INFINITE INDEX SUBGROUPS AND FINITENESS PROPERTIES OF INTERSECTIONS OF GEOMETRICALLY FINITE GROUPS

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Abstract. We explore which types of finiteness properties are possible for intersections of geometrically finite groups of isometries in negatively curved symmetric rank one spaces. Our main tool is a twist construction which takes as input a geometrically finite group containing a normal subgroup of infinite index with given finiteness properties and infinite Abelian quotient, and produces a pair of geometrically finite groups whose intersection is isomorphic to the normal subgroup.

1. Introduction

In 1954 Howson [H] proved that in a free group, the intersection of two finitely generated subgroups is finitely generated (such groups become known as Howson groups). There are many generalizations of this theorem in both geometric group theory due to A.Karras, D.Solitar, D.I.Moldavanskii, R.Gitik, E.Rips, H.Short, a.o., and in the theory of discrete groups of hyperbolic isometries due to L.Greenberg, B.Maskit, J.Hempel, J.Anderson, P.Susskind, G.Swarup, T.Soma, W.Thurston, a.o. Many of these results have geometric or topological content, see [An] for a useful survey and bibliography. One of the initial objects, surface groups correspond to discrete groups of isometries of the hyperbolic plane $\mathbb{H}^2$. They are finitely generated if and only if they are geometrically finite. Because of that most of results about Howson groups of hyperbolic isometries are for geometrically finite groups (see Section 2 for precise definition), which have very nice topological and geometrical properties. For example, they are finitely presented, and in fact are of type F_m for $m \geq 1$. It was natural to wonder whether two geometrically finite subgroups of a discrete group of hyperbolic isometries have a geometrically finite intersection. Susskind [Su1] showed that Kleinian groups on 2-plane have this property, and later Susskind and Swarup [SS]

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showed that any discrete group of isometries of the real hyperbolic space $\mathbb{H}^n$ also has it. However, as it was recently shown by Susskind [Su2], without the ambient discreteness condition the intersection of two geometrically finite subgroups of $\mathrm{Isom}\, H^4_{\mathbb{R}}$ may be an infinitely generated Fuchsian subgroup of $\mathrm{Isom}\, H^2_{\mathbb{R}} \subset \mathrm{Isom}\, H^4_{\mathbb{R}}$ with a circle as the limit set.

In this paper, we explore the possible finiteness properties of the intersection of two geometrically finite groups of isometries in negatively curved rank one symmetric spaces including the real hyperbolic ones. Our goal is to provide a simple method for producing geometrically finite groups in various negatively curved rank one symmetric spaces whose intersection is as big and wild as one may need. In particular, such intersection subgroup may have one or more of the following properties:

(1) its limit set coincides with the limit sets of initial geometrically finite groups and is arbitrary big or wild, or
(2) it is not finitely generated, or
(3) it is finitely generated but not geometrically finite, or
(4) it is finitely generated but not finitely presented.

Briefly, our method of generating examples is to start with a geometrically finite subgroup G of the isometry group of some rank one symmetric space X which preserves a totally geodesic subspace $X_G \subset X$ and has the following additional structure. There is a short exact sequence

$$1 \to G^* \to G \to \mathbb{A} \to 1 \tag{1.1}$$

where G^* is a group with one of the properties mentioned above, and $\mathbb{A}$ is an infinite Abelian group, in particular, cyclic group $\mathbb{Z}$. Then one applies a twist construction to obtain another geometrically finite group $G_T \subset \mathrm{Isom}\, X$ whose intersection with G is isomorphic to G^*.

We feel that there should be examples with intersection which is finitely presented but not of type F_3. The problem of finding such groups can be reduced to a realization problem of corresponding groups constructed in combinatorial group theory. In fact beginning with the work of Stallings [S] and Bieri [Bi], there have been many constructions of groups G which fit into the short exact sequence above with $\mathbb{A} = \mathbb{Z}$ and with a kernel G^* of type F_n but not F_{n+1} for all $n \geq 2$. Among such groups there are Gromov hyperbolic groups [G1, G2], CAT(0) and cubical groups G with kernel G^* in (1.1) of type F_2 (finitely presented) but not of type F_3. It is an interesting challenge for the future to see if any of these groups can be realized as subgroups of a geometrically finite group of isometries of some rank one symmetric space.

This work was inspired by a talk of Perry Susskind in Spring 2000 at Department of Mathematics of the University of Oklahoma, whose results

were later published in [Su2]. Author gratefully acknowledges a fruitful discussion with Perry Susskind and Noel Brady following that talk, as well as inspiring discussions in Centre de Recerca Matematica at Barcelona, especially with Oleg Bogopolski and Martin Lustig.

2. Background and preliminary results

Here we fix some notations and definitions, and briefly review some facts concerning geometry of negatively curved symmetric spaces X of rank one, Carnot groups $\mathcal{N}$ which correspond to horospheres in these spaces, see [AX2], as well as geometrical finiteness in such spaces [A1, A2, B], including some constructions of discrete geometrically infinite groups in hyperbolic spaces.

2.1. Symmetric spaces of rank one.

There are models of negatively curved symmetric spaces X (or equivalently, $\mathbb{F}$-hyperbolic spaces $H^n_{\mathbb{F}}$) in the unit balls $B^n_{\mathbb{F}}(0,1) \subset \mathbb{F}^n$ where $\mathbb{F}$ denotes either real and complex numbers, $\mathbb{R}$ and $\mathbb{C}$, or quaternions $\mathbb{H}$, or Cayley numbers $\mathbb{O}$ (octonions, in which case $n = 2$). The group of orientation preserving isometries of the real hyperbolic space $H^n_{\mathbb{R}}$ with sectional curvature $K \equiv -1$ is given by $\mathrm{PSO}(n,1)$, and the holomorphic automorphism groups of other spaces with variable sectional curvature K, $-1 \leq K \leq -1/4$ respectively are $\mathrm{PU}(n,1)$, $\mathrm{PSp}(n,1)$, and F_4^{-20}. The second components of the isometry groups $\mathrm{Isom}\, X$ of these complex spaces X have anty-holomorphic isometries (compositions of the holomorphic ones and the operator of conjugation in $\mathbb{F}^n$). Isometries $g \in \mathrm{Isom}\, X$ in these spaces may have one of the following three types which exhaust all the possibilities. If g fixes a point in X, it is called elliptic. If g has exactly one fixed point, and it lies in ∂X, g is called parabolic. If g has exactly two fixed points, and they lie in ∂X, g is called loxodromic.

A subgroup $G \subset \mathrm{Isom}\, X$ is called *discrete* if it is a discrete subset of $\mathrm{Isom}\, X$. An infinite discrete group G is called *parabolic* if it has exactly one fixed point $\{p\} = \mathrm{fix}(G)$, $p \in \partial X$; then G consists of either parabolic or elliptic elements. We denote by $\Lambda(G)$ the limit set of G, i.e. the set of all accumulation points of the orbit $G(x)$, $x \in X$, and by $\Omega(G)$ the complement $\partial X \backslash \Lambda(G)$ (the discontinuity set at infinity).

Various horospheres $X_t \subset X$, $t \in \mathbb{R}$, centered at a point $\infty \in \partial X$ can be identified with the nilpotent group $\mathcal{N}$ in the Iwasawa decomposition of $\mathrm{Isom}\, X = \mathcal{KAN}$. In its turn, for $H^n_{\mathbb{F}}$, the nilpotent group $\mathcal{N}$ can be identified with the product $\mathbb{F}^{n-1} \times \mathrm{Im}\, \mathbb{F}$ equipped with the operations:

$$(\xi, v)\cdot(\xi', v') = (\xi+\xi', v+v'+2\,\mathrm{Im}\langle \xi, \xi'\rangle) \quad \text{and} \quad (\xi, v)^{-1} = (-\xi, -v)\,, \tag{2.1}$$

where $\langle , \rangle$ is the standard Hermitian product in $\mathbb{F}^{n-1}$, $\langle z, w\rangle = \sum z_i \overline{w_i}$. The group $\mathcal{N}$ is a 2-step nilpotent Carnot group with center $\{0\}\times \operatorname{Im}\mathbb{F} \subset \mathbb{F}^{n-1}\times \operatorname{Im}\mathbb{F}$, and acts on itself by the left translations $T_h(g) = h\cdot g\,,\quad h, g \in \mathcal{N}$. We may identify $\overline{X}\backslash\{p\} = X\cup\partial X\backslash\{p\} \to \mathcal{N}\times[0,\infty)$, and call this identification the *"upper half-space model"* for our symmetric space X. This and the standard coordinates on the nilpotent Carnot group $\mathcal{N}$ give us *horospherical coordinates* on X, see [AX2].

In horospherical coordinates on $X = H^n_{\mathbb{F}}$, the left action (2.1) of the Carnot group $\mathcal{N}$ on itself extends to an isometric action (Carnot translation) on $X = H^n_{\mathbb{F}} = \mathbb{F}^{n-1}\times \operatorname{Im}\mathbb{F}\times(0,\infty)$ in the following form:

$$T_{(\xi_0,v_0)}\,:\,(\xi,v,u)\longmapsto(\xi_0+\xi\,,v_0+v+2\operatorname{Im}\langle\xi_0,\xi\rangle\,,u)\,, \tag{2.2}$$

where $(\xi, v, u) \in \mathbb{F}^{n-1}\times \operatorname{Im}\mathbb{F}\times[0,\infty)$.

Note that the group $\mathcal{K}_0\mathcal{A}$ fixing the origin of $\mathcal{N}$ and ∞ (or equivalently, fixing the geodesic in $X = H^n_{\mathbb{F}}$ ending at those two points at ∂X) is $\mathrm{U}(n-1)\times\mathbb{R}\times\mathrm{U}(1)$ in $H^n_{\mathbb{C}}$, $\mathrm{Sp}(n-1)\times\mathbb{R}\times\mathrm{Sp}(1)$ in $H^n_{\mathbb{H}}$, and $\mathrm{Spin}(7)\times\mathbb{R}$ in $H^2_{\mathbb{O}}$. The isometry group of $\mathcal{N} = \mathbb{F}^{n-1}\times\operatorname{Im}\mathbb{F}$ is the semidirect product $\mathcal{N}\rtimes C$ where C is the compact automorphism group of $\mathcal{N}$ (Carnot rotations), i.e. $\mathrm{U}(n-1)$ if $\mathbb{F}=\mathbb{C}$, $\mathrm{Sp}(n-1)$ if $\mathbb{F}=\mathbb{H}$, and $\mathrm{Spin}(7)$ if $\mathbb{F}=\mathbb{O}$.

2.2. Geometrical finiteness in rank one symmetric spaces.

Now we consider a discrete group $G\subset \operatorname{Isom}X$ acting by isometries in a symmetric rank one space X, or even in more general Hadamard space with negative pinched sectional curvature. Taking its discontinuity set $\Omega(G)\subset\partial X$, we have the quotient $M(G) = (X\cup\Omega(G))/G$, which plays important role in several equivalent definitions of geometrical finiteness, see [A1, A2, B].

First definition (**GF1**, originally due to A.Beardon and B.Maskit) is in terms of the limit set $\Lambda(G)$. A discrete group $G\subset\operatorname{Isom}X$ is geometrically finite if its limit set $\Lambda(G)$ entirely consists of conical limit points and parabolic cusps. Here a limit point $z\in\Lambda(G)$ is called a conical limit point of G if, for some (and hence every) geodesic ray $\ell\subset X$ ending at z, there is a compact set $K\subset X$ such that $g(\ell)\cap K\neq\emptyset$ for infinitely many elements $g\in G$. A parabolic fixed point $p\in\Lambda(G)$ is called a cusp point of G if the quotient $(\Lambda(G)\setminus\{p\})/G_p$ by the action of its stabilizer $G_p=\{g\in G\,:\,g(p)=p\}$ is compact.

Another definition of geometrical finiteness (**GF2**, originally due to A.Marden) is that the quotient $M(G)$ has only finitely many topological ends and each of them can be identified with the end of $M(\Gamma)$, where Γ is a maximal parabolic subgroup of G.

Two other definitions are originally due to W.Thurston. They use the minimal convex retract $C(G) \subset X/G$, where $C(G)$ is the G-quotient of the minimal convex set in X containing at infinity the limit set $\Lambda(G)$ (the convex hull of $\Lambda(G)$):

(GF3): The thick part of the minimal convex retract (=convex core) $C(G)$ of X/G is compact.

(GF4): For some $\epsilon > 0$, the uniform ϵ-neighborhood of $C(G) \subset X/G$ has a finite volume, and there is a universal bound on the orders of finite subgroups in G.

Theorem 2.1. [B] *Let X be a pinched Hadamard manifold. Then the four definitions* **GF1, GF2, GF3** *and* **GF4** *of geometrical finiteness for a discrete group $G \subset \operatorname{Isom} X$ are all equivalent.*

In the case of homogeneous spaces X, the situation can be greatly clarified by using the following Apanasov-Xie [AX2] geometric definition of parabolic cusp points and the standard cusp ends of $M(G)$. It is based on the existence of "invariant hocycles" for a parabolic subgroup $G_p \subset G$ preserving horospheres $X_p \subset X$ centered at the fixed point $p \in \partial X$, $G_p = \{g \in G : g(p) = p$. Such horospheres X_p are represented by a nilpotent Carnot group $\mathcal{N}$ in Iwasawa decomposition $\operatorname{Isom} X = \mathcal{KAN}$. Assuming that it has a compact group of automorphisms C, we have that $G_p \subset \mathcal{N} \rtimes C$ acts by isometries in $\mathcal{N} \cong \partial X \backslash \{p\}$ and, due to Apanasov-Xie [AX2] theorem, there exists a minimal affine subspace $\mathcal{N}_p \subset \mathcal{N}$ represented (up to a change of the origin) by a connected Lie subgroup of $\mathcal{N}$ and preserved by the parabolic stabilizer G_p acting there co-compactly.

Now for a given $r > 0$ and some left-invariant metric on $\mathcal{N}$ (e.g. Cygan metric ρ_c), we consider G_p-invariant sets $U_{p,r} = \{x \in \partial X \backslash \{p\} : \rho_c(x, \mathcal{N}_p) \geq 1/r\}$ and call them *standard cusp neighborhoods* at p of radius $r > 0$ if they are precisely invariant with respect to $G_p \subset G$, i.e. $\gamma(U_{p,r}) = U_{p,r}$ for $\gamma \in G_p$ and $g(U_{p,r}) \cap U_{p,r} = \emptyset$ for $g \in G \backslash G_p$. Then, due to [AX2], a parabolic fixed point $p \in \partial X$ is a cusp point if and only if it has a standard cusp neighborhood $U_{p,r}$. Clearly, the quotients $\hat{U}_{p,r}/G_p$ of such standard cusp neighborhoods provide a basis of the unique end of the quotient $M(G_p)$. Such an end is called a standard parabolic end of $M(G)$. The family $E_p = \{\hat{U}_{p,r}/G_p\}$ of closed suborbifolds in the orbifold $M(G)$ naturally defines the cusp end of $M(G)$ identified by G-orbit of the parabolic cusp point p. It is isometric to a standard cusp end, actually to the end of the almost nilpotent orbifold $M(G_p)$. Due to Apanasov [A4], such almost nilpotent locally symmetric spaces $M(G_p)$ are vector bundles over the compact infranil base $\mathcal{N}_p/G_p$ and have finite coverings by the product of Euclidean space $\mathbb{R}^k$ and a closed nil-manifold $\mathcal{N}_p/G_p^0$ (which is either a torus or the total space of a torus bundle over a torus). This shows the topological finiteness of cusp ends of non-compact geometrically finite orbifolds $M(G)$ (see [A4]). Therefore geometrically finite groups $G \subset \operatorname{Isom} X$ have finiteness type F_n for all $n \geq 1$ which generalizes the notions of finite generation (F_1) and finite presentation (F_2):

Theorem 2.2. *Let X be a non-compact symmetric space of rank one. Then any geometrically finite discrete group $G \subset \operatorname{Isom} X$ has type F_n for all $n \geq 1$ and in particular, is finitely presented.*

2.3. Some special hyperbolic groups.

Here we review some remarkable constructions of discrete hyperbolic groups.

2.3.1. Jørgensen groups.

First doubly generated geometrically infinite discrete groups of isometries of $H^3_{\mathbb{R}}$ are due to a remarkable construction by Troels Jørgensen [J] of a closed hyperbolic 3-manifold fibered over the circle. Here we shortly describe the construction, see [KAG].

Consider for $m \geq 2$ the numbers $\lambda, \varphi, \rho, x$ and y given by the equalities:

$$\lambda = e^{\frac{\pi i}{2m}}\,,\ 2\varphi = 1+(17-8\cos(\pi/m))^{\frac{1}{2}}\,,\ 2\rho = (\varphi+2)^{\frac{1}{2}}+(\varphi-2)^{\frac{1}{2}}\,,$$
$$2x(\varphi-2)^{\frac{1}{2}} = (3-\varphi)^{\frac{1}{2}}+(-\varphi-1)^{\frac{1}{2}}\,,\ 2y(\varphi-2)^{\frac{1}{2}} = -(3-\varphi)^{\frac{1}{2}}+(-\varphi-1)^{\frac{1}{2}}$$

Let $G \subset \mathrm{SL}(2,\mathbb{C})$ be a discrete group acting by isometries in the half-space model of $H^3_{\mathbb{R}}$ and generated by the matrices:

$$T = \begin{pmatrix} \rho & 0 \\ 0 & \rho^{-1} \end{pmatrix}, \quad A = \begin{pmatrix} -\lambda x & -(1+x^2) \\ 1 & \lambda^{-1}x \end{pmatrix}.$$

Its commutator subgroup $G^* \subset G$ is generated by A and $B = TA^{-1}T^{-1}A$. Consider an Abelian subgroup $\Gamma \subset G$ which acts in the half space $\mathbb{R}^3_+$ by Euclidean similarities, has an invariant axis $\ell = \{x \in \mathbb{R}^3_+ : x_1 = x_2 = 0\}$, and is generated by $K = ABA^{-1}B^{-1}$, which is a rotation by angle $2\pi/m$, and by the dilation T. It creates the following set:

$$I = \{\gamma A \gamma^{-1}\,,\ \gamma B \gamma^{-1}\,,\ \gamma A^{-1}\gamma^{-1}\,,\ \gamma B^{-1}\gamma^{-1}\ : \gamma \in \Gamma\}\,. \tag{2.1}$$

Now we define a convex hyperbolic polyhedron $Q \subset H^3_{\mathbb{R}}$ as the intersection of hyperbolic half spaces bounded by hyperbolic planes whose boundaries at infinity $\mathbb{C} \cup \{\infty\} = \partial H^3_{\mathbb{R}}$ are represented by isometric circles $\{z \in \mathbb{C} :$

$|g'(z)| = 1\}$ of all elements $g \in I$. The polyhedron Q looks like an infinite cone in the half space with vertices at 0 and∞ whose facets are isometric hyperbolic hexagons paired by hyperbolic isometries from the set I. The group Γ acts on the boundary of Q as a group of automorphisms. Intersection of Q with the dihedral angle of magnitude $2\pi/m$ (which is fundamental polyhedron for the finite cyclic group $< K >$) is a fundamental polyhedron of the doubly generated group G^*. This shows that G^* is a geometrically infinite normal subgroup of the co-compact hyperbolic group G, with infinite Abelian quotient $G/G^* = \Gamma \cong \mathbb{Z} \oplus \mathbb{Z}_n$ satisfying the exact sequence (1.1) and with the limit set $\Lambda(G^*) = \Lambda(G) = \partial H^3_{\mathbb{R}} = \mathbb{C} \cup \{\infty\}$.

Due to Selberg Lemma [A1], there is a torsion free normal subgroup of finite index $N \lhd G$ which gives a normal subgroup of finite index $N^* = N \cap G^* \lhd G^*$. For some integer $k \neq 0$, we take $C = T^k \in N$ and denote $N^*(C) =< N^*, C >$. Then N^* is a normal subgroup of $N^*(C)$ and $N^*(C)/N^* \cong \mathbb{Z}$, and we have (1.1) with infinite cyclic Abelian group. In this way one has a closed hyperbolic 3-manifold $H^3_{\mathbb{R}}/N^*(C)$ fiber over the circle with the fibers homeomorphic to a closed hyperbolic surface S, $\pi_1(S) \cong N^*$.

2.3.2. Infinite index normal subgroups with free Abelian quotients of rank ≥ 2.

Given a geometrically finite group in $H^n_{\mathbb{F}}$ with parabolic subgroups (in particular a non-uniform lattice), one can find its finite index subgroup $G \subset \mathrm{Isom}\, H^n_{\mathbb{F}}$ such that each maximal parabolic subgroup $G_p \subset G$ is either free Abelian or 2-step nilpotent. This is due to the author's Theorem 6.1 in [AX1]. In particular, parabolic subgroups G_p in each such geometrically finite group $G \subset \mathrm{Isom}\, H^n_{\mathbb{R}}$ in the real hyperbolic space $H^n_{\mathbb{R}}$ are free Abelian of some rank k, with precisely G_p-invariant horoballs $B_p \subset H^n_{\mathbb{R}}$ centered at the parabolic fixed point $p \in \partial H^n_{\mathbb{R}}$. Let $P \subset H^n_{\mathbb{R}}$ be a fundamental polyhedron of G having the vertex p at infinity. Then G_p acts by automorphisms on the boundary of the polyhedron $P^* = \cup_{g \in G_p} g(P)$ whose sides are pairwise identified by elements $g \in G \backslash G_p$. There are several cases when $G \backslash G_p$ is a subgroup of G, see [A1]. Such subgroups can be taken as normal subgroups $G^* \subset G$ in (1.1), with the free Abelian quotient $G/G^* = G_p$ and with the limit set $\Lambda(G^*) = \Lambda(G)$. In the case of a non-uniform lattice $G \subset \mathrm{Isom}\, H^n_{\mathbb{F}}$, this limit set is the whole sphere at infinity $\partial_\infty H^n_{\mathbb{F}}$.

2.3.4. Groups of hyperbolic homology 4-cobordisms with wild spherical limit set.

Let $G \subset \mathrm{Isom}\, H^4_{\mathbb{R}}$ be a convex co-compact group which uniformizes a non-trivial homology cobordism, that is a triple $(M; N_0, N_1)$ where M is a compact manifold whose interior has a complete hyperbolic structure, $\operatorname{int} M = H^4_{\mathbb{R}}/G$, and whose boundary ∂M consists of two disjoint non-homeomorphic connected closed 3-manifolds N_0 and N_1, $\partial M = N_0 \cup N_1$, and the relative homology groups are trivial, $H_*(M, N_0) = H_*(M, N_1) = 0$. The discontinuity set $\Omega(G) \subset \partial H^4_{\mathbb{R}} = S^3$ of G has two invariant components Ω_0 (a wildly embedded 3-ball in S^3) and $\Omega_1 = S^3 \backslash \overline{\Omega_0}$, and $N_i = \Omega_i/G$. The common boundary of these components is the limit set $\Lambda(G)$ which is homeomorphic to 2-sphere S^2 and is wildly knotted on a dense subset. Such groups and their non-trivial homology cobordisms were discovered by Apanasov-Tetenov [AT], see also [A1, A3]. Each of these groups was constructed as an HNN-extension of a convex co-compact finitely generated reflection group $G_0 \subset \operatorname{Isom} H^4_{\mathbb{R}}$ by a hyperbolic translation h conjugating two Fuchsian subgroups $\Gamma, \Gamma' \subset G$, $G = \langle G_0, h : h\Gamma h^{-1} = \Gamma' \rangle$. Moreover, let Σ be the set of h^k-images, $k \in \mathbb{Z}$, of the hyperbolic 3-planes which define reflections generating the reflection group G_0. Then they bound a convex hyperbolic polyhedron $P_0 \subset H^4_{\mathbb{R}}$ which is a fundamental polyhedron for an infinitely generated reflection group G^* whose generators are reflections with respect to planes $H \in \Sigma$. Clearly, the group G^* is a normal subgroup in G, $\Lambda(G^*) = \Lambda(G)$, its quotient G/G^* is the cyclic group $< h >$ which acts by automorphisms on the boundary ∂P_0, and these groups satisfy the short exact sequence (1.1).

Though the constructed groups G and $G^* \triangleleft G$ have finite order elements, the same procedure as in Section 2.3.1 gives torsion free groups with the same properties.

2.3.5. Bowditch-Mess hyperbolic groups.

Now we present another exotic 4-dimensional hyperbolic group G^* which is finitely generated normal subgroup of a geometrically finite (convex co-compact) group $G \subset \operatorname{Isom} H^4_{\mathbb{R}}$ with cyclic quotient G/G^* and satisfying the short exact sequence (1.1). Moreover, these groups are subgroups of a co-compact lattice in $\operatorname{Isom} H^4_{\mathbb{R}}$ generated by reflections in the faces of the regular hyperbolic 120-cell in $H^4_{\mathbb{R}}$ whose all dihedral angles are $\pi/2$. This group G^* is geometrically infinite and is not finitely presented. Also it has another anomaly that its action on the discontinuity set $\Omega(G^*) = \Omega(G)$ has the quotient manifold $\Omega(G^*)/G^*$ of infinite topological type. This construction is due to Bowditch and Mess [BM]; see also [A1] for other constructions of exotic subgroups of geometrically finite groups with similar "non-finiteness", including such finitely generated subgroups G^* with infinitely many conjugacy classes of Abelian rank two subgroups, as well as finitely generated subgroups G^* which are not groups of type F_n, see Remark 5.52 in [A1].

To define these groups, we start with a regular dodecahedron in $H^3_{\mathbb{R}} \subset H^4_{\mathbb{R}}$

whose all dihedral angles are $\pi/2$. It defines a hyperbolic orbifold which can be described by the following Thurston's construction. One represents this dodecahedron combinatorically as an Euclidean cube with six additional disjoint edges splitting cube faces into equal rectangles. Identifying the opposite faces of the cube so as to form a 3-torus, we convert these additional edges into three disjoint embedded circles. Our hyperbolic orbifold is defined by assigning to each of these circles a transverse cone angle equal to π. It is important to note that this orbifold fibres over the circle (given by the long diagonal of the cube). Applying the Selberg Lemma, we obtain a closed hyperbolic 3-manifold fibering over the circle and tiled by dodecahedra. It contains an immersed totally geodesic surface formed as a union of pentagonal faces. Considering a finite covering we may assume that we have such a closed hyperbolic manifold M_0 with totally geodesic 2-side surface S_0 (with a wide uniform regular neighborhood in M_0). Taking the infinite cyclic covering M of M_0 corresponding to the fibre subgroup of $\pi_1(M_0)$, we obtain a component S of the inverse image of S_0 under the covering projection. This surface $S \subset M$ is totally geodesic, has infinite topological type, and has the uniform regular neighborhood in M.

Now for $I = [-1, 1]$ and $\pi_1(M) \cong \Gamma \subset \operatorname{Isom} H^3_{\mathbb{R}} \operatorname{Isom} H^4_{\mathbb{R}}$, we realize the product $M \times I$ as a hyperbolic 4-manifold with convex boundary such that $M \times \{0\}$ is universally covered by a hyperbolic 3-space $H^3_{\mathbb{R}} \subset H^4_{\mathbb{R}}$ whose universal regular neighborhood is Γ-invariant. Take another such copy of $M \times I$ and superimpose these two copies such that the two copies of $M \times \{0\}$ sitting inside overlap orthogonally along the surface S. Identifying the superimposed pieces of $M \times I$ (after a rotation of the square, $I \times I$, through an angle of $\pi/2$), we arrive with a hyperbolic 4-manifold with boundary. It is possible to make proper choices of regular neighborhoods to smooth out the boundary locally and make it convex. It is possible to embed the 4-manifold thus obtained, as a deformation retract, inside a complete hyperbolic 4-manifold without boundary, M^4. So we have that the group of covering transformation of M^4 is our desired group $G^* \cong \Gamma *_{\pi_1(S)} \Gamma$ acting by isometries in $H^4_{\mathbb{R}}$. The cyclic extension of this group corresponding to fibering over the circle is our geometrically finite group G containing G^*. The fact that this finitely generated group G^* is not finitely presented follows from a result of B.H. Neumann [N]. It says that if the amalgamated free product $A *_C B$ of finitely generated groups A and B is finitely presented, then C is finitely generated.

3. The basic technique for constructing examples

Here we will describe the basic twisting technique for creating desired intersections of two geometrically finite groups of isometries of a symmetric rank one space $X = H^n_{\mathbb{F}}$, that is a hyperbolic space over either real, or complex, or quaternionic, or Cayley numbers $\mathbb{F}$.

We start with a pair of groups G and $G^* \triangleleft G$ which satisfy the short exact sequence (1.1), $1 \to G^* \to G \to \mathbb{A} \to 1$. In addition we can assume that G is a geometrically finite subgroup of isometries a totally geodesic subspace $Y \subset X$ with the induced metric, $G \subset \operatorname{Isom} Y \hookrightarrow \operatorname{Isom} X$ where $\hookrightarrow$ is the natural inclusion, while its normal subgroup G^* fails to have some finiteness condition. Then the normality of the infinite index subgroup $G^* \triangleleft G$ implies that these groups have the same limit set $\Lambda(G) = \Lambda(G^*) \subset \partial Y$.

Now let $\phi : G \to \mathbb{A}$ be the epimorphism in the short exact sequence above, where our Abelian group $\mathbb{A} = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_m}$ has a free Abelian subgroup of rank k, $k \geq 1$, and maybe some finite subgroups, $m \geq 0$.

In order to define the second geometrically finite group $G_2 \subset \operatorname{Isom} X$ we first assume that there is a continuous (compact) subgroup $\mathcal{U} \subset \operatorname{Isom} X$ which consists of rotations of the space X pointwise fixing the totally geodesic subspace $Y \subset X$.

For example, in the case of real hyperbolic geometry, $X = H^n_{\mathbb{R}}$, we may have Y as any subspace $H^l_{\mathbb{R}} \subset H^n_{\mathbb{R}}$, $l \leq n-2$. In this case, the compact group $\mathcal{U}$ of rotations around Y is the orthogonal group $\mathrm{SO}(n-l)$.

In the complex hyperbolic geometry, $X = H^n_{\mathbb{C}}$, we may have Y as a complex l-dimensional subspace, $l \leq n-1$. Then the rotation group $\mathcal{U}$ is the unitary group $\mathrm{U}(n-l)$. However the geodesic subspace $Y \subset H^n_{\mathbb{C}}$ may be totally real as well. Then the induced metric on Y has constant negative curvature and $Y = H^l_{\mathbb{R}}$, $l \leq n-1$. In this case compact group $\mathcal{U} = \mathrm{U}(n-l)$, that is its elements pointwise fix the whole complex l-dimensional subspace containing Y.

In quaternionic hyperbolic geometry, we have the following cases for $Y \subset H^n_{\mathbb{H}}$:

(1) Y is a quaternionic l-dimensional subspace $H^l_{\mathbb{H}}$, $l \leq n-1$. Then $\mathcal{U}$ is a subgroup of $\mathrm{Sp}(n-l)$.
(2) Y is a totally real subspace $H^l_{\mathbb{R}} \subset X = H^n_{\mathbb{H}}$, $l \leq n-1$, and $\mathcal{U} \subset \mathrm{Sp}(n-l)$, or
(3) Y is a totally geodesic subspace $H^l_{\mathbb{R}} \subset H^4_{\mathbb{R}} \cong H^1_{\mathbb{H}} \subset X = H^n_{\mathbb{H}}$, and $\mathcal{U} \subset \mathrm{Sp}(n-1)\,\mathrm{SO}(4-l)$.

Similarly, in the Cayley hyperbolic plane, if we have our totally geodesic subspace Y as a subspace of an octonionic line, $Y = H^l_{\mathbb{R}} \subset H^8_{\mathbb{R}} \cong H^1_{\mathbb{O}} \subset X = H^2_{\mathbb{O}}$, then we can rotate around Y inside of this $\mathbb{O}$-line and around it. In fact if $L_1 = \mathbb{O} \times \{0\}$ and $L_2 = \{0\} \times \mathbb{O}$ denote the coordinate $\mathbb{O}$-axes, then the stabilizer of L_1 in $\mathrm{Spin}(9)$ acts on L_1 as $\mathrm{SO}(8)$ via the even $\frac{1}{2}$-spin representation, and on L_2 as odd $\frac{1}{2}$-spin representation.

Now we have two cases. The Abelian group $\mathbb{A} = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_m}$ in the short exact sequence (1.1) either has at most one finite summand, $0 \le m \le 1$, or $m \ge 2$. Let $k \ge 1$ be the rank of the maximal free Abelian subgroup in $\mathbb{A}$.

In the first case, we pick k non-trivial elements $\alpha_1, \ldots, \alpha_k \in \mathcal{U}$ rotating the space X around the totally geodesic (pointwise fixed) subspace Y by transcendentally independent irrational angles. Also, if $m = 1$, we consider additional rotation around Y, $\alpha_{k+1} \in \mathcal{U}$ of a finite order k_1.

Note that all these rotations α_i fix the subspace Y pointwise and commute with the group $G \subset \operatorname{Isom} Y \hookrightarrow \operatorname{Isom} X$ as well as with each other. Given $g \in G$ and its image $\phi(g) = (n_1(g), \ldots, n_k(g), n_{k+1}(g)) \in \mathbb{A}$, we compose this element g with all rotations $\alpha_i^{n_i(g)}$. The resulting elements generate the group G_2. Note that since all our rotations α_i commute with G, and the composition of them is trivial if and only if $\phi(g) = 0$, the action of the group G^2 on the subspace Y is identical to the action of the initial group G, and the group G_2 is isomorphic to G. Therefore, the group G_2 contains G^* as its normal subgroup, and $G \cap G_2 \equiv G^*$. Since dynamics of the groups G and G_2 on the subspace Y are identical, Theorem 2.1 implies that the second group G_2 is geometrically finite.

In the second case of several finite summands in $\mathbb{A}$, the only difference is that one needs a bigger rotation group $\mathcal{U}$ for a faithful representation $\mathbb{A} \to \mathcal{U}$.

This technique supplies us with a large collection of pairs of geometrically finite groups whose intersection is not geometrically finite. Here are some specific examples.

Example 3.1 - Infinitely generated subgroups of surface groups. Let G be a finitely generated free group or closed surface group which acts by isometries on the hyperbolic plane $H^2_{\mathbb{R}}$, and let $\phi : G \to \mathbb{Z}$ be an epimorphism. It is easy to see that the kernel G^* of ϕ is not finitely generated. Therefore the construction above gives a pair of finitely generated (and geometrically finite) groups of isometries $H^4_{\mathbb{R}}$ which have infinitely generated intersection G^* preserving the initial hyperbolic plane $H^2_{\mathbb{R}} \subset H^4_{\mathbb{R}}$.

Example 3.2 - Double generated geometrically infinite subgroups. Let $G \subset \operatorname{Isom} H^3_{\mathbb{R}}$ be a double generated Jørgensen group from Section 2.3.1 which may have torsion or be torsion free. It acts co-compactly in a totally geodesic subspace Y (of real dimension 3) of a symmetric rank 1 space X. This space X may be either $H^k_{\mathbb{R}}$, $k \ge 5$, or $H^m_{\mathbb{C}}$, $m \ge 4$, where Y is a totally real subspace, or $H^l_{\mathbb{H}}$, $l \ge 2$, where Y is either a totally real subspace or a part of a quaternionic geodesic, or $H^n_{\mathbb{O}}$, $l \ge 1$, with Y as a part of an octonionic geodesic. In all these cases the group G isometrically acts in X as a geometrically finite group (in fact, convex co-compact, so Gromov hyperbolic), and one has a continuous subgroup $\mathcal{U} \subset \mathrm{Isom}\, X$ of rotations around the pointwise fixed subspace Y. So we can apply our twisting technique and create the second geometrically finite group $G_2 \subset \mathrm{Isom}\, X$ whose intersection with the group G is the double generated and geometrically infinite subgroup $G^* \subset \mathrm{Isom}\, H^3_{\mathbb{R}} \hookrightarrow \mathrm{Isom}\, X$ described in 2.3.1. One may have both cases of G^* with torsion or torsion free G^*.

Example 3.3 - Infinitely generated subgroups with limit k-spheres.

Let $G \subset \mathrm{Isom}\, Y$, $Y = H^n_{\mathbb{F}}$, be a geometrically finite group whose maximal parabolic subgroups $G_p \subset G$ are free Abelian. For example, G may be a finite index subgroup in a non-uniform lattice in real hyperbolic space Y, or G may be a geometrically finite group in the complex hyperbolic space, see the author's Theorem 6.1 in [AX1]. As we see in Section 2.3.2, taking a parabolic cusp point p of the group G with free Abelian stabilizer $G_p \subset G$, one may find a normal subgroup $G^* \subset G$, $G/G^* = G_p$, whose (infinitely many) generators are conjugations $\gamma g \gamma^{-1}$, $\gamma \in G_p$, of "finite" generators $g \in G \backslash G_p$ of the group G (they pairwise identify sides of a fundamental polyhedron $P \subset H^n_{\mathbb{F}}$ whose extensions do not contain the cusp p). Such infinite generated normal subgroups G^* satisfy (1.1) with free Abelian group $\mathbb{A} \cong G_p$, and have limit sets $\Lambda(G^*) = \Lambda(G)$. In the case of a non-uniform lattice $G \subset \mathrm{Isom}\, H^n_{\mathbb{R}}$, this limit set is the whole $(n-1)$-sphere at infinity $\partial_\infty H^n_{\mathbb{R}}$. Considering Y as a totally geodesic subspace in a symmetric rank 1 space X with a continuous group $\mathcal{U} \subset \mathrm{Isom}\, X$ of rotations pointwise fixing Y, one may consider the group G as a geometrically finite subgroup in $\mathrm{Isom}\, X$. So one can apply our twisting technique and create the second geometrically finite group $G_2 \subset \mathrm{Isom}\, X$ whose intersection with the group G is the infinitely generated normal subgroup G^*. In the case of a non-uniform lattice $G \subset \mathrm{Isom}\, Y$, $Y = H^n_{\mathbb{R}}$, $n \geq 2$, it has the $(n-1)$-sphere at infinity $\partial H^n_{\mathbb{R}}$ as the limit set $\Lambda(G^*)$. In this case, the ambient space X may be either $H^{n+2}_{\mathbb{R}}$, or $H^{n+1}_{\mathbb{C}}$, or $H^{k+1}_{\mathbb{H}}$ where $k = 1$ if $n \leq 4$ and $k = n$ if $n > 4$, or $H^2_{\mathbb{O}}$ if $n \leq 8$.

Example 3.4 - Infinitely generated subgroups with wild limit spheres. Let $G \subset \mathrm{Isom}\, H^4_{\mathbb{R}}$ be a finitely generated Apanasov-Tetenov group from Section 2.3.4 related to a non-trivial hyperbolic homology cobordism which may have torsion or be torsion free. It acts co-compactly in a totally geodesic subspace Y (of real dimension 4) of a symmetric rank 1 space X. This space X may be either $H^k_{\mathbb{R}}$, $k \geq 6$, or $H^m_{\mathbb{C}}$, $m \geq 5$, where Y is a totally real subspace, or $H^l_{\mathbb{H}}$, $l \geq 2$, where Y is either a totally real subspace or a quaternionic geodesic, or $H^n_{\mathbb{O}}$, $l \geq 1$, with Y as a part of an octonionic geodesic. In all these cases the group G isometrically acts in X as a geometrically finite group (in fact, convex co-compact), and one has a continuous subgroup $\mathcal{U} \subset \operatorname{Isom} X$ of rotations around the pointwise fixed subspace Y. So again we can apply our twisting technique and create the second geometrically finite group $G_2 \subset \operatorname{Isom} X$ whose intersection with the group G is the geometrically infinite (and infinitely generated) subgroup $G^* \subset \operatorname{Isom} H^3_{\mathbb{R}} \hookrightarrow \operatorname{Isom} X$ described in 2.3.4 and whose limit set is a wildly embedded 2-sphere in the 3-sphere $S^3 = \partial Y$. One may have both cases of G^* with torsion or torsion free G^*.

Example 3.5 - Finitely generated but not finitely presented subgroups.

Let G^* be one of Bowditch-Mess finitely generated normal subgroups of a geometrically finite (convex co-compact) group $G \subset \operatorname{Isom} H^4_{\mathbb{R}}$ of isometries on the real 4-dimensional hyperbolic space $H^4_{\mathbb{R}}$ presented in Section 2.3.5. The group G is a Gromov hyperbolic group with cyclic quotient $G/G^* = \mathbb{Z}$ and satisfying the short exact sequence (1.1). Its normal subgroup G^* is finitely generated (geometrically infinite) but not finitely presented. Moreover, these groups are subgroups of a co-compact lattice in $\operatorname{Isom} H^4_{\mathbb{R}}$ generated by reflections in the faces of the regular hyperbolic 120-cell in $H^4_{\mathbb{R}}$ whose all dihedral angles are $\pi/2$.

Considering $H^4_{\mathbb{R}}$ as a totally geodesic subspace Y (of real dimension 4) of a symmetric rank 1 space X, one may canonically extend the co-compact G-action to its isometric action in X. This space X may be either $H^k_{\mathbb{R}}$, $k \geq 6$; or $H^m_{\mathbb{C}}$, $m \geq 5$, where Y is a totally real subspace; or $H^l_{\mathbb{H}}$, where Y is either a quaternionic geodesic and $l \geq 2$, or a totally real subspace (then $l \geq 5$); or $H^k_{\mathbb{O}}$, $k \geq 1$, with Y as a part of an octonionic geodesic. In all these cases the group G isometrically acts in X as a geometrically finite group, and one has a continuous subgroup $\mathcal{U} \subset \operatorname{Isom} X$ of rotations around the pointwise fixed subspace Y. So we can apply our twisting technique and create the second geometrically finite group $G_2 \subset \operatorname{Isom} X$ whose action on the subspace $H^4_{\mathbb{R}} = Y \subset X$ coincides with the G-action, and whose intersection with the group G is the finitely generated but not finitely presented subgroup $G^* \subset \operatorname{Isom} H^4_{\mathbb{R}} \hookrightarrow \operatorname{Isom} X$ described above in 2.3.5.

Remark 3.6.

The above series of constructions of intersections of geometrically finite groups in negatively curved symmetric rank one spaces without some finiteness properties can be continued. As one of the possibilities we note constructions (see §5.6 in [A1]) of finitely generated normal subgroups G^* of geometrically finite groups of real hyperbolic isometries which may have infinitely many conjugacy classes of Abelian rank two subgroups, as well as finitely generated subgroups G^* which are not groups of type F_n, see Remark 5.52 in [A1].

REFERENCES

[An] James Anderson, The limit set intersection theorem for finitely generated Kleinian groups. - Math. Res. Lett. **3**(1996), 675-692.

[A1] Boris Apanasov, Conformal Geometry of Discrete Groups and Manifolds. - De Gruyter Exp. in Math., W. de Gruyter, Berlin-New York, 2000, XII + 529 pp.

[A2] _____, Geometrically finite hyperbolic structures on manifolds. - Ann. of Glob. Analysis and Geom., **1:3** (1983), 1-22.

[A3] _____, Nonstandard uniformized conformal structures on hyperbolic manifolds. - Invent. Math. **105** (1991), 137-152.

[A4] _____, Almost nilpotent manifolds with pinched negative curvature. - Centre de Recerca Matematica, **601**, Bellaterra, 2004, 1-17.

[AT] Boris Apanasov and Andrew Tetenov, Nontrivial cobordisms with geometrically finite hyperbolic structures. - J. of Diff. Geom. **28** (1988), 407-422.

[AX1] Boris Apanasov and Xiangdong Xie, Geometrically finite complex hyperbolic manifolds. - Intern. J. Math. **8** (1997), 703-757.

[AX2] _____, Discrete actions on nilpotent groups and negatively curved spaces. - J. Diff. Geometry Appl., **20**(2004), 11-29.

[Bi] Robert Bieri, Finiteness length and connectivity length for groups.- Intern. Conf. on Geometric Group Theory (Canberra, 1996), W. de Gruyter, Berlin-New York, 1999, 9-22.

[B] Brian Bowditch, Geometrical finiteness with variable negative curvature. - Duke J. Math. **77** (1995), 229-274.

[BM] Brian Bowditch and Geoffrey Mess, A 4-dimensional Kleinian group. - Trans. Amer. Math. Soc. **344** (1994), 391-405.

[G1] M. Gromov, Hyperbolic groups. - Essays in group theory, Springer Verlag, 1987.

[G2] _____, Asymptotic invariants of infinite groups.- Geometric Group Theory, vol. 2 (LMS lecture Note Series, **181**), 1993.

[H] A.G. Howson, On the intersection of finitely generated free groups. - J. Lond. Math. Soc., **29** (1954), 428-434.

[J] Troels Jørgensen, Compact 3-manifolds of constant negative curvature fibering over the circle. - Ann. of Math., **106** (1977), 61-72.

[KAG] S.L. Krushkal', B.N. Apanasov and N.A. Gusevskii, Kleinian Groups and Uniformization in Examples and Problems. - "Nauka" Acad. Publ., Novosibirsk, 1981 (Russian); Extended Engl. Transl.: Transl. Math. Monographs **62**, Amer. Math. Soc., Providence, 1986.

[N] B.H. Neumann, Some remarks on infinite groups. - J. London Math. Soc. **12** (1937), 120–127.

[S] John Stallings, On fibering certain 3-manifolds. - Topology of 3-manifolds and related topics (Proc. The Univ. of Georgia Institute, 1961), Prentice-Hall, 1962, 95-100.

[Su1] Perry Susskind, Kleinian groups with intersecting limit sets.- J.d'Anal.Math., **52**(1989), 26-38.

[Su2] _____, An infinitely generated intersection of geometrically finite hyperbolic groups. - Proc. Amer. Math. Soc., **129** (2001), 2643-2646.

[SS] Perry Susskind and Gadde A. Swarup, Limit sets of geometrically finite hyperbolic groups. - Amer. J. Math. **114** (1992), 233-250.

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