\quad+2N\beta\overline{e}_{u'}^{\frac{\beta}{2}}A_N^{1-\frac{\beta}{2}}\mathbb{E}\int_u^{u'}\rho_N(f_1-f)_s+\rho_N(f_2-f)_s 1\!\!1_{\{v_s\ \le\ N\}}ds\\
&\quad+c\beta\mathbb{E}\Big(\int_u^T\overline{e}_s^2\Delta_s^{\beta-2}\sum_{j=1}^r\Big[\sum_{i=1}^d(Y_{i,s}^1-Y_{i,s}^2)(Z_{ij,s}^1-Z_{ij,s}^2)\Big]^2ds\Big)^{\frac{1}{2}}.
\end{aligned}$$

But, there exists a positive constant C_β depending only on β such that

$$\begin{aligned}
&c\beta\mathbb{E}\Big(\int_u^{u'}\overline{e}_s^2\Delta_s^{\beta-2}\sum_{j=1}^r\Big[\sum_{i=1}^d(Y_{i,s}^1-Y_{i,s}^2)(Z_{ij,s}^1-Z_{ij,s}^2)\Big]^2ds\Big)^{\frac{1}{2}}\\
&\qquad\qquad\le\ \frac{1}{4}\mathbb{E}\sup_{u\le t\le u'}\Delta_t^{\frac{\beta}{2}}+C_\beta\mathbb{E}\int_u^{u'}\overline{e}_s\Delta_s^{\frac{\beta}{2}-1}|Z_s^1-Z_s^2|^2ds.
\end{aligned}$$

Use (3.3) and take ε small enough to obtain the existence of a positive constant $C=C(\beta,\overline{\gamma})$ such that

$$\begin{aligned}
&\mathbb{E}\sup_{u\le t\le u'}\Delta_t^{\frac{\beta}{2}}+\mathbb{E}\int_u^{u'}\overline{e}_s\Delta_s^{\frac{\beta}{2}-1}|Z_s^1-Z_s^2|^2ds\\
&\le\ C\Bigg[\overline{e}_{u'}^{\frac{\beta}{2}}\mathbb{E}|Y_{u'}^1-Y_{u'}^2|^\beta+\frac{\overline{e}_{u'}^{\frac{\beta}{2}}}{A_N^{\frac{\beta}{2}}}+\overline{e}_{u'}^{\frac{\beta}{2}}\sup_i\mathbb{E}\int_u^{u'}|f_i(s,Y_s^i,Z_s^i)|^\beta 1\!\!1_{\{\Phi_s>N\}}ds\\
&\quad+N\overline{e}_{u'}^{\frac{\beta}{2}}A_N^{1-\frac{\beta}{2}}\mathbb{E}\int_u^{u'}\rho_N(f_1-f)_s+\rho_N(f_2-f)_s 1\!\!1_{\{v_s\ \le\ N\}}ds\Bigg].
\end{aligned}$$

We shall estimate $J:=\sup_i\mathbb{E}\int_u^{u'}|f_i(s,Y_s^i,Z_s^i)|^\beta 1\!\!1_{\{\Phi_s>N\}}ds$, $i=1,2$.
Using the fact that $1\!\!1_{\{\Phi_s>N\}}\le 1\!\!1_{\{v_s>5^{-1}N\}}+1\!\!1_{\{|Y_s^1|>5^{-1}N\}}+1\!\!1_{\{|Y_s^2|>5^{-1}N\}}+1\!\!1_{\{|Z_s^1|>5^{-1}N\}}+1\!\!1_{\{|Z_s^2|>5^{-1}N\}}$ and $1\!\!1_{\{a>b\}}\le\dfrac{a^\nu}{b^\nu}$ for every $a,b,\nu>0$, we show

that for every $N > 1$

$$
\begin{aligned}
J \le & \left(\frac{5}{N}\right)^{\nu} \sup_i \mathbb{E}\int_u^{u'} |f_i(s, Y_s^i, Z_s^i)|^{\beta} v_s^{\nu} ds \\
& + \left(\frac{5}{N}\right)^{\nu} \sup_i \mathbb{E}\int_u^{u'} |f_i(s, Y_s^i, Z_s^i)|^{\beta} |Y_s^1|^{\nu} ds \\
& + \left(\frac{5}{N}\right)^{\nu} \sup_i \mathbb{E}\int_u^{u'} |f_i(s, Y_s^i, Z_s^i)|^{\beta} |Y_s^2|^{\nu} ds \\
& + \left(\frac{5}{N}\right)^{\nu} \sup_i \mathbb{E}\int_u^{u'} |f_i(s, Y_s^i, Z_s^i)|^{\beta} |Z_s^1|^{\nu} ds. \\
& + \left(\frac{5}{N}\right)^{\nu} \sup_i \mathbb{E}\int_u^{u'} |f_i(s, Y_s^i, Z_s^i)|^{\beta} |Z_s^2|^{\nu} ds.
\end{aligned}
$$

Young's inequality gives the existence of a positive constant C such that for every $N > 1$

$$
J \le \frac{C}{N^{\nu}}\left\{1+\Theta_p^1+\Theta_p^2+\sup_i \mathbb{E}\int_u^{u'} |f_i(s, Y_s^i, Z_s^i)|^{\beta(\frac{q'}{q'-\nu} \vee \frac{2}{2-\nu} \vee \frac{p}{p-\nu})} ds + \mathbb{E}\int_u^{u'} v_s^{q'} ds\right\}
$$

where $\Theta_p^i := \mathbb{E}\sup_t |Y_t^i|^p + \mathbb{E}\left(\int_0^T |Z_s^i|^2 ds\right)^{\frac{p}{2}}$.

Using Proposition 3.2, we have (since $\beta(\frac{q'}{q'-\nu} \vee \frac{2}{2-\nu} \vee \frac{p}{p-\nu}) \le \hat{\beta}$)

$$
J \le \frac{C}{N^{\nu}}\left\{1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T |\overline{\eta}_s|^q ds + \mathbb{E}\int_u^{u'} v_s^{q'} ds\right\}.
$$

Hence, for $a := (\frac{\nu}{\mu} \wedge \frac{\beta}{2}) - \beta r K"$ and N large enough we get

$$
\begin{aligned}
& \mathbb{E}\sup_{u \le t \le u'} \Delta_t^{\frac{\beta}{2}} + \mathbb{E}\int_u^{u'} \overline{e}_s \Delta_s^{\frac{\beta}{2}-1} |Z_s^1 - Z_s^2|^2 ds \\
& \le N A_N^{1+\frac{\beta}{2}} \left[\mathbb{E}|Y_{u'}^1 - Y_{u'}^2|^{\beta} + \mathbb{E}\int_0^T \rho_N(f_1 - f)_s + \rho_N(f_2 - f)_s 1\!\!1_{\{v_s \le N\}} ds\right] \\
& + \frac{1}{A_N^a}\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right].
\end{aligned}
$$

here we have used the assumption $A_N \le N^{\mu}$ (see **(H.4)**). Lemma 3.5 is proved. ■

As a consequence of lemma 3.5

Lemma 3.6. *Let $(\xi^i, f_i)_{i=1,2}$ satisfies **(H.3)** (with the same $\overline{\eta}, \alpha$ and α') and let (Y^i, Z^i) be a solution of $(E^{(\xi^i, f_i)})$. Then, there exists $\beta = \beta(p,q,\alpha,\alpha') \in]1, p\wedge 2[$ such that for every $\varepsilon > 0$ there is an integer $N_\varepsilon = N_\varepsilon(p,q,\alpha,\alpha',K',\mu,q',\varepsilon,(A_N)_N)$ such that for every function f satisfying **(H.4)***

$$\mathbb{E}\sup_{0\le t\le T}|Y_t^1 - Y_t^2|^\beta + \mathbb{E}\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}}ds$$
$$\le N_\varepsilon\left[\mathbb{E}|\xi^1 - \xi^2|^\beta + \mathbb{E}\int_0^T \rho_{N_\varepsilon}(f_1 - f)_s + \rho_{N_\varepsilon}(f_2 - f)_s ds\right]$$
$$+\varepsilon\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right].$$

Proof . Let $(u_0 = 0 < ... < u_{\ell+1} = T)$ be a subdivision of $[0,T]$ such that for every $i \in \{0,..,\ell\}$

$$u_{i+1} - u_i \le r$$

From lemma 3.5, we have for all $\varepsilon > 0$ there is an integer N_ε such that for every functional process f satisfying **(H.4)**

$$\mathbb{E}\sup_{u_\ell\le t\le T}|Y_t^1 - Y_t^2|^\beta + \mathbb{E}\int_{u_\ell}^T \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}}ds$$
$$\le N_\varepsilon\left[\mathbb{E}|\xi^1 - \xi^2|^\beta + \mathbb{E}\int_0^T \rho_{N_\varepsilon}(f_1 - f)_s + \rho_{N_\varepsilon}(f_2 - f)_s ds\right]$$
$$+\varepsilon\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right].$$

Suppose that for some $i \in \{0,..,\ell\}$ we have for all $\varepsilon > 0$ there is an integer N_ε such that for every function f satisfying **(H.4)**

$$\mathbb{E}\sup_{u_{i+1}\le t\le T}|Y_t^1 - Y_t^2|^\beta + \mathbb{E}\int_{u_{i+1}}^T \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}}ds$$
$$\le N_\varepsilon\left[\mathbb{E}|\xi^1 - \xi^2|^\beta + \mathbb{E}\int_0^T \rho_{N_\varepsilon}(f_1 - f)_s + \rho_{N_\varepsilon}(f_2 - f)_s ds\right]$$
$$+\varepsilon\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right].$$

Then for every $\varepsilon' > 0$ there is an integer $N_{\varepsilon'}$ such that for every function f satisfying **(H.4)**

$$\begin{aligned}
&\mathbb{E}\sup_{u_i\le t\le T}|Y_t^1 - Y_t^2|^\beta + \mathbb{E}\int_{u_i}^T \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}}ds\\
&\le \mathbb{E}\sup_{u_i\le t\le u_{i+1}}|Y_t^1 - Y_t^2|^\beta + \mathbb{E}\int_{u_i}^{u_{i+1}} \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}}ds\\
&+ N_{\varepsilon'}\left[\mathbb{E}|\xi^1 - \xi^2|^\beta + \mathbb{E}\int_0^T \rho_{N_{\varepsilon'}}(f_1 - f)_s + \rho_{N_{\varepsilon'}}(f_2 - f)_s ds\right]\\
&+ \varepsilon'\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right].
\end{aligned}$$

Using lemma 3.5 we obtain, for every $\varepsilon', \varepsilon" > 0$ there exist $N_{\varepsilon'} > 0$ and $N_{\varepsilon"} > 0$ such that for every function f satisfying **(H.4)**

$$\begin{aligned}
&\mathbb{E}\sup_{u_i\le t\le T}|Y_t^1 - Y_t^2|^\beta + \mathbb{E}\int_{u_i}^T \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}}ds\\
&\le N_{\varepsilon"}\left[\mathbb{E}|Y_{u_{i+1}}^1 - Y_{u_{i+1}}^2|^\beta + \mathbb{E}\int_0^T \rho_{N_{\varepsilon"}}(f_1 - f)_s + \rho_{N_{\varepsilon"}}(f_2 - f)_s ds\right]\\
&+ N_{\varepsilon'}\left[\mathbb{E}|\xi^1 - \xi^2|^\beta + \mathbb{E}\int_0^T \rho_{N_{\varepsilon'}}(f_1 - f)_s + \rho_{N_{\varepsilon'}}(f_2 - f)_s ds\right]\\
&+ 2\varepsilon'\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right]\\
&\le N_{\varepsilon'}N_{\varepsilon"}\ \ \mathbb{E}|\xi^1 - \xi^2|^\beta\\
&+ (N_{\varepsilon'}N_{\varepsilon"} + 2N_{\varepsilon'})\ \ \mathbb{E}\int_0^T \rho_{(N_{\varepsilon'}N_{\varepsilon"})}(f_1 - f)_s + \rho_{(N_{\varepsilon'}N_{\varepsilon"})}(f_2 - f)_s ds\\
&+ (2\varepsilon' + \varepsilon" N_{\varepsilon'})\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right].
\end{aligned}$$

For $\varepsilon > 0$, let $\varepsilon' := \dfrac{\varepsilon}{4}$ and $\varepsilon" := \dfrac{\varepsilon}{2N_{(\frac{\varepsilon}{4})}}$ we have then the existence of an integer N_ε such that for every function f satisfying **(H.4)**

$$\begin{aligned}
&\mathbb{E}\sup_{u_i\leq t\leq T}|Y_t^1-Y_t^2|^\beta+\mathbb{E}\int_{u_i}^T\frac{|Z_s^1-Z_s^2|^2}{(1+|Y_s^1-Y_s^2|^2)^{1-\frac{\beta}{2}}}ds\\
&\leq\; N_\varepsilon\left[\mathbb{E}|\xi^1-\xi^2|^\beta+\mathbb{E}\int_0^T\rho_{N_\varepsilon}(f_1-f)_s+\rho_{N_\varepsilon}(f_2-f)_sds\right]\\
&\quad+\varepsilon\left[1+\Theta_p^1+\Theta_p^2+\mathbb{E}\int_0^T\overline{\eta}_s^q ds+\mathbb{E}\int_0^T v_s^{q'}ds\right].
\end{aligned}$$

We complete the proof by induction ■

Proposition 3.3. *Let $(\xi^i, f_i)_{i=1,2}$ satisfies **(H.3)** (with the same $\overline{\eta}, \alpha$ and α') and let (Y^i, Z^i) be a solution of $(E^{(\xi^i, f_i)})$. Then, there exists $\beta = \beta(p, q, \alpha, \alpha') \in]1, p\wedge 2[$ such that for every $\varepsilon > 0$ there is an integer $N_\varepsilon = N_\varepsilon(p, q, \alpha, \alpha', K', \mu, q', \varepsilon, (A_N)_N)$ such that for every function f satisfying **(H.4)***

$$\begin{aligned}
&\mathbb{E}\sup_{0\leq t\leq T}|Y_t^1-Y_t^2|^\beta+\mathbb{E}\left(\int_0^T|Z_s^1-Z_s^2|^2ds\right)^{\frac{\beta}{2}}\\
&\leq\; N_\varepsilon\left[\mathbb{E}|\xi^1-\xi^2|^\beta+\mathbb{E}\int_0^T\rho_{N_\varepsilon}(f_1-f)_s+\rho_{N_\varepsilon}(f_2-f)_sds\right]\\
&\quad+\varepsilon\left[1+\Theta_p^1+\Theta_p^2+\mathbb{E}\int_0^T\overline{\eta}_s^q ds+\mathbb{E}\int_0^T v_s^{q'}ds\right],
\end{aligned}$$

where $\Theta_p^i := \mathbb{E}\sup_t|Y_t^i|^p+\mathbb{E}\left(\int_0^T|Z_s^i|^2ds\right)^{\frac{p}{2}}$.

Proof . Using Hölder's inequality, Young's inequality and the fact that $\dfrac{\beta}{2} < 1$, we obtain for all $\varepsilon' > 0$

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^T |Z_s^1 - Z_s^2|^2 ds\right)^{\frac{\beta}{2}} \\
\leq\ & \mathbb{E}\left\{ \left[\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds\right]^{\frac{\beta}{2}} \sup_{s\leq T}\left(1+|Y_s^1 - Y_s^2|^2\right)^{(1-\frac{\beta}{2})\frac{\beta}{2}} \right\} \\
\leq\ & \left[\mathbb{E}\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds\right]^{\frac{\beta}{2}} \left(1+\mathbb{E}\sup_{s\leq T}|Y_s^1 - Y_s^2|^\beta\right)^{\frac{2-\beta}{2}} \\
\leq\ & \left[\mathbb{E}\sup_{s\leq T}|Y_s^1 - Y_s^2|^\beta + \mathbb{E}\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds\right]^{\frac{\beta}{2}} \\
&+ \left[\mathbb{E}\sup_{0\leq t\leq T}|Y_t^1 - Y_t^2|^\beta + \mathbb{E}\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds\right] \\
\leq\ & \varepsilon' + (1+\varepsilon'^{\frac{\beta-2}{\beta}})\left[\mathbb{E}\sup_{0\leq t\leq T}|Y_t^1 - Y_t^2|^\beta + \mathbb{E}\int_0^T \frac{|Z_s^1 - Z_s^2|^2}{(1+|Y_s^1 - Y_s^2|^2)^{1-\frac{\beta}{2}}} ds\right].
\end{aligned}
$$

Use lemma 3.5 to conclude for every $\varepsilon', \varepsilon" > 0$

$$
\begin{aligned}
&\mathbb{E}\left(\int_0^T |Z_s^1 - Z_s^2|^2 ds\right)^{\frac{\beta}{2}} \\
&\leq \varepsilon' + (1+\varepsilon'^{\frac{\beta-2}{\beta}}) N_{\varepsilon"}\left[\mathbb{E}|\xi^1 - \xi^2|^\beta + \mathbb{E}\int_0^T \rho_{N_{\varepsilon"}}(f_1 - f)_s + \rho_{N_{\varepsilon"}}(f_2 - f)_s ds\right] \\
&+ \varepsilon"(1+\varepsilon'^{\frac{\beta-2}{\beta}})\left[1 + \Theta_p^1 + \Theta_p^2 + \mathbb{E}\int_0^T \overline{\eta}_s^q ds + \mathbb{E}\int_0^T v_s^{q'} ds\right].
\end{aligned}
$$

Letting $\varepsilon' = \dfrac{\varepsilon}{2}$ and $\varepsilon" = \dfrac{\varepsilon}{2(1+(\frac{\varepsilon}{2})^{\frac{\beta-2}{2}})}$, we finish this proof of proposition 3.3. ■

Remark 3.1. *The uniqueness of equation $(E^{(\xi,f)})$ follows by letting $f_1 = f_2 = f$ and $\xi_1 = \xi_2 = \xi$ in Proposition 3.3.*

The following stability result follows from propositions (3.3), (3.2) and (3.1)

Proposition 3.4. *Let (ξ, f) satisfies **(H.0)-(H.4)** and $(\xi^n, f_n)_n$ satisfies **(H.0)-(H.3)** uniformly on n. Assume moreover that*

$$(H^n.2)\quad \begin{cases} \textit{there exist } M^n \textit{ and } K^n \in \mathbb{L}^0(\Omega\times[0,T];\mathbb{R}_+) \textit{ satisfying} \\ M^n \leq M,\ K^n\leq K \textit{ and } (M^n,K^n)\longrightarrow (M,K) \textit{ a.e. such that:} \\ \langle y, f_n(t,y,z)\rangle \ \leq\ \eta_t + f_t^0|y| + M_t^n|y|^2 + K_t^n|y||z| \end{cases}$$

(a) $\xi^n \to \xi$ *a.s.*

(b) For every N, $\lim_n \rho_N(f_n - f) = 0$ *a.e.*

(c) for each $n\in\mathbb{N}$ $(E^{(\xi^n,f_n)})$ *has a solution* (Y^n,Z^n) *which satisfies* $\mathbb{E}\sup_{t\leq T}|Y_t^n|^p e^{\frac{p}{2}\int_0^T\lambda_s^n ds} < \infty$, *where* $\lambda_s^n := 2M_s^n + \dfrac{(K^n)_s^2}{2\gamma}$.

Then, there exists $(Y,Z)\in \mathbb{L}^p(\Omega;\mathcal{C}([0,T];\mathbb{R}^d))\times\mathbb{L}^p(\Omega;\mathbb{L}^2([0,T];\mathbb{R}^{dr}))$ *such that*

i)

$$\mathbb{E}\sup_t |\,Y_t\,|^p\, e^{\frac{p}{2}\int_0^t\lambda_s ds} + \mathbb{E}\left[\int_0^T e^{\int_0^s\lambda_r dr}\,|\,Z_s\,|^2\,ds\right]^{\frac{p}{2}}$$
$$\leq C^{p,\gamma}\left\{\mathbb{E}\,|\,\xi\,|^p\, e^{\frac{p}{2}\int_0^T\lambda_s ds} + \mathbb{E}\left(\int_0^T e^{\int_0^s\lambda_r dr}\eta_s ds\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T e^{\frac{1}{2}\int_0^s\lambda_r dr} f_s^0 ds\right)^p\right\}$$

ii) for every $p' < p$, $(Y^n,Z^n)\to(Y,Z)$ *strongly in* $\mathbb{L}^{p'}(\Omega;\mathcal{C}([0,T];\mathbb{R}^d))\times\mathbb{L}^{p'}(\Omega;\mathbb{L}^2([0,T];\mathbb{R}^{dr}))$.

iii) for every $\hat\beta < \dfrac{2}{\alpha'}\wedge\dfrac{p}{\alpha}\wedge\dfrac{p}{\alpha'}\wedge q$

$$\lim_{n\to\infty}\mathbb{E}\int_0^T |f_n(s,Y_s^n,Z_s^n) - f(s,Y_s,Z_s)|^{\hat\beta} ds = 0$$

Moreover, (Y,Z) *is the unique solution of* $(E^{(\xi,f)})$.

Proof . From Proposition 3.1, Proposition 3.2 and Proposition 3.3, we have

$$a')\ \left[\mathbb{E}\sup_t|Y_t^n|^p e^{\frac{p}{2}\int_0^t\lambda_s^n ds} + \mathbb{E}\left(\int_0^T e^{\int_0^t\lambda_s^n ds}|Z_s^n|^2 ds\right)^{\frac{p}{2}}\right]$$
$$\leq C^{p,\gamma}\left\{\mathbb{E}\,|\,\xi^n\,|^p\, e^{\frac{p}{2}\int_0^T\lambda_s ds} + \mathbb{E}\left(\int_0^T e^{\int_0^s\lambda_r dr}\eta_s ds\right)^{\frac{p}{2}} + \right.$$
$$\left. +\mathbb{E}\left(\int_0^T e^{\frac{1}{2}\int_0^s\lambda_r dr} f_s^0 ds\right)^p\right\} := D_n.$$

$b')\ \mathbb{E}\displaystyle\int_0^T |f_n(s,Y_s^n,Z_s^n)|^{\hat\beta} ds \leq C(1 + D_n + \int \bar\eta_s^q ds).$

$c')$ There exists $\beta > 1$ such that for every $\varepsilon > 0$ there exists $N_\varepsilon > 0$:

$$\begin{aligned}
&\mathbb{E}\sup_t |Y^n_t - Y^m_t|^\beta + \mathbb{E}\left(\int_0^T |Z^n_s - Z^m_s|^2 ds\right)^{\frac{\beta}{2}} \\
&\leq N_\varepsilon \left[\mathbb{E}|\xi^n - \xi^m|^\beta + \mathbb{E}\int_0^T \rho_{N_\varepsilon}(f_n - f)_s + \rho_{N_\varepsilon}(f_m - f)_s ds\right] \\
&\quad + \varepsilon\left[1 + D_n + D_m + \mathbb{E}\int_0^T \overline{\eta}^q_s ds + \mathbb{E}\int_0^T v^{q'}_s ds\right].
\end{aligned}$$

We deduce the existence of $(Y, Z) \in \mathbb{L}^p(\Omega; \mathcal{C}([0,T]; \mathbb{R}^d)) \times \mathbb{L}^p(\Omega; \mathbb{L}^2([0,T]; \mathbb{R}^{dr}))$ such that

$$\begin{aligned}
&i)\ \mathbb{E}\sup_t \mid Y_t \mid^p e^{\frac{p}{2}\int_0^t \lambda_s ds} + \mathbb{E}\left[\int_0^T e^{\int_0^s \lambda_r dr} \mid Z_s \mid^2 ds\right]^{\frac{p}{2}} \\
&\leq C^{p,\gamma}\left\{\mathbb{E} \mid \xi \mid^p e^{\frac{p}{2}\int_0^T \lambda_s ds} + \mathbb{E}\left(\int_0^T e^{\int_0^s \lambda_r dr} \eta_s ds\right)^{\frac{p}{2}} + \right. \\
&\qquad\qquad \left. + \mathbb{E}\left(\int_0^T e^{\frac{1}{2}\int_0^s \lambda_r dr} f^0_s ds\right)^p\right\}
\end{aligned}$$

$ii)$ for all $p' < p$, $(Y^n, Z^n) \to (Y, Z)$ strongly in $\mathbb{L}^{p'}(\Omega; \mathcal{C}([0,T]; \mathbb{R}^d)) \times \mathbb{L}^{p'}(\Omega; \mathbb{L}^2([0,T]; \mathbb{R}^{dr}))$.

Let us show $iii)$. To this end set $a := \limsup_{n\to\infty} \mathbb{E}\int_0^T |f(s, Y^n_s, Z^n_s) - f(s, Y_s, Z_s)|^{\hat{\beta}} ds$ and consider a subsequence n' of n such that $a := \lim_{n'\to\infty} \mathbb{E}\int_0^T |f(s, Y^{n'}_s, Z^{n'}_s) - f(s, Y_s, Z_s)|^{\hat{\beta}} ds$ and $(Y^{n'}, Z^{n'}) \to (Y, Z)$ a.e. The continuity of f and assumption **(H.3)** ensures us that $a = 0$. It remains to prove that

$$\limsup_{n\to\infty} \mathbb{E}\int_0^T |f_n(s, Y^n_s, Z^n_s) - f(s, Y^n_s, Z^n_s)|^{\hat{\beta}} ds = 0.$$

We use holder's inequality, the previous claim b'), proposition 3.2 and

Chebychev's inequality to get

$$\begin{aligned}
&\mathbb{E}\int_0^T |f_n(s,Y_s^n,Z_s^n)-f(s,Y_s^n,Z_s^n)|^{\hat\beta}ds\\
&\le \mathbb{E}\int_0^T \rho_N(f_n-f)_s^{\hat\beta}ds+\\
&+(\mathbb{E}\int_0^T |f_n(s,Y_s^n,Z_s^n)-f(s,Y_s^n,Z_s^n)|^{r\hat\beta}ds)^{\frac{1}{r}}(\mathbb{E}\int_0^T 1_{|Y_s^n|+|Z_s^n|\ge N}ds)^{\frac{r-1}{r}}\\
&\le \mathbb{E}\int_0^T \rho_N(f_n-f)_s^{\hat\beta}ds+\frac{const(r)}{N^{\frac{(r-1)(p\wedge 2)}{r}}},
\end{aligned}$$

for some reel $r>1$ such that $r\hat\beta < \dfrac{2}{\alpha'}\wedge\dfrac{p}{\alpha}\wedge\dfrac{p}{\alpha'}\wedge q$.
Let $n\longrightarrow\infty$ first and then $N\longrightarrow\infty$ to obtain assertion $iii)$.
Proposition 3.4 is proved ■

3.3. **Approximation.** Now, what we would like to do is to construct a sequence (ξ^n,f_n) which approximate (ξ,f) and satisfy properties $(a)-(f)$ below. With the help of this approximation, we can construct a solution (Y,Z) to the BSDE $(E^{(\xi,f)})$ via Proposition 3.4.

Set $\overline{\Lambda}_t := \eta_t+\overline{\eta}_t+f_t^0+M_t+K_t$ and let h_t be a predictable process such that $0<h_t\le 1$.

Proposition 3.5. *Assume that (ξ,f) satisfies **(H.0)–(H.3)**. Then there exists a sequence (ξ^n,f_n) such that*
(a) For each n, ξ^n is bounded and $|\xi^n|\le|\xi|$ and ξ^n converges to ξ a.s.
(b) For each n, f_n is bounded and globally Lipschitz in (y,z).
There exists a constant $C=C(d,r,p)$ such that for each n
(c) $|f_n(t,\omega,y,z)| \le 1\!\!1_{\{\overline{\Lambda}_t\le n\}}\{\overline{\eta}_t+|y|^{\alpha}+|z|^{\alpha'}+Ch_t\}$.
(d) $<y,f_n(t,\omega,y,z)> \le 1\!\!1_{\{\overline{\Lambda}_t\le n\}}\{\eta_t+f_t^0|y|+M_t|y|^2+K_t|y||z|+Ch_t\}$.
(e) For every N, $\rho_N(f_n-f)(t,\omega)\longrightarrow 0$ as $n\longrightarrow\infty$ a.e (t,ω).
(f) For every N, $\rho_N(f_n-f)(t,\omega) \le 2\{\overline{\eta}_t+N^{\alpha}+N^{\alpha'}+Ch_t\}$.

Proof . Let $\psi:\mathbb{R}\longrightarrow[0,\dfrac{\exp(-1)}{c_1}]$ defined by:

$$\psi(x):=\begin{cases} c_1^{-1}\exp(-\dfrac{1}{1-x^2}) & \text{if } |x|<1\\ 0 & \text{else}\end{cases}$$

where $c_1=\displaystyle\int_{-1}^{1}\exp(-\frac{1}{1-x^2})dx$.

The sequence (ξ^n, f_n) defined by $\xi^n := \xi 1_{[\xi^n \leq n]}$ and

$$f_n(t,y,z) = (c_1 e)^2 1\!\!1_{\{\overline{\Lambda}_t \ \leq \ n\}} \psi(n^{-2}|y|^2)\psi(n^{-2}|z|^2)\times$$
$$m^{(d+dr)} \int_{\mathbb{R}^d}\int_{\mathbb{R}^{dr}} f(t, y-u, z-v)\Pi_{i=1}^d \psi(mu_i)\Pi_{i=1}^d \Pi_{j=1}^r \psi(mv_{ij}) du dv,$$

with $m := \dfrac{n^{2p}}{h_t}$ satisfies the required properties ■

proofs being almost the same as

4. Application to partial differential equations (PDE)

In this section we give a Feynman-Kac formula to multidimensional PDEs for which we establish a uniqueness via the BSDEs. The new idea here, is to prove that the uniqueness for a system of semilinear PDE follows from the uniqueness of an associated system of linear PDE.

4.1. Formulation of the problem.

For this we consider the following system of semilinear PDE

$$(\mathcal{P}^{(g,F)}) \begin{cases} \dfrac{\partial u(t,x)}{\partial t} + \mathcal{L}u(t,x) + F(t,x,u(t,x),\sigma^*\nabla u(t,x)) = 0 & t \in]0,T[, x \in \mathbb{R}^k \\ u(T,x) = g(x) & x \in \mathbb{R}^k \end{cases}$$

where $\mathcal{L} := \dfrac{1}{2}\sum_{i,j}(\sigma\sigma^*)_{ij}\partial^2_{ij} + \sum_i b_i\partial_i, \quad \sigma \in \mathcal{C}^3_b(\mathbb{R}^k, \mathbb{R}^{kr}), \quad b \in \mathcal{C}^2_b(\mathbb{R}^k, \mathbb{R}^k)$ and $g : \mathbb{R}^k \to \mathbb{R}^d, \quad F : [0,T] \times \mathbb{R}^k \times \mathbb{R}^d \times \mathbb{R}^{dr} \to \mathbb{R}^d$ are measurable functions.
Let,

$$\mathcal{H}^{1+} := \bigcup_{\delta \geq 0, \beta > 1} \Big\{ v \in \mathcal{C}([0,T]; \mathbb{L}^\beta(\mathbb{R}^k, e^{-\delta|x|}dx; \mathbb{R}^d)) : \int_0^T \int_{\mathbb{R}^k} |\sigma^*\nabla v(s,x)|^\beta e^{-\delta|x|} dx ds < \infty \Big\}$$

Definition 4.1. *A (weak) solution of* $(\mathcal{P}^{(g,F)})$ *is a function* $u \in \mathcal{H}^{1+}$ *such that for every* $t \in [0,T]$ *and every* $\varphi \in C_c^1([0,T] \times \mathbb{R}^d)$

$$\begin{aligned}\int_t^T &< u(s), \frac{\partial \varphi(s)}{\partial s} > ds + < u(s), \varphi(s) >= \\ &= \; < g, \varphi(s) > + \int_t^T < F(s,.,u(s), \sigma^* \nabla u(s)), \varphi(s) > ds + \\ &\quad + \int_t^T < Lu(s), \varphi(s) > ds\end{aligned}$$

where $< f(s), h(s) >= \int_{\mathbb{R}^k} f(s,x)h(s,x)dx$.

Noticing that an integrating by part allows to see that,

$$\begin{aligned}< Lu(s), \varphi(s) >= \int_{\mathbb{R}^k} \frac{1}{2} \langle \sigma^* \nabla u(s,x); \sigma^* \nabla \varphi(s,x) \rangle dxds \\ + < u(s), div(\tilde{b}\varphi)(s) >\end{aligned}$$

where $\tilde{b}_i := b_i - \frac{1}{2} \sum_j \partial_j (\sigma\sigma^*)_{ij}$

4.2. **Assumptions.** In the sequel, we need the following assumptions:

There exist $\delta \geq 0$ and $\overline{p} > 1$ such that

(A.0) $g(x) \in \mathbb{L}^{\overline{p}}(\mathbb{R}^k, e^{-\delta|x|}dx; \mathbb{R}^d)$

(A.1) $F(t,x,.,.)$ is continuous a.e. (t,x)

(A.2) $\left\{\begin{array}{l}\text{There are } \eta' \in \mathbb{L}^{\frac{\overline{p}}{2}\vee 1}([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dtdx; \mathbb{R}_+)), \\ f^{0'} \in \mathbb{L}^{\overline{p}}([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dtdx; \mathbb{R}_+)), \text{ and } M, M' \in \mathbb{R}_+ \\ \text{such that} \\ \\ \langle y, F(t,x,y,z) \rangle \leq \\ \leq \eta'(t,x) + f^{0'}(t,x)|y| + (M + M'|x|)|y|^2 + \sqrt{M + M'|x|}|y||z|\end{array}\right.$

(A.3) $\left\{\begin{array}{l}\text{There are } \overline{\eta}' \in \mathbb{L}^q([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dtdx; \mathbb{R}_+))(\text{for some } q > 1), \\ \alpha \in]1, \overline{p}[\\ \text{and } \alpha' \in]1, \overline{p} \wedge 2[\text{ such that} \\ \\ |F(t,x,y,z)| \leq \overline{\eta}'(t,x) + |y|^\alpha + |z|^{\alpha'}\end{array}\right.$

$$
\textbf{(A.4)}\quad \begin{cases} \text{There are } K, r \in \mathbb{R}_+ \text{ such that for every } N \in \mathbb{N} \text{ and every } x, y, \\ y',\ z,\ z' \\ \text{satisfying } e^{r|x|},\ |\ y\ |,\ |\ y'\ |,\ |\ z\ |,\ |\ z'\ | \leq N, \\ \\ \langle y - y'; F(t,x,y,z) - F(t,x,y',z') \rangle \leq \\ \quad \leq K \log N \left(\dfrac{1}{N} + |y-y'|^2 \right) + \sqrt{K \log N} |y-y'||z-z'|. \end{cases}
$$

4.3. **Existence and uniqueness for** $(\mathcal{P}^{(g,F)})$**.** We consider the diffusion process with infinitesimal operator $\mathcal{L}$

$$
X_s^{t,x} = x + \int_t^s b(X_r^{t,x})dr + \int_t^s \sigma(X_r^{t,x})dW_r, \qquad t \leq s \leq T
$$

Theorem 4.1. *Under assumption* ***(A.0)-(A.4)*** *we have*
1) The PDE $(\mathcal{P}^{(g,F)})$ *has a unique solution* u *on* $[0,T]$
2) For all $t \in [0,T]$ *there exists* $D_t \subset \mathbb{R}^k$ *such that*
i) $\displaystyle\int_{D_t^c} 1\ dx = 0$
ii) for all $t \in [0,T]$ *and all* $x \in D_t$ $(E^{(\xi^{t,x}, f^{t,x})})$ *has a unique solution* $(Y^{t,x}, Z^{t,x})$ *on* $[t,T]$
where $\xi^{t,x} := g(X_T^{t,x})$ *and* $f^{t,x}(s,y,z) := 1\!\!1_{\{s>t\}} F(s, X_s^{t,x}, y, z)$
3) Moreover, for all $t \in [0,T]$

$$
\big(u(s, X_s^{t,x}), \sigma^* \nabla u(s, X_s^{t,x})\big) = \big(Y_s^{t,x}, Z_s^{t,x}\big) \qquad a.e.(s,x,\omega)
$$

Let $p \in]\alpha \vee \alpha', \overline{p}[$ *if* $M' > 0$ *and* $p = \overline{p}$ *if* $M' = 0$*. Then there is a constant* C *depending only on* $\delta, M, M', p, \overline{p}$, $|\sigma|_\infty, |b|_\infty$ *and* T

$$
\sup\nolimits_{0 \leq t \leq T} \int_{\mathbb{R}^k} |\ u(t,x)\ |^p\ e^{-\delta'|x|} dx + \int_0^T \int_{\mathbb{R}^k} |\ \sigma^* \nabla u(t,x)\ |^{p \wedge 2}\ e^{-\delta'|x|} dt dx
$$
$$
\leq C \Bigg(1\!\!1_{[M' \neq 0]} + \int_{\mathbb{R}^k} |\ g(x)\ |^{\overline{p}}\ dx + \int_{\mathbb{R}^k} \int_0^T \eta'(s,x)^{\frac{\overline{p}}{2} \vee 1} ds dx + \int_{\mathbb{R}^k} \int_0^T f^{0'}(s,x)^{\overline{p}} ds dx \Bigg)
$$

where $\delta' = \delta + \kappa' + 1\!\!1_{[M' \neq 0]}$ *and* $\kappa' := 4 \dfrac{p \overline{p} M' T}{(\overline{p} - p)}$.

4.4. **Proof of Theorem 4.1.** A) **Existence.**

Lemma 4.1. 1) *There exists* $\kappa > 0$ *depending only on* $|\sigma|_\infty, |b|_\infty$ *and* T *such that*

$$
\sup_{t,x} \mathbb{E}[\exp(\kappa \sup_{t \leq s \leq T} |\ X_s^{t,x} - x\ |^2)] < \infty. \tag{4.0}
$$

In particular for all $r > 0$ there is a constant $C(r,\kappa)$ such that for all (t,x)

$$\mathbb{E}[\exp(r \sup_{t\leq s\leq T} \mid X_s^{t,x} \mid)] \leq C(r,\kappa)\exp(r \mid x \mid)$$

2) *For all $\delta \geq 0$ there exists a constant $C_{\delta,T} > 1$ such that for all $\varphi \in \mathbb{L}^0(\mathbb{R}^k)$, all $t \in [0,T]$ and all $s \in [t,T]$*

$$C_{\delta,T}^{-1}\int_{\mathbb{R}^k} |\varphi(x)|e^{-\delta|x|}dx \leq \mathbb{E}\int_{\mathbb{R}^k} |\varphi(X_s^{t,x})|e^{-\delta|x|}dx \leq C_{\delta,T}\int_{\mathbb{R}^k} |\varphi(x)|e^{-\delta|x|}dx. \tag{4.2}$$

Moreover for all $\delta \geq 0$ there exists a constant $C_{\delta,T} > 1$ such that for all $\psi \in \mathbb{L}^0([0,T]\times\mathbb{R}^k)$, all $t \in [0,T]$ and all $s \in [t,T]$

$$\begin{aligned} C_{\delta,T}^{-1}\int_{\mathbb{R}^k}\int_t^T |\psi(s,x)|dse^{-\delta|x|}dx &\leq \mathbb{E}\int_{\mathbb{R}^k}\int_t^T \mathbb{E}|\psi(s,X_s^{t,x})|dse^{-\delta|x|}dx \\ &\leq C_{\delta,T}\int_{\mathbb{R}^k}\int_t^T |\psi(s,x)|dse^{-\delta|x|}dx. \end{aligned}$$

Proof. The first assertion is well known. Its particular case follows by using Young's inequality, indeed

$$\begin{aligned} \mathbb{E}[\exp(r \sup_{t\leq s\leq T} \mid X_s^{t,x} \mid)] &\leq \exp(r \mid x \mid)\mathbb{E}[\exp(r \sup_{t\leq s\leq T} \mid X_s^{t,x} - x \mid)] \\ &\leq \exp(r \mid x \mid)\mathbb{E}[\exp(\frac{r}{\sqrt{\kappa}}\sqrt{\kappa} \sup_{t\leq s\leq T} \mid X_s^{t,x} - x \mid)] \\ &\leq \exp(\frac{r^2}{\kappa})\exp(r \mid x \mid)\mathbb{E}[\exp(\kappa \sup_{t\leq s\leq T} \mid X_s^{t,x} - x \mid^2)]. \end{aligned}$$

For the second assertion, see [**?**] proposition 5.1. Lemma 4.1 is proved. ■

Lemma 4.2. *Let $p \in]\alpha\vee\alpha', \bar{p}[$ if $M' > 0$ and $p = \bar{p}$ if $M' = 0$. Let $t \in [0,T]$. There exists $D_t \subset \mathbb{R}^k$ such that*
i) $\displaystyle\int_{D_t^c} 1\, dx = 0$
ii) for all $x \in D_t$

$$\begin{aligned} &\mathbb{E} \mid g(X_T^{t,x}) \mid^p e^{\frac{p}{2}\int_t^T \lambda_s^{t,x}ds} + \mathbb{E}\left(\int_t^T \eta'(s,X_s^{t,x})e^{\int_t^s \lambda_r^{t,x}dr}ds\right)^{\frac{p}{2}} \\ &\quad + \mathbb{E}\left(\int_t^T f^{0'}(s,X_s^{t,x})e^{\frac{1}{2}\int_t^s \lambda_r^{t,x}dr}ds\right)^p + \mathbb{E}\int_t^T \overline{\eta}'(s,X_s^{t,x})^q ds \; < +\infty, \end{aligned}$$

where $\lambda_s^{t,x} := 4(M + M'|X_s^{t,x}|)$.

Proof . Using Young's inequality and lemma 4.1 we obtain

$$\mathbb{E} \mid g(X_T^{t,x}) \mid^p e^{\frac{p}{2}\int_t^T \lambda_s^{t,x} ds} + \mathbb{E}\left(\int_t^T \eta'(s, X_s^{t,x}) e^{\int_t^s \lambda_r^{t,x} dr} ds\right)^{\frac{p}{2}}$$
$$+\mathbb{E}\left(\int_t^T f^{0'}(s, X_s^{t,x}) e^{\frac{1}{2}\int_t^s \lambda_r^{t,x} dr} ds\right)^p + \mathbb{E}\int_t^T \overline{\eta}'(s, X_s^{t,x})^q ds$$
$$\leq C\left(\mathbb{E} \mid g(X_T^{t,x}) \mid^{\overline{p}} + \mathbb{E}\int_t^T \eta'(s, X_s^{t,x})^{\frac{\overline{p}}{2}\vee 1} + \ \mathbb{E}\int_t^T f^{0'}(s, X_s^{t,x})^{\overline{p}} + \right.$$
$$\left. + \mathbb{E}\int_t^T \overline{\eta}'(s, X_s^{t,x})^q ds + 1\!\!1_{[M'\neq 0]} e^{\kappa'|x|}\right)$$
$$:= \Gamma^{t,x}$$

for some constant C depending only on $M, M', p, \bar{p}$, $|\sigma|_\infty, |b|_\infty$ and T and where $\kappa' := 4\dfrac{p\overline{p}M'T}{(\overline{p}-p)}$.

Using Lemma 4.1-2) and assumptions **(A.0)-(A.3)**, we can show that

$$\int_{\mathbb{R}^k} \Gamma^{t,x} e^{-\delta'|x|} dx \ < \ \infty$$

where $\delta' = \delta + \kappa' + 1$. The set $D_t := \{x; \quad \Gamma^{t,x} < \infty\}$. Lemma 4.2 is proved. ∎

Lemma 4.3. *Assume* ***(A.0)-(A.4)****. Let* $p \in]\alpha \vee \alpha', \overline{p}[$ *if* $M' > 0$ *and* $p = \bar{p}$ *if* $M' = 0$*. Then, for every* $t \in [0, T]$ *and every* $x \in D_t$ $(E^{(\xi^{t,x}, f^{t,x})})$ *has a unique solution* $(Y^{t,x}, Z^{t,x})$ *which satisfies, for every* $t \in [0, T]$ *and every* $x \in D_t$*,*

$$\begin{aligned}
&\mathbb{E}\sup_{t\leq s\leq T} \mid Y_s^{t,x} \mid^p + \mathbb{E}\left(\int_t^T \mid Z_s^{t,x} \mid^2 ds\right)^{\frac{p}{2}} \\
&\leq C\left(\mathbb{E} \mid g(X_T^{t,x}) \mid^{\overline{p}} + \ \mathbb{E}\int_t^T \eta'(s, X_s^{t,x})^{\frac{\overline{p}}{2}\vee 1} ds\right. \\
&\qquad \left. + \ \mathbb{E}\int_t^T f^{0'}(s, X_s^{t,x})^{\overline{p}} ds + 1\!\!1_{[M'\neq 0]} e^{\kappa'|x|}\right)
\end{aligned} \tag{4.3}$$

for some constant C *depending only on* $M, M', p, \bar{p}$, $|\sigma|_\infty, |b|_\infty$ *and* T *.*

Proof . For all t and all $x \in D_t$, $(\xi^{t,x}, f^{t,x})$ satisfies **(H.0)-(H.4)** with $\gamma = \inf\{\dfrac{1}{4}, \dfrac{p-1}{4}\}$, $M_s = M + M'|X_s^{t,x}|$, $K_s = \sqrt{M + M'|X_s^{t,x}|}$, $\eta_s = \eta'(s, X_s^{t,x})$,

$f_s^0 = f^{0'}(s, X_s^{t,x})$, $\overline{\eta}_s = \overline{\eta}'(s, X_s^{t,x})$, $v_s = \exp(r|X_s^{t,x}|)$ and $A_N = N$. Hence Lemma 4.3 follows from Theorem 2.1 and Lemma 4.2. ■

Set $g_n(x) := g(x)1\!\!1_{\{|g(x)|\leq n\}}$,
$F_n(t,x,y,z) =$
$= (n^{2p}e^{|x|})^{(d+dr)}(c_1 e)^2 1\!\!1_{\{\eta'(t,x)+\overline{\eta}'(t,x)+f^{0'}(t,x)+|x|\leq n\}}\psi(n^{-2}|y|^2)\psi(n^{-2}|z|^2)\times$
$\int_{\mathbb{R}^d}\int_{\mathbb{R}^{dr}} F(t,x,y-u,z-v)\Pi_{i=1}^d\psi(n^{2p}e^{|x|}u_i)\Pi_{i=1}^d\Pi_{j=1}^r\psi(n^{2p}e^{|x|}v_{ij})dudv,$
$\xi_n^{t,x} := g_n(X_T^{t,x})$ and $f_n^{t,x}(s,y,z) := 1\!\!1_{\{s>t\}}F_n(s, X_s^{t,x}, y, z)$.
The sequence (g_n, F_n) satisfies **(A.0)-(A.3)** uniformly in n, hence $(\xi_n^{t,x}, f_n^{t,x})$ satisfies **(H.0)-(H.3)** uniformly in n . Moreover, for every n : $(\xi_n^{t,x}, f_n^{t,x})$ is bounded and $f_n^{t,x}$ is globally Lipschitz.

Let $(Y^{t,x,n}, Z^{t,x,n})$ be the unique solution of $(E^{(\xi_n^{t,x}, f_n^{t,x})})$. It is not difficult to show that for every t, $x \in D_t$ and every n

$$\begin{aligned}
&\mathbb{E}\sup_{t\leq s\leq T} | Y_s^{t,x,n} |^p + \mathbb{E}\left(\int_t^T | Z_s^{t,x,n} |^2 ds\right)^{\frac{p}{2}} \\
&\leq C\left(\mathbb{E}\int_t^T e^{-(\frac{\overline{p}}{2}\vee 1)|X_s^{t,x}|}ds + \mathbb{E} | g(X_T^{t,x}) |^{\overline{p}} + \right.\\
&\left. + \mathbb{E}\int_t^T \eta'(s, X_s^{t,x})^{\frac{\overline{p}}{2}\vee 1}ds + \mathbb{E}\int_t^T f^{0'}(s, X_s^{t,x})^{\overline{p}}ds + 1\!\!1_{[M'\neq 0]}e^{\kappa'|x|}\right)
\end{aligned} \tag{4.4}$$

for some constant C not depending on (t, x, n). To see this, use proposition 3.5 (with $h_t := e^{-|X_s^{t,x}|}$), proposition 3.1) and the proof of proposition 3.4. a').

From the result of [5] (see also [6]) we have

Lemma 4.4. *There exists a unique solution u^n of*

$$(\mathcal{P}^{(g_n,F_n)})\begin{cases} \dfrac{\partial u^n(t,x)}{\partial t} + \mathcal{L}u^n(t,x) + F_n(t,x,u^n(t,x), \sigma^*\nabla u^n(t,x)) = 0, \\ \qquad\qquad\qquad\qquad t \in]0,T[,\ x \in \mathbb{R}^k \\ u^n(T,x) = g_n(x) \quad x \in \mathbb{R}^k \end{cases}$$

such that for all t

$$u^n(s, X_s^{t,x}) = Y_s^{t,x,n} \quad \textit{and} \quad \sigma^*\nabla u^n(s, X_s^{t,x}) = Z_s^{t,x,n} \qquad a.e\ (s,\omega,x).$$

From Proposition 3.4-(ii) we have

Lemma 4.5. *[Stability] For all $t \in [0,T]$, all $x \in D_t$ and all $p' < p$*

$$\lim_n \left[\mathbb{E} \sup_{0\le s\le T} \mid Y_s^{t,x,n} - Y_s^{t,x} \mid^{p'} + \mathbb{E}\left(\int_t^T \mid Z_s^{t,x,n} - Z_s^{t,x} \mid^2 ds \right)^{\frac{p'}{2}} \right] = 0.$$

Using Lemma 4.1 assertion 2), inequality 4.4, Lemma 4.5 and the Lebesgue dominated convergence theorem, we obtain

Lemma 4.6. *[Covergence of PDE]*

$$\lim_{n,m} \sup_{0\le t\le T} \int_{\mathbb{R}^k} \mid u^n(t,x) - u^m(t,x) \mid^{p'} e^{-\delta'|x|} dx = 0$$
$$\lim_{n,m} \int_0^T \int_{\mathbb{R}^k} \mid \sigma^*\nabla u^n(t,x) - \sigma^*\nabla u^m(t,x) \mid^{p'\wedge 2} e^{-\delta'|x|} dt dx = 0.$$

Using Lemma 4.1, Lemma 4.6 and the fact that $\mathcal{H}^{1+}$ is complete, we show that there exists $u \in \mathcal{H}^{1+}$ such that

$i)$ $\sup_{0\le t\le T}\int_{\mathbb{R}^k}| u(t,x) |^{p'} e^{-\delta'|x|}dx + \int_0^T\int_{\mathbb{R}^k}| \sigma^*\nabla u(t,x) |^{p'\wedge 2} e^{-\delta'|x|} dt dx < \infty$
$ii)$ $\lim_n \sup_{0\le t\le T} \int_{\mathbb{R}^k} \mid u^n(t,x) - u(t,x) \mid^{p'} e^{-\delta'|x|} dx = 0$
$iii)$ $\lim_n \mathbb{E} \int_{\mathbb{R}^k} \left(\int_t^T \mid \sigma^*\nabla u^n(s, X_s^{t,x}) - \sigma^*\nabla u(s, X_s^{t,x}) \mid^2 e^{-\delta'|x|} ds \right)^{\frac{p'}{2}} dx = 0$
$\forall t \in [0,T]$
$iv)$ $\qquad (u(s, X_s^{t,x}), \sigma^*\nabla u(s, X_s^{t,x})) = (Y_s^{t,x}, Z_s^{t,x}) \quad$ a.e.

On the other hand, we use Proposition 3.2 and Proposition 3.4 we respectively have for every $t \in [0,T]$ and every $x \in D_t$

$$\mathbb{E} \int_t^T |F_n(s, X_s^{t,x}, u^n(s, X_s^{t,x}), \sigma^*\nabla u^n(s, X_s^{t,x})|^{\hat{\beta}} ds \le$$
$$\le C \left(1 + \Theta_p^{t,x,n} + \mathbb{E} \int_t^T |\overline{\eta}'(s, X_s^{t,x})|^q ds \right)$$

and

$$\lim_n \mathbb{E} \int_t^T |F_n(s, X_s^{t,x}, u^n(s, X_s^{t,x}), \sigma^*\nabla u^n(s, X_s^{t,x})) -$$
$$- F(s, X_s^{t,x}, u(s, X_s^{t,x}), \sigma^*\nabla u(s, X_s^{t,x}))|^{\hat{\beta}} ds = 0$$

where $\Theta_p^{t,x,n} = \mathbb{E}\sup_s |Y_s^{t,x,n}|^p + \mathbb{E}\left(\int_t^T |Z_s^{t,x,n}|^2 ds\right)^{\frac{p}{2}}$, $\hat{\beta}$ is some real in $]1,\infty[$ and C is a constant not depending on (t,x,n).
We deduce from Lemma 4.1, the Lebesgue dominated convergence theorem and the inequality (4.4) that

$$\lim_n \int_0^T \int_{\mathbb{R}^d} |F_n(s,x,u^n(s,x),\sigma^*\nabla u^n(s,x))-$$

$$-F(s,x,u(s,x),\sigma^*\nabla u(s,x))|^{\hat{\beta}} e^{-(1+\delta')|x|} dxds = 0.$$

As a consequence of Lemma 4.3 and the proof of Proposition 3.2, we have the following existence result

Proposition 4.1. *Under assumptions **(A.0)-(A.4)**, the PDE $(\mathcal{P}^{(g,F)})$ has a unique solution u such that $u(s,X_s^{t,x}) = Y_s^{t,x}$ and $\sigma^*\nabla u(s,X_s^{t,x}) = Z_s^{t,x}$. Moreover, letting $p \in]\alpha\vee\alpha', \overline{p}[$ if $M'>0$ and $p=\bar{p}$ if $M'=0$, then there is a constant C depending only on $\delta', M, M', p, \bar{p}$, $|\sigma|_\infty, |b|_\infty$ and T such that s*

$$\sup_{0\le t\le T}\int_{\mathbb{R}^k} \mid u(t,x)\mid^p e^{-\delta'|x|}dx + \int_0^T\int_{\mathbb{R}^k} \mid \sigma^*\nabla u(t,x)\mid^{p\wedge 2} e^{-\delta'|x|}dtdx$$
$$\le C\left(1+\int_{\mathbb{R}^k}\mid g(x)\mid^{\overline{p}} dx + \int_{\mathbb{R}^k}\int_0^T \eta(s,x)^{\frac{\overline{p}}{2}\vee 1}dsdx + \int_{\mathbb{R}^k}\int_0^T f^{0'}(s,x)^{\overline{p}}dsdx\right)$$

where $\delta' = \delta+\kappa'+1$ and $\kappa' := 4\dfrac{p\overline{p}M'T}{(\overline{p}-p)}$.

B) **Uniqueness:**

In order to prove uniqueness we need the following lemmas

Lemma 4.7. *Let $\varepsilon \in]0,1[$, $g\in\mathcal{C}_c^\infty([0,T]\times\mathbb{R}^k;\mathbb{R})$. Then, there exists a unique solution $\phi^\varepsilon \in \cap_{q>\frac{3}{2}}\mathcal{W}_q^{1,2}([0,T]\times\mathbb{R}^k;\mathbb{R})\cap\mathcal{C}^{1,2}([0,T]\times\mathbb{R}^k;\mathbb{R})$ of the following PDE*

$$(\mathcal{P}_\varepsilon(g))\begin{cases} \dfrac{\partial\phi^\varepsilon(t,x)}{\partial t} - \dfrac{1}{2}div(\sigma\sigma^*\nabla\phi^\varepsilon) - \varepsilon\triangle\phi^\varepsilon(t,x) + \langle\tilde{b}(x); \\ \nabla\phi^\varepsilon(t,x)\rangle = g(t,x) \\ \phi^\varepsilon(0,x) = 0 \qquad x\in\mathbb{R}^k \end{cases}$$

Moreover the solution ϕ^ε satisfies the following bounds

$$\sup_{(\varepsilon,t,x)}\left\{|\frac{\partial\phi^\varepsilon}{\partial t}(t,x)|+|\nabla\phi^\varepsilon(t,x)|+|\phi^\varepsilon(t,x)|\right\}<\infty.$$

Proof . The existence and uniqueness, of the solution ϕ^ε, follow from [23] (p. 318 and pp. $341-342$). We shall prove the uniform bounds for ϕ^ε and for their first derivatives. These bounds can be established by adapting the proof given in Krylov [22] pp. $330-344$. However, we give here a probabilistic proof which is very simple. We assume that the dimension k is 1. Let $X_t^\varepsilon(x)$ denote the diffusion process associated to the problem $(\mathcal{P}_\varepsilon(g))$. For the simplicity, and without loss of the generality, we assume that g does not depend from t and the drift coefficient of $X_t^\varepsilon(x)$ is zero. The process $X_t^\varepsilon(x)$ is then the unique (strong) solution of the following SDE

$$X_t^\varepsilon(x)=x+\int_0^t\sigma_\varepsilon(X_s^\varepsilon(x))dW_s,\qquad 0\le t\le T$$

Let $M:=\sup_{(\varepsilon,t,x)}(|g'(X_t^\varepsilon(x))|+|\sigma(t,x)|+|\sigma'(t,x)|)$. Since the coefficients σ_ε, $\tilde{b}$ are smooth and σ_ε is uniformly elliptic, then the solution ϕ^ε belongs to $\mathcal{C}^{1,2}$. Hence, Itô's formula shows that,

$$\phi^\varepsilon(t,x)=-\mathbb{E}\int_t^T g(X_s^\varepsilon(x))ds.$$

Since $g\in\mathcal{C}_c^\infty$, we immediately get

$$\sup_{(\varepsilon,t,x)}\left\{|\frac{\partial\phi^\varepsilon}{\partial t}(t,x)|+|\phi^\varepsilon(t,x)|\right\}<\infty.$$

Since $\sigma_\varepsilon\in\mathcal{C}_b^3$, we then have

$$|\frac{\partial\phi^\varepsilon(t,x)}{\partial x}|\le M\mathbb{E}\int_t^T|\frac{\partial X_s^\varepsilon(x)}{\partial x}|ds$$

It remains to show that $\sup_{(\varepsilon,t,x)}\mathbb{E}(|\frac{\partial X_t^\varepsilon(x)}{\partial x}|)<\infty$.

Since $|\sigma'_\varepsilon(t,x)|\le|\sigma'(t,x)|\le\sup_{(t,x)}|\sigma'(t,x)|\le M$, we have

$$\begin{aligned}\mathbb{E}(|\frac{\partial X_t^\varepsilon(x)}{\partial x}|^2)&\le 1+\mathbb{E}\int_0^t|\sigma'_\varepsilon(X_s^\varepsilon(x))|^2|\frac{\partial X_s^\varepsilon(x)}{\partial x}|^2ds\\&\le 1+M^2\mathbb{E}\int_0^t|\frac{\partial X_s^\varepsilon(x)}{\partial x}|^2ds\end{aligned}$$

The Gronwall Lemma gives now the desired result.
In multidimensional case, the proof can be performed similarly since it is based on the fact that the first derivative of σ_ε is bounded uniformly in ε. which is valid in multidimensional case also, see [16] pp. 198-201. Lemma 4.7 is proved. ■

Lemma 4.8. *0 is the unique solution of*

$$(\mathcal{P}^{(0,-div(\tilde{b})(x)y)})\begin{cases} \dfrac{\partial w(t,x)}{\partial t} + \mathcal{L}w(t,x) - div(\tilde{b})(x)w(t,x) = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad t \in]0,T[,\ x \in \mathbb{R}^k \\ w(T,x) = 0 \quad x \in \mathbb{R}^k \end{cases}$$

satisfying for some $\beta > 1$

$$\sup_{0\le t\le T}\int_{\mathbb{R}^k} | w(t,x) |^\beta + | w(t,x) | \, dx + \tag{4.1}$$
$$+\int_0^T\int_{\mathbb{R}^k} | \sigma^*\nabla w(t,x) |^\beta + | \sigma^*\nabla w(t,x) | \, dtdx < \infty.$$

Proof . Let w be a solution of $(\mathcal{P}(0, -div(\tilde{b})(x)y))$ satisfying (4.1) and consider $w_n \in \mathcal{C}_c^\infty(R^k)$ such that

$$\int_0^T\int_{\mathbb{R}^k} |w(s,x) - w_n(s,x)| dxds + \int_0^T\int_{\mathbb{R}^k} |\sigma^*\nabla(w(s,x) - w_n(s,x))| dxds \to 0.$$

Let $\varepsilon \in]0,1[$, $g \in \mathcal{C}_c^\infty([0,T]\times\mathbb{R}^k;\mathbb{R})$ and consider the unique solution $\phi^\varepsilon \in \cap_{q>\frac{3}{2}}\mathcal{W}_q^{1,2}([0,T]\times\mathbb{R}^k;\mathbb{R}) \cap \mathcal{C}^{1,2}([0,T]\times\mathbb{R}^k;\mathbb{R})$ of the following problem

$$(\mathcal{P}_\varepsilon(g))\begin{cases} \dfrac{\partial \phi^\varepsilon(t,x)}{\partial t} - \dfrac{1}{2}div(\sigma\sigma^*\nabla\phi^\varepsilon) - \varepsilon \triangle \phi^\varepsilon(t,x) + \langle \tilde{b}(x); \\ \nabla\phi^\varepsilon(t,x)\rangle = g(t,x)\rangle \\ \phi^\varepsilon(0,x) = 0 \quad x \in \mathbb{R}^k \end{cases}$$

the existence and uniqueness of ϕ^ε follows from Lemma 4.7.

Let $(\psi_i)_{i\in\mathbb{N}} \subset \mathcal{C}_c^\infty(R^k)$ be such that $\psi_i \in [0,1]$, $\psi_i \to 1$ uniformly on every compact set and $\nabla\psi_i \to 0$ uniformly on $\mathbb{R}^k$.

By considering $\phi^\varepsilon\psi_i$ as a test function, we have

$$\int_0^T\int_{\mathbb{R}^k}\left[w\frac{\partial\phi^\varepsilon}{\partial t} + \frac{1}{2}\langle\sigma^*\nabla w; \sigma^*\nabla\phi^\varepsilon\rangle + w\langle\tilde{b}; \nabla\phi^\varepsilon\rangle\right]\psi_i dxdt +$$
$$\int_0^T\int_{\mathbb{R}^k}\frac{1}{2}\langle\sigma^*\nabla w; \sigma^*\nabla\psi_i\rangle\phi^\varepsilon + \langle\tilde{b}; \nabla\psi_i\rangle\phi^\varepsilon w dxdt = 0.$$

Introducing w_n, we obtain

$$\int_0^T \int_{\mathbb{R}^k} w_n \psi_i \left[\frac{\partial \phi^\varepsilon}{\partial t} - \frac{1}{2} div(\sigma\sigma^* \nabla \phi^\varepsilon) + \langle \tilde{b}; \nabla \phi^\varepsilon \rangle \right] dt dx = \chi_1^{\varepsilon,i}(n) + \chi_2^{\varepsilon,n}(i),$$

where

$$\chi_1^{\varepsilon,i}(n) := -\int_0^T \int_{\mathbb{R}^k} \Big[(w - w_n) \frac{\partial \phi^\varepsilon}{\partial t} + \frac{1}{2} \langle \sigma^* \nabla (w - w_n); \sigma^* \nabla \phi^\varepsilon \rangle + (w - w_n) \langle \tilde{b}; \nabla \phi^\varepsilon \rangle \Big] \psi_i dx dt$$

and

$$\chi_2^{\varepsilon,n}(i) := -\int_0^T \int_{\mathbb{R}^k} \langle \frac{1}{2} \phi^\varepsilon \sigma\sigma^* \nabla w + \phi^\varepsilon w \tilde{b} - \frac{1}{2} w_n \sigma\sigma^* \nabla \phi^\varepsilon \ ; \ \nabla \psi_i \rangle dx dt.$$

From Lemma 4.7, we have

$$\sup_\varepsilon \sup_{(t,x)} \left\{ |\frac{\partial \phi^\varepsilon}{\partial t}(t,x)| + |\nabla \phi^\varepsilon(t,x)| + |\phi^\varepsilon(t,x)| \right\} < \infty.$$

Hence

$$\sup_{\varepsilon,i} |\chi_1^{\varepsilon,i}(n)| \to_{n\to\infty} 0$$

and

$$\sup_{\varepsilon,n} |\chi_2^{\varepsilon,n}(i)| \to_{i\to\infty} 0.$$

Since by integrating by part we have, $\int_0^T \int_{\mathbb{R}^k} w_n \psi_i \triangle \phi^\varepsilon dx dt = -\int_0^T \int_{\mathbb{R}^k} \nabla (w_n \psi_i) \nabla \phi^\varepsilon dx dt$, then using the Lebesgue dominated convergence theorem we deduce that

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^k} w g(t,x) dx dt &= \lim_n \lim_i \lim_\varepsilon \int_0^T \int_{\mathbb{R}^k} w_n \psi_i (g(t,x) + \varepsilon \triangle \phi^\varepsilon) dx dt \\ &= \lim_n \lim_i \lim_\varepsilon (\chi_1^{\varepsilon,i}(n) + \chi_2^{\varepsilon,n}(i)) = 0. \end{aligned}$$

Lemma 4.8 is proved. ■

Proof of uniqueness for $(\mathcal{P}^{(g,F)})$**.** The proof is divided into two steps:

Step1. 0 *is the unique solution of* $(\mathcal{P}^{(0,0)})$ *satisfying (4.1).*

Let w_1 be a solution of $(\mathcal{P}^{(0,0)})$ satisfying (4.1) then by Lemma 4.8 it is also the unique solution of $(\mathcal{P}^{(0,-div\tilde{b}(x)y + div\tilde{b}(x)w_1(t,x))})$ satisfying (4.1) [since if u is a solution of $(\mathcal{P}^{(0,-div\tilde{b}(x)y + div\tilde{b}(x)w_1(t,x))})$, then $u - w_1$ is a solution of $(\mathcal{P}^{(0,-div\tilde{b}(x)y)})$ and hence $u - w = 0$ by Lemma 4.8]. From Proposition 4.1 we have $(w_1(s, X_s^{t,x}), \sigma^* \nabla w_1(s, X_s^{t,x}))$ is the unique solution of the BSDE $(E^{(0,-div\tilde{b}(X_s^{t,x})y + div\tilde{b}(X_s^{t,x})w_1(s,X_s^{t,x}))})$. Thanks to the uniqueness of BSDE and Lemma 4.1, we get $w_1 = 0$.

Step2. *0 is the unique solution of* $(\mathcal{P}^{(0,0)})$.

Let w_1 be a solution of $(\mathcal{P}^{(0,0)})$. Since $w_1 \in \mathcal{H}^{1+}$, then there exist $\beta' > 1, \delta' \geq 0$ be such that,

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^k} \mid w_1(t,x)\mid^{\beta'} e^{-\delta'|x|}dx+\int_0^T\int_{\mathbb{R}^k} \mid \sigma^*\nabla w_1(t,x)\mid^{\beta'} e^{-\delta'|x|}dxdt < \infty.$$

Let $\delta > \delta'$ and set $\tilde{w_1} = w_1 f(x)$, where $f \in \mathcal{C}^2(\mathbb{R}^k;\mathbb{R}_+^*)$ such that $f(x) = e^{-\delta|x|}$ if $\mid x\mid > 1$, then by Lemma 4.8 $\tilde{w}_1$ is the unique solution of

$$(\mathcal{P}_1^{(0,0)}) \begin{cases} \dfrac{\partial w(t,x)}{\partial t} + \mathcal{L}w(t,x) - div(\tilde{b})(x)w(t,x) + H(x)\tilde{w}_1(t,x) + \\ \qquad\qquad + \langle \overline{H}(x), \sigma^*\nabla\tilde{w}_1(t,x)\rangle = 0 \\ w(T,x) = 0 \end{cases}$$

satisfying (4.1)

where H and $\overline{H}$ are bounded and continuous functions.
Proposition 4.1 implies that $(\tilde{w}_1(s,X_s^{t,x}), \sigma^*\nabla\tilde{w}_1(s,X_s^{t,x}))$ is the unique solution of the
BSDE $(E^{(0,-div(\tilde{b})(X_s^{t,x})H(X_s^{t,x})\tilde{w}_1(s,X_s^{t,x})+\langle\overline{H}(X_s^{t,x}),\sigma^*\nabla\tilde{w}_1(s,X_s^{t,x})\rangle)})$, hence $\tilde{w}_1 = 0$ which implies that $w_1 = 0$.

Step 3. $(\mathcal{P}^{(g,F)})$ *has a unique solution if and only if* 0 *is the unique solution of* $(\mathcal{P}^{(0,0)})$.

By Proposition 4.1, there exists a unique solution u of the problem $(\mathcal{P}^{(g,F)})$ such that $u(s,X_s^{t,x}) = Y_s^{t,x}$ and $\sigma^*\nabla u(s,X_s^{t,x}) = Z_s^{t,x}$. Let u' be another solution of $(\mathcal{P}^{(g,F)})$ and set

$$\hat{F}(t,x) = F(s,x,u(s,x),\sigma^*\nabla u(s,x)) - F(s,x,u'(s,x),\sigma^*\nabla u'(s,x)).$$

$w := u - u'$ is then a solution of the problem

$$(\mathcal{P}^{(0,\hat{F})}) \qquad \begin{cases} \dfrac{\partial w(t,x)}{\partial t} + \mathcal{L}w(t,x) + \hat{F}(t,x) = 0 \quad t \in]0,T[,\ x\in\mathbb{R}^k \\ w(T,x) = 0 \quad x\in\mathbb{R}^k \end{cases}$$

On the other hand $(0,\hat{F})$ satisfies assumptions **(A.0)-(A.4)**, then Proposition 4.1 ensures the existence of a unique solution $\tilde{w}$ of $(\mathcal{P}^{(0,\hat{F})})$ such that $\tilde{w}(s,X_s^{t,x}) = \tilde{Y}_s^{t,x}$ and $\sigma^*\nabla\tilde{w}(s,X_s^{t,x}) = \tilde{Z}_s^{t,x}$ where $(\tilde{Y}_s^{t,x},\tilde{Z}_s^{t,x})$ is the unique solution of

$$\tilde{Y}_s^{t,x} = \int_s^T \hat{F}(r, X_r^{t,x})dr - \int_s^T \tilde{Z}_r^{t,x} dW_r$$

The uniqueness of $(\mathcal{P}^{(0,\hat{F})})$ (which follows from step 1) allows us to deduce that

$$u'(s, X_s^{t,x}) = Y_s^{t,x} - \tilde{Y}_s^{t,x} \qquad \text{and} \qquad \sigma^*\nabla u'(s, X_s^{t,x}) = Z_s^{t,x} - \tilde{Z}_s^{t,x}.$$

This implies that $u'(t, X_s^t)$ is solution to the BSDE $(E^{(g,F)})$. The uniqueness of this BSDE shows that $u'(t, X_s^t) = u(t, X_s^t)$. We get then that $u(t,x) = u'(t,x)$ *a.e.* by using Lemma 4.1. Theorem 4.1 is proved. ■

Consequence: Let $g \in \mathbb{L}^p([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dx; \mathbb{R}^d)$ for some $p > 1$ and $\delta \geq 0$. Let $A : [0,T] \times \mathbb{R}^k \longrightarrow \mathbb{R}^{d\times d}$, $B : [0,T] \times \mathbb{R}^k \longrightarrow (\mathbb{R}^d)^{dr}$ and $C : [0,T] \times \mathbb{R}^k \longrightarrow \mathbb{R}^{d\times d}$ be measurable functions such that there is a constant K, for all (t,x)

$$\|A(t,x)\| + \|B(t,x)\|^2 \leq K(1 + |x|), \;\; \|C(t,x)\| \leq K \;\text{ and }\; C(t,x) \geq 0.$$

Under this consideration we have

Proposition 4.2. *The PDE*

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} + \mathcal{L}w(t,x) + A(t,x)w(t,x) + \langle\langle\, B(t,x);\, \sigma^*\nabla w(t,x)\,\rangle\rangle - \\ \qquad\qquad\qquad\qquad -C(t,x)w(t,x)\log|w(t,x)| = 0, \\ w(T,x) = g(x) \qquad x \in \mathbb{R}^k \end{cases}$$

has a unique solution w and $(w(s, X_s^{t,x}), \sigma^*\nabla w(s, X_s^{t,x}))$ is the unique solution of

$$E(g(X_T^{t,x}),\; A(s, X_s^{t,x})y \;+ \langle\langle B(s, X_s^{t,x}); z\rangle\rangle \;- C(s, X_s^{t,x})y\log|y|),$$

where $\langle\langle B; z\rangle\rangle := \displaystyle\sum_{i=1}^{d}\sum_{j=1}^{r} B_{ij}Z_{ij}$.

Set

$$F(t,x,y,z) := A(t,x)y \;+ \langle\langle B(t,x); z\rangle\rangle \;- C(t,x)y\log|y|.$$

Arguing as in the introductory examples, we show the following claims 1) and 3). The claim 2) follows by using Young's inequality.

1) $\langle y, F(t,x,y,z)\rangle \leq K + (K + K|x|)|y|^2 + \sqrt{K + K|x|}|y||z|$

2) for all $\varepsilon > 0$ there is a constant C_ε such that

$|F(t,x,y,z)| \leq C_\varepsilon(1+|x|^{C_\varepsilon}+|y|^{1+\varepsilon}+|z|^{1+\varepsilon})$

3) for every $N>3$ and every $x, y,\ y'\ z,\ z'$ satisfying $e^{|x|},\ |\ y\ |,\ |\ y'\ |$ $,\ |\ z\ |,\ |\ z'\ |\leq N$:

$\langle y - y'; F(t,x,y,z) - F(t,x,y',z')\rangle \leq K'\log N\left(\frac{1}{N}+|y-y'|^2\right)+$
$+\sqrt{K'\log N}|y-y'||z-z'|$,
where $K' := 1+4Kd+K^2$.

So assumptions **(A.0)-(A.4)** are satisfied for (g,F). ■

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References

[1] K. Bahlali, Backward stochastic differential equations with locally Lipschitz coefficient, *C. R. Acad. Sci.*, Paris, **3**33, no. 5, 481–486, (2001).

[2] K. Bahlali, Existence and uniqueness of solutions for BSDEs with locally Lipschitz coefficient. *Electron. Comm. Probab.*, **7**, 169–179, (2002).

[3] K. Bahlali, E.H. Essaky, M. Hassani, E. Pardoux, Existence, uniqueness and stability of backward stochastic differential equations with locally monotone coefficient. *C. R. Acad. Sci.*, Paris **3**35, no. 9, 757–762, (2002).

[4] K. Bahlali, B. Mezerdi, M. N'zi, Y. Ouknine, Weak solutions and a Yamada-Watanabe theorem for BSDEs. *Preprint* (2001), *submited.*

[5] V. Bally, A. Matoussi : Weak solutions for SPDEs and backward doubly stochastic differential equations. J. Theoret. Probab. 14 (2001), no. 1, 125–164.

[6] G. Barles, E. Lesigne: SDE, BSDE and PDE. IN "Backward Stochastic Differentail Equations", N. El Karoui and L. Mazliak, editors, Pitman Research Notes in Math. Series. 364, Longman, 1997.

[7] Ph. Briand, B. Delyon, Y. Hu, E. Pardoux, L. Stoica, L^p solutions of backward stochastic differential equations. *Stochastic Process. Appl.*, **1**08, no. 1, 109–129, (2003).

[8] R. Buckdahn, H.J. Engelbert, A. Rascanu, On weak Solution of Stochastic Differential Equations, *to appear.*

[9] R. Buckdahn, Y. Hu, Probabilistic approach to homogenizations of systems of quasilinear parabolic PDEs with periodic structures. *Nonlinear Anal.*, **3**2, no. 5, 609–619, (1998).

[10] R. Buckdahn, Y. Hu, S. Peng, Probabilistic approach to homogenization of viscosity solutions of parabolic PDEs. *NoDEA. Nonlinear Differential Equations Appl.*, **6**, no. 4, 395–411, (1999).

[11] F. Delarue, Thèse de Doctorat de l'Université de Provence, Marseille, (2001).

[12] A. Dermoune, S. Hamadène and Y. Ouknine, Backward stochastic differential equation with local time. *Stoc. Stoc. Reports.* **6**6, 103-119, (1999).

[13] R. Darling, É. Pardoux, Backward SDE with monotonicity and random terminal time, *Ann. of Probab.* 25, 1135-1159, (1997).

[14] N. El Karoui, L Mazliak Edts, Backward Stochastic Differentail Equations", *Pitman Research Notes in Mathematics, Series 364, N. El-Karoui and S. Mazliak edts.* (1997).

[15] N. El Karoui, S. Peng and M.C. Quenez, Backward stochastic differential equations in finance. *Mathematical Finance.* **7**, 1-71, (1997).

[16] M. Freidlin, Functional integration and partial differential equations. *Annals of Mathematics Studies*, 109, Princeton University Press, Princeton, (1985).

[17] S. Hamadène, Multidimentional Backward SDE's with uniformly continuous coefficients. *Bernoulli* **9**, no. 3, 517–534, (2003).

[18] S. Hamadène, Equations differentielles stochastiques retrogrades, le cas localement lipschitzien. *Ann. Inst. Henri Poincaré.* **3**2, 645-660.

[19] S. Hamadène, J.P. Lepeletier, S. Peng, BSDE With continuous coefficients and applications to Markovian nonzero sum stochastic differential games, *Pitman Research Notes in Mathematics, Series 364, N. El-Karoui and S. Mazliak edts.*, (1997)

[20] M. Hassani, Y. Ouknine, On a general result for backward stochastic differential equations. *Stochastic and Stochastic Reports,*, **7**3 , no. 3-4, 219-240, (2002).

[21] M. Kobylanski, Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann. Probab.*, bf 28, no. 2, 558–602, (2000).

[22] N. V. Krylov, Nonlinear Elliptic and Parabolic Equations of the Second Order, D.Reidel Publishing Company.

[23] O. A. Ladyženskaja, V. A. Solonnikov, N. N. Ural'ceva, Linear and Quasilinear Equations Of Parabolic Type.

[24] A. Lejay, A probabilistic representation of the solution of some quasi-linear PDE with a divergence form operator. Application to existence of weak solutions of FBSDE, *to appear.*

[25] J.P. Lepeltier, J. San Martin, (1998), Existence for BSDE with Superlinear-Quadratic coefficients. *Stoc. Stoc. Reports* 63, 227-240.

[26] X. Mao, Adapted solutions of backward stochastic differential equations with non-Lipschitz coefficient. *Stoch. Proc. Appl.* 58, 281-292, (1995).

[27] E. Pardoux, Homogenization of linear and semilinear second order parabolic PDEs with periodic coefficients: a probabilistic approach. J. Funct. Anal. 167 (1999), no. 2, 498–520.

[28] E. Pardoux, BSDE's, weak convergence and homogenization of semilinear PDEs, in *Nonlin. Analy., Diff. Equa. and Control*, F. Clarke and R. Stern (eds), Kluwer Acad. Publi., Dordrecht, 503-549, (1999).

[29] E. Pardoux, S. Peng, Adapted solution of a backward stochastic differential equation. System Control Lett. **14**, 55-61, (1990).

[30] E. Pardoux, S. Peng, Backward stochastic differential equations and quasilinear parabolic partial differential equations. Stochastic partial differential equations and their applications (Charlotte, NC, 1991), *Lecture Notes in Control and Inform. Sci.*, **1**76, 200–217, Springer, Berlin, 1992.

[31] S. Peng, Probabilistic interpretation for systems of quasilinear parabolic partial differential equations. *Stochastics Stochastics Rep.*, **3**7, no. 1-2, 61–74, (1991).

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