QUASI-POTENCY AND CYCLIC SUBGROUP SEPARABILITY

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Abstract. We study the profinite topology on discrete groups and in particular the property of cyclic subgroup separability. We investigate the class of quasi-potent, cyclic subgroup separable groups, producing many examples and showing how it behaves with respect to certain group constructions.

Introduction

The profinite topology on a discrete group G has, as a basis neighbourhood of 1, the finite index subgroups. We then say that G is *residually finite* if the trivial subgroup is closed in this topology. Similarly, we say that G is *subgroup separable* if every finitely generated subgroup of G is closed and *cyclic subgroup separable* if every cyclic subgroup is closed.

Another way of defining separability is to say that a subgroup H of G is separable (or closed in the profinite topology) if for any $g \in G - H$ there exists a finite quotient of G in which the images of H and g are disjoint. Thus it is clear that residual finiteness gives a solution to the word problem for G and that any separable subgroup of G has solvable membership problem.

However, subgroup separability is a very strong condition that is satisfied by very few classes of groups. For instance, it is well known that $F_2 \times F_2$ has subgroups with unsolvable membership problem so they cannot be subgroup separable. There are even examples of 3-manifold groups and knot groups which are not subgroup separable (see [4] and [12]). Hence it seems reasonable to ask whether groups are separable with respect to a certain class of subgroups, and here we shall be concerned with the case of cyclic subgroups and cyclic separability.

It turns out to be useful to impose the further restriction of *quasi-potency* (defined below) which allows one to construct finite quotients in some regular way. We thus define the class $\mathcal{C}_n$ to consist of those cyclic subgroup separable groups which are also quasi-potent. It is fairly easy to show that

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this class contains all finitely generated virtually free and virtually nilpotent groups. Moreover, this class is closed under taking amalgamated free products over virtually cyclic groups (Theorem 3.6).

In fact, this class contains many examples and in general behaves well with respect to group constructions. For instance, it is fairly straightforward to show that the example shown in [4], of a 3-manifold group which is not subgroup separable,

$$\langle a, b, t \ : \ [a, b], t^{-1}atb \rangle,$$

lies in our class $\mathcal{C}_n$.

The class also contains all right-angled Artin groups (Theorem 3.8), all finitely generated residually free groups (Corollary 4.2) and all polycyclic groups (Corollary 5.5). It is also closed under taking direct sums and under certain extensions (Corollary 4.3 and Theorem 5.1). For instance, finitely generated free-by-free or surface-by-free groups are in the class $\mathcal{C}_n$. One consequence of this is that, since there are examples of free-by-free groups with unsolvable conjugacy problem (see [10]), the class $\mathcal{C}_n$ contains groups which are not *conjugacy separable*. That is, they do not all have the property that conjugacy classes may be distinguished in finite quotients.

1. Notations and Definitions

Throughout the paper, we will write $H \leq G$ to indicate that H is a subgroup of G, $H \trianglelefteq G$ to indicate that H is a normal subgroup of G and $H \,\mathrm{ch}\, G$ to indicate it is a characteristic subgroup. Additionally, we write $H \leq_f G$, $H \trianglelefteq_f G$ and $H \,\mathrm{ch}_f\, G$ to denote that H is, respectively a finite-index subgroup, a finite-index normal subgroup and a finite-index characteristic subgroup of G.

The properties under study in this paper will be all strengthenings of the notion of residual finiteness. Recall that a group G is called residually finite when for every element $g \neq 1$ there exists a finite-index, normal subgroup $H \trianglelefteq_f G$ such that $g \notin H$, i.e. g has nontrivial image in G/H. One can specialize this definition by requiring certain values of the orders or indices for the objects involved in the definition of residual finiteness.

One such specialisation is the condition of *potency*. For a group to be potent, there must be finite quotients where a given non-torsion element has arbitrary order.

Definition 1.1. *A group G is called* potent *when for every element $g \neq 1$ there exists a family of finite-index, normal subgroups $H_n \trianglelefteq_f G$ such that $\langle g \rangle \cap H_n = \langle g^n \rangle$, i.e. if g has infinite order, the image of g in the quotient G/H_n has order n.*

This notion can be generalised by requiring that only some orders need to be realised, namely, those which are multiple of a fixed number.

Definition 1.2. *A group G is called* quasi-potent *when for every element g there exist a positive integer k_g, which depends on g, and a family of finite-index, normal subgroups $H_n \trianglelefteq_f G$ such that $\langle g\rangle \cap H_n = \langle g^{nk_g}\rangle$, i.e. the image of g in the quotient G/H_n has order nk_g.*

The concept of quasi-potency for a group G admits the following equivalent characterisation, which is sometimes more useful:

Remark 1.3. *A group G is quasi-potent if and only if the following two conditions hold:*

- *G is residually finite.*
- *For each element g of infinite order, there exist a positive integer k_g, which depends on g, and a family of finite quotients Q_n of G such that g has order nk_g in Q_n.*

Clearly, a potent group is quasi-potent, and both are residually finite.

All these definitions can be modified by using characteristic subgroups instead of merely normal ones. Recall that a subgroup $H \leq G$ is called characteristic if it is invariant under any automorphism of G. Clearly, every characteristic subgroup is normal. In this manner, we can define groups which are characteristically residually finite, characteristically potent and characteristically quasi-potent if the finite-index subgroups are required to be characteristic.

Observe that if a group G is finitely generated, then G is characteristically residually finite if and only if it is residually finite, since one can construct a finite-index characteristic subgroup by taking the intersection of all automorphic images of a given finite-index normal subgroup. For arbitrary groups this is no longer true: The free abelian group of countably infinite rank is residually finite but it has no characteristic subgroups of finite index. The conditions of potency and quasi-potency and their characteristic counterparts are not related since any construction, such as the one above, changes the orders of elements in the quotients in complicated ways.

Another strengthening is obtained when the quotient is required to be a p-group for a given prime number p. Hence, a group is called *residually a finite p-group* if, for every element $g \in G$, there exists a finite-index, normal subgroup H such that $g \notin H$, and G/H is a p-group.

Recall that the profinite topology of a group is the topology that has as a neighbourbood basis of the identity all subgroups of finite index. Namely, a basis for the topology is given by all cosets of all finite-index subgroups.

Residual finiteness admits an easy interpretation using the profinite topology: a group is residually finite if and only if its profinite topology is Hausdorff, or, equivalently, if the subgroup $\{1\}$ is closed in the profinite topology. Analogously the notion of being residually a finite p-group can be read in terms of the pro-p topology. The notions of potency and quasi-potency, since they involve orders of elements in the quotient, are not easily described in terms of the profinite and pro-p topologies. The notion of residual finiteness has received considerable attention as have its generalisations to subgroup and cyclic subgroup separability, which is the subject of this paper.

Definition 1.4. *A group G is called* subgroup separable *(also referred to as* LERF*), if every finitely generated subgroup is closed in the profinite topology. The group G is called* cyclic subgroup separable *if every cyclic subgroup of G is closed in the profinite topology.*

We note that a subgroup $K \leq G$ is closed in the profinite topology if for every element $g \in G - K$ one can find a finite-index subgroup H such that $K \subset H$ and $g \notin H$. Equivalently, $gH \cap K = \varnothing$. Or, to put it yet in another way, elements lying outside a closed subgroup can be separated from the subgroup in some finite quotient.

The next proposition is straightforward but we spell it out because of its usefulness.

Proposition 1.5. *Let H be a subgroup of a group G.*

- *If $G_0 \leq_f G$, then H is separable in G if and only if $H \cap G_0$ is separable in G_0.*
- *If $H_0 \leq_f H$, then if H_0 is separable in G, so is H.*

2. The classes $\mathcal{C}_n$ and $\mathcal{C}_c$

Definition 2.1. *The class $\mathcal{C}_n$ consists of all groups which are quasi-potent and whose cyclic subgroups are separable. The class $\mathcal{C}_n$ admits as a subclass the class $\mathcal{C}_c$ of groups which are characteristically quasi-potent and also whose cyclic subgroups are separable.*

Clearly, due to the quasi-potency condition, both classes $\mathcal{C}_n$ and $\mathcal{C}_c$ are subclasses of the class of residually finite groups. It is also clear that the class $\mathcal{C}_n$ is closed under taking subgroups, though the same is not apparent for $\mathcal{C}_c$.

It is our goal to prove that some known families of groups belong to the classes $\mathcal{C}_n$ and $\mathcal{C}_c$. The main tool to accomplish this fact will be the following result:

Proposition 2.2. *Let G be a finitely generated group which is residually a finite p-group for every prime number p. Then, G is characteristically potent.*

Before we begin the proof, we will need some lemmas.

Lemma 2.3. *A finitely generated solvable group of finite exponent is finite.*

Proof. We argue by induction on the derived length of our solvable group, G. If G has derived length 1, then G is abelian and the result is clear, so we shall assume that G has derived length n and the lemma holds for all solvable groups of derived length at most $n-1$. Then G/G' is a finitely generated abelian group of finite exponent and is therefore finite. Moreover, G' must also be finitely generated (since it has finite index in G), has finite exponent, and is itself a solvable group of derived length at most $n-1$. By the induction hypothesis, G' and hence G is finite. □

Lemma 2.4. *Let G be a group, K a normal subgroup of G and π the projection from G to G/K. Let $H \leq G$ be a subgroup which contains K. If H/K is a characteristic subgroup of G/K then H is a normal subgroup of G. If K is characteristic, then so is H.*

Lemma 2.5. *Let G be a finite p-group and $g \in G$ an element of order $p^k \geq p$. Then there exists a characteristic subgroup K of G such that gK has order p^{k-1} in G/K.*

Proof. As G has order p^m, we proceed by induction on m, the case $m = 1$ being trivial. G is a finite p-group and hence nilpotent. Thus its center $Z(G)$ is a non-trivial characteristic abelian subgroup of G. The set of elements of order p in $Z(G)$ forms a characteristic subgroup H of $Z(G)$, and hence also of G. Note that gH has order equal to either p^{k-1} or p^k in G/H. If gH has order p^{k-1}, by taking $K = H$ we are done. If gH has order p^k in G/H, then, since G/H has order smaller than p^m, the induction hypothesis applies to find a subgroup K/H in G/H such that the image of gH in G/K has order p^{k-1}. Use Lemma 2.4 to finish the proof. □

Proof of Proposition 2.2. If K_1, K_2 are finite index characteristic subgroups of G then so is $K_1 \cap K_2$. Moreover, the order of $g(K_1 \cap K_2)$ in $G/(K_1 \cap K_2)$ is the least common multiple of the orders of gK_1 and gK_2, since $G/(K_1 \cap K_2)$ is isomorphic to the image of the diagonal subgroup of $G \times G$ in $G/K_1 \times G/K_2$.

Hence, it is sufficient to prove that, given $g \in G$ there exists a characteristic subgroup K such that gK has order p^m in G/K, for all primes p and for all positive integers m.

During our proof we will find a family of characteristic subgroups of finite index of G, such that their intersection will be trivial. Consider the descending central series of G, denoted by $\Gamma_i(G)$, defined recursively by $\Gamma_0(G) = G$ and $\Gamma_i(G) = [\Gamma_{i-1}(G), G]$. Observe that $\Gamma_i(G)$ is a characteristic subgroup of G. Consider also the subgroup G^{p^m}, for a given m, which is the subgroup generated by all the p^m-th powers of the elements of G. Clearly, the subgroup G^{p^m} is characteristic as well. Define the subgroup $G_i = \Gamma_i(G)G^{p^i}$, and observe that, as the product of characteristic subgroups, it is also characteristic. We have that the subgroup G_i has finite index in G, as a consequence of Lemma 2.3. Moreover, G/G_i is a finite p-group.

Since G is residually a finite p-group, and every map from g to a finite p-group factors through some G/G_i, we deduce that

$$\bigcap_i G_i = \{1\}.$$

By Lemmas 2.4 and 2.5, it is sufficient to show that for a given $g \neq 1$ in G, the orders of the element gG_i in G/G_i are unbounded. This can only fail if for some integer k, $g^{p^k} \in G_i$ for all i. But since the intersection of the G_i is trivial, this would imply that $g^{p^k} = 1$, which contradicts the fact that G is torsion-free. □

From these results, one can now deduce that some groups which are residually a finite p-group for all p are in the classes $\mathcal{C}_c$ and $\mathcal{C}_n$, namely, finitely generated free groups and nilpotent groups.

Proposition 2.6. *Let F be a finitely generated free group. Then*

(1) *F is residually a finite p-group for every p.*
(2) *Cyclic subgroups in F are separable*

Proof. These properties are both well known and in fact free groups are subgroup separable. See [7] or [11] for 1. Part 2 is a consequence of Marshall Hall's Theorem, first proved in [6]. □

In the same way, we can deduce that finitely generated nilpotent groups are in these classes:

Proposition 2.7. *Let G be a finitely generated, torsion-free nilpotent group. Then*

(1) *G is residually a finite p-group for every p.*
(2) *Cyclic subgroups of G are separable.*

Proof. These results are elementary. In fact, finitely generated nilpotent groups are subgroup separable. See, for instance, [15], Theorem 4, page 19 and Exercise 11, page 18. □

The following result will allow to extend the result to groups that are virtually free or virtually nilpotent.

Lemma 2.8. *Let G be a finitely generated group, which belongs to the class $\mathcal{C}_c$, and such that any subgroup of finite index $H \leq_f G$ also belongs to the class $\mathcal{C}_c$. Then, a group which is virtually G, that is, a group which has G as a subgroup of finite index, also belongs to $\mathcal{C}_c$.*

Proof. Let K be a group such that $G \leq_f K$. Cyclic subgroup separability and residual finiteness of K are immediate, so it only remains to show that K is characteristically quasi-potent. To do this we note that since G is finitely generated and so is K, there are only finitely many subgroups in K with the same index as G. Their intersection will give a subgroup $G_0 \leq_f G$ which is characteristic in K. By hypothesis, $G_0 \in \mathcal{C}_c$. Suppose now that we have a $g \in K$ of infinite order. Then $g^n \in G_0$, for some n and there exists an integer k_g and a sequence of subgroups $K_n \operatorname{ch}_f G_0$ such that $K_0 \cap \langle g^n \rangle = \langle g^{nk_g} \rangle$. Since the K_n are also characteristic of finite index in K, and the proof is complete. □

3. Invariance under amalgams over virtually cyclic subgroups

The goal for this section is to show that an amalgamated free product of groups in $\mathcal{C}_n$ over a virtually cyclic group is again in $\mathcal{C}_n$. Some of this material has already appeared in [14] and [13], though we present it in order to give a slightly more general result, especially in view of our extension to certain HNN extensions in Theorem 3.7, and in order to present an elementary proof.

We shall make use of the classical reduced form theorem for amalgamated free products. We shall follow the formulation in [5], Theorems 25 and 26.

Theorem 3.1. *Let $G = G_1 *_H G_2$ be an amalgamated free product.*
1. Any $g \in G - H$ can be written as $x_1 \dots x_n$, where the x_i are alternately from $G_1 - H$ and $G_2 - H$. This expression is called a reduced form for g. Such a product can never be in H unless $n = 1$ and $x_1 \in H$. If $g = x_1 \in H$, then x_1 is the reduced form for g.
2. If we can also write $g = y_1 \dots y_m$, with the y_j being alternately from $G_1 - H$ and $G_2 - H$, then $m = n$, $y_1 \in x_1 H$, $y_n \in H g_n$ and $y_i \in H g_i H$ for all other i.
3. The number n is called the length of g and denoted by $l_G(g)$ or $l(g)$ if G is understood; the length is well defined. We have that $g \in G_1 \cup G_2$ if and only if $l(g) = 1$.
4. If $g, h \in G$ then $l(gh) \geq l(g) - l(h)$ and $l(g^{-1}) = l(g)$.

An element $g \in G_1 *_H G_2$ with reduced form $g = x_1 \dots x_n$ is called cyclically reduced if either $n = 1$ or else if x_1 and x_n belong to different factors. The following lemma establishes some easy facts for cyclically reduced elements.

Lemma 3.2. *Let $g \in G_1 *_H G_2$.*

- *If g is cyclically reduced then either $l(g) = 1$ or $l(g^k) = |k|l(g)$ for all integers k. In the former case, every power of g lies in $G_1 \cup G_2$.*
- *Every element of $G_1 *_H G_2$ is conjugate to a cyclically reduced element.*

The strategy to prove that the class $\mathcal{C}_n$ is closed under free amalgams over virtually cyclic groups is the following: take an element $g \in G_1 *_H G_2$ and write it in reduced form. By the residual finiteness of G_i we can find finite-index subgroups $H_i \leq_f G_i$ which do not contain the elements involved in the reduced form of g. Then, consider the image of g in a possible amalgam of G_1/H_1 and G_2/H_2. The only problem is that to construct this amalgam, one needs to make sure that $H \cap H_1 = H \cap H_2$. This is the purpose of the following technical lemma.

Lemma 3.3. *Let $G_1, G_2 \in \mathcal{C}_n$, let H be a virtually cyclic subgroup of both G_1 and G_2 and suppose that $\langle x \rangle$ is a finite index subgroup of H. Then, there exists an integer m, and sequences $A_k \trianglelefteq_f G_1$, $B_k \trianglelefteq_f G_2$ such that*

$$A_k \cap H = \langle x^{km} \rangle = B_k \cap H.$$

Moreover, if N_1, N_2 are normal, finite-index subgroups of G_1 and G_2 respectively, then there exist subgroups, $M_1 \trianglelefteq_f G_1$, $M_2 \trianglelefteq_f G_2$, such that $M_i \leq N_i$ and $M_1 \cap H = M_2 \cap H$.

Proof. Let $1 = h_1, \ldots, h_n$ be coset representatives of $\langle x \rangle$ in H. As G_1 is cyclic subgroup separable, there exists a $P_1 \trianglelefteq_f G_1$ such that $h_i P_1 \cap \langle x \rangle = \varnothing$, or, equivalently, $h_i \notin P_1 \langle x \rangle$, for all $i \neq 1$. Clearly, this implies that $P_1 \cap H \leq \langle x \rangle$. By symmetry, we can also find a $P_2 \trianglelefteq_f G_2$ such that $P_2 \cap H \leq \langle x \rangle$.

Also, as $G_1 \in \mathcal{C}_n$, due to quasi-potency, there exists an integer m_x, and a sequence, $A_k \trianglelefteq_f G_1$ such that $A_k \cap \langle x \rangle = \langle x^{km_x} \rangle$. As is customary, and to ease the notation, we will replace the sequence A_k by a subsequence when needed, without changing the name A_k, and thus the number m_x can be replaced by any of its multiples.

Combining the last two observations, we can assume, without loss of generality, that $A_k \cap \langle x \rangle \leq P_1 \cap H$. Thus, by replacing each A_k by $A_k \cap P_1$, again changing notation, we can assume that $A_k \cap H = \langle x^{km_x} \rangle$.

By symmetry, there exists an integer n_x and a sequence, $B_k \trianglelefteq_f G_2$ such that $B_k \cap H = \langle x^{kn_x} \rangle$. By taking further subsequences, we can then assume that $m_x = n_x = m$, by just replacing m_x and n_x by their least common multiple. This concludes the proof of the first assertion.

To prove the second statement, suppose that $N_1 \cap \langle x \rangle = \langle x^t \rangle$ and $N_2 \cap \langle x \rangle = \langle x^s \rangle$. Then we let $M_1 = N_1 \cap A_{st}$ and $M_2 = N_2 \cap B_{st}$. Clearly, $M_1 \cap H = \langle x^{stm} \rangle = M_2 \cap H$. □

Due to this last result, we shall use a particular construction: given an amalgamated free product $G_1 *_H G_2$, we will take two subgroups $N_1 \trianglelefteq_f G_1$ and $N_2 \trianglelefteq_f G_2$ with $N_1 \cap H = N_2 \cap H$. In this situation, we will denote by Γ_{N_1,N_2} the amalgamated free product of the quotients, i.e., the product $\overline{G_1} *_{\overline{H}} \overline{G_2}$ where $\overline{G_i} = G_i/N_i$ and $\overline{H} = H/(N_i \cap H) \cong HN_i/N_i$ for both $i = 1, 2$. Also, we have the natural map

$$\pi_{N_1,N_2} : G_1 *_H G_2 \longrightarrow \Gamma_{N_1,N_2}$$

onto the amalgamated free product of the quotients. Observe that, in the case which interests us more, which is when the $\overline{G_i} = G_i/N_i$ are finite, the group Γ_{N_1,N_2} is virtually free, and hence, has many of the properties which we are interested in, and which will be suitable to pull back to $G_1 *_H G_2$ with the map π_{N_1,N_2}. The next result is a statement on the consistency of the definitions of length when we pass from G to Γ_{N_1,N_2}.

Proposition 3.4. *Let $G_1, G_2 \in \mathcal{C}_n$ and let H be virtually cyclic. Given $1 \neq g \in G = G_1 *_H G_2$ then there exists a Γ_{N_1,N_2} with $\pi_{N_1,N_2}(g) \neq 1$ and $l_{\Gamma_{N_1,N_2}}(g) = l_G(g)$. Moreover, for any Γ_{M_1,M_2}, if $M_i \leq N_i$ then we similarly find that $l_{\Gamma_{M_1,M_2}}(g) = l_G(g)$.*

Proof. The proof divides into two cases depending on whether $l_G(g)$ is bigger than one or equal to one. So suppose first that g has length greater than one. Thus we can write it as a reduced product $g = x_1 \ldots x_n$ where no x_i lies in H. Let $S_1 = \{x_i : x_i \in G_1\}$ and $S_2 = \{x_i : x_i \in G_2\}$. Since H is separable in both G_1 and G_2 and since the sets S_i are finite we can find $A_1 \trianglelefteq_f G_1$ and $A_2 \trianglelefteq_f G_2$ such that each $S_i \cap HA_i$ is empty. By Lemma 3.3 we can find $N_1 \trianglelefteq_f G_1$, $N_2 \trianglelefteq_f G_2$ with $N_i \leq A_i$ and $N_1 \cap H = N_2 \cap H$. Clearly, each of $S_i \cap HN_i$ is empty. This means that if we look at the image of $x_1 \ldots x_n$ in Γ_{N_1,N_2} we get a reduced product of the same length.

On the other hand, if $l_G(g) = 1$ then assume that $g \in G_1$. We can find an $A_1 \trianglelefteq_f G_1$ such that $g \notin A_1$. By applying Lemma 3.3 again, we get $N_i \trianglelefteq_f G_i$ such that $\pi_{\Gamma_{N_1,N_2}}(g)$ is different from 1 and has length one.

The second assertion of this Lemma is obvious. □

The next result will imply separability for cyclic subgroups in G.

Proposition 3.5. *Let $G_1, G_2 \in \mathcal{C}_n$ and H be virtually cyclic. Consider a $g \in G = G_1 *_H G_2$ which is cyclically reduced and an $h \in G$ such that $h \notin \langle g \rangle$. Then we can find a Γ_{N_1,N_2} such that $\pi_{N_1,N_2}(h) \notin \langle \pi_{N_1,N_2}(g) \rangle$.*

Proof. Again, the proof divides into two cases depending on the length of g. Suppose first that $l_G(g) > 1$. By repeated applications of Proposition 3.4 and Lemma 3.3 we can find a Γ_{N_1,N_2} such that $\pi_{N_1,N_2}(g^r h) \neq 1$ for all integers r such that $|r| \leq l_G(h)$. Moreover, we can ensure that

$l_G(g) = l_{\Gamma_{N_1,N_2}}(g)$ and $l_G(h) = l_{\Gamma_{N_1,N_2}}(h)$. (In fact this second condition isn't necessary since we would always have that $l_G(h) \geq l_{\Gamma_{N_1,N_2}}(h)$.)

Proceeding, this must imply that, since g is cyclically reduced, so is $\pi_{N_1,N_2}(g)$. Hence, $l_{\Gamma_{N_1,N_2}}(\pi_{N_1,N_2}(g^k)) = |k| l_{\Gamma_{N_1,N_2}}(\pi_{N_1,N_2}(g)) \geq 2|k|$. Thus, for any r such that $|r| \leq l_G(h)$, $\pi_{N_1,N_2}(g^r h) \neq 1$ by construction, while for any such that $|r| > l_G(h)$, $l_{\Gamma_{N_1,N_2}}(\pi_{N_1,N_2}(g^r h)) \geq 2|r| - l_G(h) > 1$. Thus, as required, $\pi_{N_1,N_2}(h) \notin \langle \pi_{N_1,N_2}(g) \rangle$.

Consiedr now the case where g has length one. We shall assume, without loss of generality, that $g \in G_1$. If we also have that $h \in G_1$ then since cyclic subgroups of G_1 are separable, we can find a $A_1 \trianglelefteq_f G_1$ such that $g^r h \notin A_i$. An application of Lemma 3.3 would finish off this case.

The only cases that remain are where either $l_G(h) \geq 2$ or $h \in G_2 - H$. In the first of these, we can find Γ_{N_1,N_2} such that $l_{\Gamma_{N_1,N_2}}(\pi_{N_1,N_2}(h)) = l_G(h)$ and since $l_{\Gamma_{N_1,N_2}}(\pi_{N_1,N_2}(g^r)) = l_G(g^r) = 1$, it is clear that $\pi_{N_1,N_2}(h) \notin \langle \pi_{N_1,N_2}(g) \rangle$. In the second of these, where $h \in G_2 - H$, we can utilise the separability of H to find an $A_2 \trianglelefteq_f G_2$ such that $h \notin HA_2$. By Lemma 3.3, we can find Γ_{N_1,N_2} such that $\pi_{N_1,N_2}(h) \notin H/(N_1 \cap H) = H/(N_2 \cap H)$ and hence $\pi_{N_1,N_2}(h) \neq \pi_{N_1,N_2}(g^r) \in \pi_{N_1,N_2}(G_1)$. □

The main result of the section is then the following.

Theorem 3.6. *Let $G_1, G_2 \in \mathcal{C}_n$ and H be a virtually cyclic common subgroup. Then $G = G_1 *_H G_2$ is also in $\mathcal{C}_n$.*

Proof. Consider a $g \in G$, $g \neq 1$. First we need to find an integer m_g and a sequence $L_k \trianglelefteq_f G$ such that $L_k \cap \langle g \rangle = \langle g^{km_g} \rangle$. Equivalently, we require G to be residually finite and for each infinite order element, g, we need a sequence of maps $G \to F_k$ where each F_k is a finite group in which the image of g has order km_g. Note that Lemma 3.4 already shows G to be residually finite so we may assume that g has infinite order.

Clearly, one can freely replace g by a conjugate, so we shall assume that g is cyclically reduced.

Suppose first that $l_G(g) > 1$. Then Proposition 3.4 allows us to find a Γ_{N_1,N_2} such that the image of g in Γ_{N_1,N_2} has the same length as g. Clearly, this also means that $\pi_{N_1,N_2}(g)$ must be cyclically reduced and hence that $\pi_{N_1,N_2}(g)$ has infinite order. However, Γ_{N_1,N_2} is virtually free and hence in $\mathcal{C}_n$. So we can find a sequence of maps $\Gamma_{N_1,N_2} \to F_k$ where each F_k is finite and the image of g in F_k has order km_g for some positive integer, m_g. Composing these maps with the map $G \to \Gamma_{N_1,N_2}$ gives the required property for g.

Now suppose that g has length 1 and we shall, without loss of generality, assume that $g \in G_1$. Since $G_1 \in \mathcal{C}_n$ we can find a sequence $M_k \trianglelefteq_f G_1$ and a positive integer m_g such that $M_k \cap \langle g \rangle = \langle g^{km_g} \rangle$. Also, by Lemma 3.3, we can find an $A_1 \trianglelefteq_f G_1$ and a sequence $B_i \trianglelefteq_f G_2$ such that $A_1 \cap H \geq B_i \cap H$ for all i. In fact, the lemma implies that *any* subgroup of $A_1 \cap H$ is equal to $B_i \cap H$ for some i.

Now, there is also an integer t such that $A_1 \cap \langle g \rangle = \langle g^t \rangle$. By replacing M_k by a subsequence and m_g by some multiple, which we shall continue to denote with the same symbols, we can assume that t divides m_g.

Let $L_k = M_k \cap A_1$. Then $L \cap \langle g \rangle = \langle g^{km_g} \rangle$. Moreover, there exists an integer, i_k such that $L \cap H = B_{i_k} \cap H$. Hence the image of g in $\Gamma_{L_k, B_{i_k}}$ has order km_g. Since this last group is virtually free, meaning that it has a normal, finite index, torsion free subgroup, there exists a finite group, F_k and a map $G \to F_k$ such that the image of g in this group has order km_g. This verifies the first of the properties to check whether G lies in $\mathcal{C}_n$.

For the second property, we need to show that all cyclic subgroups are separable. However, since we know that all cyclic subgroups in virtually free groups are separable, this directly follows from Proposition 3.5. □

3.1. **HNN extensions.** The previous result raises the question as to whether it might be strengthened, to which the answer is generally negative. To be more precise, there are examples (see [1]) of amalgamated free products of finitely generated torsion free nilpotent groups, amalgamating a cycle, which fail to be subgroup separable. Theorem 3.6 implies they are cyclic subgroup separable and this example show that the conclusion cannot be strengthened in general.

Another natural question to ask is whether Theorem 3.6 is true for HNN extensions. However, there are examples (see [9]) of HNN extensions of finitely generated free nilpotent groups, amalgamating a cycle, which fail to be residually finite. Hence, one cannot expect a full generalisation of Theorem 3.6.

However, the proof of Theorem 3.6 clearly works for certain special classes of HNN extensions.

Theorem 3.7. *Let $H \in \mathcal{C}_n$ and suppose that $A \leq H$ is closed in the profinite topology of H. Then the HNN extension, $G = \langle H, t \ : \ t^{-1}at = a, \text{ for all } a \in A \rangle$ is in $\mathcal{C}_n$.*

Proof. Given any $N \trianglelefteq_f H$, we can construct a homomorphism from G to the group $\overline{G} = \langle \overline{H}, \overline{t} \ : \ \overline{t}^{-1}\overline{a}\overline{t} = \overline{a}, \text{ for all } \overline{a} \in \overline{A} \rangle$, where $\overline{H}$ denotes H/N, $\overline{A}$ denotes A/N and $\overline{a}$ denotes aN.

The group $\overline{G}$ is an HNN extension (and it is here that we are using the fact that t does no more than commute with the edge group) of a finite group and is thus virtually free. The proofs of Propositions 3.3, 3.5 and Theorem 3.6 go through with essentially the same proofs, using the normal form for HNN extensions and Britton's Lemma in the place of the normal form for amalgamated free products. □

One application of the above Theorem is in showing that the class of *right-angled Artin groups* lies in $\mathcal{C}_n$. Recall that a right-angled Artin group is a finitely presented group which is given by a presentation in which the only relators are commutators in some of the generators. Usually one starts with a finite simplicial graph Γ, with vertex set X, and associate the right-angled Artin group $G = G_\Gamma$,

$$G = \langle X \ : \ [x,y] \text{ if there is an edge joining } x \text{ to } y \text{ in } \Gamma\rangle.$$

The result is the following:

Theorem 3.8. *Right-angled Artin groups are in* $\mathcal{C}_n$.

Proof. The idea is to prove this by induction on the number of generators, using Theorem 3.7. Clearly, any right-angled Artin group is either free, or can be written as an HNN extension of the type given in Theorem 3.7, where the vertex group is another right-angled Artin group with fewer generators. Moreover, using the standard presentation above, the edge group will be generated by a subset of the generators (vertices).

More precisely, given a finite simplicial graph Γ and a vertex x, we construct the graph Γ_x by deleting x and all adjacent edges. If we let $star(x)$ denote the set of vertices adjacent to x in Γ (not including x itself), we get the following description of G_Γ,

$$G_\Gamma = \langle G_{\Gamma_x}, x \ : \ x^{-1}yx = y \text{ for all } y \in star(x)\rangle.$$

Hence it is sufficient to show that, given a subset of vertices $X_0 \subset X$, the subgroup $H = \langle X_0\rangle$ is separable in G_Γ. We do this by induction on the number of vertices in Γ, noting that the cyclic case is trivial. Choose some $x \in X - X_0$ and let $A = \langle star(x)\rangle$. By the induction hypothesis, A is separable in G_{Γ_x}, using the notation above.

Now consider some $g \notin H$. Our goal is to find a $N \trianglelefteq_f G_\Gamma$ such that $g \notin HN$. If $g \in G_{\Gamma_x}$ the inductive hypothesis lets us find an $N_0 \trianglelefteq_f G_{\Gamma_x}$ such that $g \notin HN_0$. We can then consider the quotient of G_{Γ_x},

$$\langle G_{\Gamma_x}/N_0, x \ : \ \ x^{-1}yN_0x = yN_0 \text{ for all } y \in star(x)\rangle.$$

As discussed in the proof to Theorem 3.7, this is an HNN extension and a virtually free group and it is easy to see that g and H have disjoint images in this quotient. Since the image of H is finite, the residual finiteness of virtually free groups would separate g from H in some finite quotient and hence we would be done.

The other case to consider is when $g \notin G_{\Gamma_x}$. In this case we can use the normal form for HNN extensions and write,

$$g = x^{n_1} h_1 x^{n_2} h_2 \dots h_k x^{n_k},$$

where each n_i is a non-zero integer (except possibly n_1 and n_k which could be zero) and $h_i \in G_{\Gamma_x}$. Additionally, we may insist that we have at least one non-zero n_i and that such a product is reduced in the sense that if $n_i n_{i+1} < 0$ then $h_i \in G_{\Gamma_x} - A$.

Now, as A is separable, we can find a $N_0 \trianglelefteq_f G_{\Gamma_x}$ such that if $h_i \notin A$ then $h_i \notin AN_0$. As before, we can then construct a quotient to the HNN extension,

$$\langle G_{\Gamma_x}/N_0, x \ : \ \ x^{-1} y N_0 x = y N_0 \text{ for all } y \in star(x) \rangle,$$

and the definition of N_0 ensures that the image of g in this quotient maps to a reduced product involving the letter x. In particular, the image of g lies outside G_{Γ_x}/N_0 and hence is disjoint from the image of H. We invoke the residual finiteness of virtually free groups again to complete the proof. □

4. More groups in $\mathcal{C}_n$ and $\mathcal{C}_c$

4.1. **Direct Products.** We will establish now the invariance of the class $\mathcal{C}_n$ under finite direct products.

Proposition 4.1. *Let $H, K \in \mathcal{C}_n$. Then $H \times K$ is cyclic subgroup separable.*

Proof. Let $h_1, h \in H$ and $k_1, k \in K$ such that $(h_1, k_1) \notin \langle (h, k) \rangle$. Our goal is to find a normal finite-index subgroup L of $H \times K$ such that (h_1, k_1) does not belong to the cyclic subgroup generated by $(h, k)L$ in $(H \times K)/L$.

Now if $h_1 \notin \langle h \rangle$ (in H) then by hypothesis we can find a normal subgroup $N \trianglelefteq_f H$ such that $h_1 \notin \langle h \rangle N$. Hence, $(h_1, k_1) \notin \langle (h, k) \rangle (N \times K)$ and we are done.

Thus, by symmetry, we only need to consider the case where $h_1 \in \langle h \rangle$ and $k_1 \in \langle k \rangle$. Hence we can find integers n, m such that

$$h_1 = h^n \qquad k_1 = k^m.$$

By replacing (h, k) with its inverse if necessary, we shall assume that n is positive. Moreover, since $H \times K$ is clearly residually finite, we can further assume that not both h, k have finite order. Thus, without loss of generality, we shall assume that k has infinite order.

The argument is slightly different, depending upon whether h has infinite or finite order.

Case (i): h has infinite order.

Now as $H, K \in \mathcal{C}_n$, due to quasi-potency we can find $N \trianglelefteq_f H$ and $M \trianglelefteq_f K$ such that
1) hN has order t in H/N,
2) kM has order t in K/M,
3) $t > |n| + |m|$.

We claim that $N \times M$ is our required subgroup. It is clearly normal of finite index in $H \times K$ so it is only a matter of checking whether $(h_1, k_1) \in \langle (h,k) \rangle (N \times M)$. We argue by contradiction. Suppose that there exists an integer p such that

$$h_1 \in h^p N, k_1 \in k^p M,$$

Clearly, since $(h^t, k^t) \in N \times M$ we can assume that $0 \leq p < t$. Now, since $h_1 = h^n$ and $k_1 = k^m$ we have that $h^{n-p} \in N$ and $k^{m-p} \in M$. Since $0 < n < t$ we have that $|n - p| < t$ and hence $n - p = 0$. Thus $|m - p| = |m - n| \leq |n| + |m| < t$ which contradicts the definition of t, unless $m = p = n$. However, this last condition implies that $(h_1, k_1) = (h,k)^p \in \langle (h,k) \rangle$ which contradicts our hypotheses.

Case (ii): h has finite order r.

Since $h_1 = h^n$ we shall assume that $0 < n < r$ and, moreover, since $H, K \in \mathcal{C}_n$, we shall find $N \trianglelefteq_f H$ and $M \trianglelefteq_f K$ such that
1) hN has order r in H/N,
2) kM has order tr in K/M for some positive integer t.

Again, we claim that $N \times M$ is the required subgroup and we argue by contradiction. Thus, for some integer p, $h_1 \in h^p N$ and $k_1 \in k^p M$. In particular, we have that $n - p \equiv m - p \equiv 0 \bmod r$. Hence $m \equiv n \bmod r$ and $(h,k)^m = (h^n, k^m) = (h_1, k_1)$ which again contradicts our hypotheses. □

Proposition 4.2. *Let $H, K \in \mathcal{C}_n$. Then $H \times K$ is quasi-potent.*

Proof. Since $H \times K$ is residually finite we can consider an element $(h,k) \in H \times K$ of infinite order. Without loss of generality, we assume that h has infinite order. Thus we can find a sequence $N_k \trianglelefteq_f H$ such that hN_k has order km_h in H/N for some $m_h > 0$. Let $L_k = N_k \times K$. Then $L_k \trianglelefteq_f H \times K$ and $(h,k)L_k$ has order km_h in $(H \times K)/L_k$. □

Corollary 4.3. *The class $\mathcal{C}_n$ is closed under arbitrary direct sums.*

This result indicates that all direct sums of free groups and nilpotent groups are in C_n. We will prove that some other classes of groups are in $\mathcal{C}_n$, so their direct products will be as well.

4.2. **Residually free groups.** The next family of groups we will study are (finitely generated) residually free groups. The definition is analogous to the other residual properties we are using, i.e., a group is residually free when for any element (different from 1) one can find a normal subgroup which does not contain it, and such that the quotient by it is free.

The fact that free groups are residually a finite p-group for all primes p, means that residually free groups satisfy the same property: the map into a free group can be composed with the maps onto finite groups that exist for free groups. So by Proposition 2.2, they are quasi-potent. So, to be able to conclude then that residually free groups are in the class $\mathcal{C}_c$, we only need to prove that they are cyclic subgroup separable.

Theorem 4.4. *Let G be a finitely generated residually free group. Then G is cyclic subgroup separable.*

Proof. Suppose that $x, y \in G$ such that $y \notin \langle x \rangle$. Now, if the commutator $[x, y] \neq 1$ then, by hypothesis, we can construct a map $\pi : G \to F$ to a free group such that $\pi[x, y] \neq 1$. In particular, $\pi(x), \pi(y) \neq 1$ and $\langle \pi(x), \pi(y) \rangle$ is a free group of rank 2. Hence $\pi(y) \notin \langle \pi(x) \rangle$ and there exists an $H \leq_f F$ containing $\pi(x)$ but not $\pi(y)$. Then $\pi^{-1}(H) \leq_f G$ contains x but not y.

Suppose now that x and y commute and construct a map from G to a free group,

$$\pi_1 : G \longrightarrow F_1$$

such that $\pi_1(x) \neq 1$. As x and y commute, $\pi_1(x)$ and $\pi_1(y)$ must also commute. Hence we can find an integer r such that $\pi_1(y) = \pi_1(x^r)$. Moreover, as F_1 is torsion free, this integer r is unique.

Now we construct another map to a free group,

$$\pi_2 : G \longrightarrow F_2,$$

this time with the property that $\pi_2(x^{-r}y) \neq 1$. So we have:

$$\pi_1(x^{-r}y) = 1 \qquad \pi_2(x^{-r}y) \neq 1.$$

Now consider the map $\pi : G \to F_1 \times F_2$, given by $\pi(g) = (\pi_1(g), \pi_2(g))$. We claim that $\pi(y) \notin \langle \pi(x) \rangle$. For, if it were, then we could find an integer n such that $\pi(x^{-n}y) = 1$ and $\pi_2(x^{-n}y) = 1$. But the first equation can only be satisfied if $n = r$ and the second if $n \neq r$. Thus $\pi(y) \notin \langle \pi(x) \rangle$ and since, by Proposition 4.1, $F_1 \times F_2$ is cyclic subgroup separable, we can find an $H \leq_f F_1 \times F_2$ containing $\pi(x)$ but not $\pi(y)$. As before, $\pi^{-1}(H) \leq_f G$ contains x but not y and completes the proof of cyclic separability of G. □

Corollary 4.5. *A finitely generated residually free group is in the class $\mathcal{C}_c$.*

Proof. A residually free group is also residually a finite p-group for all primes p so by Proposition 2.2, we are done. □

Corollary 4.6. *A finitely generated virtually residually free group is in the class $\mathcal{C}_c$.*

Proof. This is a corollary of Lemma 2.8, because a finite-index subgroup of a finitely generated residually free group is also finitely generated and residually free. □

Corollary 4.7. *Surface groups are in $\mathcal{C}_c$.*

Proof. Surface groups are residually free (see [2]).

□

5. Extensions

Theorem 5.1. *Consider a short exact sequence of groups,*

$$1 \to N \to G \to K \to 1.$$

Suppose that $N \in \mathcal{C}_c$ is finitely generated and $K \in \mathcal{C}_n$. Further assume that for every $N_0 \leq_f N$ which is normal in G, the quotient G/N_0 is residually finite. Then $G \in \mathcal{C}_n$.

If, additionally, $K \in \mathcal{C}_c$ is finitely generated and N is a characteristic subgroup of G then $G \in \mathcal{C}_c$.

Proof. The first thing to observe is that if N_0 is a *characteristic* subgroup of finite index in N, then there exists a finite index subgroup G_0 of G such that $G_0 \cap N = N_0$. This is merely a consequence of the residual finiteness of G/N_0, but immediately implies that G is residually finite.

Also note that we can always ensure that G_0 is normal in G and if N is characteristic in G and K is finitely generated we can further ensure that G_0 is characteristic in G.

Quasi-potency is then fairly straightforward. Taking $g \in G$ we need to find a sequence of finite quotients Q_k, and a positive integer, m_g, such that g has order km_g in Q_k. Since we already know that G is residually finite, we can assume that g has infinite order in G.

If the image of g also has infinite order in K, then the quasi-potency of K will produce the sequence Q_k. Note that if N is characteristic in G and $K \in \mathcal{C}_c$ then we can also ensure that these quotients are by characteristic subgroups.

If the image of g has finite order in K, we can replace g by a power and hence, without loss of generality, assume that $g \in N$. Hence we can find a sequence $N_k \operatorname{ch}_f N$ such that g has order km_g in N/N_k for some positive integer m_g. The remarks above ensure that we can find $G_k \trianglelefteq_f G$ such that $G_k \cap N = N_k$ and this demonstrates quasi-potency. Further, if K is finitely generated we can ensure that the G_k are characteristic. Thus G is always quasi-potent, and is characteristically quasi-potent under the extra assumptions that N is characteristic in G and K is finitely generated and in $\mathcal{C}_c$.

Next we need to prove the separability of cyclic subgroups in G. So consider $g, h \in G$ such that $h \notin \langle g \rangle$. Since $\langle g \rangle$ is closed in the profinite topology if some finite index subgroup of $\langle g \rangle$ is closed, we can replace g by a power. In particular we can assume that either $g \in N$ or the image of g has infinite order in K.

Case 1: The image of g has infinite order in K.

Case 1 (i): $g^r h \in N$.

Notice that since we are assuming g to have infinite order in K, the integer r is unique. Now, by hypothesis, $g^r h \neq 1$ so there exists a finite index subgroup $N_0 \leq_f N$ such that $g^r h \notin N_0$. As N is finitely generated, we can assume that N_0 is characteristic in N and hence normal in G.

By hypothesis, G/N_0 is residually finite and hence there exists a finite index subgroup of G/N_0 which is isomorphic to a finite index subgroup of K. Thus G/N_0 is cyclic subgroup separable. By construction, we have that $hN_0 \notin \langle gN_0 \rangle$ and hence we can find a $G_0 \trianglelefteq_f G$ such that $h \notin \langle g \rangle G_0$.

Case 1 (ii): $h \notin \langle g \rangle N$.

Here the cyclic separability of K ensures that there exists a $K_0 \trianglelefteq_f K$ such that $hN \notin \langle g \rangle N K_0$. Taking G_0 to be the preimage of K_0 in G we get that $G_0 \trianglelefteq_f G$ and $h \notin \langle g \rangle G_0$ and we are done.

Case 2: $g \in N$.

Case 2 (i): $h \in N$.

Then, by the cyclic separability of N, there exists a subgroup $N_0 \,\mathrm{ch}_f\, N$ such that $h \notin \langle g \rangle N_0$. As above, we can find a $G_0 \trianglelefteq_f G$ such that $G_0 \cap N = N_0$. Then we are done since it is clear that $h \notin \langle g \rangle G_0$.

Case 2 (ii): $h \notin N$.

As K is residually finite, we can find a $K_0 \leq_f K$ not containing hN. Let G_0 be the preimage of K_0 in G. Then $G_0 \leq_f G$ and it contains g but not h. This completes the proof of cyclic separability and the proof of the theorem. □

5.1. **Applications.** Theorem 5.1 provides criteria for when extensions of groups in $\mathcal{C}_n$ (or $\mathcal{C}_c$) also belong to this class. However, the hypothesis that G/N_0 is residually finite is rather technical, so we will state several cases where it is satisfied.

First, a definition. Given a short exact sequence

$$1 \to N \to G \to K \to 1,$$

and a finite index subgroup G_1 of G, there is an induced short exact sequence,

$$1 \to N \cap G_1 \to G_1 \to G_1/(G_1 \cap N) \to 1.$$

We shall say that the former short exact sequence *virtually splits* if there exists some $G_1 \leq_f G$ such that the latter sequence splits.

Proposition 5.2. *If the short exact sequence in Theorem 5.1 virtually splits, then G/N_0 is always residually finite. That is, the technical hypothesis is always satisfied.*

Proof. Let $G_1 \leq_f G$ be a subgroup for which the induced sequence splits. Then, $G_1 = N_1 \rtimes K_1$, where $N_1 = (N \cap G_1)$ and $K_1 \cong G_1/(G_1 \cap N)$. Note that K_1 is isomorphic to G_1N/N which is a subgroup of K and hence residually finite. If $N_0 \leq_f N$ and is normal in G then,

$$G_1/(G_1 \cap N_0) = G_1/(N_1 \cap N_0) \cong N_1/(N_1 \cap N_0) \rtimes K_1,$$

since $K_1 \cap N_1 = 1$. Thus $G_1/(G_1 \cap N_0)$ must be residually finite, since it has K_1 as a subgroup of finite index. However,

$$G_1/(G_1 \cap N_0) \cong G_1N_0/N_0 \leq_f G/N_0,$$

and thus G/N_0 has a subgroup isomorphic to $G_1/(G_1 \cap N_0)$ as a subgroup of finite index so it is also residually finite.□

As an application we get the following, though it is clear for elementary reasons.

Corollary 5.3. *The hypothesis of Theorem 5.1 is satisfied if K is virtually free.*

Another situation where we get the same behaviour is with surface groups,

Proposition 5.4. *The hypothesis of Theorem 5.1 is satisfied if K is virtually a surface group.*

Proof. By [8], Exercise 4.7, page 60, any group G which fits into a short exact sequence,

$$1 \to F \to G \to \Gamma \to 1,$$

where F is finite and Γ is a hyperbolic surface group is residually finite. The case of a torus is dealt with below, Theorem 5.10. □

5.2. **Polycyclic.** We can apply Theorem 5.1 to polycyclic groups. By [15], Theorem 4, page 35, given a polycyclic group P, there exists a finite index subgroup P_0 which is nilpotent-by-abelian. In particular, it has a nilpotent derived subgroup. Moreover, any subgroup of P_0 also has a nilpotent derived subgroup. Thus we can consider the short exact sequence

$$1 \to [P_0, P_0] \to P_0 \to P_0/[P_0, P_0] \to 1,$$

where both the group on the left and the group on the right are in $\mathcal{C}_c$ and moreover, the group $[P_0, P_0]$ is characteristic in P_0. Since $P_0/[P_0, P_0]$ is abelian, we immediately see that the hypotheses for Theorem 5.1 are satisfied and hence that $P_0 \in \mathcal{C}_c$. Moreover, the same argument shows that every subgroup of P_0 is also in $\mathcal{C}_c$. Thus, by Lemma 2.8, P is also in $\mathcal{C}_c$.

Theorem 5.5. *All polycyclic subgroups are in $\mathcal{C}_c$.*

We shall now show that the hypotheses of Theorem 5.1 are satisfied whenever the group K is a polycyclic group.

Recall that the ascending chain condition for a group G is the property that all ascending chains of subgroups of G are eventually stable. Equivalently, it states that all subgroups of G are finitely generated.

Recall also that polycyclic groups are precisely those solvable groups which satisfy the ascending chain condition.

Thus the free abelian group of countably rank is solvable but not polycyclic. For a finitely generated example, take the free metabelian group $F/[F', F']$ where F is the free (non-abelian) group of rank 2 and F' the derived subgroup. It is well known that F' is a free group of countably infinite rank and thus that $F'/[F', F']$ is free abelian of countably infinite rank. Hence $F/[F', F']$ is finitely generated, solvable but not polycyclic.

Proposition 5.6. *Virtually polycyclic groups are subgroup separable.*

Proof. [15], Exercise 11, page 18. □

We will need the following result about polycyclic groups. Its proof can be found in [3].

Lemma 5.7. *Let G be a finitely generated group with a finite, normal subgroup N such that G/N is abelian. Then G is virtually abelian.*

We will also need the following result about the ascending chain condition:

Lemma 5.8. *Consider a short exact sequence of groups,*

$$1 \to A \to B \to C \to 1.$$

If A and C are finitely generated, then so is B. Moreover, if both A and C satisfy the ascending chain condition, then so does B.

Proof. The first claim is clear. For the second claim consider a subgroup $H \leq A$. Then we get a short exact sequence,

$$1 \to H \cap A \to H \to H/(H \cap A) \to 1.$$

The second assertion of the lemma now follows once we recall the second isomorphism theorem,

$$H/(H \cap A) \cong HA/A \leq B/A \cong C.$$

□

We finally return to polycyclic groups to establish our final results:.

Proposition 5.9. *Let G be a group with a finite normal subgroup, N, such that G/N is polycyclic. Then G has a finite-index polycyclic subgroup.*

Proof. By Lemma 5.8, all subgroups of G are finitely generated. We argue by induction on the derived length of G/N. For derived length equal one we appeal to Lemma 5.7. Otherwise, we can apply an induction hypothesis to deduce that G', the derived subgroup of G, contains a finite-index polycyclic subgroup H_1 . Here it is where we are using that G satisfies the ascending chain condition.

Without loss of generality, we can assume that H_1 is a normal subgroup of G. Hence G/H_1 is a finitely generated group, whose quotient by G'/H_1 is abelian. Thus, again by Lemma 5.7, G contains a finite index subgroup H_0 such that H_0/H_1 is abelian. Since H_0 is finitely generated and H_1 is polycyclic, H_0 must also be polycyclic. □

Corollary 5.10. *The hypotheses of Theorem 5.1 are satisfied whenever $N \in \mathcal{C}_c$ and K is polycyclic-by-finite.*

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