INTERMEDIATE ASYMPTOTICS BEYOND HOMOGENEITY AND SELF-SIMILARITY: LONG TIME BEHAVIOR FOR $u_t = \Delta\phi(u)$

JOSÉ A. CARRILLO, MARCO DI FRANCESCO AND GIUSEPPE TOSCANI

ABSTRACT. We investigate the long time asymptotics in $L^1_+(\mathbf{R})$ for solutions of general nonlinear diffusion equations $u_t = \Delta\phi(u)$. We describe, for the first time, the intermediate asymptotics for a very large class of non-homogeneous nonlinearities ϕ for which long time asymptotics cannot be characterized by self-similar solutions. Scaling the solutions by their own second moment (temperature in the kinetic theory language) we obtain a universal asymptotic profile characterized by fixed points of certain maps in probability measures spaces endowed with the euclidean Wasserstein distance d_2. In the particular case of $\phi(u) \sim u^m$ at first order when $u \sim 0$, we also obtain an optimal rate of convergence in L^1 towards the asymptotic profile identified, in this case, as the Barenblatt self–similar solution corresponding to the exponent m. This second result holds for a larger class of nonlinearities compared to results in the existing literature and is achieved by a variation of the entropy dissipation method in which the nonlinear filtration equation is considered as a perturbation of the porous medium equation.

1. Introduction

We consider the Cauchy problem for a general nonlinear diffusion equation, namely

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta\phi(u) \\ u(\cdot, 0) = u_0. \end{cases} \tag{1}$$

Here, $(x,t) \in \mathbf{R}^N \times \mathbf{R}_+$, while the initial datum u_0 is taken to be nonnegative and belonging in $L^1(\mathbf{R}^N)$. We shall assume throughout this paper that the nonlinearity function ϕ satisfies the following assumptions:

(NL1) $\phi \in C[0,+\infty) \cap C^1(0,+\infty)$, $\phi(0) = 0$ and $\phi'(u) > 0$ for all $u > 0$.

(NL2) $\exists C > 0$ and $m > \frac{N-2}{N}$ such that $\phi'(u) \geq Cu^{m-1}$ for all $u > 0$.

(NL3) $\dfrac{\phi(u)}{u^{1-1/N}}$ is nondecreasing on $u \in (0, \infty)$.

Under assumption **(NL1)**, it is well–known that the Cauchy problem (1) is well–posed for any initial datum in $L^1_+(\mathbf{R}^N)$. We refer to [Bén76, Vér79] for the existence and regularity theory for equation (1) with initial data in L^1, obtained by means of the Crandall–Ligget formula for nonlinear semigroups. Well-posedness and properties of the solutions were further developed in [BC79, BC81, Vér79]. Solutions are mass-preserving, i.e., the following conservation law holds

$$\int_{\mathbf{R}^N} u(x,t)dx = \int_{\mathbf{R}^N} u_0(x)dx.$$

Moreover, the unique generalized solution to (1) may not be classical in general. In case of *slow* diffusion, i.e. $\phi'(0) = 0$, the support of the solution travels with finite speed. This is due to the degeneracy of the parabolic operator as u approaches zero (see [Kal87, Kne77] and the references therein for the general diffusion equation (1); see also the survey paper by Vazquez [Váz03] and the references therein for the porous medium equation $\phi(u) = u^m$ particularly [Váz83]). Concerning the evolution of the integral norms, given $u_0 \in L^p(\mathbf{R}^N)$, the solution $u(\cdot,t)$ of (1) belongs to $L^p(\mathbf{R}^N)$ at any time $t > 0$, and we have

$$\|u(\cdot,t)\|_{L^p(\mathbf{R}^N)} \le \|u_0\|_{L^p(\mathbf{R}^N)}. \tag{2}$$

Under the extra assumption on the nonlinearity ϕ **(NL2)**, the equation (1) enjoys an L^1–L^∞ regularizing property. Indeed, it was proved that the solution to (1) with initial datum in L^1 satisfies the following temporal decay estimate (see [AB79] for the power law case, [Vér79] for the general nonlinear case).

Theorem 1.1 ($\mathbf{L^1}$–$\mathbf{L^\infty}$ **regularizing effect**). *Let $u(x,t)$ be the solution to the Cauchy problem* (1)*, with $u_0 \in L^1_+(\mathbf{R}^N)$. Let the nonlinearity ϕ satisfy assumptions* **(NL1)-(NL2)** *above. Then, at any $t > 0$, $u(x,t) \in L^\infty(\mathbf{R}^N)$ and the following estimate hold*

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^N)} \le C\, t^{-\frac{N}{N(m-1)+2}} \|u_0\|_{L^1(\mathbf{R}^N)}. \tag{3}$$

Assumption **(NL3)** implies that the entropy functional associated to equation (1) is displacement convex [McC97] and thus, the flow map of the nonlinear diffusion (1) is a non-expansive contraction in time with respect to the euclidean Wasserstein distance d_2 in probability measures [Ott01, Agu02, CMV04].

The first result in this work proves the existence of a universal asymptotic profile for general nonlinearities satisfiying assumptions **(NL1)-(NL3)** above. Let us define the "temperature" of a solution u of (1) as its second moment, i.e.,

$$\theta_u(t) = \int_{\mathbf{R}^N} \frac{|x|^2}{2} u(x,t)\, dx.$$

This definition is natural from a kinetic theory point of view, see for instance [CIP94], and is nothing else than the variance of the density $u(x,t)$ from a probabilistic point of view.

Time-dependent rescalings have been used to translate the Cauchy problem for homogeneous nonlinearities $\phi(u) = u^m$ onto nonlinear Fokker-Planck type equations [CT00, Ott01, DdP02]. These changes of variables translate particular self-similar Barenblatt-Pattle solutions onto stationary states for the corresponding nonlinear Fokker-Planck equations. In fact, it is quite easy to check that the change of variables is defined in terms of the explicit value of the temperature of the expected asymptotic profile, i.e., the Barenblatt-Pattle self-similar solution. This fact has been noted recently in [Tos04].

We propose to study the behavior of the solution when we perform a nonlinear time-dependent rescaling depending on the own temperature of the solution. More precisely, given a solution of (1) we will study the long time behavior of

$$\theta_u(t)^{N/2} u(\theta_u(t)^{1/2} x, t). \tag{4}$$

Similar scalings have been used in the analysis of homogeneous cooling states in granular media equations (see for instance [BCP97, Tos03] and the references therein). The nonlinear time dependent scaling (4) can be also seen as the projection of the solution $u(\cdot,t)$ onto the manifold of probability measures with unit temperature (see also [CG03])

$$\mathcal{M} = \left\{ \mu \in \mathcal{P}_2(\mathbf{R}^N), \frac{1}{2}\int_{\mathbf{R}^N} |x|^2 d\mu(x) = 1 \right\}.$$

In order to have the above scaling (4) well defined for all positive times t, in addition we must require the following natural assumption on the nonlinearity ϕ, namely

(FT) For any solution $u(x,t)$ to (1), the following relation holds,

$$\int |x|^2 u(x,0) dx < \infty, \quad \Rightarrow \quad \int |x|^2 u(x,t) dx < \infty \quad \text{for all } t > 0$$

We postpone to section 3 the discussion of providing reasonable sufficient conditions on ϕ such that condition **(FT)** is satisfied. Let us briefly point out that for the slow diffusion case is clearly satisfied while for the fast diffusion case the control of moments is delicate even in the power-law case [CV03], and thus, this case has to be analyzed carefully.

We focus our next section on proving the main result of this work:

Theorem 1.2 (**Asymptotic profile for general nonlinear diffusions**). *Given ϕ verifying the hypotheses* **(NL1)**–**(NL3)**–**(FT)**, *there exists $t_* > 0$ and a one parameter curve of probability densities v_t^∞, with unit temperature defined for $t \geq t_*$ such that, for any solution of* (1) *with initial data $(1 + |x|^2)u_0 \in L_+^1(\mathbf{R}^N)$ of unit mass and temperature,*

$$d_2\left(\theta_u(t)^{N/2}u(\theta_u(t)^{1/2}\ \cdot\ , t), v_t^\infty\right) \longrightarrow 0 \quad \textit{as} \quad t \to \infty.$$

Moreover, the asymptotic profile v_t^∞ is characterized as the unique fixed point of the renormalized *flow map $S(t)$*

$$S(t)u_0 := \theta_u(t)^{N/2}u(\theta_u(t)^{1/2}\cdot, t),$$

where $u(\cdot, t)$ is the solution to (1) *with initial datum u_0.*

The main ingredient of previous theorem is the proof of a contraction property for the maps $S(t)$, $t \geq t_* > 0$, obtained as compositions of the flow map for the nonlinear diffusion equation $u_t = \Delta\phi(u)$ and the projection of the solution onto the unit second moment manifold $\mathcal{M}$. The asymptotic profile v_t^∞ is nothing else that the Barenblatt-Prattle solution at the time in which it has unit temperature in the case of the homogeneous nonlinearities $\phi(u) = u^m$ [Tos04]. Indeed, thanks to our point of view we can generalize the classical notions of self–similarity and source–type solution by means of the idea of *invariance of the solution orbit after projection onto the subset* $\mathcal{M}$. The contraction property makes use of the $L^1 - L^\infty$ regularizing effect to control the behavior of the temperature of the solution.

Let us mention that the existence of 'caloric' self–similar profiles for general diffusion equations has been the subject of [vDP77] in a context of solutions attaining different end–states at infinity. Moreover, the asymptotic behavior of nonlinear diffusion equations has been studied in [BP85] in a bounded domain context. Even if far from our purposes, we also mention [BV04] which contains very recent development in general nonlinear diffusion equations applied to image contour enhancement. As far as we are concerned, our theorem is the first result concerning L^1 asymptotic profiles for such general nonlinearities. We can obtain more detailed information of the asymptotics in specific cases. Let us consider the particular case of asymptotically homogeneous nonlinearities, i.e., let us assume that the nonlinearity ϕ satisfies the additional assumptions

(NL4) $\phi(u) = u^m\psi(u)$ for some $m > \max(\frac{N-1}{N}, \frac{N}{N+2})$, with ψ satisfying:
- **(P1)** $\exists \lim_{u\to 0^+} \psi(u) = l \in (0, +\infty)$ (for simplicity we assume $l = 1$).
- **(P2)** $\psi \in C[0, +\infty) \cap C^1(0, +\infty)$.
- **(P3)** $\psi'(u) = O(u^k)$ as $u \to 0^+$, for some $k > -1$.

(NL5) $h(\cdot) \in L^1_{loc}([0, +\infty))$.

where h is given by

$$h(u) = \int_1^u \frac{\phi'(\eta)}{\eta} d\eta. \tag{5}$$

In this case, we can use a variant of the entropy method for nonlinear homogeneous equations [CT00, Ott01, DdP02] based on the $L^1 - L^\infty$ regularizing effect and on the generalized Log-Sobolev inequalities [CJM+01]. More precisely, we show in section 4:

Theorem 1.3 (**Asymptotically homogeneous nonlinearities**).
Let $u(x,t)$ be the solution to the Cauchy problem (1)*, with nonlinearity ϕ satisfying hypotheses* **(NL1)-(NL5)-(FT)** *above and with initial datum $u_0 \geq 0$ such that*

$$\int_{\mathbf{R}^N} \left[u_0^m(x) + |x|^2 u_0(x) + F(u_0(x)) \right] dx < +\infty$$

with $F(u) = uh(u) - \phi(u)$. Let $u^\infty(x,t)$ be the Barenblatt self–similar function such that u^∞ has the same mass as u_0. Then, the following estimate holds for all $t \geq 0$

$$\|u(t) - u^\infty(t)\|_{L^1(\mathbf{R}^N)} \leq C\,(t+1)^{-\frac{\lambda\delta}{\alpha}},$$

where C, λ, δ and α are explicitly given in section 4.

Let us remark that this particular case was first treated in [Kam75, Kam76] obtaining L^∞ decay and L^1 convergence without rate to the corresponding Barenblatt-Pattle self-similar solution. Recently, another variant of the entropy method was proposed in [BDE02] but the range of nonlinearities in there has some stricter hypotheses that we can avoid by the perturbation argument proposed in section 3 reminiscent of arguments done for convection-diffusion equations in [CF04].

2. Intermediate asymptotics for general nonlinearities

Throughout this section the finite temperature hypothesis **(FT)** is assumed, further discussion about sufficient conditions ensuring **(FT)** is postponed to section 3. We start by performing the following *nonlinear* time dependent scaling

$$u(x,t) = \theta_u(t)^{-N/2} v(y,t), \qquad y = \theta_u(t)^{-1/2} x. \tag{6}$$

Let us remark that the above scaling (6) has the advantage of fixing the temperature or variance of the solution at any time $t > 0$ to be one, namely

$$\int_{\mathbf{R}^N} \frac{|y|^2}{2} v(y,t)\, dy = 1.$$

Let us then analyze the time evolution of the the temperature $\theta_u(t)$ and show that it diverges as $t \to \infty$ with a precise rate.

Lemma 2.1. *Assume the nonlinearity* ϕ *satisfies* **(NL1)–(NL2)** *and* **(FT)**, *then*

$$\theta_u(t) \geq \frac{1}{4} t^{\frac{1}{N(m-1)+2}}. \tag{7}$$

Proof. We have, for an arbitrary $R(t)$,

$$\begin{aligned}\theta_u(t) &= \int_{\mathbf{R}^N} \frac{|x|^2}{2} u(x,t)\,dx = \int_{|x|\geq R(t)} \frac{|x|^2}{2} u(x,t)\,dx + \int_{|x|\leq R(t)} \frac{|x|^2}{2} u(x,t)\,dx \\ &\geq \frac{R(t)^2}{2} \int_{|x|\geq R(t)} u(x,t)\,dx = \frac{R(t)^2}{2} \left[1 - \int_{|x|\leq R(t)} u(x,t)\,dx\right] \\ &\geq \frac{R(t)^2}{2} \left[1 - C_N \|u(t)\|_{L^\infty} R(t)^N\right],\end{aligned}$$

where C_N is the volume of the unit sphere in $\mathbf{R}^N$. Taking into account the smoothing effect in Theorem 1.1, we have

$$\|u(t)\|_{L^\infty} \leq C_0 t^{-\frac{N}{N(m-1)+2}}, \tag{8}$$

where the constant C_0 depends on the mass of the initial datum, and thus, by choosing

$$R(t) = \frac{1}{2C_0} t^{\frac{1}{N(m-1)+2}},$$

we obtain the desired below estimate (7). $\diamond$

We then recall the definition of euclidean Wasserstein distance between any two probability densities u_1 and u_2,

$$d_2(u_1,u_2)^2 = \inf_{u_2 = T_\sharp u_2} \int_{\mathbf{R}^N} |x - T(x)|^2 u_1(x)\,dx,$$

where $u_2 = T_\sharp u_2$ if and only if

$$\int_{\mathbf{R}^N} \psi(x) u_2(x)\,dx = \int_{\mathbf{R}^N}\int_{\mathbf{R}^N} \psi(T(x)) u_1(x)\,dx, \quad \text{for all} \quad \psi \in L^1(\mathbf{R}^N, u_2\,dx),$$

with $T : \mathbf{R}^N \longrightarrow \mathbf{R}^N$ is any measurable map. Previous definition can be extended properly to space of probability measures with bounded second moment $\mathcal{P}_2(\mathbf{R}^N)$ (see for instance [Vil03]).

The flow map of the nonlinear difussion equation (1) enjoys the following property with respect to the euclidean Wasserstein distance. This property is hidden in the arguments in [Ott01, Agu02] and fully proved in a much more general setting in [CMV04].

Theorem 2.2. *Let $u_1(x,t)$ and $u_2(x,t)$ be solutions to* (1) *with the nonlinearity ϕ verifying* **(NL1)**–**(NL3)**–**(FT)** *and with initial data $u_{1,0}$ and $u_{2,0}$ of unit mass and finite temperature, then*

$$d_2(u_1(t), u_2(t)) \leq d_2(u_{1,0}, u_{2,0}), \quad \text{for all } \ t \geq 0. \tag{9}$$

Given $u_1(x,t)$ and $u_2(x,t)$ the solutions to (1) with the nonlinearity ϕ verifying **(NL1)** and **(NL3)** and with initial data $u_{1,0}$ and $u_{2,0}$ of unit mass and finite temperature, we set, for all $t \geq 0$,

$$\theta_1(t) = \int_{\mathbf{R}^N} \frac{|x|^2}{2} u_1(x,t)\, dx, \quad \theta_2(t) = \int_{\mathbf{R}^N} \frac{|x|^2}{2} u_2(x,t)\, dx$$

and $\theta(t) = \min\{\theta_1(t), \theta_2(t)\}$. We define $\overline{u}_i(t)$, $i = 1,2$ at time $t \geq 0$ as follows. Suppose for simplicity that, for $t \geq 0$, $\theta(t) = \theta_1(t)$. Then,

$$\overline{u}_1(x,t) = u_1(x,t) \quad \overline{u}_2(x,t) = \left(\frac{\theta_2(t)}{\theta_1(t)}\right)^{d/2} u_2\left(\left(\frac{\theta_2(t)}{\theta_1(t)}\right)^{1/2} x, t\right)$$

(of course, a symmetric definition holds if $\theta(t) = \theta_2(t)$). It is easily seen that $\overline{u}_1$ and $\overline{u}_2$ have the same temperature $\theta(t) = \theta_1(t)$ at any time $t \geq 0$.

Lemma 2.3.

$$d_2(\overline{u}_1(t), \overline{u}_2(t)) \leq d_2(u_1(t), u_2(t)).$$

Proof. Let $t \geq 0$ be fixed. Suppose $\theta_1(t) = \theta(t)$. Set $\frac{\theta_2(t)}{\theta_1(t)} = 1 + a$, for some $a \geq 0$. Throughout this lemma we use the notation

$$\begin{aligned} u_1(x,t) = \overline{u}_1(x,t) \ &\longrightarrow \ u(x,t) \\ \overline{u}_2(x,t) \ &\longrightarrow \ v(x,t) \\ u_2(x,t) = (1+a)^{-N/2} v((1+a)^{-1/2}x, t) \ &\longrightarrow \ v^a(x,t). \end{aligned}$$

Then, the proof of the lemma comes from the following claim,

$$d_2(u,v) = \inf_{a \geq 0} d_2(u, v^a). \tag{10}$$

To prove (10), we have by definition of d_2,

$$d_2(u, v^a)^2 = \inf_{T_\sharp u = v^a} \int_{\mathbf{R}^N} |x - T(x)|^2 u(x)\, dx.$$

For a nonnegative a, let us denote the dilation

$$d_a(x) = (1+a)^{1/2} x.$$

We claim the following assertion,

$$\left\{T : \mathbf{R}^N \to \mathbf{R}^N, T_\sharp u = v^a\right\} = \left\{d_a \circ T, T : \mathbf{R}^N \to \mathbf{R}^N, T_\sharp u = v\right\}. \tag{11}$$

To prove (11), let $T_\sharp u = v$. Then, we claim that $T^a(x) = (1+a)^{1/2}T(x)$ is such that $T^a_\sharp u = v^a$. To see this, let $\psi \in L^1(\mathbf{R}^N, v^a dx)$, we have (we skip the dependence on time t which is fixed)

$$\int_{\mathbf{R}^N} \psi(x) v^a(x)\, dx = \int_{\mathbf{R}^N} \psi(x)(1+a)^{-d/2} v((1+a)^{-1/2}x)\, dx$$
$$= \int_{\mathbf{R}^N} \psi(x(1+a)^{1/2}) v(x)\, dx = \int_{\mathbf{R}^N} \psi((1+a)^{1/2}T(x)) u(x)\, dx.$$

By inverting the roles of v and v^a, we conclude that the assertion (11) is true.

We go back to the proof of (10). From (11) we have

$$d_2(u, v^a)^2 = \inf_{T_\sharp u = v} \int_{\mathbf{R}^N} \left| x - (1+a)^{1/2}T(x) \right|^2 u(x)\, dx = \inf_{T_\sharp u = v} f_T(a), \quad (12)$$

with

$$f_T(a) = \int_{\mathbf{R}^N} |x|^2 u(x)\, dx + (1+a)\int_{\mathbf{R}^N} |T(x)|^2 u(x) dx$$
$$-2(1+a)^{1/2} \int_{\mathbf{R}^N} x \cdot T(x) u(x)\, dx.$$

Computing directly the first derivative of $f_T(a)$ w.r.t. a yields

$$f_T'(a) = \int_{\mathbf{R}^N} |T(x)|^2 u(x) - (1+a)^{-1/2} \int_{\mathbf{R}^N} x \cdot T(x) u(x)\, dx$$
$$\geq \int_{\mathbf{R}^N} |T(x)|^2 u(x) dx - (1+a)^{-1/2} \left[2\theta(t)\right]^{1/2} \left[\int_{\mathbf{R}^N} |T(x)|^2 u(x)\, dx \right]^{1/2}$$

and since u and v have the same (finite) temperature, we have by definition of push–forward

$$\int_{\mathbf{R}^N} |T(x)|^2 u(x)\, dx = \int_{\mathbf{R}^N} |x|^2 v(x)\, dx = \int_{\mathbf{R}^N} |x|^2 u(x)\, dx = 2\theta(t).$$

Therefore, $f_T'(a) \geq 0$ for all $a \geq 0$. Hence, the minimum is attained at $a = 0$, which proves (10) and the lemma. $\diamond$

Let us consider again any two solutions u_1 and u_2 of (1). By performing the time dependent scaling (6), with v_i, $i = 1, 2$, denoting the two rescaled solutions, it is easily seen that

$$v_i(y, t) = \theta(t)^{N/2} \overline{u}_i(\theta(t)^{1/2} y, t), \quad i = 1, 2,$$

where again $\theta(t)$ is the minimum between the two temperatures θ_1 and θ_2. Hence, by a well known scaling property of the Wasserstein distance (see

[Vil03]), we obtain

$$d_2^2(v_1(t), v_2(t)) = \theta(t)^{-1} d_2^2(\overline{u}_1(t), \overline{u}_2(t)). \tag{13}$$

Our first result is a straightforward consequence of collecting lemmas 2.3 and 2.1 and identity (13).

Theorem 2.4. *Let $u_1(x,t)$ and $u_2(x,t)$ be solutions to* (1) *with the nonlinearity ϕ verifying* **(NL1)–(NL3)–(FT)** *and with initial data $u_{1,0}$ and $u_{2,0}$ of unit mass and finite temperature, Let v_1 and v_2 be their* projections onto the 'unit second moment sphere' *in $\mathcal{P}_2(\mathbf{R}^N)$ as in* (6)*. Then, for all $t \geq 0$,*

$$d_2^2(v_1(t), v_2(t)) \leq \theta(t)^{-1} d_2^2(u_{1,0}, u_{2,0}) \to 0 \quad \text{as} \quad t \to \infty. \tag{14}$$

Our intention now is to recover the existence of some fixed points from the result in Theorem 2.4. To perform this task, we consider the complete metric space of probability measures

$$\mathcal{M} = \left\{ \mu \in \mathcal{P}_2(\mathbf{R}^N), \frac{1}{2} \int_{\mathbf{R}^N} |x|^2 d\mu(x) = 1 \right\},$$

endowed with the 2–Wasserstein distance d_2 (see [Vil03]).

Let us define a 'renormalized' flow map $S(t) : \mathcal{M} \longrightarrow \mathcal{M}$ as follows. For u_0 belonging to the subspace of absolutely continuous with respect to Lebesgue measures $\mathcal{M} \cap L^1(\mathbf{R}^N)$, we consider the unique solution $u(x,t)$ to equation (1) with initial datum u_0 and its projection $v(y,t)$ given by the scaling (6). We then set $S(t)u_0 := v(t)$ by definition. Theorem 2.4 and Lemma 2.1 implies that the map $S(t) : \mathcal{M} \cap L^1(\mathbf{R}^N) \longrightarrow \mathcal{M} \cap L^1(\mathbf{R}^N)$ verifies

$$d_2^2(S(t)(u_{1,0}), S(t)(u_{2,0})) \leq 4t^{-\frac{1}{N(m-1)+2}} d_2^2(u_{1,0}, u_{2,0}) \tag{15}$$

We now trivially extend the map $S(t)$ to the whole space $\mathcal{M}$ by using property (15) of $S(t)$.

Consider $\mu \in \mathcal{M}$. There exists a sequence $u_n \in \mathcal{M} \cap L^1(\mathbf{R}^N)$ such that $d_2(u_n, \mu) \to 0$ as $n \to \infty$. It suffices to consider a normalized regularization via standard mollifiers ρ_n

$$v_n = \rho_n * \mu.$$

We then perform the mass preserving scaling $u_n(x) = \alpha_n v_n(\alpha_n x)$, where α_n is the temperature of v_n. Since the sequence u_n is tight and converges in the sense of measures to μ, then $\alpha_n \to 1$ (see [Vil03]). Moreover, it is easily seen than u_n converges also to μ in the sense of measures. Therefore, we conclude that $d_2(u_n, \mu) \to 0$.

Since the map $S(t)$ verifies (15), then the sequence $\{S(t)u_n\}_n$ is a Cauchy sequence in $\mathcal{M}$. By completeness of $\mathcal{M}$, $S(t)u_n \to \nu$. We define $\nu = S(t)\mu$. It is easy to see based again on (15) that this definition is independent on the choice of the sequence u_n.

Once the map $S(t)$ is defined properly for all $t > 0$, we can show the proof of Theorem 1.2.

Proof of Theorem 1.2. Let us consider $t_\star$ large enough such that $\theta(t) < 1$ for $t \geq t_\star$. Let us remark that $t_\star$ does not depend on the initial data due to Lemma 2.1. Therefore, the maps $S(t) : \mathcal{M} \longrightarrow \mathcal{M}$ are global contractions on the complete metric space $\mathcal{M}$. By Banach fixed point theorem, there exists a family of measures $\{\mu_t^\infty,\ t > t_\star\} \subset \mathcal{M}$ such that $S(t)\mu_t^\infty = \mu_t^\infty$.

We claim that the family of fixed points is actually a family of probability densities. This is easily proven by a uniform estimate of this family in some L^p norm, $p > 1$, and then by weak compactness in L^1. The uniform estimate in L^p, $p > 1$ is obtained as follows. Each of the fixed points μ_t^∞ can be obtained as a limit of an iteration sequence of densities v_n with mass one. By the L^1–L^∞ smoothing effect, Theorem 1.1, this sequence is uniformly bounded in L^p, $p > 1$ (we recall that the time t is fixed) and thus, the measure μ_t^∞ is absolutely continuous with respect to the Lebesgue measure with density $v_t^\infty \in \mathcal{M} \cap L^1(\mathbf{R}^N)$.

Moreover, given any solution $u(x,t)$ to the Cauchy problem

$$\begin{cases} u_t = \Delta\phi(u) \\ u(x,0) = u_0(x) \end{cases} \tag{16}$$

where $u_0 \in L^1_+(\mathbf{R}^N)$, $\int_{\mathbf{R}^N} u_0 = 1$, $\int_{\mathbf{R}^N} u_0|x|^2 = 1$, we have

$$d_2\left(\theta_u(t)^{N/2} u(\theta_u(t)^{1/2}\ \cdot\ , t), v_t^\infty\right) \longrightarrow 0 \quad \text{as} \quad t \to \infty, \tag{17}$$

where $\theta_u(t)$ is the temperature of $u(t)$ by a direct application of property (15) and the trivial estimate of $d_2(u_1, u_2)$ in terms of second moments of the measures u_1 and u_2. $\diamond$

The above procedure can be easily generalized to the case of initial datum with arbitrary temperature by simply defining the space $\mathcal{M}_{\theta_0}$ to be the space of probability measures with temperature equal the temperature of the initial datum θ_0, and the projection onto this space instead of the rescaled function $v(t)$ as above. Thus, one obtains a generalized family of maps $S_{\theta_0}(t)$, and a family of fixed points $v_{\theta_0,t}^\infty \in \mathcal{M}_{\theta_0}$ satisfying that

$$d_2\left(\left(\frac{\theta_u(t)}{\theta_0}\right)^{N/2} u\left(\left(\frac{\theta_u(t)}{\theta_0}\right)^{1/2}\ \cdot\ , t\right), v_{\theta_0,t}^\infty\right) \longrightarrow 0 \quad \text{as} \quad t \to \infty,$$

for all solutions of (1) with initial temperature θ_0.

Next, by projecting the two arguments of the left hand side above onto $\mathcal{M}$, by means of elementary scaling properties of the Wasserstein distance, one easily obtains

$$d_2\left(\theta_u(t)^{N/2}u(\theta_u(t)^{1/2}\ \cdot\ ,t), \theta_0^{N/2}v^\infty_{\theta_0,t}(\theta_0^{1/2}\ \cdot)\right) \longrightarrow 0 \quad \text{as} \quad t\to\infty, \tag{18}$$

for all solutions to the Cauchy problem (16) with nonnegative initial datum with unit mass and temperature θ_0. Therefore, we can write the following corollary:

Corollary 2.5. *There exists a one parameter curve of probability densities $v^\infty_{\theta_0,t}$, with temperature θ_0 and $t \geq t_* > 0$ such that, for any solution of* (1) *with ϕ verifying* **(NL1)**–**(NL3)**–**(FT)** *and with initial data $(1+|x|^2)u_0 \in L^1_+(\mathbf{R}^N)$ of unit mass and temperature θ_0,*

$$d_2\left(\theta_u(t)^{N/2}u(\theta_u(t)^{1/2}\ \cdot\ ,t), \theta_0^{N/2}v^\infty_{\theta_0,t}(\theta_0^{1/2}\ \cdot)\right) \longrightarrow 0 \quad \textit{as} \quad t\to\infty.$$

Moreover, the asymptotic profile $v^\infty_{\theta_0,t}$ is uniquely characterized.

Final Remarks.

(1) Let us finally stress that in the homogeneous case $\phi(u) = u^m$, the family $v^\infty_{\theta_0,t}$ does not depend on t and coincides with the Barenblatt self-similar profile at the time in which it has temperature θ_0. This result is a consequence of the main theorem in [Tos04] and the uniqueness of the fixed points $v^\infty_{\theta_0,t}$.
(2) It is an open problem to obtain qualitative behavior of these asymptotic profiles $v^\infty_{\theta_0,t}$ in the full generality of hypotheses **(NL1)**–**(NL3)**.
(3) The results in the next section suggests the Barenblatt profile as the unique limit of the family $v^\infty_{\theta_0,t}$ as $t\to\infty$ in case of ϕ satisfying the additional assumptions **(NL4)**–**(NL5)**. In general, it is an open problem to detect an eventual limit, see for instance the case $\phi(u) = u\log u + 4u^2$, for which it is not possible to detect a power–like behavior at $u = 0$.
(4) Some of the arguments above like the projection onto the unit sphere of $\mathcal{P}_2(\mathbf{R}^N)$ and the fixed point arguments are reminiscent of ideas in [CG03, CMV04].

3. Evolution of the temperature

The aim of this section is to provide reasonable sufficient conditions for the global finiteness of the temperature of solutions to the equation

$$u_t = \Delta\phi(u), \tag{19}$$

under assumption **(NL1)**. For the sake of clearness, we first analyze the well known homogeneous case $\phi(u) = u^m$ in the range of adiabatic exponents

$m > 1 - \frac{1}{N}$. We recall that this range of nonlinearities includes the porous medium case $m > 1$, the fast diffusion case with finite entropy equilibrium $\max\left\{1 - \frac{1}{N}, \frac{N}{N+2}\right\} \leq m \leq 1$, and the very fast diffusion case $0 < m < \frac{1}{3}$, $N = 1$. It is well known that the typical asymptotic states for those cases are given by the self–similar Barenblatt–Prattle profiles. All of these self–similar profiles attain a centered Dirac delta at time zero and they have finite temperature at any time in the range $m \geq \max\left\{1 - \frac{1}{N}, \frac{N}{N+2}\right\}$. On the other hand, the temperature of the self–similar profile in the very fast diffusion case blows up instantaneously after time $t = 0$.

Proposition 3.1. *Let* $m \geq \max\left\{1 - \frac{1}{N}, \frac{N}{N+2}\right\}$. *Given* $u(x,t)$ *the solution to the Cauchy problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta u^m \\ u(\cdot, 0) = u_0, \end{cases} \tag{20}$$

with initial datum $(1 + |x|^2)u_0 \in L^1_+(\mathbf{R}^N)$, *then the temperature is finite at any time, i.e.* $\int |x|^2 u(x,t)dx < \infty$ *for all times* $t > 0$.

Proof. We recall the contraction property of the 2–Wasserstein distance, Theorem 2.2,

$$d_2(u_1(t), u_2(t)) \leq d_2(u_1(0), u_2(0)),$$

for any pair of solutions u_1 and u_2). Since the Barenblatt profile $\overline{u}(t)$ has finite temperature at any time, therefore, we have by triangulation (δ_0 is the Dirac mass centered at zero)

$$\begin{aligned} &\left[\int u(x,t)|x|^2 dx\right]^{1/2} = d_2(u(t), \delta_0) \leq d_2(u(t), \overline{u}(t)) + d_2(\overline{u}(t), \delta_0) \\ &\leq d_2(u_0, \overline{u}(0)) + \left[\int |x|^2 \overline{u}(t) dx\right]^{1/2} \\ &\leq \left[2\int u_0(x)|x|^2 dx + 2\int |x|^2 \overline{u}(0) dx\right]^{1/2} + \left[\int |x|^2 \overline{u}(t) dx\right]^{1/2} < \infty. \diamond \end{aligned}$$

We now attack the problem in the full generality of assumption **(NL1)**. We start by computing the time evolution of the temperature as follows,

$$\frac{d}{dt}\int |x|^2 u dx = -2\int_{\mathbf{R}^N} x \cdot \nabla\phi(u) dx = 2N\int_{\mathbf{R}^N} \phi(u) dx. \tag{21}$$

Hence, a straightforward computation yields the following first result for the *slow* diffusion case.

Proposition 3.2. *Let ϕ satisfy **(NL1)** and the following generalized slow diffusion assumption*

(SD1) $\phi(u) \leq Cu^\alpha$, $C > 0$, $\alpha \geq 1$, *for u in some compact interval* $[0, K]$.

*Then, condition **(FT)** is satisfied.*

Proof. The proof is a straightforward consequence of the L^p control (3) and the regularization procedure to construct the solutions of the nonlinear filtration equation. $\diamond$

In case condition **(SD1)** is not satisfied, identity (21) does not provide a direct estimate of the temperature. However, by repeating the triangulation argument in the proof of proposition 3.1 we can easily deduce the following general property for diffusion equations (19).

Proposition 3.3. *Suppose that there exist at least one solution $\overline{u}(x,t)$ to (19) with initial datum having finite temperature such that $\int u(x,t)|x|^2 dx < \infty$ for all times $t > 0$. Then, condition **(FT)** is satisfied.*

As a consequence of the above proposition, we only need to find one solution $\overline{u}(t)$, even for a restrictive initial data, in order to control the temperature of any solution with initial second moment bounded.

We perform this task by requiring some further assumption on the initial data. Let us first recall the definition of the enthalpy

$$h(u) = \int_1^u \frac{\phi'(\xi)}{\xi} d\xi.$$

In the sequel we shall assume that condition **(NL5)** holds, so that the $H(u) = \displaystyle\int_0^u h(\xi) d\xi$ is well defined. Trivial calculations show that

$$H(u) = uh(u) - \phi(u)$$

(see [CJM$^+$01]). We then define the following generalized entropy functional

$$E(u) = \int H(u) + \frac{1}{2}\int |x|^2 u dx. \tag{22}$$

We recall that the (convex) functional (22) attains its minimum under the constraint of unit mass on the unique ground state u^∞ provided the following assumption, in the fast diffusion case $h(0_+) = -\infty$, is met (see [CJM$^+$01])

(FD1) If $h(0_+) = -\infty$ then there is $C \in \mathbf{R}$ such that $u^\infty = h^{-1}(C - \frac{|x|^2}{2}) \in L^1(\mathbf{R}^N)$, with unit integral and $E(u^\infty) < \infty$.

For more properties of the functional (22) we refer to [CJM$^+$01]. The main result of this section is the following.

Theorem 3.4. *Let ϕ be such that condition* **(FD1)** *holds and let us also impose*

(FD2) *There exists a positive α such that $\phi'(u)u^\alpha$ is nondecreasing on* **R**

Then, condition **(FT)** *is satisfied.*

Proof. Let us compute the evolution of the entropy functional $E(u)$ suitably combined with the term $\int \phi(u)$. Given α by condition **(FD2)**, after integration by parts we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\int \frac{1}{\alpha}\int \phi(u(t))dx + \int H(u(t))dx + \frac{1}{2}\int |x|^2 u(t)dx\right] \\
&= \frac{1}{\alpha}\int \phi'(u)\Delta\phi(u)dx + \int h(u)\Delta\phi(u)dx + \frac{1}{2}\int |x|^2\Delta\phi(u)dx \\
&= -\frac{1}{\alpha}\int \phi''(u)\phi'(u)\left|\nabla u\right|^2 dx - \int \frac{\phi'(u)^2}{u}\left|\nabla u\right|^2 dx - \int x\cdot\nabla\phi(u)dx \\
&= \int\left(-\frac{1}{\alpha}\frac{\phi''(u)}{\phi'(u)} - \frac{1}{u}\right)(\phi'(u))^2\left|\nabla u\right|^2 dx + d\int \phi(u)dx.
\end{aligned}$$

We compute

$$\left(-\frac{1}{\alpha}\frac{\phi''(u)}{\phi'(u)} - \frac{1}{u}\right) = -\frac{d}{du}\left(\frac{1}{\alpha}\log\phi'(u) + \log u\right) = -\frac{d}{du}\log\left(u\phi'(u)^{1/\alpha}\right) \le 0,$$

where the last inequality is justified by **(FD2)** and by the monotonicity of the log. Then, by denoting the relative entropy

$$RE(u|u^\infty) = E(u) - E(u^\infty) \ge 0,$$

we deduce

$$\begin{aligned}
\frac{d}{dt}&\left[\int \frac{1}{\alpha}\phi(u(t))dx + RE(u(t)|u^\infty)\right] \le d\int \phi(u(t))dx \\
&\le d\alpha\left[\int \frac{1}{\alpha}\phi(u(t))dx + RE(u|u^\infty)\right],
\end{aligned}$$

which implies

$$\left[\int \frac{1}{\alpha}\phi(u(t))dx + RE(u|u^\infty)\right] \le Ae^t$$

by Gronwall inequality, and consequently

$$\int \phi(u(t))dx \le Ae^t$$

for some positive constant A. We finally compute

$$\frac{d}{dt}\int |x|^2 u(t)dx = d\int \phi(u)dx \le Ae^t$$

which implies that the temperature of u is finite at any positive time.

Some of the computations above are formal since we have disregarded the control of the tails at infinity of the solutions but the arguments can be riguroulsy proved by means of standard approximations procedures [Váz92, Váz03, CV03]. $\diamond$

4. Long time asymptotics for asymptotically homogeneous nonlinearity

In this section we will assume that the nonlinearity function ϕ satisfies the following additional assumptions:

(NL4) $\phi(u) = u^m\psi(u)$ for some $m > \max(\frac{N-1}{N}, \frac{N}{N+2})$, with ψ satisfying:
- **(P1)** $\exists \lim_{u\to 0^+} \psi(u) = l \in (0, +\infty)$ (for simplicity we assume $l = 1$).
- **(P2)** $\psi \in C[0, +\infty) \cap C^1(0, +\infty)$.
- **(P3)** $\psi'(u) = O(u^k)$ as $u \to 0^+$, for some $k > -1$.

(NL5) $h(\cdot) \in L^1_{loc}([0, +\infty))$, where h is given by (5).

We are interested in the study of the time–asymptotic behaviour of the solution to (1). It is already known that such behaviour is well described by the Barenblatt–Pattle self–similar solutions to the porous medium equation $u_t = \Delta u^m$ (where m is the same exponent as in conditions **(NL2)** and **(NL4)** above, see [Kam75, BDE02]).

We first perform the following, by now standard, time–dependent scaling, in order to put in evidence the role of the term u^m in the nonlinearity ϕ.

$$u(x,t) = R(t)^{-N\lambda} v(y,s) y = xR(t)^{-\lambda} \qquad s = \lambda \log R(t)$$
$$R(t) = \left(1 + \frac{t}{\lambda}\right)\lambda = \frac{1}{N(m-1)+2} \tag{23}$$

As usual in this framework, equation (1) turns into the following non linear (time–dependent) Fokker–Planck type equation

$$\frac{\partial v}{\partial s} = \nabla\cdot(yv) + e^{Nms}\Delta\phi(e^{-Ns}v), \tag{24}$$

with initial datum $v(y, 0) = u_0(y)$. In the sequel it will be useful to write equation (24) as follows

$$\frac{\partial v}{\partial s} = \nabla\cdot\left[v\nabla\left(\frac{|y|^2}{2} + e^{N(m-1)s}h(e^{-Ns}v)\right)\right], \tag{25}$$

We close these preliminaries by re–stating the contraction property (2) and temporal decay estimate (3) in terms of the new unknown function v and the new independent variables (y, s). The estimate for the L^p–norm, for $p \in [1, +\infty)$, follows by interpolation.

Proposition 4.1.

(a) *Let $v(y,s)$ be the solution to (24) with initial datum $u_0 \in L^1_+(\mathbf{R}^N)$. Then, for all $p \in [1,+\infty)$ the following estimate holds for some fixed constant $C > 0$*

$$\|v(\cdot,s)\|_{L^p(\mathbf{R}^N)} \le Ce^{N\left(1-\frac{1}{p}\right)s}\left[\lambda\left(e^{\frac{s}{\lambda}}-1\right)\right]^{-\lambda N\left(1-\frac{1}{p}\right)}\|u_0\|_{L^1(\mathbf{R}^N)}, \tag{26}$$

while, for $p = +\infty$ we have

$$\|v(\cdot,s)\|_{L^\infty(\mathbf{R}^N)} \le Ce^{Ns}\left[\lambda\left(e^{\frac{s}{\lambda}}-1\right)\right]^{-\lambda N}\|u_0\|_{L^1(\mathbf{R}^N)}. \tag{27}$$

Hence, for any fixed $s_0 > 0$ and for all $p \in [0,+\infty]$, we have

$$\sup_{s\ge s_0}\|v(\cdot,s)\|_{L^p(\mathbf{R}^N)} \le C(s_0)\|u_0\|_{L^1(\mathbf{R}^N)}, \tag{28}$$

where $C(s_0)$ doesn't depend on p.

(b) *Let $v(y,s)$ be the solution to (24) with initial datum $u_0 \in L^p(\mathbf{R}^N)$. Then, the following local stability property holds at any $s \ge 0$*

$$\|v(\cdot,s)\|_{L^p(\mathbf{R}^N)} \le e^{N\left(1-\frac{1}{p}\right)s}\|u_0\|_{L^p(\mathbf{R}^N)}. \tag{29}$$

A note about the notation. In the sequel we denote by C a generic positive constant. Sometimes we shall indicate its dependence on some parameters by means of the expressions $C(\ldots)$ or $C_{\ldots}$.

4.1. **Statement of the problem and result.** Let v be the solution to the Cauchy problem

$$\begin{cases} \dfrac{\partial v}{\partial s} = \nabla\cdot(yv) + e^{Nms}\Delta\phi(e^{-Ns}v) \\ v(y,0) = u_0(y). \end{cases} \tag{30}$$

Due to the conservation of the mass, we can set

$$\int_{\mathbf{R}^N} u_0(y)dy = \int_{\mathbf{R}^N} u(y,s)dy = M.$$

We expect the solution $v(y,s)$ to behave like the rescaled Barenblatt similarity function

$$v^\infty(y) = \begin{cases} \left(C_M - \lambda\dfrac{|y|^2}{2}\right)_+^{\frac{1}{m-1}} & \text{if } \ m \ne 1 \\ C_M e^{-\frac{|y|^2}{2}} & \text{if } \ m = 1, \end{cases} \tag{31}$$

as $s \to +\infty$, where the constant C_M is chosen in such a way that $v^\infty(y)$ has mass M, and λ depends only on m and on the space dimension N. We emphasize that v^∞ is *not* a solution to equation (30). Let us then define our entropy functional

$$H(v) = \begin{cases} \dfrac{1}{m-1}\displaystyle\int_{\mathbf{R}^N} v(y)^m dy + \dfrac{1}{2}\int_{\mathbf{R}^N} |y|^2 v(y) dy & \text{if } m \neq 1 \\ \displaystyle\int_{\mathbf{R}^N} v(y)\log v(y) dy + \dfrac{1}{2}\int_{\mathbf{R}^N} |y|^2 v(y) dy & \text{if } m = 1 \\ \dfrac{1}{1-m}\displaystyle\int_{\mathbf{R}^N} \left(mv(y) - v(y)^m\right) dy + \dfrac{1}{2}\int_{\mathbf{R}^N} |y|^2 v(y) dy & \text{if } m < 1. \end{cases} \tag{32}$$

Here m is the exponent describing the behavior of the nonlinearity at zero, given by condition **(NL4)**. We recall that the convex functional $H(v)$ attains its minimum over $L^1_+(\mathbf{R}^N)$, under the constraint $\int_{\mathbf{R}^N} v =$ constant, at the state v^∞ with mass M (see [CT00]). The *relative* entropy is defined, as usual, by

$$H(v|v^\infty) = H(v) - H(v^\infty).$$

The above functional $H(v|v^\infty)$ is related to a Dirichlet–type integral, the so–called *entropy production* or *generalized Fisher information*, by means of the following generalized Log-Sobolev–type inequality (see [CT00, CJM$^+$01]).

Theorem 4.2. *Let $v \in L^1_+(\mathbf{R}^N)$ such that $\int_{\mathbf{R}^N} v(y)dy = M$, let v^∞ be the ground state defined in (31) with mass M. Then, the following inequality holds,*

$$H(v) - H(v^\infty) \leq \frac{1}{2} I(v|v^\infty), \tag{33}$$

where

$$I(v|v^\infty) = \int_{\mathbf{R}^N} v \left| \nabla \left(\frac{m}{m-1} v^{m-1} + \frac{|y|^2}{2} \right) \right|^2 dy,$$

where m *is the same exponent as in the definition (32) of the entropy functional .*

We also recall the generalized *Csiszár–Kullback* inequality, which provides an upper bound of the L^1 norm of the difference between any positive density v and the ground state v^∞ having the same mass as v, in terms of their relative entropy. More precisely, we have

Theorem 4.3. *Let $v \in L^1_+(\mathbf{R}^N)$, with $\int_{\mathbf{R}^N} v(y)dy = \int_{\mathbf{R}^N} v^\infty(y)dy$. Then, the following holds*

$$\|v - v^\infty\|^\alpha_{L^1(\mathbf{R}^N)} \leq C[H(v) - H(v^\infty)], \tag{34}$$

where

$$\alpha = \begin{cases} 2 & \textit{if } m \leq 2 \\ m & \textit{if } m \geq 2, \end{cases} \tag{35}$$

where m *is again the exponent in the definition (32) of the entropy functional.*

In the sequel we shall also need the *generalized* entropy functional (see [BDE02])

$$E(v,s) = e^{mNs}\int_{\mathbf{R}^N} F(e^{-Ns}v)dy + \frac{1}{2}\int_{\mathbf{R}^N}|y|^2 v dy, \tag{36}$$

where

$$F(u) = \int_0^u h(\theta)d\theta, \tag{37}$$

and h is the enthalpy defined in (5). The function F is well–defined thanks to condition **(NL5)**.

We observe that the following identity holds

$$F(u) = uh(u) - \phi(u).$$

In what follows, we shall assume the solution to enjoy enough regularity in order to be treated as a classical solution; the rigorous justification of our result then directly follows by an approximation argument used in [Ott01], subsequently generalized for equations of the form (24) in [CJM$^+$01].

Let us restate the main result of this section already announced as Theorem 1.3.

Theorem 4.4. *Let $v(y,s)$ be the solution to the Cauchy problem (30). Let the initial datum $v_0 \in L^1_+(\mathbf{R}^N)$ be such that*

$$\int_{\mathbf{R}^N} F(v_0(y))dy + \frac{1}{2}\int_{\mathbf{R}^N}|y|^2 v_0(y)dy + \int_{\mathbf{R}^N} v_0(y)^m dy < +\infty, \tag{38}$$

with F given by (37). Then, the relative entropy functional $H(v(s)|v^\infty)$ satisfies

$$H(v(s)|v^\infty) \le Ce^{-\delta s}, \quad \text{for all} \ \ s \ge 0, \tag{39}$$

*where $\delta = \min\{2, N(k+1)\}$, k is the exponent in the structural condition **(NL4)**, and $C > 0$ is a constant depending on the mass of v_0 and on the bounded quantity in (38).*

The Csiszár Kullback inequality (34) above and the time dependent scaling (23), then, provides the rate of convergence in L^1 for the solution to the original problem (1) towards the Barenblatt self–similar functions

$$u^\infty(x,t) = \Big(C - \lambda|x|^2 t^{-\frac{2}{N(m-1)+2}}\Big)_+^{\frac{1}{m-1}}. \tag{40}$$

Corollary 4.5. *Let $u(x,t)$ be the solution to the Cauchy problem (1), with initial datum $u_0 \geq 0$ such that*

$$\int_{\mathbf{R}^N} \left[u_0^m(x) + |x|^2 u_0(x) + F(u_0(x))\right] dx < +\infty.$$

Let $u^\infty(x,t)$ be the Barenblatt self–similar function given by (40) with the constant C such that $\int u_\infty = \int u_0$. Then, the following estimate holds for all $t \geq 0$

$$\|u(t) - u^\infty(t)\|_{L^1(\mathbf{R}^N)} \leq C(t+1)^{-\frac{\lambda\delta}{\alpha}}, \tag{41}$$

where λ is defined in (23) and α is defined by (35).

Remark 4.6. The rate of convergence in (41) has been already obtained in [BDE02]. However, our result includes more general nonlinearity functions. In particular, we don't need to require the technical condition $(m-1)uh(u) - mf(u) \leq 0$, which is not satisfied (for instance) by the nonlinearity function $\phi(u) = u^2 + P(u)$, where $P(u)$ is a suitable standard regularization of the function

$$p(u) = \begin{cases} -u^3 & \text{if } 0 \leq u \leq 1/2 \\ -1/8 & \text{if } u \geq 1/2. \end{cases}$$

Moreover, our approach seems to be much simpler than that in [BDE02], since we make use the usual nonlinear version of the Log–Sobolev inequality, and we use directly the Csiszar–Kullback inequality (34) in order to get a rate of convergence in L^1.

4.2. **Proof of Theorem 4.4.** In order to prove theorem 4.4, we perform an estimate for large times s, which is obtained basically by means of the regularizing effect (27) and of the Log-Sobolev–type inequality (33). Then we use the generalized entropy functional (36)in order to control the evolution of the entropy in a finite time interval.

Proposition 4.7. *Under the assumptions **(NL1)**–**(NL5)** on the nonlinearity ϕ, there exist an $s_0 > 0$ and a positive constant $C(s_0)$ depending on s_0 such that, if $H(v(s_0)) < +\infty$, then we have*

$$H(v(s)|v^\infty) \leq C(s_0)e^{-\min\{2,N(k+1)\}s} \tag{42}$$

for all $s \geq s_0$.

Proof. We recall that the equation (30) can be written in the alternative way (25). Then, integration by parts yields

$$\begin{aligned}\frac{d}{ds}H(v(s)) =& \int_{\mathbf{R}^N}\left[\frac{mv^{m-1}}{m-1} + \frac{|y|^2}{2}\right]\nabla\cdot\left[v\nabla\left(\frac{|y|^2}{2} + e^{N(m-1)s}h(e^{-Ns}v)\right)\right]dy \\ =& -\int_{\mathbf{R}^N} v\nabla\left(\frac{mv^{m-1}}{m-1} + \frac{|y|^2}{2}\right)\cdot\nabla\left(\frac{|y|^2}{2} + e^{N(m-1)s}h(e^{-Ns}v)\right)dy.\end{aligned}$$

Hence, we employ the structural condition **(NL4)** to obtain

$$
\begin{aligned}
&\frac{d}{ds}H(v(s)) = \\
&= \; -\int_{\mathbf{R}^N} \psi(e^{-Ns}v)\left|\nabla\left(\frac{m}{m-1}v^{m-1}-\frac{|y|^2}{2}\right)\right|^2 dy + \\
&\quad -me^{-Ns}\int_{\mathbf{R}^N} v^{2m-2}\psi'(e^{-Ns}v)\,|\nabla v|^2\, dy + \\
&\quad -e^{-Ns}\int_{\mathbf{R}^N} v^m\psi'(e^{-Ns}v)\left[y\cdot\nabla\left(\frac{m}{m-1}v^{m-1}\right)\right] dy \\
&= \; -\int_{\mathbf{R}^N} v\left(\psi(e^{-Ns}v)+\frac{1}{m}\psi'(e^{-Ns}v)e^{-Ns}v\right)\left|\nabla\left(\frac{mv^{m-1}}{m-1}-\frac{|y|^2}{2}\right)\right|^2 dy + \\
&\quad +e^{Nms}\int_{\mathbf{R}^N} \left(e^{-Ns}v\right)^m \psi'(e^{-Ns}v)\left[y\cdot\nabla(e^{-Ns}v)\right] dy + \\
&\quad +\frac{1}{m}\, e^{-Ns}\int_{\mathbf{R}^N} v^2\psi'(e^{-Ns}v)|y|^2 dy = \sum_{j=1}^{3} I_j. \qquad (43)
\end{aligned}
$$

We first compute the term I_2. We observe that, from the hypothesis **(P3)** on the nonlinearity, the function $g(u) = u^m\psi'(u)$ is is summable over any interval $[0, L)$, $L > 0$. Hence, the primitive

$$G(u) = \int_0^u \theta^m\psi'(\theta)d\theta$$

is well defined on $[0, +\infty)$. As a consequence of that, and after integration by parts, I_2 may be written as follows

$$I_2 = -Ne^{Nms}\int_{\mathbf{R}^N} G(e^{-Ns}v)dy.$$

Again from the structural hypothesis **(P3)**, it follows easily that $G(u)$ is a $(m+k+1)$–Hölder function on a neighborhood of $u = 0$. This fact, together with estimate (27), yields

$$
\begin{aligned}
I_2 &\leq C(M, s_0)e^{-N(k+1)s}\int_{\mathbf{R}^N} v^{m+k+1}dy \\
&\leq C(M, s_0)\|v\|^{k+1}_{L^\infty(\mathbf{R}^N)}e^{-N(k+1)s}\int_{\mathbf{R}^N} v^m dy \\
&\leq C(M, s_0)e^{-N(k+1)s}H(v(s)),
\end{aligned}
$$

for some $s_0 > 0$ (chosen in order to have $e^{-Ns}v$ small) and for any $s \geq s_0$. In a very similar way, we estimate I_3

$$I_3 \leq C(M, s_0)e^{-N(k+1)} \int_{\mathbf{R}^N} v^{2+k}|y|^2 \leq C(M, s_0)e^{-N(k+1)} H(v(s)),$$

for s_0 large and $s \geq s_0$.

The integral term I_1 may be written as follows.

$$I_1 = -\int_{\mathbf{R}^N} \alpha(s)v \left| \nabla \left(\frac{m}{m-1} v^{m-1} + \frac{|y|^2}{2} \right) \right|^2 dy,$$

where $\alpha(s) = \psi(e^{-Ns}v) + \frac{1}{m}\psi'(e^{-Ns}v)e^{-Ns}v$. By means of the structural condition **(NL2)** and the first order Taylor expansion, we have

$$\alpha(s) = 1 + e^{-Ns}v\psi'(e^{-Ns}\eta) + \frac{1}{m}e^{-Ns}v\psi'(e^{-Ns}v), \quad \eta \in [0, v(s)].$$

We employ once again the regularizing effect (27) to get

$$\alpha(s) \geq 1 - C(M, s_0)e^{-N(k+1)s}, \quad s \geq s_0,$$

and we can choose s_0 large enough in order to have $\alpha(s) \geq 0$ for any $s \geq s_0$. Hence, we put the above estimates into (43) and we use the Sobolev–type inequality (33) to recover

$$\begin{aligned}
\frac{d}{ds} H(v(s)|v^\infty) &\leq -2\left(1 - C(M, s_0)e^{-N(k+1)s}\right) H(v(s)|v^\infty) + \\
&\quad + C(M, s_0)e^{-N(k+1)s} H(v(s)) \\
&\leq -2\left(1 - C(M, s_0)e^{-N(k+1)s}\right) H(v(s)|v^\infty) + \\
&\quad + C(M, s_0)e^{-N(k+1)s} H(v(s_0)).
\end{aligned}$$

Finally, we use the variation of constants formula and obtain the desired estimate (42). □

Next, we perform a local–in–time estimate of the entropy, in order to control the constant $C(s_0)$ in the inequality (42).

Lemma 4.8. *Suppose that the initial datum u_0 satisfies (38). Then, the following inequality holds at any $s \geq 0$,*

$$E(v(s)) \leq e^{(m-1)s} E(v_0), \tag{44}$$

where $E(u)$ is defined in (36). In particular, the entropy $H(v(s))$ is uniformly bounded on any finite time interval $[0, s_0]$.

Proof. We calculate the evolution in time of the functional $E(v(s), s)$ defined in (36) (see [BDE02]). After integration by parts, we get

$$\begin{aligned}\frac{d}{ds}E(v(s)) = &-\int_{\mathbf{R}^N} v\left|y + e^{(m-1)Ns}\nabla h(e^{-Ns}v)\right|^2 dy+\\ &+ e^{mNs}\int_{\mathbf{R}^N}\left[(m-1)ve^{-Ns}h(e^{-Ns}v) - m\phi(e^{-Ns}v)\right] dy\\ &\leq e^{mNs}(m-1)\int F(e^{-Ns}v)dy \leq (m-1)E(v(s)),\end{aligned}$$

which proves (44). The last assertion comes directly from (29) in case $m > 1$. In case $m \leq 1$, we observe that the absolute entropy $H(v)$ is bounded from below (we are in the finite entropy equilibrium range, see section 3). Moreover, $H(v) \leq \int v$, and therefore the absolute entropy is uniformly bounded. □

Remark 4.9. The entropy dissipation method has been successfully used in [CF04] in order to prove convergence towards Barenblatt solutions for diffusion dominated convection–diffusion equations. By means of the same approach in the present paper, one can generalized the results in [CF04] to a generalized convection–diffusion model where the power law in the diffusion term is replaced by a more general ϕ satisfying the hypothesis above.

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José A. Carrillo,
Institució Catalana de Recerca i Estudis Avançats (ICREA) and
Departament de Matemàtiques,
Universitat Autònoma de Barcelona,
E-08193 Bellaterra, Spain.
E-mail: carrillo@mat.uab.es

Marco Di Francesco,
Dipartimento di Matematica,
Università di L'Aquila,
I-67100 L'Aquila, Italy.
E-mail: difrance@univaq.it

Giuseppe Toscani,
Dipartimento di Matematica,
Università di Pavia,
I-27100 Pavia, Italy.
E-mail: toscani@dimat.unipv.it