EMBEDDING THEOREMS OF FUNCTION CLASSES, I

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Abstract. In this paper we study embedding theorems of function classes, which are subclasses of L_p, $1 \le p \le \infty$. To define these classes, we use the notion of best trigonometric approximation as well as that of a (λ, β)-derivative, which is the generalization of a fractional derivative. Estimates of best approximations of transformed Fourier series are obtained.

1. Introduction.

It is well-known that if $f \in L_p$, $1 \le p \le \infty$, and $\sum\limits_{k=1}^{\infty} k^{r-1} E_k(f)_p < \infty$ for $r \in \mathbf{N}$, then $f^{(r)} \in L_p$ and

$$\left\| f^{(r)} \right\|_p \le C_1(r) \sum_{k=1}^{\infty} k^{r-1} E_k(f)_p.$$

For $p = \infty$ this fact was proved by Bernstein in [Be1], for other p we refer to [p.209, De-Lo] and [Ch. 5,6, Ti]. As a corollary (see [Steč]) we have the following inequality

$$E_n(f^{(r)})_p \le C_2(r) \left(n^r E_n(f)_p + \sum_{k=n+1}^{\infty} k^{r-1} E_k(f)_p \right). \tag{1}$$

On the other hand, one can write the inverse inequality (see [p. 206, De-Lo]):

$$n^r E_n(f)_p \le C_3(r) E_n(f^{(r)})_p.$$

Thus, for $\alpha \in (0,1)$ and $\varepsilon = \left\{ \varepsilon_n = n^{-(r+\alpha)} \right\}$, $\delta = \{\delta_n = n^{-\alpha}\}$ the following function classes coincide:

$$E_p[\varepsilon] = \left\{ f \in L_p : E_n(f)_p = O[\varepsilon_n] \right\}, \tag{2}$$

$$W_p^r E[\delta] = \left\{ f \in L_p : f^{(r)} \in L_p,\ E_n\left(f^{(r)}\right)_p = O[\delta_n] \right\}. \tag{3}$$

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We shall obtain the necessary and sufficient conditions for embedding theorems of some function classes which are more general than (2) and (3). We shall use the concept of a (λ, β)-derivative, which allows us to consider $f^{(r)}$ as well as $\widetilde{f}^{(r)}$.

As an r-th derivative we shall consider the fractional derivative in the sense of Weyl. We would like to mention earlier papers [Ha-Li], [Kr], [Mu], [Og] in which this concept was used to examine the question mentioned above. Also we mention papers [Be1], [Ch-Zh], [Ha-Sh], [Mo], [Steč] where the results were obtained in the necessity part.

The paper is organized in the following way. Section 2 contains some definitions and preliminaries. In section 3, we present our main theorems. Section 4 contains lemmas. Sections 5 and 6 include the proofs of the main results for the cases $1 < p < \infty$ and $p = 1, \infty$, respectively.

Finally, we mention the paper by Stepanets [Step] where the analogues of inequality (1) for (λ, β)-derivatives were obtained. The advantage of our findings compared to the results of Stepanets is that our theorems are stronger for the case of $1 < p < \infty$.

2. Definition and notation

Let $L_p = L_p[0, 2\pi]$ $(1 \le p < \infty)$ be a space of 2π-periodic functions for which $|f|^p$ is integrable, and $L_\infty \equiv C[0, 2\pi]$ be the space of 2π-periodic continuous functions with $\|f\|_\infty = \max\{|f(x)|, 0 \le x \le 2\pi\}$.
Let a function $f(x) \in L_1$ have the Fourier series

$$f(x) \sim \sigma(f) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_\nu(f) \cos \nu x + b_\nu(f) \sin \nu x) \equiv \sum_{\nu=0}^{\infty} A_\nu(f, x). \tag{4}$$

By the transformed Fourier series of (4) we mean the series

$$\sigma(f, \lambda, \beta) := \sum_{\nu=1}^{\infty} \lambda_\nu \left[a_\nu \cos\left(\nu x + \frac{\pi\beta}{2}\right) + b_\nu \sin\left(\nu x + \frac{\pi\beta}{2}\right)\right],$$

where $\beta \in \mathbf{R}$ and $\lambda = \{\lambda_n\}$ is a given sequence of positive numbers. The sequence $\lambda = \{\lambda_n\}$ satisfies $\triangle_2$-condition if $\lambda_{2n} \le C\lambda_n$ for all $n \in \mathbf{N}$. For $\lambda = \{\lambda_n\}_{n\in\mathbf{N}}$ we define $\triangle\lambda_n := \lambda_n - \lambda_{n+1}$; $\triangle^2\lambda_n := \triangle(\triangle\lambda_n)$.

Let $S_n(f)$ denote the n-th partial sum of (4), $V_n(f)$ denote the de la Vallée-Poussin sum and $K_n(x)$ be the Fejér kernel, i.e.

$$S_n(f) = \sum_{\nu=0}^{n} A_\nu(x), \quad V_n(f) = \frac{1}{n} \sum_{\nu=n}^{2n-1} S_\nu(f),$$

$$K_n(x) = \frac{1}{n+1}\sum_{\nu=0}^{n}\left(\frac{1}{2} + \sum_{m=1}^{\nu}\cos mx\right).$$

Let $E_n(f)_p$ be the best approximation of a function f by trigonometric polynomials of order no more than n, i.e.

$$E_n(f)_p = \inf_{\alpha_k,\beta_k\in\mathbf{R}}\left\|f(x) - \sum_{k=0}^{n}(\alpha_k\cos kx + \beta_k\sin kx)\right\|_p.$$

Let Φ be the class of all decreasing null-sequences. For $\beta \in \mathbf{R}$ and $\lambda = \{\lambda_n > 0\}$ we define the following function class :

$$W_p^{\lambda,\beta} = \Big\{f \in L_p : \quad \exists\, g \in L_p, \quad g(x) \sim \sigma(f,\lambda,\beta)\Big\}. \tag{5}$$

We call the function $g(x) \sim \sigma(f,\lambda,\beta)$ the (λ,β)-derivative of the function $f(x)$ and denote it by $f^{(\lambda,\beta)}(x)$. Also, we define for $\varepsilon \in \Phi$

$$W_p^{\lambda,\beta}E[\varepsilon] = \Big\{f \in W_p^{\lambda,\beta} : \quad E_n\left(f^{(\lambda,\beta)}\right)_p = O[\varepsilon_n]\Big\}. \tag{6}$$

In the case $\lambda_n \equiv 1$ and $\beta = 0$ the class $W_p^{\lambda,\beta}E[\varepsilon]$ coincides with the class $E_p[\varepsilon]$ (see (2)).

It is clear that if $\lambda_n = n^r, r > 0$, $\beta = r$, then the class $W_p^{\lambda,\beta}E[\varepsilon]$ coincides with the class $W_p^rE[\varepsilon]$ (see (3) where $f^{(r)}$ denotes a fractional derivative in the Weyl sense) and if $\lambda_n = n^r, r > 0$, $\beta = r+1$, then the class $W_p^{\lambda,\beta}E[\varepsilon]$ coincides with the class

$$\widetilde{W}_p^rE[\varepsilon] := \Big\{f \in L_p : \widetilde{f}^{(r)} \in L_p,\, E_n\left(\widetilde{f}^{(r)}\right)_p = O[\varepsilon_n]\Big\}.$$

Here and further, $\widetilde{f}$ is a conjugate function to f.

By $C(s,t,\cdots)$ we denote the positive constants that are dependent only on $s,t,\cdots$ and may be different in different formulas.

3. Main Results

Theorem 1. *Let $1 < p < \infty$, $\theta = \min(2,p)$, $\beta \in \mathbf{R}$, and $\lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers satisfying $\triangle_2$-condition. Let*

$\varepsilon = \{\varepsilon_n\}, \omega = \{\omega_n\} \in \Phi$. *Then*

$$E_p[\varepsilon] \subset W_p^{\lambda,\beta} \iff \sum_{n=1}^{\infty} \left(\lambda_{n+1}^{\theta} - \lambda_n^{\theta}\right) \varepsilon_n^{\theta} < \infty, \tag{7}$$

$$E_p[\varepsilon] \subset W_p^{\lambda,\beta} E[\omega] \iff \left\{ \sum_{\nu=n+1}^{\infty} \left(\lambda_{\nu+1}^{\theta} - \lambda_{\nu}^{\theta}\right) \varepsilon_{\nu}^{\theta} \right\}^{\frac{1}{\theta}} + \lambda_n \varepsilon_n = O\left[\omega_n\right], \tag{8}$$

$$W_p^{\lambda,\beta} \subset E_p[\varepsilon] \iff \frac{1}{\lambda_n} = O\left[\varepsilon_n\right], \tag{9}$$

$$W_p^{\lambda,\beta} E[\omega] \subset E_p[\varepsilon] \iff \frac{\omega_n}{\lambda_n} = O\left[\varepsilon_n\right]. \tag{10}$$

Theorem 2. *Let* $p = 1, \infty$, $\beta \in \mathbf{R}$, *and* $\lambda = \{\lambda_n\}$ *be a non-decreasing sequence of positive numbers satisfying* $\triangle_2$*-condition. Let* $\varepsilon = \{\varepsilon_n\}, \omega = \{\omega_n\} \in \Phi$.
A. *If* $\triangle \lambda_n \le C \triangle \lambda_{2n}$ *and* $\triangle^2 \lambda_n \ge 0$ *(or* ≤ 0*), then*

$$E_p[\varepsilon] \subset W_p^{\lambda,\beta} \iff \left|\cos\frac{\beta\pi}{2}\right| \sum_{n=1}^{\infty} \left(\lambda_{n+1} - \lambda_n\right) \varepsilon_n + \left|\sin\frac{\beta\pi}{2}\right| \sum_{n=1}^{\infty} \lambda_n \frac{\varepsilon_n}{n} < \infty, \tag{11}$$

$$E_p[\varepsilon] \subset W_p^{\lambda,\beta} E[\omega] \iff \left|\cos\frac{\beta\pi}{2}\right| \sum_{\nu=n+1}^{\infty} \left(\lambda_{\nu+1} - \lambda_{\nu}\right) \varepsilon_{\nu} + \lambda_n \varepsilon_n + \left|\sin\frac{\beta\pi}{2}\right| \sum_{\nu=n+1}^{\infty} \lambda_{\nu} \frac{\varepsilon_{\nu}}{\nu} = O\left[\omega_n\right]. \tag{12}$$

B. *If for* $\beta = 2k$, $k \in \mathbf{Z}$ *the condition* $\triangle^2\left(1/\lambda_n\right) \ge 0$ *holds, and for* $\beta \ne 2k$, $k \in \mathbf{Z}$ *conditions* $\triangle^2\left(1/\lambda_n\right) \ge 0$ *and* $\sum\limits_{\nu=n+1}^{\infty} \frac{1}{\nu\lambda_{\nu}} \le \frac{C}{\lambda_n}$ *are fulfilled, then*

$$W_p^{\lambda,\beta} \subset E_p[\varepsilon] \iff \frac{1}{\lambda_n} = O\left[\varepsilon_n\right], \tag{13}$$

$$W_p^{\lambda,\beta} E[\omega] \subset E_p[\varepsilon] \iff \frac{\omega_n}{\lambda_n} = O\left[\varepsilon_n\right]. \tag{14}$$

One can draw many conclusions from the inequalities which we use in proofs of Theorems 1 and 2. The simplest ones are
Corollary 1. *Let* $1 < p < \infty$, $\theta = \min(2, p)$, *and* $r > 0$, $A \ge 0$. *If for*

$f \in L_p$ the series

$$\sum_{k=1}^{\infty} k^{r\theta-1} \ln^{A\theta} k \, E_k^{\theta}(f)_p$$

converges, then there exists $f^{(\lambda,\beta)} \in L_p$ with $\lambda = \{n^r \ln^A n\}$ and $\beta \in \mathbf{R}$, and

$$E_n(f^{(\lambda,\beta)})_p \le$$
$$\le C(r,A,p) \left(n^r \ln^A n \, E_n(f)_p + \left\{ \sum_{k=n+1}^{\infty} k^{r\theta-1} \ln^{A\theta} k \, E_k^{\theta}(f)_p \right\}^{\frac{1}{\theta}} \right).$$

Corollary 2. *Let $p = 1, \infty$, and $r > 0$, $A \ge 0$. If for $f \in L_p$ the series*

$$\sum_{k=1}^{\infty} k^{r-1} \ln^{A} k \, E_k(f)_p$$

converges, then there exist $f^{(\lambda,\beta)}$, $\tilde{f}^{(\lambda,\beta)} \in L_p$ with $\lambda = \{n^r \ln^A n\}$ and $\beta \in \mathbf{R}$, and

$$E_n(f^{(\lambda,\beta)})_p + E_n(\tilde{f}^{(\lambda,\beta)})_p \le$$
$$\le C(r,A) \left(n^r \ln^A n \, E_n(f)_p + \sum_{k=n+1}^{\infty} k^{r-1} \ln^{A} k \, E_k(f)_p \right).$$

Corollary 3. *Let $1 \le p \le \infty$, and $r > 0$, $A \ge 0$. If for $f \in L_p$ there exist $f^{(\lambda,\beta)}$, $\tilde{f}^{(\lambda,\beta)} \in L_p$ with $\lambda = \{n^r \ln^A n\}$ and $\beta \in \mathbf{R}$, then*

$$n^r \ln^A n \, E_n(f)_p \le C(r,A,p) E_n(f^{(\lambda,\beta)})_p.$$
$$n^r \ln^A n \, E_n(f)_p \le C(r,A,p) E_n(\tilde{f}^{(\lambda,\beta)})_p.$$

We note that if $\lambda = \{n^r \ln^A n\}$, then $f^{(\lambda,\beta)}$ is a fractional-logarithmic derivative of f (see, for example, [Ku]).

4. Auxiliary Results.

Lemma 1. ([V. 1, p. 215, Zy]) *Let $f(x)$ have the Fourier series $\sum_{\nu=1}^{\infty} (a_\nu \cos n_\nu x + b_\nu \sin n_\nu x)$, where $n_{\nu+1}/n_\nu \ge q > 1$ and $\sum_{\nu=1}^{\infty} \left(a_\nu^2 + b_\nu^2\right) < \infty$. Then for $1 \le p < \infty$*

$$C_1(p,q) \left\{ \sum_{\nu=1}^{\infty} \left(a_\nu^2 + b_\nu^2\right) \right\}^{\frac{1}{2}} \le \|f\|_p \le C_2(p,q) \left\{ \sum_{\nu=1}^{\infty} \left(a_\nu^2 + b_\nu^2\right) \right\}^{\frac{1}{2}}.$$

Lemma 2. ([V.2, p. 269, Ba2]) *Let $f(x) \in L_\infty$ have the Fourier series $\sum\limits_{\nu=1}^{\infty} (a_\nu \cos n_\nu x + b_\nu \sin n_\nu x)$, where $a_\nu, b_\nu \ge 0$ and $n_{\nu+1}/n_\nu \ge q > 1$. Then*

$$C_1(q) \sum_{n_\nu > n} (a_\nu + b_\nu) \le E_n(f)_\infty \le C_2(q) \sum_{n_\nu > n} (a_\nu + b_\nu).$$

Lemma 3. ([Steč]) *Let $f(x) \in L_p$, $p = 1, \infty$, and let $\sum\limits_{n=1}^{\infty} n^{-1} E_n(f)_p < \infty$ be true. Then $\tilde{f}(x) \in L_p$ and*

$$E_n(\tilde{f})_p \le C \left(E_n(f)_p + \sum_{k=n+1}^{\infty} k^{-1} E_k(f)_p \right) \qquad (k \in \mathbf{N}).$$

Lemma 4. ([V. 1, p. 182, Zy]) *Let $\varepsilon_n \downarrow 0$. The condition $\nu \varepsilon_\nu \to 0$ is both necessary and sufficient for $\sum\limits_{\nu=1}^{\infty} \varepsilon_\nu \sin \nu x$ to be the Fourier series of a continuous function.*

Lemma 5. ([Te]) *Let $f(x) \in L_1$ have a Fourier series (4). Then*

$$E_n(f)_1 \ge C \sum_{\nu=n+1}^{\infty} \frac{b_\nu}{\nu}.$$

Lemma 6. *Let $1 \le p \le \infty$ and $E_p[\varepsilon] \subset W_p^{\lambda,\beta} E[\omega]$. Then*

$$\lambda_n \varepsilon_n = O(\omega_n) \qquad n \to \infty. \tag{15}$$

Proof. We presume that (15) does not hold. Then there exists a sequence $\{m_n\}$ such that $\lambda_{m_n} \varepsilon_{m_n} \ge C_n \omega_{m_n}$ and $C_n \uparrow \infty$ as $n \to \infty$. One can also choose a subsequence $\{m_{n_k}\}$ such that

$$\frac{m_{n_{k+1}}}{m_{n_k}} \ge 2, \qquad \varepsilon_{m_{n_k}} \ge \frac{1}{2} \varepsilon_{m_{n_k}} + \varepsilon_{m_{n_{k+1}}} \quad \text{and} \quad \lambda_{m_{n_k}} \varepsilon_{m_{n_k}} \ge C_{n_k} \omega_{m_{n_k}}.$$

Let us consider the case $1 \le p < \infty$. We consider the series

$$\sum_{k=0}^{\infty} \Big(\varepsilon_{m_{n_k}}^2 - \varepsilon_{m_{n_{k+1}}}^2 \Big)^{\frac{1}{2}} \cos\Big((m_{n_k} + 1)x - \frac{\pi\beta}{2} \Big). \tag{16}$$

Since

$$\sum_{k=0}^{\infty} \Big(\varepsilon_{m_{n_k}}^2 - \varepsilon_{m_{n_{k+1}}}^2 \Big) = \varepsilon_{m_{n_0}}^2,$$

by Lemma 1, the series (16) is the Fourier series of a function $f_0(x) \in L_p$ and $E_s(f_0)_p \leq C\varepsilon_s$, i.e. $f_0 \in E[\varepsilon]$. Then $f_0 \in W_p^{\lambda,\beta}E[\omega]$. On the other hand,

$$E_{m_{n_k}}(f_0^{(\lambda,\beta)})_p \geq C\lambda_{m_{n_k}+1}\left(\varepsilon^2_{m_{n_k}} - \varepsilon^2_{m_{n_{k+1}}}\right)^{\frac{1}{2}} =$$
$$= C\lambda_{m_{n_k}+1}\left[\left(\varepsilon_{m_{n_k}} - \varepsilon_{m_{n_{k+1}}}\right)\left(\varepsilon_{m_{n_k}} + \varepsilon_{m_{n_{k+1}}}\right)\right]^{\frac{1}{2}} \geq C\lambda_{m_{n_k}}\varepsilon_{m_{n_k}} \geq$$
$$\geq C_{n_k}\omega_{m_{n_k}}.$$

Thus, $f_0 \notin W_p^{\lambda,\beta}E_p[\omega]$. This contradiction implies (15).

Let $p = \infty$. Let us consider the series

$$\sum_{k=0}^{\infty}\left(\varepsilon_{m_{n_k}} - \varepsilon_{m_{n_{k+1}}}\right)\cos\Big((m_{n_k}+1)x - \frac{\pi\beta}{2}\Big). \tag{17}$$

By Lemma 2, there exists $f_1 \in L_p$ with Fourier series (17) and $E_s(f_1)_p \leq C\varepsilon_s$, i.e. $f_1 \in E_p[\varepsilon] \subset W_p^{\lambda,\beta}E[\omega]$. On the other hand,

$$E_{m_{n_k}}(f_1^{(\lambda,\beta)})_p \geq C\lambda_{m_{n_k}+1}\left(\varepsilon_{m_{n_k}} - \varepsilon_{m_{n_{k+1}}}\right) \geq C\lambda_{m_{n_k}}\varepsilon_{m_{n_k}} \geq C\,C_{n_k}\omega_{m_{n_k}},$$

i.e. $f_1 \notin W_p^{\lambda,\beta}E[\omega]$. $\square$

Lemma 7. ([Si-Ti]) *Let* $p = 1, \infty$ *and* $\{\lambda_n\}$ *be monotonic concave* (*or convex*) *sequence. Let*

$$T_n(x) = \sum_{\nu=0}^{n} a_\nu \cos\nu x + b_\nu \sin\nu x,$$
$$T_n(\lambda, x) = \sum_{\nu=0}^{n} \lambda_\nu\left(a_\nu \cos\nu x + b_\nu \sin\nu x\right).$$

Then for $M > N \geq 0$ *one has*

$$\|T_M(\lambda,x) - T_N(\lambda,x)\|_p \leq \mu(M,N)\,\|T_M(x) - T_N(x)\|_p,$$

where

$$\mu(M,N) =$$
$$= \begin{cases} 2M(\lambda_M - \lambda_{M-1}) + \lambda_{N+1} - (N+1)(\lambda_{N+2} - \lambda_{N+1}), & \text{if } \lambda_n\uparrow,\ \triangle^2\lambda_n \geq 0; \\ 2\lambda_M + (N+1)(\lambda_{N+2} - \lambda_{N+1}) - \lambda_{N+1}, & \text{if } \lambda_n\uparrow,\ \triangle^2\lambda_n \leq 0; \\ (N+1)(\lambda_{N+1} - \lambda_{N+2}) + \lambda_{N+1}, & \text{if } \lambda_n\downarrow,\ \triangle^2\lambda_n \geq 0. \end{cases}$$

Lemma 8. *Let* $p = 1, \infty$. *Set*

$$T_{2^n,2^{n+1}}(x) = \sum_{\nu=2^n}^{2^{n+1}}\Big(c_\nu \cos\nu x + d_\nu \sin\nu x\Big).$$

Then

$$C_1 \left\|\widetilde{T}_{2^n,2^{n+1}}(\cdot)\right\|_p \le \left\|T_{2^n,2^{n+1}}(\cdot)\right\|_p \le C_2 \left\|\widetilde{T}_{2^n,2^{n+1}}(\cdot)\right\|_p . \tag{18}$$

Proof. We rewrite $T_{2^n,2^{n+1}}(x)$ in the following way

$$T_{2^n,2^{n+1}}(x) = \sum_{\nu=2^n}^{2^{n+1}} \frac{1}{\nu}\Big(\nu c_\nu \cos\nu x + \nu d_\nu \sin\nu x\Big).$$

Applying Lemma 7 and the Bernstein inequality we have

$$\begin{aligned}\left\|T_{2^n,2^{n+1}}(\cdot)\right\|_p &\le C\frac{1}{2^n}\left\|\sum_{\nu=2^n}^{2^{n+1}}\Big(\nu c_\nu \cos\nu x + \nu d_\nu \sin\nu x\Big)\right\|_p \\ &= C\frac{1}{2^n}\left\|\left(\sum_{\nu=2^n}^{2^{n+1}} -d_\nu \cos\nu x + c_\nu \sin\nu x\right)'\right\|_p \\ &\le C\left\|\widetilde{T}_{2^n,2^{n+1}}(\cdot)\right\|_p .\end{aligned}$$

The same reasoning for $\widetilde{T}_{2^n,2^{n+1}}(x)$ implies the correctness of the left-hand side of (18). □

5. Proof of Theorem 1.

We divide the proof of Theorem 1 into two parts.

5.1. Proof of sufficiency.

Step 1. Let us prove the sufficiency part in (7). First, if $\lambda_n \equiv 1$, the Riesz inequality ([V. 1, p. 253, Zy]) $\|\tilde{f}\|_p \le C(p)\|f\|_p$ implies

$$\|f^{(\lambda,\beta)}\|_p \le C(p,\beta)\|f\|_p \tag{19}$$

Let the series in the right part of (7) be convergent and $f \in E_p[\varepsilon]$. We use the following representation

$$\lambda_{2^n}^\theta = \lambda_1^\theta + \sum_{\nu=2}^{n+1}\left(\lambda_{2^{\nu-1}}^\theta - \lambda_{2^{\nu-2}}^\theta\right).$$

Applying the Minkowski's inequality we get $\Big($ here and further $\triangle_1 := A_1(f,x), \triangle_{n+2} := \sum\limits_{\nu=2^n+1}^{2^{n+1}} A_\nu(f,x), n = 0,1,2,\cdots,$ where $A_\nu(f,x)$ is from

(4) $\Big)$

$$I_1 :=$$

$$:= \left\{ \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \lambda_{2^{n-1}}^2 \triangle_n^2 \right]^{\frac{p}{2}} dx \right\}^{\frac{1}{p}}$$

$$\leq \ C(\lambda) \left(\int_0^{2\pi} \left\{ \lambda_1^2 \triangle_1^2 + \lambda_2^2 \triangle_2^2 + \sum_{n=3}^{\infty} \lambda_{2^{n-2}}^2 \triangle_n^2 \right\}^{\frac{p}{2}} dx \right)^{\frac{1}{p}}$$

$$\leq C(p,\lambda) \left(\int_0^{2\pi} \left\{ \lambda_1^2 \triangle_1^2 + \lambda_2^2 \triangle_2^2 + \sum_{n=3}^{\infty} \triangle_n^2 \left[\lambda_1^\theta + \sum_{\nu=3}^{n} \left(\lambda_{2^{\nu-2}}^\theta - \lambda_{2^{\nu-3}}^\theta \right) \right]^{\frac{2}{\theta}} \right\}^{\frac{p}{2}} dx \right)^{\frac{1}{p}}$$

$$\leq C(p,\lambda) \left(\lambda_1^\theta \left\{ \int_0^{2\pi} \left[\sum_{n=1}^{\infty} \triangle_n^2 \right]^{\frac{p}{2}} dx \right\}^{\frac{\theta}{p}} + \sum_{s=3}^{\infty} \left(\lambda_{2^{s-2}}^\theta - \lambda_{2^{s-3}}^\theta \right) \left\{ \int_0^{2\pi} \left[\sum_{n=s}^{\infty} \triangle_n^2 \right]^{\frac{p}{2}} dx \right\}^{\frac{\theta}{p}} \right)^{\frac{1}{\theta}}.$$

By the Littlewood-Paley theorem ([V. II, p. 233, Zy]) and

$$\|f - S_n(f)\|_p \leq C(p)\, E_n(f)_p, \tag{20}$$

we get

$$I_1 \leq C(p,\lambda) \left\{ \lambda_1^\theta E_0^\theta(f)_p + \sum_{s=1}^{\infty} \left(\lambda_{2^s}^\theta - \lambda_{2^{s-1}}^\theta \right) E_{2^s}^\theta(f)_p \right\}^{\frac{1}{\theta}}.$$

Since $f \in E_p[\varepsilon]$ we have $I_1 < \infty$. Thus, by the Littlewood-Paley theorem, there exists a function $g \in L_p$ with Fourier series

$$\sum_{n=1}^{\infty} \lambda_{2^{n-1}} \triangle_n, \tag{21}$$

and $\|g\|_p \leq C(p) I_1$. We rewrite series (21) in the form of $\sum_{n=1}^{\infty} \gamma_n A_n(f,x)$, where $\gamma_i := \lambda_i$, $i = 1, 2$ and $\gamma_\nu := \lambda_{2^n}$ for $2^{n-1} + 1 \leq \nu \leq 2^n$ $(n = 2, 3, \cdots)$. Further, we write the series

$$\sum_{n=1}^{\infty} \lambda_n A_n(f,x) = \sum_{n=1}^{\infty} \gamma_n \Lambda_n A_n(f,x), \tag{22}$$

where $\Lambda_1 := \Lambda_2 := 1$, $\Lambda_\nu := \lambda_\nu / \gamma_n = \lambda_\nu / \lambda_{2^n}$ for $2^{n-1} + 1 \leq \nu \leq 2^n$ $(n = 2, 3, \cdots)$. The sequence $\{\Lambda_n\}$ satisfies the conditions of the Marcinkiewicz multiplier theorem ([V.II, p. 232, Zy]), i.e. the series (22) is the Fourier series of a function $f^{(\lambda,0)} \in L_p$, $\|f^{(\lambda,0)}\|_p \le C(p,\lambda)\|g\|_p$.

Using the properties of $\{\lambda_n\}$ and (19), we write

$$\begin{aligned}\|f^{(\lambda,\beta)}\|_p &\le C(p,\lambda)\left\{\lambda_1^\theta E_0^\theta(f)_p + \sum_{s=1}^{\infty} E_{2^s}^\theta(f)_p \sum_{n=2^{s-1}}^{2^s-1}\left(\lambda_{n+1}^\theta - \lambda_n^\theta\right)\right\}^{\frac{1}{\theta}} \\ &\le C(p,\lambda)\left\{\lambda_1^\theta E_0^\theta(f)_p + \sum_{n=1}^{\infty}\left(\lambda_{n+1}^\theta - \lambda_n^\theta\right) E_n^\theta(f)_p\right\}^{\frac{1}{\theta}}. \qquad (23)\end{aligned}$$

Thus, the sufficiency in (7) has been proved.

Step 2. Let the relation in the right-hand side of (8) hold, and $f \in E_p[\varepsilon]$. Let us prove $f \in W_p^{\lambda,\beta}E[\omega]$. We have

$$E_n(f^{(\lambda,\beta)})_p \le \|f^{(\lambda,\beta)} - S_n(f^{(\lambda,\beta)})\|_p = \|\left(f - S_n\right)^{(\lambda,\beta)}\|_p.$$

Applying (23) for the function $(f - S_n)$ we get

$$\begin{aligned}E_n(f^{(\lambda,\beta)})_p &\le C(p,\lambda)\left\{\lambda_1^\theta E_0^\theta\left(f - S_n\right)_p + \sum_{m=1}^{\infty}\left(\lambda_{m+1}^\theta - \lambda_m^\theta\right) E_m^\theta\left(f - S_n\right)_p\right\}^{\frac{1}{\theta}} \\ &\le C(p,\lambda)\left\{\lambda_1^\theta E_n^\theta(f)_p + E_n^\theta(f)_p \sum_{m=1}^{n}\left(\lambda_{m+1}^\theta - \lambda_m^\theta\right) + \sum_{m=n+1}^{\infty}\left(\lambda_{m+1}^\theta - \lambda_m^\theta\right) E_m^\theta(f)_p\right\}^{\frac{1}{\theta}} \\ &\le C(p,\lambda)\left\{\lambda_n^\theta E_n^\theta(f)_p + \sum_{m=n+1}^{\infty}\left(\lambda_{m+1}^\theta - \lambda_m^\theta\right) E_m^\theta\left(f\right)_p\right\}^{\frac{1}{\theta}} \le C(p,\lambda)\omega_n.\end{aligned}$$

This proves the sufficiency in (8).

Step 3. Now we shall prove that conditions $\frac{1}{\lambda_n} = O\left[\varepsilon_n\right]$ and $\frac{\omega_n}{\lambda_n} = O\left[\varepsilon_n\right]$ are sufficient for $W_p^{\lambda,\beta} \subset E_p[\varepsilon]$ and $W_p^{\lambda,\beta}E[\omega] \subset E_p[\varepsilon]$, respectively.

Let $f \in W_p^{\lambda,\beta}$. From the properties of the sequence $\{\lambda_n\}$, using the Littlewood-Paley and the Marcinkiewicz multiplier theorem, we get

$$\begin{aligned}E_n(f)_p &\le \|f - S_n(f)\|_p \le \frac{C(p,\lambda)}{\lambda_n}\|f^{(\lambda,\beta)} - S_n(f^{(\lambda,\beta)})\|_p \\ &\le \frac{C(p,\lambda)}{\lambda_n} E_n(f^{(\lambda,\beta)})_p.\end{aligned}$$

If $\frac{1}{\lambda_n} = O\left[\varepsilon_n\right]$, then $E_n(f)_p \le \frac{C}{\lambda_n} = O\left[\varepsilon_n\right]$, and if $\frac{\omega_n}{\lambda_n} = O\left[\varepsilon_n\right]$, then $E_n(f)_p \le \frac{C}{\lambda_n}E_n(f^{(\lambda,\beta)})_p \le C\frac{\omega_n}{\lambda_n} = O\left[\varepsilon_n\right]$, i.e. $f \in E_p[\varepsilon]$.

The proof of the sufficiency part in (9) and (10) is complete.

5.2. Proof of necessity.

Step 4. Let us prove the necessity part in (7). Let $E_p[\varepsilon] \subset W_p^{\lambda,\beta}$ but the series in (7) be divergent.

Let $2 \le p < \infty$. We consider the series

$$\sum_{\nu=0}^{\infty} \left(\varepsilon_{2^\nu-1}^2 - \varepsilon_{2^{\nu+1}-1}^2\right)^{\frac{1}{2}} \cos 2^\nu x. \tag{24}$$

Since $\sum\limits_{\nu=0}^{\infty} \left(\varepsilon_{2^\nu-1}^2 - \varepsilon_{2^{\nu+1}-1}^2\right) = \varepsilon_0^2$, by Lemma 1, the series (24) is the Fourier series of a function $f_2(x) \in L_p$.

Let $2^\nu - 1 \le n < 2^{\nu+1} - 1$. Then $E_n(f_2)_p \le \|f_2 - S_n(f_2)\|_p \le C\varepsilon_n$, i.e. $f_2 \in E_p[\varepsilon] \subset W_p^{\lambda,\beta}$. On the other hand,

$$\begin{aligned}
\|f_2^{(\lambda,\beta)}\|_p &\ge C\left\{\sum_{\nu=0}^{\infty} \left(\varepsilon_{2^\nu-1}^2 - \varepsilon_{2^{\nu+1}-1}^2\right)\lambda_{2^\nu}^2\right\}^{\frac{1}{2}} \\
&\ge C\left\{\sum_{\nu=0}^{\infty} \left(\varepsilon_{2^\nu-1}^2 - \varepsilon_{2^{\nu+1}-1}^2\right)\left[\sum_{n=0}^{\nu} \left(\lambda_{2^{n+1}}^2 - \lambda_{2^n}^2\right) + \lambda_1^2\right]\right\}^{\frac{1}{2}} \\
&= C\left\{\lambda_1^2\varepsilon_0^2 + \sum_{n=0}^{\infty} \left(\lambda_{2^{n+1}}^2 - \lambda_{2^n}^2\right)\varepsilon_{2^n-1}^2\right\}^{\frac{1}{2}} \\
&= C\left\{\lambda_1^2\varepsilon_0^2 + \sum_{n=0}^{\infty} \sum_{\nu=2^n}^{2^{n+1}-1} \left(\lambda_{\nu+1}^2 - \lambda_\nu^2\right)\varepsilon_{2^n-1}^2\right\}^{\frac{1}{2}} \\
&\ge C\left\{\lambda_1^2\varepsilon_0^2 + \sum_{\nu=1}^{\infty} \left(\lambda_{\nu+1}^2 - \lambda_\nu^2\right)\varepsilon_\nu^2\right\}^{\frac{1}{2}} = \infty
\end{aligned}$$

This contradiction implies the convergence of series in (7).

Let now $1 < p < 2$. We shall consider the series

$$\left(\varepsilon_0^p - \varepsilon_1^p\right)^{\frac{1}{p}} \cos x + \sum_{\nu=0}^{\infty} 2^{\nu(\frac{1}{p}-1)} \left(\varepsilon_{2^{\nu+1}-1}^p - \varepsilon_{2^{\nu+2}-1}^p\right)^{\frac{1}{p}} \sum_{\mu=2^\nu+1}^{2^{\nu+1}} \cos \mu x. \tag{25}$$

By Jensen inequality and

$$C_1(p)2^{\nu(p-1)} \le \left\|\sum_{\mu=2^{\nu-1}+1}^{2^\nu} \cos \mu x\right\|_p^p \le C_2(p)2^{\nu(p-1)}, \tag{26}$$

we have

$$\int_0^{2\pi}\left\{\sum_{\nu=0}^{\infty}\left[\left(\varepsilon^p_{2^{\nu+1}-1}-\varepsilon^p_{2^{\nu+2}-1}\right)^{\frac{1}{p}}2^{\nu(\frac{1}{p}-1)}\sum_{\mu=2^{\nu}+1}^{2^{\nu+1}}\cos\mu x\right]^2\right\}^{\frac{p}{2}}dx$$
$$\leq\int_0^{2\pi}\left[\sum_{\nu=0}^{\infty}2^{\nu(1-p)}\left(\varepsilon^p_{2^{\nu+1}-1}-\varepsilon^p_{2^{\nu+2}-1}\right)\left|\sum_{\mu=2^{\nu-1}+1}^{2^{\nu}}\cos\mu x\right|^p\right]dx\leq C(p)\varepsilon_1^p.$$

By the Littlewood-Paley theorem, there exists a function $f_3\in L_p$ with Fourier series (25). Let $n=2^{\nu}$. Then

$$\begin{aligned}&\|f_3-S_n(f_3)\|_p=\\&=\left\|\sum_{m=\nu}^{\infty}2^{m(\frac{1}{p}-1)}\left(\varepsilon^p_{2^{m+1}-1}-\varepsilon^p_{2^{m+2}-1}\right)^{\frac{1}{p}}\sum_{\mu=2^m+1}^{2^{m+1}}\cos\mu x\right\|_p\\&\leq\left\{\int_0^{2\pi}\left[\sum_{m=\nu}^{\infty}2^{m(1-p)}\left(\varepsilon^p_{2^{m+1}-1}-\varepsilon^p_{2^{m+2}-1}\right)\left|\sum_{\mu=2^m+1}^{2^{m+1}}\cos\mu x\right|^p\right]dx\right\}^{\frac{1}{p}}\\&\leq C(p)\varepsilon_{2^{\nu+1}-1}\leq C(p)\varepsilon_{2^{\nu}}.\end{aligned}$$

Let $n=0$. Then $E_0(f_3)_p\leq C(p)\left(\varepsilon_0^p-\varepsilon_1^p+\varepsilon_1^p\right)^{\frac{1}{p}}=C(p)\varepsilon_0$.

Let $2^{\nu}<n<2^{\nu+1}$. Then

$$\begin{aligned}&\|f_3-S_n(f_3)\|_p=\\&=\left\|2^{\nu(\frac{1}{p}-1)}\left(\varepsilon^p_{2^{\nu+1}-1}-\varepsilon^p_{2^{\nu+2}-1}\right)^{\frac{1}{p}}\sum_{\mu=n+1}^{2^{\nu+1}}\cos\mu x\right.\\&+\left.\sum_{m=\nu+1}^{\infty}2^{m(\frac{1}{p}-1)}\left(\varepsilon^p_{2^{m+1}-1}-\varepsilon^p_{2^{m+2}-1}\right)^{\frac{1}{p}}\sum_{\mu=2^m+1}^{2^{m+1}}\cos\mu x\right\|_p\leq C(p)\varepsilon_n.\end{aligned}$$

Therefore, one has $f_3\in E_p[\varepsilon]$. By our assumption, this implies $f_3(x)\in W_p^{\lambda,\beta}$.

On the other hand, Paley's theorem on Fourier coefficients [V.2, p. 121, Zy] implies

$$\begin{aligned}
&\left\| f_3^{(\lambda,\beta)} \right\|_p^p \geq \\
&\geq \; C(p) \left\{ \left(\varepsilon_0^p - \varepsilon_1^p\right) \lambda_1^p + \sum_{\nu=0}^{\infty} \left(\varepsilon_{2^{\nu+1}-1}^p - \varepsilon_{2^{\nu+2}-1}^p \right) 2^{\nu(1-p)} \sum_{\mu=2^\nu+1}^{2^{\nu+1}} \lambda_\mu^p \mu^{p-2} \right\} \\
&\geq \; C(\lambda,p) \left\{ \left(\varepsilon_0^p - \varepsilon_1^p\right) \lambda_1^p + \sum_{\nu=0}^{\infty} \left(\varepsilon_{2^\nu-1}^p - \varepsilon_{2^{\nu+1}-1}^p \right) \lambda_{2^\nu}^p \right\} \\
&\geq \; C(\lambda,p) \left\{ \varepsilon_0^p \lambda_1^p + \sum_{\nu=1}^{\infty} \left(\lambda_{\nu+1}^p - \lambda_\nu^p \right) \varepsilon_\nu^p \right\} \; = \; \infty.
\end{aligned}$$

This contradicts $f_3(x) \in W_p^{\lambda,\beta}$. The series in (7) converges.

Step 5. Now we shall prove the necessity in (8). Let $2 \leq p < \infty$ and $2^\nu \leq n < 2^{\nu+1}, \nu = 0, 1, 2, \cdots$. We consider

$$\left(\varepsilon_n^2 - \varepsilon_{2^{\nu+1}-1}^2\right)^{\frac{1}{2}} \cos(n+1)x + \sum_{\nu=0}^{\infty} \left(\varepsilon_{2^\nu-1}^2 - \varepsilon_{2^{\nu+1}-1}^2\right)^{\frac{1}{2}} \cos 2^\nu x. \qquad (27)$$

Repeating the argument we used for series (24) we can see that the series (27) is the Fourier series of a function $f_{4,n} \in L_p$ and $f_{4,n} \in E_p[\varepsilon]$. Therefore, $f_{4,n} \in W_p^{\lambda,\beta} E[\omega]$.

Let us show that the positive constant C_1 in the inequality $E_m(f_{4,n}^{(\lambda,\beta)})_p \leq C_1 \omega_m$ $(m = 0, 1, 2, \cdots)$ is independent of m and n. Indeed, for the function

$$f_{4,n}(x) = f_2(x) + \left(\varepsilon_n^2 - \varepsilon_{2^{\nu+1}-1}^2\right)^{\frac{1}{2}} \cos(n+1)x$$

one has

$$E_m(f_{4,n}^{(\lambda,\beta)})_p \leq \left\| f_{4,n}^{(\lambda,\beta)} - S_m(f_{4,n}^{(\lambda,\beta)}) \right\|_p \leq \left\| f_2^{(\lambda,\beta)} - S_m(f_2^{(\lambda,\beta)}) \right\|_p +$$

$$+ \left\| \left[\left(\varepsilon_n^2 - \varepsilon_{2^{\nu+1}-1}^2\right)^{\frac{1}{2}} \cos(n+1)x \right]^{(\lambda,\beta)} - S_m\left(\left[\left(\varepsilon_n^2 - \varepsilon_{2^{\nu+1}-1}^2\right)^{\frac{1}{2}} \cos(n+1)x \right]^{(\lambda,\beta)} \right) \right\|_p .$$

Since $f_2 \in E_p[\varepsilon] \subset W_p^{\lambda,\beta} E[\omega]$, $E_m(f_2^{(\lambda,\beta)})_p = O(\omega_m)$.

Then for $m \geq n+1$: $E_m(f_{4,n}^{(\lambda,\beta)})_p = E_m(f_2^{(\lambda,\beta)})_p \leq C(f_2, p, \lambda, \beta)\omega_m$ and for $0 \leq m \leq n : E_m(f_{4,n}^{(\lambda,\beta)})_p \leq C(f_2, p, \lambda, \beta)\omega_m + C(p, \lambda)\lambda_n \varepsilon_n$. By Lemma 6, we have

$$E_m(f_{4,n}^{(\lambda,\beta)})_p \leq C(f_2, p, \lambda, \beta)\omega_m + C(p, \lambda)\lambda_m \varepsilon_m \leq C(f_2, p, \lambda, \beta)\omega_m.$$

Thus, $E_m(f_{4,n}^{(\lambda,\beta)})_p \leq C_1 \omega_m$ where C_1 does not depend on n and m.

On the other hand,

$$
\begin{aligned}
C\omega_n &\geq \\
&\geq\ E_n(f_{4,n}^{(\lambda,\beta)})_p \\
&\geq C(p,\lambda)\left[\left(\varepsilon_n^2-\varepsilon_{2^{\nu+1}-1}^2\right)\lambda_n^2+\sum_{m=\nu+1}^{\infty}\left(\varepsilon_{2^m-1}^2-\varepsilon_{2^{m+1}-1}^2\right)\lambda_{2^m}^2\right]^{\frac{1}{2}} \\
&\geq C(p,\lambda)\left[\left(\varepsilon_n^2-\varepsilon_{2^{\nu+1}-1}^2\right)\lambda_n^2+\lambda_{2^{\nu+1}-1}^2\varepsilon_{2^{\nu+1}-1}^2+\sum_{m=2^{\nu+1}}^{\infty}\left(\lambda_{m+1}^2-\lambda_m^2\right)\varepsilon_m^2\right]^{\frac{1}{2}} \\
&\geq C(p,\lambda)\left[\varepsilon_n^2\lambda_n^2+\left(\sum_{m=n+1}^{2^{\nu+1}-1}\left(\lambda_{m+1}^2-\lambda_m^2\right)+\lambda_{n+1}^2\right)\varepsilon_n^2+\sum_{m=2^{\nu+1}}^{\infty}\left(\lambda_{m+1}^2-\lambda_m^2\right)\varepsilon_m^2\right]^{\frac{1}{2}} \\
&\geq C(p,\lambda)\left[\varepsilon_n^2\lambda_n^2+\sum_{m=n+1}^{\infty}\left(\lambda_{m+1}^2-\lambda_m^2\right)\varepsilon_m^2\right]^{\frac{1}{2}}.
\end{aligned}
$$

Thus, the relation in the right-hand side of (8) holds.

Let $1<p<2$. For $n=0$ we consider the series (25). For $2^m\leq n<2^{m+1}, m=0,1,2,\cdots$ we define

$$
\begin{aligned}
&(\varepsilon_0^p-\varepsilon_1^p)^{\frac{1}{p}}\cos x+ \\
&+\left(\sum_{\nu=0}^{m-1}+\sum_{\nu=m+1}^{\infty}\right)2^{\nu(\frac{1}{p}-1)}\left(\varepsilon_{2^{\nu+1}-1}^p-\varepsilon_{2^{\nu+2}-1}^p\right)^{\frac{1}{p}}\sum_{\mu=2^\nu+1}^{2^{\nu+1}}\cos\mu x+ \\
&+\left(\varepsilon_n^p-\varepsilon_{2^{m+2}-1}^p\right)^{\frac{1}{p}}\cos(n+1)x. \qquad (28)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\left\|(\varepsilon_0^p-\varepsilon_1^p)^{\frac{1}{p}}\cos x+\sum_{\nu=0}^{m-1}2^{\nu(\frac{1}{p}-1)}\left(\varepsilon_{2^{\nu+1}-1}^p-\varepsilon_{2^{\nu+2}-1}^p\right)^{\frac{1}{p}}\sum_{\mu=2^\nu+1}^{2^{\nu+1}}\cos\mu x\right\|_p^p \\
&+\left(\varepsilon_n^p-\varepsilon_{2^{m+2}-1}^p\right)+\left\|\sum_{\nu=m+1}^{\infty}2^{\nu(\frac{1}{p}-1)}\left(\varepsilon_{2^{\nu+1}-1}^p-\varepsilon_{2^{\nu+2}-1}^p\right)^{\frac{1}{p}}\sum_{\mu=2^\nu+1}^{2^{\nu+1}}\cos\mu x\right\|_p^p \\
&\leq C(p)\left(\varepsilon_0^p-\varepsilon_1^p+\varepsilon_1^p-\varepsilon_{2^{m+1}-1}^p+\varepsilon_n^p-\varepsilon_{2^{m+2}-1}^p+\varepsilon_{2^{m+2}-1}^p-\varepsilon_{2^{m+3}-1}^p+\cdots\right) \\
&\leq C(p)\varepsilon_0^p,
\end{aligned}
$$

there exists a function $f_{5,n}(x)\in L_p$ with Fourier series (28). One can verify that

$E_0(f_{5,n})_p \le C(p)\varepsilon_0$, $E_k(f_{5,n})_p \le C(p)\varepsilon_k$. Therefore, $f_{5,n} \in E_p[\varepsilon] \subset W_p^{\lambda,\beta}E[\omega]$.

Let us show that the positive constant C_2 in the inequality $E_m(f_{5,n}^{(\lambda,\beta)})_p \le C_2\omega_m$ $(m = 0, 1, 2, \cdots)$ is independent of m and n. We note

$$f_{5,n}(x) = f_3(x) - \sum_{\mu=2^m+1}^{2^{m+1}} 2^{m(\frac{1}{p}-1)}\left(\varepsilon^p_{2^{m+1}-1} - \varepsilon^p_{2^{m+2}-1}\right)^{\frac{1}{p}}\cos\mu x + \left(\varepsilon^p_n - \varepsilon^p_{2^{m+2}-1}\right)^{\frac{1}{p}}\cos(n+1)x.$$

If $n = 0, 1$, then $E_k(f_{5,n}^{(\lambda,\beta)})_p = E_k(f_3^{(\lambda,\beta)})_p \le C(f_3, p, \lambda, \beta)\omega_k$. If $n = 2, 3, \cdots$, and $2^m \le n < 2^{m+1}$, then for $k = 0, 1, \cdots, n$

$$\begin{aligned} E_k(f_{5,n}^{(\lambda,\beta)})_p &\le \left\|f_{5,n}^{(\lambda,\beta)} - S_k(f_{5,n}^{(\lambda,\beta)})\right\|_p \le \left\|f_3^{(\lambda,\beta)} - S_k(f_3^{(\lambda,\beta)})\right\|_p \\ &+ \left\|\left(\sum_{\mu=2^m+1}^{2^{m+1}} 2^{m(\frac{1}{p}-1)}\left(\varepsilon^p_{2^{m+1}-1} - \varepsilon^p_{2^{m+2}-1}\right)^{\frac{1}{p}}\cos\mu x\right)^{(\lambda,\beta)}\right\|_p \\ &+ \left\|\left(\left(\varepsilon^p_n - \varepsilon^p_{2^{m+2}-1}\right)^{\frac{1}{p}}\cos(n+1)x\right)^{(\lambda,\beta)}\right\|_p. \end{aligned}$$

By Lemma 6, using $f_3 \in W^{\lambda,\beta}E[\omega]$, we have $E_k(f_{5,n}^{(\lambda,\beta)})_p \le C(f_3, p, \lambda, \beta)\omega_k$ for $k = 0, 1, 2, \cdots, n$. For $k > n$ we write $E_k(f_{5,n}^{(\lambda,\beta)})_p \le \left\|f_3^{(\lambda,\beta)} - S_k(f_3^{(\lambda,\beta)})\right\|_p \le C(f_3, p, \lambda, \beta)\omega_k$.

Thus, $E_k(f_{5,n}^{(\lambda,\beta)})_p \le C_2\omega_k$ for any k and C_2 is independent of m and n. On the other hand,

$$
\begin{aligned}
&C_2\omega_n \geq \\
&\geq E_n(f_{5,n}^{(\lambda,\beta)})_p \geq C(p)\left\|f_{5,n}^{(\lambda,\beta)} - S_k(f_{5,n}^{(\lambda,\beta)})\right\|_p \\
&\geq C(p)\left(\int_0^{2\pi}\left\{\left\langle\left[\left(\varepsilon_n^p - \varepsilon_{2^{m+2}-1}^p\right)^{\frac{1}{p}}\cos(n+1)x\right]^{(\lambda,\beta)}\right\rangle^2\right.\right. \\
&+ \left.\left.\sum_{\nu=m+1}^{\infty}\left\langle\left[\left(\varepsilon_{2^{\nu+1}-1}^p - \varepsilon_{2^{\nu+2}-1}^p\right)^{\frac{1}{p}} 2^{\nu(\frac{1}{p}-1)}\sum_{\mu=2^\nu+1}^{2^{\nu+1}}\cos\mu x\right]^{(\lambda,\beta)}\right\rangle^2\right\}^{\frac{p}{2}} dx\right)^{\frac{1}{p}} \\
&\geq C(p)\left\{\left[\left(\varepsilon_n^p - \varepsilon_{2^{m+2}-1}^p\right)\lambda_n^p\int_0^{2\pi}|\cos(n+1)x|^p\,dx\right]^{\frac{1}{p}}\right. \\
&+ \left.\left[\int_0^{2\pi}\left(\sum_{\nu=m+1}^{\infty}\left\langle\left[\left(\varepsilon_{2^{\nu+1}-1}^p - \varepsilon_{2^{\nu+2}-1}^p\right)^{\frac{1}{p}} 2^{\nu(\frac{1}{p}-1)}\sum_{\mu=2^\nu+1}^{2^{\nu+1}}\cos\mu x\right]^{(\lambda,\beta)}\right\rangle^2\right)^{\frac{p}{2}} dx\right]^{\frac{1}{p}}\right\} \\
&\geq C(p)\left\{\left(\varepsilon_n^p - \varepsilon_{2^{m+2}-1}^p\right)\lambda_n^p + \sum_{\nu=m}^{\infty}\left(\varepsilon_{2^{\nu+2}-1}^p - \varepsilon_{2^{\nu+3}-1}^p\right)\lambda_{2^\nu}^p\right\}^{\frac{1}{p}} \\
&\geq C(p)\left\{\left(\varepsilon_n^p - \varepsilon_{2^{m+1}-1}^p\right)\lambda_n^p + \sum_{\nu=m+1}^{\infty}\left(\varepsilon_{2^{\nu}-1}^p - \varepsilon_{2^{\nu+1}-1}^p\right)\lambda_{2^\nu}^p\right\}^{\frac{1}{p}} \\
&\geq C(p)\left\{\varepsilon_n^p\lambda_n^p + \sum_{\nu=n+1}^{\infty}\left(\lambda_{\nu+1}^p - \lambda_\nu^p\right)\varepsilon_\nu^p\right\}^{\frac{1}{p}}.
\end{aligned}
$$

This implies the necessity in (8) for $1 < p < 2$. The proof of the necessity part in (8) is complete.

Step 6. Now we shall prove that $W_p^{\lambda,\beta} \subset E_p[\varepsilon]$ implies $\frac{1}{\lambda_n} = O\left[\varepsilon_n\right]$.

First, we note that the last condition is equivalent to the following one: $\forall\ \gamma = \{\gamma_n\} \in \Phi$ one has $\frac{\gamma_n}{\lambda_n} = O\left[\varepsilon_n\right]$. We shall obtain only nontrivial part which is : $\frac{\gamma_n}{\lambda_n} = O\left[\varepsilon_n\right]$ implies $\frac{1}{\lambda_n} = O\left[\varepsilon_n\right]$.

Let us assume $\frac{1}{\lambda_n} = O\left[\varepsilon_n\right]$ does not hold. Then there exists a sequence $\{C_k \uparrow \infty\}$ such that $\frac{1}{\lambda_{n_k}\varepsilon_{n_k}} \geq C_k$. Using $\frac{1}{\lambda_n} = O\left[\varepsilon_n\right]$ we have $\frac{C}{\gamma_{n_k}} \geq C_k$. Choosing $\gamma_{n_k} := \frac{1}{\sqrt{C_k}} \to 0$, we write $C \geq \sqrt{C_k} \to \infty$. This contradiction gives

$$\frac{1}{\lambda_n} = O\left[\varepsilon_n\right] \qquad \Longleftrightarrow \qquad \forall\, \gamma = \{\gamma_n\} \in \Phi \qquad \frac{\gamma_n}{\lambda_n} = O\left[\varepsilon_n\right]. \tag{29}$$

Let us assume $\frac{\gamma_n}{\lambda_n} = O\left[\varepsilon_n\right]$ does not hold for all $\gamma \in \Phi$, but $W_p^{\lambda,\beta} \subset E_p[\varepsilon]$. Then there exist $\gamma = \{\gamma_n\} \in \Phi$ and $\{C_n \uparrow \infty\}$ such that $\frac{\gamma_{m_n}}{\lambda_{m_n}} \geq C_n \varepsilon_{m_n}$. Further, we choose a subsequence $\{m_{n_k}\}$ such that $\frac{m_{n_{k+1}}}{m_{n_k}} \geq 2$ and $\gamma_{m_{n_k}} \leq 2^{-k}$. Consider the series

$$\sum_{k=0}^{\infty} \frac{\gamma_{m_{n_k}}}{\lambda_{m_{n_k}}} \cos(m_{n_k}+1)x. \tag{30}$$

Since $\sum\limits_{k=0}^{\infty} \frac{\gamma_{m_{n_k}}}{\lambda_{m_{n_k}}} \leq \frac{1}{\lambda_{m_{n_0}}} \sum\limits_{k=0}^{\infty} \frac{1}{2^k} < \infty$, there exists a function $f_6 \in L_p$ with Fourier series (30). Because $\sum\limits_{k=0}^{\infty} \gamma_{m_{n_k}} \leq \sum\limits_{k=0}^{\infty} \frac{1}{2^k} < \infty$, we have $f_6^{(\lambda,\beta)} \in L_p$, i.e. $f_6 \in W_p^{\lambda,\beta}$. By (20) and by Lemma 1,

$$E_{m_{n_k}}(f_6)_p \geq C(p) \left\| f_6 - S_{m_{n_k}}(f_6) \right\|_p \geq C(p) \left(\sum_{s=k}^{\infty} \frac{\gamma^2_{m_{n_s}}}{\lambda^2_{m_{n_s}}} \right)^{\frac{1}{2}} \geq C(p) \frac{\gamma_{m_{n_k}}}{\lambda_{m_{n_k}}} \geq C(p) C_{n_k} \varepsilon_{m_{n_k}},$$

i.e. $f_6 \notin E_p[\varepsilon]$. This contradiction implies that the condition $\frac{1}{\lambda_n} = O\left[\varepsilon_n\right]$ is necessary for $W_p^{\lambda,\beta} \subset E_p[\varepsilon]$. The proof of the necessity part in (9) is complete.

Step 7. Let us prove that $W_p^{\lambda,\beta} E[\omega] \subset E_p[\varepsilon]$ implies $\frac{\omega_n}{\lambda_n} = O\left[\varepsilon_n\right]$. If the last condition does not hold, then there exists $\{C_n \uparrow \infty\}$ such that $\frac{\omega_{m_n}}{\lambda_{m_n}} \geq C_n \varepsilon_{m_n}$. We choose a subsequence $\{m_{n_k}\}$ such that $\frac{m_{n_{k+1}}}{m_{n_k}} \geq 2$ and $\omega_{m_{n_k}} \geq \frac{1}{2}\omega_{m_{n_k}} + \omega_{m_{n_{k+1}}}$. Since $\sum\limits_{k=0}^{\infty} \frac{\omega^2_{m_{n_k}} - \omega^2_{m_{n_{k+1}}}}{\lambda^2_{m_{n_k}}} \leq \frac{\omega^2_{m_{n_0}}}{\lambda^2_{m_{n_0}}}$, by Lemma 1, the series

$$\sum_{k=0}^{\infty} \frac{\left(\omega^2_{m_{n_k}} - \omega^2_{m_{n_{k+1}}}\right)^{\frac{1}{2}}}{\lambda_{m_{n_k}}} \cos(m_{n_k}+1)x \tag{31}$$

is the Fourier series of a function $f_7 \in L_p$. We have also $f_7^{(\lambda,\beta)} \in L_p$, $E_n(f_7^{(\lambda,\beta)})_p \leq C\omega_n$.

On the other hand,

$$\begin{aligned}
E_{m_{n_k}}(f_7)_p &\geq C(p)\left\|f_7 - S_{m_{n_k}}(f_7)\right\|_p \\
&\geq C(p)\left(\sum_{s=k}^{\infty}\frac{\omega_{m_{n_s}}^2 - \omega_{m_{n_{s+1}}}^2}{\lambda_{m_{n_s}}^2}\right)^{\frac{1}{2}} \\
&= C(p)\left(\sum_{s=k}^{\infty}\frac{(\omega_{m_{n_s}} - \omega_{m_{n_{s+1}}})(\omega_{m_{n_s}} + \omega_{m_{n_{s+1}}})}{\lambda_{m_{n_s}}^2}\right)^{\frac{1}{2}} \\
&\geq C(p)\frac{\omega_{m_{n_k}}}{\lambda_{m_{n_k}}} \geq C(p)C_{n_k}\varepsilon_{m_{n_k}},
\end{aligned}$$

i.e. $f_7 \notin E_p[\varepsilon]$. Therefore, $\frac{\omega_n}{\lambda_n} = O\left[\varepsilon_n\right]$. The proof of the necessity part in (10) is complete. □

6. Proof of Theorem 2.

We divide the proof of Theorem 2 into two parts.

6.1. Proof of sufficiency.

Step 1. Let us show that if the series in (11) converges and $f \in E_p[\varepsilon]$, then $f \in W_p^{\lambda,\beta}$.

We consider the series

$$\cos\frac{\pi\beta}{2}V_1(\lambda, f) - \sin\frac{\pi\beta}{2}\widetilde{V_1}(\lambda, f) \tag{32}$$
$$+\sum_{n=1}^{\infty}\left\{\cos\frac{\pi\beta}{2}(V_{2^n}(\lambda, f) - V_{2^{n-1}}(\lambda, f)) - \sin\frac{\pi\beta}{2}\left(\widetilde{V_{2^n}}(\lambda, f) - \widetilde{V_{2^{n-1}}}(\lambda, f)\right)\right\},$$

where $V_1(\lambda, f) := \lambda_1 A_1(f, x)$,

$$\begin{aligned}
V_n(\lambda, f) &:= \sigma(\lambda, V_n(f)) = \\
&= \sum_{m=1}^{n}\lambda_m A_m(f, x) + \sum_{m=n+1}^{2n-1}\lambda_m\left(1 - \frac{m-n}{n}\right)A_m(f, x)\ (n \geq 2).
\end{aligned}$$

Let $M > N > 0$. From the inequality $\|f - V_n(f)\|_p \le CE_n(f)_p$ and Lemma 7, using the properties of $\{\lambda_n\}$, we get

$$A := \left\| \sum_{n=N}^{M} \left[\cos\frac{\pi\beta}{2}(V_{2^{n+1}}(\lambda,f) - V_{2^n}(\lambda,f)) - \sin\frac{\pi\beta}{2}\left(\widetilde{V_{2^{n+1}}}(\lambda,f) - \widetilde{V_{2^n}}(\lambda,f)\right)\right]\right\|_p$$

$$\le \sum_{n=N}^{M} \left[|\cos\frac{\pi\beta}{2}| \, \|V_{2^{n+1}}(f) - V_{2^n}(f)\|_p \right.$$

$$\cdot \left(\sum_{m=2^{n-1}-1}^{2^{n+2}+3} |\triangle^2 \lambda_{m+2}|(m+1) + (2^{n+2}-1)|\triangle\lambda_{2^{n+2}-1}| \right)$$

$$+ |\sin\frac{\pi\beta}{2}| \left\| \widetilde{V_{2^{n+1}}}(f) - \widetilde{V_{2^n}}(f) \right\|_p$$

$$\left. \cdot \left(\sum_{m=2^{n-1}-1}^{2^{n+2}+3} |\triangle^2 \lambda_{m+2}|(m+1) + (2^{n+2}-1)|\triangle\lambda_{2^{n+2}-1}| \right) \right]$$

$$+ |\cos\frac{\pi\beta}{2}| \left\| \sum_{n=N}^{M} \lambda_{2^{n+1}-1} \left(V_{2^{n+1}} - V_{2^n}\right)(f) \right\|_p$$

$$+ |\sin\frac{\pi\beta}{2}| \left\| \sum_{n=N}^{M} \lambda_{2^{n+1}-1} \left(\widetilde{V_{2^{n+1}}} - \widetilde{V_{2^n}}\right)(f) \right\|_p$$

$$\le C \left\{ \lambda_{2^{N-1}} \left(|\cos\frac{\pi\beta}{2}| E_{2^{N-1}}(f)_p + |\sin\frac{\pi\beta}{2}| E_{2^{N-1}}(\tilde{f})_p \right) \right.$$

$$\left. + \sum_{n=2^{N-1}}^{\infty} (\lambda_{n+1} - \lambda_n) \left(|\cos\frac{\pi\beta}{2}| \, E_n(f)_p + |\sin\frac{\pi\beta}{2}| \, E_n(\tilde{f})_p \right) \right\}.$$

Further, we apply Lemma 3. Then the convergence of series in (11) and $f \in E_p[\varepsilon]$ imply that there exists $\varphi \in L_p$ such that the series (32) converges to φ in L_p.

Let us show that $\sigma(\varphi) = \sigma(f^{(\lambda,\beta)})$. If F_n is the n-th partial sum of (32), then, say for cosine coefficients, $a_n(\varphi) = a_n(\varphi - F_{N+n}) + a_n(F_{N+n}) = a_n(\varphi - F_{N+n}) + a_n(f^{(\lambda,\beta)})$, and

$$\begin{aligned} a_n(\varphi - F_{N+n}) &= \frac{1}{\pi} \int_{-\pi}^{\pi} (\varphi - F_{N+n})(x) \cos nx \, dx \\ &\le C(p) \|\varphi - F_{N+n}\|_p \longrightarrow 0 \ (N \to \infty). \end{aligned}$$

The proof of the sufficiency part in (11) is complete.
Step 2. Let us prove the sufficiency part in (12).

Let $\sin\frac{\pi\beta}{2}=0$. If $n=0,1$, then the proof comes from (11). If $2^{m-1}+1\le n<2^m, m\in\mathbf{N}$, we consider the best approximant $T_n^*(x)=T_n^*(f,x)$, i.e. $E_n(f)_p=\|f(\cdot)-T_n^*(f,\cdot)\|_p$.

$$\begin{aligned}
E_n(f^{(\lambda,\beta)})_p &\le \left\|f^{(\lambda,\beta)}-T_n^{*(\lambda,\beta)}+V_{2^{m+2}}(f^{(\lambda,\beta)})-V_{2^{m+2}}(f^{(\lambda,\beta)})\right\|_p \\
&\le \left\|T_n^{*(\lambda,\beta)}-V_{2^{m+2}}(f^{(\lambda,\beta)})\right\|_p+\left\|f^{(\lambda,\beta)}-V_{2^{m+2}}(f^{(\lambda,\beta)})\right\|_p.
\end{aligned}$$

By Lemma 7, we obtain

$$\left\|T_n^{*(\lambda,\beta)}-V_{2^{m+2}}(f^{(\lambda,\beta)})\right\|_p\le C(\lambda)\lambda_n\left\|T_n^*-V_{2^{m+2}}(f)\right\|_p\le C(\lambda)\lambda_nE_n(f)_p.$$

Further, applying two times the Abel's transformation and Lemma 7 we write for $M>N$

$$\begin{aligned}
A &= \left\|\sum_{n=N}^{M}\Big(V_{2^{n+1}}(f^{(\lambda,\beta)})-V_{2^n}(f^{(\lambda,\beta)})\Big)\right\|_p \\
&\le \sum_{n=N}^{M}\left\|V_{2^{n+1}}(f)-V_{2^n}(f)\right\|_p\cdot \\
&\cdot \left(\sum_{m=2^n+1}^{2^{n+2}-3}|\triangle^2\lambda_{m+2}|(m+1)+(2^{n+2}-1)|\lambda_{2^{n+2}-1}-\lambda_{2^{n+2}}|\right) \\
&+ \left\|\sum_{n=N}^{M}\lambda_{2^{n+2}-1}\left(V_{2^{n+1}}-V_{2^n}\right)(f)\right\|_p \\
&\le C(\lambda)\left\{\lambda_{2^{N+1}}E_{2^N}(f)_p+\sum_{n=N}^{\infty}E_{2^n}(f)_p\left(\lambda_{2^{n+2}-1}-\lambda_{2^{n+1}-1}\right)\right\}.
\end{aligned}$$

Then

$$\begin{aligned}
&\left\|f^{(\lambda,\beta)}-V_{2^{m+2}}(f^{(\lambda,\beta)})\right\|_p\le \qquad (33)\\
&\le C(\lambda)\left\{\lambda_nE_n(f)_p+\sum_{\nu=n+1}^{\infty}\left(\lambda_{\nu+1}-\lambda_\nu\right)E_\nu(f)_p\right\}.
\end{aligned}$$

Let $\sin \frac{\pi\beta}{2} \neq 0$. Then the convergence of series in (11) implies that there exist $\widetilde{f} \in L_p$, $f^{(\lambda,0)} \in L_p$ and $\widetilde{f}^{(\lambda,0)} \in L_p$. Then

$$\begin{aligned} E_n(f^{(\lambda,\beta)})_p &\leq E_n\left(\cos\frac{\pi\beta}{2} f^{(\lambda,0)} - \sin\frac{\pi\beta}{2}\widetilde{f}^{(\lambda,0)}\right)_p \\ &\leq |\cos\frac{\pi\beta}{2}| E_n\left(f^{(\lambda,0)}\right)_p + |\sin\frac{\pi\beta}{2}| E_n\left(\widetilde{f}^{(\lambda,0)}\right)_p. \end{aligned}$$

Applying (33), we get

$$E_n(f^{(\lambda,0)})_p \leq C(\lambda)\left\{\lambda_n E_n(f)_p + \sum_{\nu=n+1}^{\infty} (\lambda_{\nu+1} - \lambda_\nu)\, E_\nu(f)_p\right\}.$$

By Lemma 3, (33) implies

$$\begin{aligned} E_n\left(\widetilde{f}^{(\lambda,0)}\right)_p &\leq C(\lambda)\left\{\lambda_n E_n(\widetilde{f})_p + \sum_{\nu=n+1}^{\infty} (\lambda_{\nu+1} - \lambda_\nu)\, E_\nu(\widetilde{f})_p\right\} \\ &\leq C(\lambda)\left\{\lambda_n E_n(f)_p + \sum_{\nu=n+1}^{\infty} \frac{\lambda_\nu}{\nu} E_\nu(f)_p\right\}. \end{aligned}$$

Therefore, for all $\beta \in \mathbf{R}$ we have

$$\begin{aligned} E_n(f^{(\lambda,\beta)})_p \leq C(\lambda,\beta)\Bigg\{\lambda_n E_n(f)_p &+ |\cos\frac{\pi\beta}{2}| \sum_{\nu=n+1}^{\infty} (\lambda_{\nu+1} - \lambda_\nu)\, E_\nu(f)_p \\ &+ |\sin\frac{\pi\beta}{2}| \sum_{\nu=n+1}^{\infty} \frac{\lambda_\nu}{\nu} E_\nu(f)_p\Bigg\}. \end{aligned}$$

If $f \in E_p[\varepsilon]$, from (12) we obtain $E_n(f^{(\lambda,\beta)})_p = O(\omega_n)$, i.e. $f \in W_p^{\lambda,\beta}E[\omega]$. The proof of the sufficiency part in (12) is complete.

Step 3. Let us show that if $\frac{1}{\lambda_n} = O(\varepsilon_n)$ and $f \in W_p^{\lambda,\beta}$, then $f \in E_p[\varepsilon]$. Also we shall verify that if $\frac{\omega}{\lambda_n} = O(\varepsilon_n)$ and $f \in W_p^{\lambda,\beta}E[\omega]$, then $f \in E_p[\varepsilon]$. Set $\frac{1}{\lambda} := \left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \cdots\right\}$.

Let $\sin\frac{\pi\beta}{2} = 0$. Applying two times the Abel's transformation and Lemma 7 we have for $M > N$

$$A := \left\| \sum_{n=N}^{M} [V_{2^{n+1}}(f) - V_{2^n}(f)] \right\|_p$$
$$= \left\| \sum_{n=N}^{M} \left[\left(V_{2^{n+1}}(f^{(\lambda,\beta)})\right)^{(\frac{1}{\lambda},0)} - \left(V_{2^n}(f^{(\lambda,\beta)})\right)^{(\frac{1}{\lambda},0)} \right] \right\|_p$$
$$\le \sum_{n=N}^{M} \left\| V_{2^{n+1}}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)}) \right\|_p$$
$$\times \left(\sum_{m=2^n+1}^{2^{n+2}-3} |\triangle^2 \lambda_{m+2}^{-1}|(m+1) + (2^{n+2}-1)|\lambda_{2^{n+2}-2}^{-1} - \lambda_{2^{n+2}-1}^{-1}| \right)$$
$$+ \left\| \sum_{n=N}^{M} \lambda_{2^{n+2}-1}^{-1} \left[V_{2^{n+1}}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)}) \right] \right\|_p \le \frac{C(\lambda)}{\lambda_{2^N}} \left\| f^{(\lambda,\beta)} \right\|_p . \quad (34)$$

Then, from (34), one has

$$E_n(f)_p \le \frac{C(\lambda)}{\lambda_{2^N}} \left\| f^{(\lambda,\beta)} \right\|_p . \quad (35)$$

Let $T_n^*(x)$ be the best approximant for $f^{(\lambda,\beta)}$, i.e. $E_n(f^{(\lambda,\beta)})_p = \|f^{(\lambda,\beta)}(\cdot) - T_n^*(\cdot)\|_p$. Using (35) for $f - T_n^{*(\frac{1}{\lambda},-\beta)}$, we get

$$E_n(f)_p = E_n(f^{(\lambda,\beta)} - T_n^{*(\frac{1}{\lambda},-\beta)})_p \le \frac{C(\lambda)}{\lambda_n} \left\| f^{(\lambda,\beta)}(\cdot) - T_n^*(\cdot) \right\|_p$$
$$= \frac{C(\lambda)}{\lambda_n} E_n(f^{(\lambda,\beta)})_p. \quad (36)$$

Let $\sin\frac{\pi\beta}{2} \ne 0$. Then

$$A = \left\| \sum_{n=N}^{M} \left[\cos\frac{\pi\beta}{2} \left(\left(V_{2^{n+1}}(f^{(\lambda,\beta)})\right)^{(\frac{1}{\lambda},0)} - \left(V_{2^n}(f^{(\lambda,\beta)})\right)^{(\frac{1}{\lambda},0)} \right) \right.\right.$$
$$\left.\left. + \sin\frac{\pi\beta}{2} \left(\left(\widetilde{V}_{2^{n+1}}(f^{(\lambda,\beta)})\right)^{(\frac{1}{\lambda},0)} - \left(\widetilde{V}_{2^n}(f^{(\lambda,\beta)})\right)^{(\frac{1}{\lambda},0)} \right) \right] \right\|_p$$
$$\le \left\| \cos\frac{\pi\beta}{2} \sum_{n=N}^{M} \left[\left(V_{2^{n+1}}(f^{(\lambda,\beta)})\right)^{(\frac{1}{\lambda},0)} - \left(V_{2^n}(f^{(\lambda,\beta)})\right)^{(\frac{1}{\lambda},0)} \right] \right\|_p$$
$$+ \left\| \sin\frac{\pi\beta}{2} \sum_{n=N}^{M} \left[\left(\widetilde{V}_{2^{n+1}}(f^{(\lambda,\beta)})\right)^{(\frac{1}{\lambda},0)} - \left(\widetilde{V}_{2^n}(f^{(\lambda,\beta)})\right)^{(\frac{1}{\lambda},0)} \right] \right\|_p$$

To estimate the first item we apply (34) and to estimate the second one we use Lemmas 7, 8, as well as the condition $\sum\limits_{\nu=n+1}^{\infty} \frac{1}{\nu\lambda_\nu} \le \frac{C}{\lambda_n}$. We get

$$\left\| \sum_{n=N}^{M} V_{2^{n+1}}(f^{(\lambda,\beta)}) - V_{2^n}(f^{(\lambda,\beta)}) \right\|_p \le \frac{C(\lambda,\beta)}{\lambda_{2^N}} \left\| f^{(\lambda,\beta)} \right\|_p .$$

Repeating the argument we used in proving (36) we arrive at the inequality

$$E_n(f)_p \le \frac{C(\lambda,\beta)}{\lambda_n} E_n(f^{(\lambda,\beta)})_p.$$

From this it is clear that $\frac{1}{\lambda_n} = O(\varepsilon_n)$ and $\frac{\omega_n}{\lambda_n} = O(\varepsilon_n)$ are necessary conditions for $W_p^{\lambda,\beta} \subset E_p[\varepsilon]$ and $W_p^{\lambda,\beta}E(\omega) \subset E_p[\varepsilon]$, respectively. The proves of sufficiency parts in (13) and (14) are complete.

6.2. Proof of necessity.

Step 4. Let us show that if $E_p[\varepsilon] \subset W_p^{\lambda,\beta}$, then the series in (11) converges. We suppose the inverse, i.e. that the series in (11) diverges.

Step 4(a): $p = \infty$. We start with the case $\sin \frac{\pi\beta}{2} = 0$.Then we define the series

$$\sum_{\nu=1}^{\infty} (\varepsilon_{\nu-1} - \varepsilon_\nu) \Big(\cos \nu x + \sin \nu x \Big),$$

which converges to the function $f_8 \in L_p$. It can be easily found: $E_n(f_8)_p \le \varepsilon_n$, i.e. $f_8 \in E_p[\varepsilon] \subset W_p^{\lambda,\beta}$. On the other hand,

$$\begin{aligned}
\left\| f_8^{(\lambda,\beta)} \right\|_p &\ge \sum_{\nu=1}^{\infty} \lambda_\nu \left(\varepsilon_{\nu-1} - \varepsilon_\nu\right) \\
&= \lambda_1 \left(\varepsilon_0 - \varepsilon_1\right) + \sum_{\nu=2}^{\infty} \left(\varepsilon_{\nu-1} - \varepsilon_\nu\right) \left[\sum_{m=2}^{\nu} \left(\lambda_m - \lambda_{m-1}\right) + \lambda_1 \right] \\
&= \sum_{m=2}^{\infty} \left(\lambda_m - \lambda_{m-1}\right) \varepsilon_{m-1} + \lambda_1 \varepsilon_0 \\
&= \sum_{n=1}^{\infty} \left(\lambda_{n+1} - \lambda_n\right) \varepsilon_n + \lambda_1 \varepsilon_0 = \infty.
\end{aligned}$$

This contradiction implies the convergence the series in (11).

Let $\sin\frac{\pi\beta}{2}\neq 0$. Since

$$\begin{aligned}\sum_{n=1}^{\infty}(\lambda_{n+1}-\lambda_n)\,\varepsilon_n &\le C\sum_{\nu=0}^{\infty}\lambda_{2^\nu}\varepsilon_{2^\nu}\\ &\le C\left(\sum_{\nu=1}^{\infty}\varepsilon_{2^\nu}\sum_{m=2^{\nu-1}}^{2^\nu-1}\frac{\lambda_m}{m}+\lambda_1\varepsilon_1\right)\le C\sum_{n=1}^{\infty}\frac{\lambda_n}{n}\varepsilon_n,\end{aligned}$$

we have

$$|\cos\frac{\beta\pi}{2}|\sum_{n=1}^{\infty}(\lambda_{n+1}-\lambda_n)\,\varepsilon_n+|\sin\frac{\beta\pi}{2}|\sum_{n=1}^{\infty}\lambda_n\frac{\varepsilon_n}{n}\asymp\sum_{n=1}^{\infty}\lambda_n\frac{\varepsilon_n}{n}. \tag{37}$$

We consider the series $\sum\limits_{n=1}^{\infty}\frac{\varepsilon_{n-1}}{n}\sin nx$. Then, by Lemma 4, the sum of this series, say, $f_9(x)$ is from L_∞. From [Ba1], $E_n(f_9)_p\le C\varepsilon_n$, i.e. $f_9\in E_p[\varepsilon]\subset W_p^{\lambda,\beta}$. We also have

$$\left\|f_9^{(\lambda,\beta)}\right\|_p\ge C\sum_{n=1}^{\infty}\frac{\varepsilon_{n-1}}{n}\lambda_n\ge C\sum_{n=1}^{\infty}\frac{\varepsilon_n}{n}\lambda_n=\infty.$$

Thus, the series in (11) converges.

Step 4(b): $p=1$. First let $\sin\frac{\pi\beta}{2}=0$. We define the series

$$\sum_{\nu=1}^{\infty}(\varepsilon_{\nu-1}-\varepsilon_\nu)\,\tau_\nu(x), \tag{38}$$

where $\tau_{\nu+1}(x)=\sum\limits_{j=1}^{\nu+1}\alpha_j^\nu\sin jx$ and $\alpha_j^\nu=\begin{cases}\frac{j}{\nu+2}, & 1\le j\le\frac{\nu+2}{2}\\ 1-\frac{j}{\nu+2}, & \frac{\nu+2}{2}\le j\le\nu+1\end{cases}.$

The series (38) converges to a $f_{10}\in L_p$ and $E_n(f_{10})_p\le C\varepsilon_n$ (see [Ge]). Then $f_{10}\in E_p[\varepsilon]\subset W_p^{\lambda,\beta}$. One can rewrite (38) in the following way :

$$\sum_{\nu=1}^{\infty}b_\nu\sin\nu x,\quad\text{where}$$

$$b_\nu=\sum_{j=\nu}^{2\nu-2}\left(1-\frac{\nu}{j+1}\right)(\varepsilon_{\nu-1}-\varepsilon_\nu)+\sum_{j=2\nu-1}^{\infty}\frac{\nu}{j+1}\,(\varepsilon_{\nu-1}-\varepsilon_\nu)\,.$$

By Lemma 5, we get

$$\left\|f_{10}^{(\lambda,\beta)}\right\|_1\ge C\sum_{\nu=1}^{\infty}\lambda_\nu\frac{b_\nu}{\nu}\ge C\left(\sum_{\nu=2}^{\infty}(\lambda_{\nu+1}-\lambda_\nu)\varepsilon_\nu+\lambda_3\varepsilon_3\right)=\infty.$$

This contradicts the divergence of the series in (11). Thus, the series in (11) converges.

Let $\sin\frac{\pi\beta}{2}\neq 0$. As we saw in (37), the divergence of the series in (11) in this case is equivalent to the divergence of $\sum\limits_{n=1}^{\infty}\frac{\lambda_n}{n}\varepsilon_n$.

We consider the series

$$\sum_{\nu=1}^{\infty}(\varepsilon_{\nu-1}-\varepsilon_{\nu})\,K_{\nu}(x). \tag{39}$$

This series is convergent in L_1 (see [Ge]) to a function $f_{11}(x)$, and $E_n(f_{11})_p = O\left(\varepsilon_n\right)$. Therefore, $f_{10}\in E_p[\varepsilon]\subset W_p^{\lambda,\beta}$.

One can rewrite (39) in the following way :

$$\sum_{\nu=1}^{\infty}a_{\nu}\cos\nu x,\qquad \text{where}\qquad a_{\nu}=\varepsilon_{\nu-1}-\nu\sum_{j=\nu}^{\infty}\frac{\varepsilon_{j-1}-\varepsilon_j}{j+1}.$$

We note that $\{a_\nu\}$ is monotonic null sequence. Indeed,

$$a_{\nu}-a_{\nu+1}=\sum_{j=\nu}^{\infty}\frac{\varepsilon_{j-1}-\varepsilon_j}{j+1}\geq 0. \tag{40}$$

By Lemma 5, using monotonicity of $\{a_\nu\}$ and conditions on $\{\lambda_\nu\}$, we have

$$\begin{aligned}
\left\|f_{11}^{(\lambda,\beta)}\right\|_1 &\geq C\sum_{\nu=1}^{\infty}\frac{\lambda_\nu}{\nu}a_\nu \geq C\sum_{\nu=1}^{\infty}\lambda_{2^{\nu+1}}a_{2^\nu}\\
&= C\sum_{\nu=1}^{\infty}a_{2^\nu}\left[\sum_{n=1}^{\nu}(\lambda_{2^{n+1}}-\lambda_{2^n})+\lambda_2\right]\\
&\geq C\left(\lambda_1 a_1+\sum_{n=1}^{\infty}(\lambda_{2^{n+1}}-\lambda_{2^n})a_{2^n}\right)\\
&\geq C\left(\lambda_1 a_1+\sum_{n=2}^{\infty}a_{2^n}\sum_{\nu=2^n}^{2^{n+1}-1}(\lambda_{\nu+1}-\lambda_\nu)\right)\\
&\geq C\left(\lambda_1 a_1+\sum_{n=1}^{\infty}(\lambda_{n+1}-\lambda_n)a_n\right).
\end{aligned} \tag{41}$$

On the other hand, by (40), we have

$$\begin{aligned}
\left\|f_{11}^{(\lambda,\beta)}\right\|_1 &\geq \\
&\geq C\sum_{\nu=1}^{\infty}\frac{\lambda_\nu}{\nu}a_\nu = C\left(\sum_{\nu=1}^{\infty}\frac{\lambda_\nu}{\nu}\varepsilon_{\nu-1}-\sum_{\nu=1}^{\infty}\lambda_\nu\sum_{j=\nu}^{\infty}\frac{\varepsilon_{j-1}-\varepsilon_j}{j+1}\right)\\
&\geq C\left(\sum_{\nu=1}^{\infty}\frac{\lambda_\nu}{\nu}\varepsilon_{\nu-1}-\sum_{\nu=1}^{\infty}(a_\nu-a_{\nu+1})\lambda_{\nu+1}\right)\\
&= C\left(\sum_{\nu=1}^{\infty}\frac{\lambda_\nu}{\nu}\varepsilon_{\nu-1}-\sum_{\nu=1}^{\infty}(a_\nu-a_{\nu+1})\left[\sum_{n=1}^{\nu}(\lambda_{n+1}-\lambda_n)+\lambda_1\right]\right)\\
&= C\left(\sum_{\nu=1}^{\infty}\frac{\lambda_\nu}{\nu}\varepsilon_{\nu-1}-\left[\lambda_1a_1+\sum_{n=1}^{\infty}(\lambda_{n+1}-\lambda_n)a_n\right]\right).
\end{aligned}$$

Combining this inequality and (41), we get

$$\left\|f_{11}^{(\lambda,\beta)}\right\|_1 \geq C\sum_{\nu=1}^{\infty}\lambda_\nu\nu^{-1}\varepsilon_{\nu-1}=\infty,$$

and so, the series in (11) converges. The proof of the necessity part in (11) is complete.

Step 5. Let us show the correctness of the necessity part in (12).

Step 5(a): $p=\infty$. We consider the series

$$\begin{aligned}
&\varepsilon_n\cos\Big((n+1)x-\frac{\pi\beta}{2}\Big)+\\
+\;&\sum_{\nu=1}^{\infty}\Big(\cos\frac{\pi\beta}{2}\,(\varepsilon_{\nu-1}-\varepsilon_\nu)\cos\nu x+\sin\frac{\pi\beta}{2}\frac{\varepsilon_{\nu-1}}{\nu}\sin\nu x\Big).
\end{aligned}\tag{42}$$

There exists a function $f_{12,n}\in L_p$ with the Fourier series (42). One can see $E_m(f_{12,n})_p\leq\varepsilon_m$, i.e. $f_{12,n}\in E_p[\varepsilon]\subset W_p^{\lambda,\beta}E[\omega]$. Therefore, $f_{12,n}^{(\lambda,\beta)}\in L_p$. Note that the series

$$\sum_{\nu=1}^{\infty}\Big(\cos\frac{\pi\beta}{2}\,(\varepsilon_{\nu-1}-\varepsilon_\nu)\cos\nu x+\sin\frac{\pi\beta}{2}\frac{\varepsilon_{\nu-1}}{\nu}\sin\nu x\Big)$$

is the Fourier series of a function $f_{13}\in L_p$ and $f_{13}\in E_p[\varepsilon]\subset W_p^{\lambda,\beta}E[\omega]$, i.e. $E_m(f_{13})_p=O(\omega_m)$.

Let us show that the positive constant C_1 in the inequality $E_m(f_{12,n}^{(\lambda,\beta)})_p\leq C_1\omega_m$ $(m=0,1,2,\cdots)$ does not depend on m and n. We have

$$f_{12,n}(x)=f_{13}(x)+\varepsilon_n\cos\Big((n+1)x-\frac{\pi\beta}{2}\Big).$$

Let $m > n$. It is easy to see that

$$f_{12,n}^{(\lambda,\beta)}(x) = f_{13}^{(\lambda,\beta)}(x) + \lambda_n \varepsilon_n \cos(n+1)x.$$

Then, $E_m(f_{12,n}^{(\lambda,\beta)})_p \le E_m(f_{13}^{(\lambda,\beta)})_p \le C_1(f_{13},\lambda,\beta)\omega_m$. We write for $0 \le m \le n$:

$$\begin{aligned} E_m(f_{12,n}^{(\lambda,\beta)})_p &\le E_m(f_{13}^{(\lambda,\beta)})_p + E_m\Big(\lambda_n\varepsilon_n\cos(n+1)x\Big)_p \\ &\le C_1(f_{13},\lambda,\beta)\omega_m + C_2\lambda_n\varepsilon_n. \end{aligned}$$

Hence, by Lemma 6, $E_m(f_{13,n}^{(\lambda,\beta)})_p \le C\omega_m$ where C does not depend on n and m.

Then

$$\begin{aligned} & C\omega_n \\ & \ge E_n(f_{12,n}^{(\lambda,\beta)})_p \\ & \ge C\left[\lambda_n\varepsilon_n + \cos^2\frac{\pi\beta}{2}\sum_{\nu=n+1}^{\infty}(\varepsilon_{\nu-1}-\varepsilon_\nu)\,\lambda_\nu + \sin^2\frac{\pi\beta}{2}\sum_{\nu=n+1}^{\infty}\frac{\lambda_\nu}{\nu}\varepsilon_{\nu-1}\right] \\ & \ge C\Bigg[\lambda_n\varepsilon_n + \cos^2\frac{\pi\beta}{2}\lambda_n\left(\sum_{\nu=n}^{2n}(\varepsilon_\nu-\varepsilon_{\nu+1}) + \varepsilon_{2n+1}\right) + \\ & \qquad + \cos^2\frac{\pi\beta}{2}\sum_{\nu=2n+1}^{\infty}\lambda_\nu\,(\varepsilon_{\nu-1}-\varepsilon_\nu) + \\ & \qquad + \sin^2\frac{\pi\beta}{2}\lambda_n\sum_{\nu=n}^{2n}\frac{\varepsilon_\nu}{\nu} + \sin^2\frac{\pi\beta}{2}\sum_{\nu=2n+1}^{\infty}\lambda_\nu\frac{\varepsilon_{\nu-1}}{\nu}\Bigg] \\ & \ge C\left[\lambda_n\varepsilon_n + |\cos\frac{\pi\beta}{2}|\sum_{\nu=n+1}^{\infty}(\lambda_{\nu+1}-\lambda_\nu)\,\varepsilon_\nu + |\sin\frac{\pi\beta}{2}|\sum_{\nu=n+1}^{\infty}\lambda_\nu\frac{\varepsilon_\nu}{\nu}\right]. \end{aligned}$$

Thus, the relation in the right-hand side of (12) holds.

Step 5(b): $p = 1$. In this case we define the series

$$\begin{aligned} &\varepsilon_n\sin\Big((n+1)x - \frac{\pi\beta}{2}\Big) + \\ &+ \sum_{\nu=1}^{\infty}\Big(-\sin\frac{\pi\beta}{2}\,(\varepsilon_{\nu-1}-\varepsilon_\nu)\,K_\nu(x) + \cos\frac{\pi\beta}{2}\,(\varepsilon_{\nu-1}-\varepsilon_\nu)\,\tau_\nu(x)\Big). \end{aligned} \tag{43}$$

Then there exists a function $f_{14,n} \in L_p$ such that (43) is the Fourier series of $f_{14,n}$ (see [Ge]). Also, $E_m(f_{14,n})_p = O(\varepsilon_m)$, i.e. $f_{14,n} \in E_p[\varepsilon] \subset W_p^{\lambda,\beta}E[\omega]$.

Note that the series

$$\sum_{\nu=1}^{\infty}\Big(-\sin\frac{\pi\beta}{2}\,(\varepsilon_{\nu-1}-\varepsilon_\nu)\,K_\nu(x)+\cos\frac{\pi\beta}{2}\,(\varepsilon_{\nu-1}-\varepsilon_\nu)\,\tau_\nu(x)\Big)$$

is the Fourier series of a function $f_{15}\in L_p$ and $f_{15}\in E_p[\varepsilon]\subset W_p^{\lambda,\beta}E[\omega]$, i.e. $E_m(f_{15})_p=O(\omega_m)$.

We shall prove that the positive constant C_2 in the inequality $E_m(f_{14,n}^{(\lambda,\beta)})_p\le C_2\omega_m$ $(m=0,1,2,\cdots)$ does not depend on m and n. We note

$$f_{14,n}^{(\lambda,\beta)}(x)=f_{15}^{(\lambda,\beta)}(x)+\lambda_n\varepsilon_n\sin(n+1)x.$$

Let $m>n$. Then $E_m(f_{14,n}^{(\lambda,\beta)})_p=E_m(f_{15}^{(\lambda,\beta)})_p\le C(f_{15},\lambda,\beta)\omega_m$. For $0\le m\le n$ we write

$$\begin{aligned}E_m(f_{14,n}^{(\lambda,\beta)})_p &\le E_m(f_{15}^{(\lambda,\beta)})_p+E_m\Big(\lambda_n\varepsilon_n\sin(n+1)x\Big)_p\\ &\le C(f_{15},\lambda,\beta)\omega_m+C\lambda_n\varepsilon_n.\end{aligned}$$

Therefore, we have $E_m(f_{14,n}^{(\lambda,\beta)})_p\le C_2\omega_m$ from Lemma 6.

We rewrite the series (43) in the following way :

$$\varepsilon_n\sin\Big((n+1)x-\frac{\pi\beta}{2}\Big)+\sum_{\nu=1}^{\infty}\Big(-\sin\frac{\pi\beta}{2}a_\nu\cos\nu x+\cos\frac{\pi\beta}{2}b_\nu\sin\nu x\Big),$$

where

$$\begin{aligned}a_\nu &= \varepsilon_{\nu-1}-\nu\sum_{j=\nu}^{\infty}\frac{\varepsilon_{j-1}-\varepsilon_j}{j+1},\\ b_\nu &= \sum_{j=\nu}^{2\nu-2}\left(1-\frac{\nu}{j+1}\right)(\varepsilon_{j-1}-\varepsilon_j)+\sum_{j=2\nu-1}^{\infty}\frac{\nu}{j+1}\,(\varepsilon_{\nu-1}-\varepsilon_\nu)\,.\end{aligned}$$

We have

$$\begin{aligned}f_{14,n}^{(\lambda,\beta)}(x) &\sim \lambda_n\varepsilon_n\sin(n+1)x+\sum_{\nu=1}^{\infty}\lambda_\nu\Big(-\sin\frac{\pi\beta}{2}\cos\frac{\pi\beta}{2}a_\nu\cos\nu x\\ &\quad+\ \sin^2\frac{\pi\beta}{2}a_\nu\sin\nu x+\cos^2\frac{\pi\beta}{2}b_\nu\sin\nu x+\sin\frac{\pi\beta}{2}\cos\frac{\pi\beta}{2}b_\nu\cos\nu x\Big).\end{aligned}$$

We note that if $f(x)\sim\frac{c_0(f)}{2}+\sum\limits_{\nu=1}^{\infty}(c_\nu(f)\cos\nu x+d_\nu(f)\sin\nu x)$, then $E_n(f)_p\ge Cd_{n+1}(f)$. Then

$$\begin{aligned}&E_n(f_{14,n}^{(\lambda,\beta)})_p\ge \qquad (44)\\ &\ge\ C\left(\lambda_n\varepsilon_n+\sin^2\frac{\pi\beta}{2}\lambda_{n+1}a_{n+1}+\cos^2\frac{\pi\beta}{2}\lambda_{n+1}b_{n+1}\right)\ge C\lambda_n\varepsilon_n.\end{aligned}$$

Also $E_n(f_{14,n}^{(\lambda,\beta)})_p \geq C\sin^2\frac{\pi\beta}{2}\lambda_{n+1}a_{n+1}$ and, by Lemma 5, we write for $2^{m-1}\leq n<2^m$

$$\begin{aligned}
E_n(f_{14,n}^{(\lambda,\beta)})_p &\geq C\sin^2\frac{\pi\beta}{2}\sum_{\nu=n+1}^{\infty}\lambda_\nu\frac{a_\nu}{\nu}\\
&\geq C\sin^2\frac{\pi\beta}{2}\left(\sum_{\nu=n+1}^{2^m-1}\lambda_\nu\frac{a_\nu}{\nu}+\sum_{k=m}^{\infty}\lambda_{2^{k+1}}a_{2^k}\right)\\
&\geq C\sin^2\frac{\pi\beta}{2}\sum_{\nu=m}^{\infty}(\lambda_{2^{\nu+1}}-\lambda_{2^\nu})\sum_{k=\nu}^{\infty}a_{2^k}\\
&\geq C\sin^2\frac{\pi\beta}{2}\sum_{\nu=m}^{\infty}(\lambda_{2^{\nu+1}}-\lambda_{2^\nu})a_{2^\nu}\\
&\geq C\sin^2\frac{\pi\beta}{2}\sum_{\nu=2^m}^{\infty}(\lambda_{\nu+1}-\lambda_\nu)a_\nu.
\end{aligned}$$

Hence,

$$E_n(f_{14,n}^{(\lambda,\beta)})_p \geq C\sin^2\frac{\pi\beta}{2}\left(\lambda_{n+1}a_{n+1}+\sum_{\nu=n+1}^{\infty}(\lambda_{\nu+1}-\lambda_\nu)a_\nu\right). \quad (45)$$

On the other hand,

$$\begin{aligned}
&E_n(f_{14,n}^{(\lambda,\beta)})_p \geq\\
&\geq C\sin^2\frac{\pi\beta}{2}\sum_{\nu=n+1}^{\infty}\lambda_\nu\frac{a_\nu}{\nu}\\
&\geq C\sin^2\frac{\pi\beta}{2}\left(\sum_{\nu=n+1}^{\infty}\frac{\lambda_\nu}{\nu}\varepsilon_{\nu-1}-\sum_{\nu=n+1}^{\infty}(a_\nu-a_{\nu+1})\lambda_{\nu+1}\right)\\
&\geq C\sin^2\frac{\pi\beta}{2}\left(\sum_{\nu=n+1}^{\infty}\frac{\lambda_\nu}{\nu}\varepsilon_{\nu-1}-C_1\left[\lambda_{n+1}a_{n+1}+\sum_{\nu=n+1}^{\infty}(\lambda_{\nu+1}-\lambda_\nu)a_\nu\right]\right).
\end{aligned}$$

Applying (45), we get

$$E_n(f_{14,n}^{(\lambda,\beta)})_p \geq C\sin^2\frac{\pi\beta}{2}\sum_{\nu=n+1}^{\infty}\frac{\lambda_\nu}{\nu}\varepsilon_{\nu-1}. \quad (46)$$

Hence, using (44), (46), and Lemma 5, we have

$$\begin{aligned} E_n(f_{14,n}^{(\lambda,\beta)})_p \geq \\ \geq \ C\left(\lambda_n\varepsilon_n+\cos^2\frac{\pi\beta}{2}\sum_{\nu=n+1}^{\infty}\lambda_\nu\frac{b_\nu}{\nu}+\sin^2\frac{\pi\beta}{2}\sum_{\nu=n+1}^{\infty}\lambda_\nu\frac{a_\nu}{\nu}\right) \\ \geq \ C\left(\lambda_n\varepsilon_n+\cos^2\frac{\pi\beta}{2}\sum_{\nu=n+1}^{\infty}\varepsilon_\nu\left(\lambda_{\nu+1}-\lambda_\nu\right)+\sin^2\frac{\pi\beta}{2}\sum_{\nu=n+1}^{\infty}\frac{\lambda_\nu}{\nu}\varepsilon_\nu\right). \end{aligned}$$

This proves the necessity part in (12).

Step 6. Let us show that $W_p^{\lambda,\beta}\subset E_p[\varepsilon]$ implies $\frac{1}{\lambda_n}=O\left[\varepsilon_n\right]$.
As we noticed above (see (29)) it is enough to show that $W_p^{\lambda,\beta}\subset E_p[\varepsilon]$ implies $\frac{\gamma_n}{\lambda_n}=O\left[\varepsilon_n\right]$ for all $\gamma=\{\gamma_n\}\in\Phi$.

We suppose that the last condition does not hold. Then there exist $\gamma=\{\gamma_n\}\in\Phi$ and $\{C_n\uparrow\infty\}$ such that $\frac{\gamma_{m_n}}{\lambda_{m_n}}\geq C_n\varepsilon_{m_n}$. We can choose a subsequence $\{m_{n_k}\}$ such that $\frac{m_{n_{k+1}}}{m_{n_k}}\geq 2$ and $\gamma_{m_{n_k}}\leq 2^{-k}$. Further, we consider the series (30), which is the Fourier series of $f_6\in L_p$ and $f_6^{(\lambda,\beta)}\in L_p$. On the other hand,

$$E_{m_{n_k}}(f_6)_p\geq C\frac{\gamma_{m_{n_k}}}{\lambda_{m_{n_k}}}\geq CC_{n_k}\varepsilon_{m_{n_k}},$$

i.e. $f_6\notin E_p[\varepsilon]$. This contradicts $W_p^{\lambda,\beta}\subset E_p[\varepsilon]$. The proof of the necessity part in (13) is complete.

Step 7. We shall prove that $W_p^{\lambda,\beta}E[\omega]\subset E_p[\varepsilon]$ implies $\frac{\omega_n}{\lambda_n}=O\left[\varepsilon_n\right]$.
If the last condition does not hold, then there exists $\{C_n\uparrow\infty\}$ such that $\frac{\omega_{m_n}}{\lambda_{m_n}}\geq C_n\varepsilon_{m_n}$. We choose a subsequence $\{m_{n_k}\}$ such that $\frac{m_{n_{k+1}}}{m_{n_k}}\geq 2$ and $\omega_{m_{n_k}}\geq\frac{1}{2}\omega_{m_{n_k}}+\omega_{m_{n_{k+1}}}$. The series

$$\sum_{k=0}^{\infty}\frac{\omega_{m_{n_k}}-\omega_{m_{n_{k+1}}}}{\lambda_{m_{n_k}}}\cos(m_{n_k}+1)x$$

is the Fourier series of a function $f_{16}\in L_p$, and $f_{16}^{(\lambda,\beta)}\in W_p^{\lambda,\beta}E[\omega]$. On the other hand,

$$E_{m_{n_k}}(f_{16})_p\geq C\frac{\omega_{m_{n_k}}-\omega_{m_{n_{k+1}}}}{\lambda_{m_{n_k}}}\geq C\frac{\omega_{m_{n_k}}}{\lambda_{m_{n_k}}}\geq CC_{n_k}\varepsilon_{m_{n_k}},$$

i.e. $f_{16}\notin E_p[\varepsilon]$. This contradicts our conjecture $W_p^{\lambda,\beta}E[\omega]\subset E_p[\varepsilon]$. The proof of the necessity part in (14) is complete. □

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