COMBINATORIAL AND METRIC PROPERTIES OF THOMPSON'S GROUP T

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Abstract. We discuss metric and combinatorial properties of Thompson's group T, such as the normal forms for elements and uniqueness of tree pair diagrams. We relate these properties to those of Thompson's group F when possible, and highlight combinatorial differences between the two groups. We define a set of unique normal forms for elements of T arising from minimal factorizations of elements into convenient pieces. We show that the number of carets in a reduced representative of T estimates the word length, that F is undistorted in T, and that cyclic subgroups of T are undistorted. We show that every element of T has a power which is conjugate to an element of F and describe how to recognize torsion elements in T.

1. Introduction

Thompson's groups F, T and V are a remarkable family of infinite, finitely-presentable groups studied for their own interesting properties as well as for their connections with questions in logic, homotopy theory and measure theory of discrete groups.

Cannon, Floyd and Parry give an excellent introduction to these groups in [5]. These three groups can be studied either algebraically, analytically or geometrically. Algebraically, each has both finite and infinite presentations. Geometrically, an element in each group can be viewed as a *tree pair diagram*; that is, as a pair of finite binary rooted trees with the same number of leaves, with a numbering system pairing the leaves in the two trees. Analytically, an element of each group can be viewed as a self map of the unit interval:

- in F as a piecewise linear homeomorphism,
- in T as a homeomorphism of the unit interval with the endpoints identified, and thus of S^1,

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- in V as a right-continuous bijection which is locally orientation preserving.

Thompson's group F in particular has been studied extensively. The group F has a standard infinite presentation in which every element has a unique normal form, and a standard two-generator finite presentation. Fordham [8] presented a method of computing the word length of $w \in F$ with respect to the standard finite generating set directly from the tree pair diagram representing w. Regarding F as a diagram group, Guba [10] also obtained an effective geometric method for computing the word metric with respect to the standard finite generating set. Belk and Brown [1] have similar results which arise from viewing elements of F as forest diagrams.

In this paper, we discuss analogues for T of some properties of F, such as normal forms for elements. We consider metrically how F is contained as a subgroup of T, and show that the number of carets in a reduced tree pair diagram representing $w \in T$ estimates the word length of w with respect to a particular generating set. We show that cyclic subgroups of T are undistorted and that every element of T has a power which is conjugate to an element of F. The groups T and V, unlike F, contain torsion elements, and we describe how to recognize these torsion elements from their tree pair diagrams.

2. Background on Thompson's groups F and T

2.1. **Presentations and tree pair diagrams.** Thompson's groups F and T both have representations as groups of piecewise-linear homeomorphisms. The group F is the group of orientation-preserving homeomorphisms of the interval $[0,1]$, where each homeomorphism is required to have only finitely many discontinuities of slope, called *breakpoints*, have slopes be powers of two and have the coordinates of the breakpoints all lie in the set of dyadic rationals. Similarly, the group T consists of orientation-preserving homeomorphisms of the circle S^1 satisfying the same conditions where we represent the circle S^1 as the unit interval $[0,1]$ with the two endpoints identified.

Cannon, Floyd and Parry give an excellent introduction to Thompson's groups F, T and V in [5]. We refer the reader to this paper for full details on results mentioned in this section. Both F and T can be studied either through finite or infinite presentations. With respect to the infinite presentation

$$\langle x_i, i \geq 0 \,|\, x_j x_i = x_i x_{j+1}, i < j \rangle$$

for F, group elements have simple normal forms which are unique. It is easy to see that F can be generated by x_0 and x_1, which form the standard finite generating set for F, and yield the finite presentation

$$\langle x_0, x_1 \mid [x_0x_1^{-1}, x_0^{-1}x_1x_0], [x_0x_1^{-1}, x_0^{-2}x_1x_0^2]\rangle.$$

The group T also has both a finite and an infinite presentation. The infinite presentation is given by two families of generators, $\{x_i, i \geq 0\}$, the same generators as in the infinite presentation of F, a family $\{c_i, i \geq 0\}$ of torsion elements, and the following relators:

(1) $x_jx_i = x_ix_{j+1}$, if $i < j$
(2) $c_nx_k = x_{k+1}c_{n+1}$, if $k < n$
(3) $c_n = x_0c_{n+1}^2$
(4) $c_nx_n = c_{n+1}$.

Using the first three relators, we see that only the generators x_0, x_1 and c_1 are required to generate the group, since the other generators can be obtained from these three. In the following, we will use c to denote the generator c_1. The group T is finitely presented using the following relators, both with respect to the infinite generating set and the finite generating set $\{x_0, x_1, c\}$:

(1) $[x_0x_1^{-1}, x_0^{-1}x_1x_0]$
(2) $[x_0x_1^{-1}, x_0^{-2}x_1x_0^2]$
(3) $c_2x_1 = x_2c_3$, (that is, $x_1^{-1}cx_0 = cx_1$)
(4) $c_1 = x_0c_2^2$, (that is, $c = x_0(x_1^{-1}cx_0)^2$)
(5) $c_1x_1 = c_2$, (that is, $x_1^{-1}cx_0x_1 = x_0^{-1}x_1x_0x_1^{-2}cx_0^2$)
(6) $c^3 = 1$.

As with Thompson's group F, we will frequently work with the more convenient infinite set of generators when constructing normal forms for elements and performing computations in the group. We will need to express elements with respect to the finite generating set when discussing word length.

A convenient representation for an element w in F or T is a tree pair diagram, as discussed in [5]. A *tree pair diagram* is a pair of finite rooted binary trees with the same number of vertices, together with a numbering of the valence one vertices. A node of the tree together with its two downward directed edges is called a *caret*. Valence one vertices of these trees are called *exposed leaves*. In F, we insist that both leaf numberings begin at 0 and increase from left to right. In T, the numberings need only increase cyclically from left to right.

The *left side* of the tree consists of the root caret, and all carets connected to the root by a path of left edges; the *right side* of the tree is defined analogously. A caret is called a *left caret* if its left leaf lies on the left side of the tree. A caret is called a *right caret* if it is not the root caret and its right leaf lies on the right side of the tree. All other carets are called *interior*.

A caret is called *exposed* if it contains two exposed leaves. We write $w = (T_-, T_+)$ to express w as a tree pair diagram, and refer to T_- as the *source* tree and T_+ as the *target* tree. Such a tree pair diagram is not unique. There are many possible diagrams representing a given element. We can choose the cyclic ordering for elements of T to always begin with 0 on the leftmost leaf of the source tree, and furthermore we impose a natural reduction condition: if $w = (T_-, T_+)$ and both trees contain a caret with two exposed leaves numbered n and $n+1$, then we remove these carets and renumber the leaves, thus forming a representative for w with fewer carets and leaves. A tree pair diagram which admits no such reductions is called a *reduced tree pair diagram*, and any element of F is represented by a unique reduced tree pair diagram. When we write $w = (T_-, T_+)$ below, we are assuming that the tree pair diagram is reduced unless otherwise specified.

When $w \in F$ or $w \in T$, we denote the number of carets in either tree of a tree pair diagram representing w by $N(w)$. When p is a word in the generators of F or T, then p represents an element w in either F or T, and we write $N(p)$ interchangeably with $N(w)$.

If $w = (T_-, T_+) \in F$, then the leaves in both trees are numbered from left to right, beginning with zero. In this case, the subdivisions of the interval are paired in increasing order, so that the intervals with zero as their left endpoint are paired, and the intervals with one as their right endpoint are paired. We may omit leaf numberings for elements of F for brevity without any ambiguity. If $w \in T$, then w corresponds to a homeomorphism of S^1 rather than $[0, 1]$. In elements of T, we can omit most leaf numbers for brevity by adopting the following convention from [5]: we understand the leaves in the source tree to be numbered from 0 to n beginning with the leftmost leaf, and indicate by a circle or zero which of the leaves in the target tree is paired with the first leaf in the source tree. Other leaf numberings can be deduced from this single mark using the cyclic order. For this reason, we often refer to tree pair diagrams representing elements of T as *marked tree pair diagrams.*

For example, the element c corresponds to the homeomorphism of S^1 given by

$$c(t) = \begin{cases} \frac{1}{2}t + \frac{3}{4} & \text{if } 0 \leq t < \frac{1}{2} \\ 2t - 1 & \text{if } \frac{1}{2} \leq t < \frac{3}{4} \\ t - \frac{1}{4} & \text{if } \frac{3}{4} \leq t \leq 1 \end{cases}$$

and has the marked tree pair diagram given in Figure 1.

2.2. **Group Multiplication in F and T.**

Group multiplication in F and T corresponds to composition of homeomorphisms, which we can interpret on the level of tree pair diagrams as well. First, we consider $u, v \in F$, where

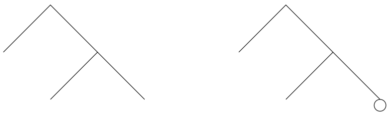

FIGURE 1. The tree pair diagram for the generator c in T.

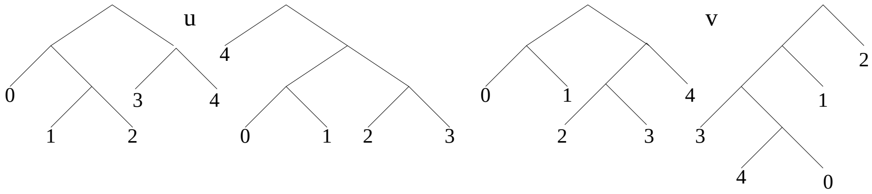

FIGURE 2. The tree pair diagram for sample elements u and v in T.

$u = (T_-, T_+)$ and $v = (S_-, S_+)$. To compute the tree pair diagram corresponding to the product vu, we create unreduced representatives (T'_-, T'_+) and (S'_-, S'_+) of the two elements in which $T'_+ = S'_-$. Then the product is represented by the possibly unreduced tree pair diagram (T'_-, S'_+).

To multiply tree pair diagrams representing elements of T we follow a similar procedure. We let $u, v \in T$, where $u = (T_-, T_+)$ and $v = (S_-, S_+)$. To compute the tree pair diagram corresponding to the product vu, we create unreduced representatives (T'_-, T'_+) and (S'_-, S'_+) of the two elements in which $T'_+ = S'_-$ as trees. The product vu will be represented by the pair (T'_-, S'_+) of trees. To decide which leaf in S'_+ to mark with the zero, we just note that it should be the leaf which is mapped onto by the zero leaf in T'_-. To identify this leaf, we find the zero leaf in T'_+. Since $T'_+ = S'_-$ as trees, this leaf viewed as a leaf in S'_- will be labelled m. Then the leaf labelled m in S'_+ will be the new zero leaf in the tree pair diagram (T'_-, S'_+) for vu. Alternately, we can follow the composition in both pairs of trees to see how the leaves map to each other. This constructed tree pair diagram will represent vu and is not necessarily reduced. For an example of this multiplication, see Figures 2, 3 and 4.

3. WORDS AND DIAGRAMS

3.1. Normal forms and tree pair diagrams in F.

With respect to the infinite presentation for F given above, every element of F has a unique normal form. To describe these, we first observe that any w can be written in the form

$$w = x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_k}^{r_k} x_{j_l}^{-s_l} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

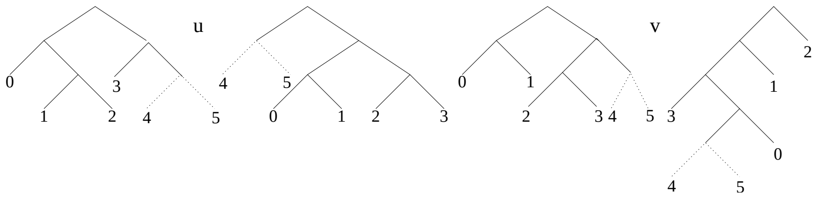

FIGURE 3. Unreduced versions of u and v for the multiplication vu in T, with carets added to perform the multiplication indicated with dashes. Now the target tree of u has the same shape as the source tree of v, allowing the composition.

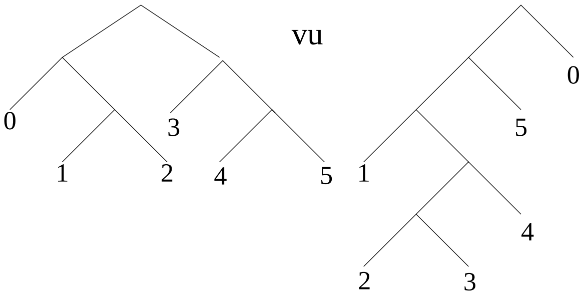

FIGURE 4. The tree pair diagram representing the product vu.

where $r_i, s_i > 0$, $0 \leq i_1 < i_2 \ldots < i_k$ and $0 \leq j_1 < j_2 \ldots < j_l$. However, this expression is not unique. Uniqueness is guaranteed by the addition of the following condition: when both x_i and x_i^{-1} occur in the expression, so does x_{i+1} or x_{i+1}^{-1}, as discussed by Brown and Geoghegan [2]. When we refer to elements of F in normal form, we mean this unique normal form.

If the normal form for $w \in F$ contains no generators with negative exponents, we refer to w as a *positive word* and similarly, we say a normal form is a *negative word* if there are no generators with positive exponents.

We call any word which has the form

$$w = x_{i_1}^{r_1} x_{i_2}^{r_2} \ldots x_{i_k}^{r_k} x_{j_l}^{-s_l} \ldots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

where $r_i, s_i > 0$, $0 \leq i_1 < i_2 \ldots < i_k$ and $0 \leq j_1 < j_2 \ldots < j_l$, a word in *pq form*, where p is the positive part of the normal form and q the negative part. The normal form for an element of F is the shortest word among all words in pq form representing the given element.

To any (not necessarily reduced) tree pair diagram (T_-, T_+) for an element of F we may associate a word in pq form representing the element, using the *leaf exponents* in the target and source trees. When the leaves of a finite rooted binary tree are numbered from left to right, beginning with zero, the leaf exponent of leaf k is the integer length of the longest string of left edges of carets which originates at leaf k and does not reach the right side of the tree. A tree pair diagram then gives the word

$$x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n} x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$$

precisely when leaf i_k in T_+ has exponent r_k, leaf j_k in T_- has leaf exponent s_k, and generators which do not appear in the word correspond to leaves with exponent zero. We think of this word as the pq factorization of the element given by the particular tree pair diagram. On the other hand, any word in pq form can be translated into a tree pair diagram. Furthermore, under this correspondence for F, reduced tree pair diagrams correspond exactly to normal forms. For examples of this correspondence, see [5, 6, 7].

We observe that if an exposed caret has leaves numbered i and $i+1$, then leaf $i+1$ must have leaf exponent zero, since it is a right leaf. If both trees in a tree pair diagram have exposed carets with leaves numbered i and $i+1$, then the corresponding normal form, computed via leaf exponents, contains the generators x_i to both positive and negative indices, but no instances of the generator x_{i+1}. This is precisely the situation when the normal form can be reduced by a relator of F. Thus the condition that the normal form is unique is exactly the condition that the tree pair diagram is reduced. This correspondence will be extended to elements of T in the next section.

3.2. **Tree pair diagrams for elements of T.** We now discuss the relationship between words in T and tree pair diagrams. The relationship is more complicated in T than it is in F. The representation of elements of T by marked tree pair diagrams suggests a way to decompose an element of T into a product of three elements: the positive and negative parts together with a torsion part in the middle, as described in [5].

Definition 3.1. *Let the marked tree pair diagram (T_-, T_+) represent $g \in T$. If T_- and T_+ each have $i+1$ carets, then we let R be the* all-right tree *which has $i+1$ carets, all of which lie on the right side of the tree. We can write g as a product of:*

(1) *$a \in F$ with tree pair diagram (T_-, R) and a has negative normal form u,*
(2) *a cyclic permutation c_i^j for some i where $1 \leq j \leq i+1$ (which permutes the leaf numbering in R), and*

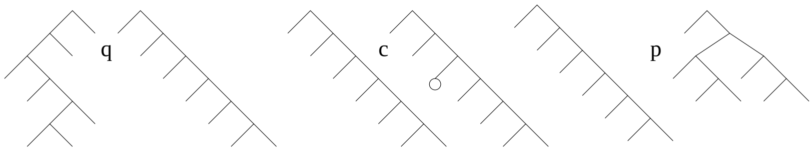

FIGURE 5. Three tree pair diagrams representing the word $x_1x_2c_5^5x_2^{-2}x_1^{-1}x_0^{-2}$ factorized as *pcq*.

(3) *$b \in F$ represented by (R, T_+), where b has positive normal form v, ignoring the leaf numbering on T_+.*

Then the word $w = vc_i^j u$ is called the pcq factorization *of g associated to the marked tree pair diagram (T_-, T_+). In the special case where $g \in F \subset T$, the pcq factorization will just be the usual pq factorization, as we consider the c part of the word to be empty.*

Figure 5 illustrates an example of an element of T decomposed in this way.

The following theorem follows from the existence of these decompositions, and an algebraic proof of this result is found in [5].

Theorem 3.2 ([5], Theorem 5.7)**.** *Any element $x \in T$ admits an expression of the form*

$$x_{i_1}^{r_1}x_{i_2}^{r_2}\dots x_{i_n}^{r_n}\,c_i^j\,x_{j_m}^{-s_m}\dots x_{j_2}^{-s_2}x_{j_1}^{-s_1},$$

where $0 \le i_1 < i_2 < \cdots < i_n$ and $0 \le j_1 < j_2 < \cdots < j_m$ and either $1 \le j < i+2$ or c_i^j is not present.

We refer to any word satisfying the hypotheses of Theorem 3.2 as a word in *pcq* form for an element of T (just as words of this form with no c_i^j term are called words in *pq* form in the group F) . Neither proof of the existence of *pcq* forms gives an easy explicit method for transforming a general word in the generators $x_i^{\pm 1}, c_i$ into *pcq* form without resorting to drawing tree pair diagrams, so we will outline such a method below. We recall that the four types of relators we are using in T are:

1. $x_jx_i = x_ix_{j+1}$, if $i < j$
2. $c_nx_k = x_{k+1}c_{n+1}$, if $k < n$
3. $c_n = x_0c_{n+1}^2$
4. $c_nx_n = c_{n+1}$

Lemma 3.3 (Pumping Lemma)**.** *The generators x_i and c_j of T satisfy*

$$c_n^m = x_{m-1}c_{n+1}^{m+1} \qquad \text{if } 1 \le m < n+2.$$

Proof. This follows immediately from the relators of the types (2) and (3). □

We consider a word $w \in T$ written in the generators $\{x_i, c_j\}$. To put w into *pcq* form, we must move all the positive powers of the x_i to the left and all negative powers to the right, leaving only generators c_i in the middle. We can use relators of types (1), (2) and (4) to accomplish that in the following way. If there is a c_n followed by an x_k, we apply a relator of type (2) to switch them if $n > k$. If $n \leq k$, then we use the Pumping Lemma to increase the index of the c_n until it is large enough, and then we use the equation

$$c_k^m x_k = c_k^{m-1} c_{k+1} = x_{m-2} c_{k+1}^{m+1},$$

where the first equality is a relator of type (4) and the second equality follows from the Pumping Lemma. The same procedure can be used for negative powers of the x_i by taking inverses in the relators and the Pumping Lemma. During this process, if two generators of the type x_n need to be moved past each other, we use the relators of type (1).

The result of this process is a product of a positive word in F, several powers of different elements of the form c_j, and a negative word in F. To combine a product of several c_j into a single element c_k, we use the Pumping Lemma. We can always combine $c_i c_j$ to obtain an expression with a single c_k by raising the lower index via the Pumping Lemma, at the cost of potentially accumulating some positive powers of x_i generators at the front of the expression or negative powers of x_i at the back of the expression. For example,

$$c_3^2 c_5 = x_1 c_4^3 c_5 = x_1 x_2 c_5^4 c_5 = x_1 x_2 c_5^5.$$

Once the resulting expression contains a single c_i generator, we can again use the relators of type (1), the relators in F, to arrange the positive and negative words so that the generators have the appropriate increasing or decreasing order by index.

The relationship between words in *pq* form and tree pair diagrams in F is different than the relationship between *pcq* forms and tree pair diagrams in T. In F, every tree pair diagram has a *pq* factorization associated to it, and any word in *pq* form is in fact the *pq* factorization associated to a (not necessarily unique) tree pair diagram. Given any word in *pq* form, then we can form a tree pair diagram for this element as follows. We consider reduced tree pair diagrams for p and q, and construct a tree pair diagram for the product pq as described in Section 2.2. The middle trees of the four trees involved in the product are all-right trees. The all-right trees in this decomposition may not have the same number of carets, so in forming the diagram for pq we simply enlarge the smaller of the two of these all-right trees (as well as the other tree in that diagram). Since only right carets are ever added during this process, all of whose leaves have leaf exponent zero, this results in a tree pair diagram whose *pq* factorization is precisely the word *pq* we began with.

In T, the correspondence between *pcq* factorizations and general *pcq* words is not so straightforward. Although each tree pair diagram has a *pcq* factorization associated to it, general words in algebraic *pcq* form are not always the *pcq* factorizations associated to a tree pair diagram. The difficulty arises when the tree pair diagram for c does not have as many carets as those for p or q, as adding right carets to enlarge c appropriately necessitates adding generators to the normal forms for p and q, so the tree pair diagram one obtains by multiplying as in F will not necessarily have the original word as its factorization. For example, the word x_1c_1 is in algebraic *pcq* form, yet it is not the *pcq* factorization associated to some tree pair diagram. There is a different representative for this element of T which is the *pcq* factorization associated to the reduced tree pair diagram for this group element: $x_1c_2x_1^{-1}$. We prefer to work with words which are *pcq* factorizations associated to tree pair diagrams, which will lead us to unique normal forms.

We can algebraically characterize the words of type *pcq* which are *pcq* factorizations associated to tree pair diagrams. The important condition is that the reduced tree pair diagram for c should have at least as many carets as those for p and q. We say that words in T with this property satisfy the *factorization condition.*

Theorem 3.4. *For elements in $T \setminus F$, the word*

$$x_{i_1}^{r_1}x_{i_2}^{r_2}\dots x_{i_n}^{r_n}\,c_i^j\,x_{j_m}^{-s_m}\dots x_{j_2}^{-s_2}x_{j_1}^{-s_1},$$

where $i_1 < i_2 < \cdots < i_n$, $j_1 < j_2 < \cdots < j_m$, and $1 \le j < i+2$, is the pcq factorization associated to a tree pair diagram if and only if the number of carets in the reduced tree pair diagram for c_i^j is greater than or equal to the number of carets in the reduced tree pair diagram for either of the words $x_{i_1}^{r_1}x_{i_2}^{r_2}\dots x_{i_n}^{r_n}$ or $x_{j_m}^{-s_m}\dots x_{j_2}^{-s_2}x_{j_1}^{-s_1}$ in F.

Proof. Given a tree pair diagram, by construction, the *pcq* factorization associated to it satisfies the factorization condition. Given a word that satisfies the factorization condition, we can easily construct the corresponding tree pair diagram as described above. We see that the tree pair diagram for the c part has enough carets so that only right carets need to be added to the trees in the diagrams for p and q, so that the diagram constructed will indeed have the original word as its *pcq* factorization. □

We can compute the number of carets of a word $w \in F$ algebraically from the normal form of w [4].

Proposition 3.5 ([4])**.** *Given a positive word in $w \in F$ in the form*

$$w = x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n},$$

then the number of carets $N(w)$ in either tree of a reduced tree diagram representing w is

$$N(w) = \max\{i_k + r_k + \ldots + r_n + 1, \text{ for } k = 1, 2, \ldots, n\}.$$

When $w \in F$ is not a positive word, $N(w)$ is the maximum of the two numbers obtained by applying Proposition 3.5 to the positive and negative parts of the normal form for w. When considering elements of T, we recall that the number of carets in a tree pair diagram for c_i^j is equal to $i + 1$. Thus it is always possible to decide algebraically when $w \in T$ written in *pcq* form corresponds to a tree pair diagram, using Proposition 3.5 to count the carets for the positive and negative parts of the word.

4. Normal forms in T

In T, we will declare the words in *pcq* form which are *pcq* factorizations associated to reduced diagrams to be the normal forms for elements of T, similar to the approach used in F. However, it is no longer true that these words cannot be shortened by applying a relation. As we saw with the normal form $x_1 c_2 x_1^{-1}$ in T, a word may be the shortest word representing an element which satisfies the factorization condition, yet there may be shorter words we can obtain by applying a relator which do not satisfy the factorization condition.

Thus, when algebraically characterizing the normal form for elements of T, we restrict ourselves to words of *pcq* form which satisfy the factorization condition, regardless of whether or not a relator may reduce the length of the word. We next need to specify algebraic conditions which characterize the *pcq* forms that correspond to normal forms.

Theorem 4.1. *Let w be a pcq factorization for an element $g \in T$ associated to a marked tree pair diagram in which each tree has $i + 1$ carets, where the c part of the word is c_i^j with $1 \leq j \leq i + 1$. A reduction of a pair of carets from the tree pair diagram occurs only if the word w satisfies one of the following conditions:*

1. *There exists a pair of generators x_{k+j} and x_k^{-1}, with $0 \leq k \leq i - j - 1$, and neither of the two generators x_{k+j+1} and x_{k+1}^{-1} appear. The reduction corresponds to applying the relation*

   $$x_{k+j} c_i^j x_k^{-1} = c_{i-1}^j$$

   after applying relations from F in the p and q parts of the word, if necessary, to make x_{k+j} and x_k^{-1} adjacent to c_i^j.

(2) *The generator* x_k^{-1} *with* $k = i - j$ *appears, and* x_{k+1}^{-1} *does not. The reduction corresponds to applying*

$$c_i^j x_k^{-1} = c_{i-1}^j$$

after possibly using relations from F *as in (1).*

(3) *There exists a pair of generators* $x_{k-i+j-2}$ *and* x_k^{-1} *for* $i > k \geq i-j+2$ *and neither one of the generators* $x_{k-i+j-1}$ *or* x_{k+1}^{-1} *appear. The reduction corresponds to applying*

$$x_{k-i+j-2} c_i^j x_k^{-1} = c_{i-1}^{j-1}$$

after possibly applying relations from F*.*

(4) *The generator* x_{j-2} *appears, and the generator* x_{j-1} *does not appear. The reduction corresponds to*

$$x_{j-2} c_i^j = c_{i-1}^{j-1}$$

after possibly applying relations from F*.*

Proof. Let $g \in T$ be represented by a tree pair diagram (T_-, T_+). If both trees have an exposed caret whose leaves are identically numbered, then we call that a *reducible caret*, as it must be removed in order to obtain the reduced tree pair diagram representing g. We now consider algebraic conditions corresponding to a reducible caret in a tree pair diagram.

In the tree pair diagram (T_-, T_+) for $g \in T$, there are two ways of labelling the leaves in the target tree T_+. The first labelling corresponds to the order in which the intervals in the subdivisions determined by these trees are paired in the homeomorphism, and is called the cyclic labelling. The cyclic labelling gives the marked leaf in the target tree the number zero, and the other leaves are given increasing labels from left to right around the leaves of the tree. The second labelling ignores the marking and puts the leaves in increasing order from left to right, beginning with zero. The first labelling is used to determine which leaves in T_- are paired with which leaves in T_+, and the second labelling is used in the computation of leaf exponents to determine the powers of the generators that appear in the word.

Suppose that the tree pair diagram for $g \in T$ is not reduced. The four cases above correspond to the following four possible locations of a reducible caret relative to the marked leaf in the target tree.

(1) Case (1) corresponds to the case when the left leaf of the reducible caret is to the left of the marked leaf in T_+, but the reducible caret is not the rightmost caret in T_-.

(2) Case (2) corresponds to the special case when the reducible caret is a right caret in T_- (in which case necessarily the left leaf is to the

left of the marked leaf in T_+). Leaf exponents from right carets will always be zero and thus right carets cannot contribute generators to the normal form. They may still result in an exposed reducible caret, which occurs exactly in this case, and the reduction will only affect the q part of the normal form.

(3) Case (3) corresponds to the case when the left leaf of the reducible caret is either to the right of or coincides with the marked leaf in T_+, but the reducible caret is not the rightmost caret in T_+.

(4) Case (4) corresponds to the special case when the reducible caret is a right caret in T_+ (in which case it cannot be to the left of the marked caret in T_+). As in Case (2), the exposed caret in this case is a right caret and does not contribute a generator to the normal form, but may still be reduced. This cancellation affects only the p part of the normal form.

We will prove case (1), and the proofs in the other cases are analogous. We consider an element $g \in T$ represented by a tree pair diagram. If $g \notin F$ then the two labellings of the leaves of T_+ do not coincide. It is easy to see that if w is the pcq word satisfying the factorization condition, where the middle expression for c is c_i^j, then the marked leaf, with leaf number zero in the first labelling, always corresponds to the generator x_{i-j+2}. Thus the leaf in T_+ which corresponds to the generator x_k^{-1} is the one numbered $k+j$ in the cyclic labelling, and hence the exposed caret in T_- corresponds to the generator x_{k+j}. Since the caret in question is not a right caret in T_-, the generator x_{k+j} will appear in w. Since the caret in question is to the left of the marked caret in T_+, it cannot be a right caret in T_+, and hence the generator x_k^{-1} appears in w. The fact that the generator x_{k+j+1} (respectively x_{k+1}^{-1}) does not appear in w follows from the fact that the caret is exposed in the source (respectively target) tree. This proves case (1).

We note that in case (2), the generator x_k^{-1} is the highest index generator with a negative exponent. This x_k^{-1} generator must correspond to a caret in T_+ which is immediately before the marked caret, and its corresponding caret in T_- is the rightmost caret. Since right carets do not correspond to algebraic generators in the normal form, there is no generator in the positive part of the word involved in this reduction.

Finally, we observe one impossible situation for a marked tree pair diagram, which does not appear in the classification above. It is impossible to have a caret in T_+ corresponding to the generator x_{i-j+1}^{-1}. The leaf numbered $i-j+1$ in the left to right labelling of T_+ is labelled $i+1$ in the cyclic labelling, since leaf $i-j+2$ is the marked leaf in T_+, so the corresponding caret cannot be exposed in T_-. $\square$

The conditions in Theorem 4.1 together with the factorization condition algebraically characterize our normal forms. The normal forms for elements in F have already been characterized, so we restrict to elements not in F in our description.

Theorem 4.2. *Any element $g \in T$ which is not an element of F admits an expression of the form pcq where*

$$p = x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n} \qquad c = c_i^j \qquad q = x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1},$$

$0 \leq i_1 < i_2 < \cdots < i_n$, $0 \leq j_1 < j_2 < \cdots < j_m$, *and* $1 \leq j < i + 2$. *Among all the words in this form representing an element, there is a unique one satisfying the following conditions, and it is the normal form.*

- *The factorization condition, which we now state as* $i + 1 \geq \max\{N(p), N(q)\}$.
- *The word does not admit any reductions. Namely, this word satisfies the following conditions:*
  - *If there exists a pair of generators x_{k+j} and x_k^{-1} simultaneously, for $k \leq i - j - 1$, then one of the generators x_{k+j+1} or x_{k+1}^{-1} must appear as well.*
  - *If there is a generator x_k^{-1} with $k = i - j$, then x_{k+1}^{-1} must exist too.*
  - *If there exists a pair of generators $x_{k-i+j-2}$ and x_k^{-1} for $k \geq i - j + 2$, then one of the generators $x_{k-i+j-1}$ or x_{k+1}^{-1} must appear as well.*
  - *If there exists a generator x_{j-2}, then a generator x_{j-1} must also appear.*

Proof. We claim that the conditions above precisely describe a set of unique normal forms for T. A *pcq* word satisfying the factorization condition is the *pcq* factorization associated to a marked tree pair diagram. However, if the *pcq* word satisfies all four reduction conditions, we have just shown in the previous theorem that this diagram is in fact the unique reduced diagram, and hence the word is in fact a normal form. □

We remark that the Pumping Lemma together with the reductions in Theorem 4.1 give an explicit way of algebraically transforming any word in the generators of T into a normal form. Namely, given any word, we rewrite it in *pcq* form using the process described following the Pumping Lemma. If the resulting word does not satisfy the factorization condition, then we iterate the Pumping Lemma until we obtain a word for which the factorization condition is satisfied. The Pumping Lemma increases the number of carets for c and the number of carets for one of the words p and q. Once a word is obtained which satisfies the factorization condition, there must be a corresponding tree pair diagram for the element. Now, if the word satisfies any of the reduction conditions in Theorem 4.1, we apply them successively using the relations described there. This method thus produces the unique normal form.

5. THE WORD METRIC IN T

5.1. **Estimating the word metric.** For metric questions concerning T, we must consider a finite generating set instead of the one used to obtain the normal form for elements. We now approximate the word length of an element of T with respect to the generating set $\{x_0, x_1, c_1\}$, using information contained in the normal form and the tree pair diagram. These estimates are similar to those for the word metric in F with respect to the generating set $\{x_0, x_1\}$ found in [3] and [4].

Theorem 5.1. *Let $w \in T$ have normal form*

$$w = x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_n}^{r_n} \, c_i^j \, x_{j_m}^{-s_m} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}.$$

We define

$$D(w) = \sum_{k=1}^{n} r_k + \sum_{l=1}^{m} s_l + i_n + j_m + i.$$

Let $|w|$ denote the word metric in T with respect to the generating set $\{x_0, x_1, c_1\}$. There exists a constant $C > 0$ so that for every $w \in T$,

$$\frac{D(w)}{C} \leq |w| \leq C \, D(w)$$

and similarly, for $N(w)$ the number of carets in a reduced tree pair diagram representing w,

$$\frac{N(w)}{C} \leq |w| \leq C \, N(w).$$

Proof. These inequalities follow from the correspondence between the normal form and the tree pair diagram for an element $w \in T$. It is clear, from Proposition 3.5, that $N(w) \geq \sum_{k=1}^{n} r_k$, $N(w) \geq \sum_{l=1}^{m} s_l$, $N(x) \geq i_n$, and $N(w) \geq j_m$. The inequality $N(w) \geq i$ is clear from the fact that c_i has $i+1$ carets. These inequalities prove that

$$D(w) \leq 5 \, N(w).$$

We rewrite the generators x_i and c_j in terms of x_0, x_1 and c and look at the lengths of these words to obtain the inequality

$$|w| \leq C \, D(w)$$

for some constant $C > 0$. Combining the two inequalities above we have

$$|w| \leq C' \, N(w).$$

To obtain lower bound on the word length, one only needs to observe that the tree pair diagram for each generator has either two or three carets. If u is a word in x_0, x_1 and c with length n, then as these generators are multiplied together, each product may add at most 3 carets to the tree pair diagram. Thus the diagram for u will have at most $3n$ carets. It then follows that

$$N(w) \leq 3|w|.$$

Combining this with the above inequality, we obtain the desired bounds. □

We use Theorem 5.1 to show that the inclusion of F in T is a quasi-isometric embedding. This means that there are constants $K > 0$ and C so that for any $w, z \in F$ we have

$$\frac{1}{K} d_F(w,z) - C \leq d_T(w,z) \leq K d_F(w,z) + C$$

where d_F and d_T represent the word metric in F and T respectively, with regard to the generating set $\{x_0, x_1\}$ of F and $\{x_0, x_1, c\}$ of T.

When considering whether the inclusion of a finitely generated subgroup H into a finitely generated group G is a quasi-isometric embedding, we can instead equivalently show that the distortion function is bounded. The distortion function is defined by

$$h(r) = \frac{1}{r} \max\{|x|_H \text{ such that } x \in H,\ |x|_G \leq r\}.$$

Word length in F is comparable to the number of carets in the reduced tree pair diagram representing the word, as seen in [4, 8]. This, combined with Theorem 5.1 easily shows that the distortion function is bounded, and thus proves the following corollary.

Corollary 5.2. *The inclusion of F in T is a quasi-isometric embedding.*

5.2. **Comparing word length in F and T.** Although Corollary 5.2 shows that F is quasi-isometrically embedded in T, we now show that the word length of certain elements of F does not change when these elements are considered as elements of T, with respect to natural finite generating sets. For this, we use the standard finite generating set $\{x_0, x_1\}$ for F and the generating set $\{x_0, x_1, t\}$ for T, where $t = c_0$ is the non-identity element of T in which each tree has a single caret. This element corresponds to a rotation of the unit circle of order 2. We use t instead of $c = c_1$ for the third generator because we are interested in understanding how multiplication by generators can change the number of carets, which is more straightforward using t than c.

To find elements which have the same length in F and T with respect to these generating sets, we consider the process by which they are constructed.

A geodesic word $a_n \cdots a_2 a_1$ in the generators $\{x_0, x_1, t\}$ representing $w \in F \subset T$ describes how a tree pair diagram for $w = (T_-, T_+)$ is created by successively applying the generators a_k to the tree pair diagram for the word $a_{k-1} \cdots a_2 a_1$, as k increases from 1 to n. We begin with the identity element of T and its reducible tree pair diagram consisting of two identical trees with one caret each, and multiply first by a_1. This changes the tree pair diagram of the identity element by creating some new carets or adding a marking. As each successive generator is added to the product, the number of carets in the existing tree pair diagram may increase, decrease or remain the same. If the number of carets increases, it can increase by at most two since each generator has at most two carets in addition to the root caret. In other cases, the number of carets may remain the same or decrease. When all generators in the sequence $a_n \cdots a_2 a_1$ have been added to the product, the resulting tree pair diagram is (T_-, T_+). We now carefully analyze the circumstances under which a single generator in this product can add two carets to an existing tree pair diagram.

Lemma 5.3. *If $w \in T$ is a non-identity element, and $w \neq t$, then $N(tw) = N(w)$.*

Proof. Since the tree pair diagram for t contains a single caret, no new carets must be added to the tree pair diagram for w in order to perform the multiplication tw. It is easy to see that no reduction can occur after multiplication by t. □

Lemma 5.4. *Let $w = (T_-, T_+) \in T$ be nontrivial and $\alpha \in \{x_0^{\pm1},\ x_1^{\pm1}, t\}$. Then $N(\alpha w) = N(w) + 2$ if and only if $\alpha = x_1^{\pm1}$ and the right subtree of the root caret of T_+ is empty.*

Proof. Since the source tree of the tree pair diagram for $x_1^{\pm1}$ contains two carets in the right subtree of the root caret, it is clear that these carets must be added to (T_-, T_+) in order to perform the multiplication, and that the resulting product does not have reducible carets in the right subtree of the root caret in either tree. The tree pair diagram for $x_0^{\pm1}$ contains two carets in each tree, but one is the root caret, so the maximum number of carets that could be added to (T_-, T_+) in order to multiply by $x_0^{\pm1}$ is one. Lemma 5.3 completes the proof. □

We consider the process of constructing the tree pair diagram for a product $a_n \cdots a_2 a_1$ where $a_i \in \{x_0^{\pm1},\ x_1^{\pm1},\ t\}$. Let $P_i = a_i a_{i-1} \cdots a_2 a_1$ for $1 \leq i \leq n$. We construct the tree pair diagrams for the successive products P_i for $i = 1, \cdots, n$. Each additional generator may either reduce, leave unchanged, increase by one, or increase by two the number of carets in the tree pair diagram corresponding to the suffix P_i of P_n. To distinguish those generators which add two carets to the tree pair diagram, we will use the letter b, while for other generators we will use the letter a. So we represent an element $w \in T$ by a string of generators

$$a_r \cdots a_h b_k \cdots a_{s+1} b_1 a_s \cdots a_1$$

where if a_t is the generator immediately to the right of b_j, then $N(b_j a_t \cdots a_1) = N(a_t \cdots a_1) + 2$, and otherwise we have $N(a_s \cdots a_1) \leq N(a_{s-1} \cdots a_1) + 1$.

Lemma 5.5. *If $w \in F$ and $w = a_r \cdots a_h b_k \cdots a_{s+1} b_1 a_s \cdots a_1$ where not all $a_i \in \{x_0^{\pm 1}, x_1^{\pm 1}\}$, then there must be at least two indices i and j so that $a_i = a_j = t$.*

Proof. Suppose that in the expression above for $w \in F$, there was a single letter t. Then we easily obtain $t \in F$, a contradiction. ☐

Lemma 5.6. *Let $w \in T$ be given by an expression of the form $a_r \cdots a_h b_k \cdots a_{s+1} b_1 a_s \cdots a_1$. Then b_{j+1} and b_j cannot be adjacent in the expression. Note that possibly $s = 0$ in which case the word ends with b_1 and that possibly $h - 1 = r$ in which case the word begins with b_k.*

Proof. Let v be the suffix of the word w to the right of the generator b_j. We know from Lemma 5.4 that in a tree pair diagram (R_-, R_+) for v, the right subtree of the root caret of R_+ is empty. Since $b_j = x_1^{\pm 1}$, the right subtree of the root caret of S_+ has one of two forms: two right carets or a right caret with a left child. In either case, the right subtree of the root caret is nonempty, and thus the next generator in the multiplication cannot add two carets to the tree pair diagram, so is not b_{j+1}. ☐

We now characterize one type of element of F whose word length is unchanged when viewed as an element of T, using the generating set $\{x_0, x_1\}$ for F and $\{x_0, x_1, t\}$ for T. These are elements $w \in F$ for which $N(w)$ exceeds the word length $|w|_F$. Fordham [8] computes $|w|_F$ by assigning an integer weight between zero and four to each pair of carets in the tree pair diagram representing w. In a given word there are at most two weights of zero. Here we investigate words in which most weights are one. Such words, for example, are represented by tree pair diagrams with no interior carets having right children.

Theorem 5.7. *If $w \in F$ with $N(w) \geq |w|_F + 1$ then $|w|_T = |w|_F$, where word length if computed with respect to the generating set $\{x_0, x_1\}$ for F and $\{x_0, x_1, t\}$ for T.*

We immediately obtain the following corollary, since $|x_0^n|_F = |x_1^n|_F = n$, while $N(x_0^n) = n + 1$ and $N(x_1^n) = n + 3$.

Corollary 5.8. *The elements x_0^n and x_1^n have word length n in both F and T with respect to the finite generating sets $\{x_0, x_1\}$ and $\{x_0, x_1, t\}$ respectively.*

We now prove Theorem 5.7.

Proof. Suppose $w \in F$ can be written as $w = a_r \cdots a_2 a_1$ where $a_i \in \{x_0^{\pm 1}, x_1^{\pm 1}, t\}$, and $r < n = |w|_F$. If a_i is a generator which adds two carets to the tree pair diagram, then rename this generator b_j. So we rewrite the expression for w as

$$w = a_p \cdots b_k \cdots a_{s+1} b_1 a_s \cdots a_1$$

with $r = p + k$ where k is the number of b_j generators. By Lemma 5.6 we have $k - 1 \leq p$, and we know from Lemma 5.4 that $b_i = x_1^{\pm 1}$.

We first prove the following lemma relating to this expression for w:

Lemma 5.9. *Let $a_p \cdots a_h b_k \cdots b_j a_t \cdots a_{s+1} b_1 a_s \cdots a_1$ be an expression for a word $w \in T$, where each $b_j = x_i^{\pm 1}$ is a generator which adds two carets to the tree pair diagram, and each a_i is a generator from the set $\{x_0^{\pm 1}, x_1^{\pm 1}, t\}$ which adds at most one caret to the tree pair diagram. We consider the generators a_k which appear between b_{j+1} and b_j.*

1. *There is at least one generator between b_{j+1} and b_j which does not increase the number of carets.*
2. *Let a_J be the generator between b_{j+1} and b_j and closest to b_{j+1} such that if v_J is the suffix of w immediately to the right of a_J, then the right subtree of the the target tree of v_J is nonempty, but the right subtree of the target tree of $a_J v_J$ is empty.*
   If $a_J = t$ and b_j is not the rightmost generator in the expression for w, then there is another generator a_L between b_{j+1} and b_j which does not add carets to the tree pair diagram.

Proof. We know that $b_{j+1} = x_1^{\pm 1}$, and for multiplication by b_{j+1} to add two carets to the tree pair diagram, the target tree of the tree pair diagram for the suffix of w immediately following b_{j+1} must have an empty right subtree. We also know that the target tree of the tree pair diagram for the suffix of w beginning with b_j does not have an empty right subtree, according to Lemma 5.4. Thus the generator a_J in the statement of the lemma must exist.

As in the statement of the lemma, let v_J denote the suffix of w immediately to the right of a_J. We now claim that $N(v_J) \geq N(a_J v_J)$. If $a_J = t$ then the claim follows from Lemma 5.3. If $a_J = x_0^{-1}$ or $a_J = x_1^{\pm 1}$, and $N(a_J v_J) = N(v_J) + 1$, then the right subtree of the target tree of $a_J v_J$ can never be empty. Thus if a_J is one of the above three generators, it can never add a caret to the tree pair diagram of v_J, because we know that the right subtree of the target tree of $a_J v_J$ must be empty.

Suppose that $a_J = x_0$. If $N(a_Jv_J) = N(v_J) + 1$, then the right subtree of the target tree of the tree pair diagram for v_J must be empty. But this contradicts the definition of a_J, and thus $N(a_Jv_J) \leq N(v_J)$, and the claim is again true.

Now consider the case $a_J = t$ when b_j is not the rightmost generator in the expression for w. Let $b_j \cdots a_1 = (R_-, R_+)$ and $v_J = (S_-, S_+)$. Since $b_j = x_1^{\pm 1}$, the right subtree of the root caret of R_+ has one of two forms: two right carets or a right caret with a left child. Let A denote the left subtree of the root caret of this tree. Denote the right subtree of S_+ by A'. Since b_j is not the rightmost generator in the expression for w, we know that both A and A' are nonempty. Thus there must be at least one generator between a_J and b_j. We claim that there must be a generator between a_J and b_j which does not increase the number of carets in the tree pair diagram.

Since the right subtree of the root caret of R_+ is nonempty, as is the left subtree A of the root caret, we see immediately that $N(x_0^{\pm 1} b_j \cdots a_1) \leq N(b_j \cdots a_1)$. Similarly, multiplication by b_j^{-1} cannot increase the number of carets in the tree pair diagram (R_-, R_+), and multiplication by t never increases the number of carets in any tree pair diagram. Thus the only generator which can precede b_j and increase the number of carets in the tree pair diagram is b_j itself. Repeat occurrences of $b_j = x_1^{\pm 1}$ are written as a_i in the expression for w, since they do not increase the number of carets by 2. But the target tree in the tree pair diagram for $b_j^m \cdots a_1$ will always have left subtree A and a nonempty right subtree. Since $v_J = (S_-, S_+)$ has a tree S_+ with empty right subtree of the root caret, we see that there must be another generator not equal to b_j between $b_j^m \cdots a_1$ and the leftmost letter in v_J. By the above argument, this generator must not increase the number of carets in the tree pair diagram. Hence there must be another generator between a_J and b_j which does not increase the number of carets in the tree pair diagram. ☐

According to Lemma 5.9, between each pair of generators b_{j+1} and b_j there is a generator which does not add carets to the tree pair diagram. There are at least $k-1$ such generators in the expression for w, between b_{j+1} and b_j for $j = 1, 2, \cdots k-1$. In the proof of Lemma 5.9, these generators are denoted a_J. Lemma 5.9 also shows the existence of at least one additional generator which does not increase the number of carets in the tree pair diagram, for the following reason.

We can assume that the expression for w contains at least one generator t, otherwise we would have a word in F. In F, we know that the word length of w is $n > r$. It follows from Lemma 5.5 that there must be at least two t generators in the expression for w. We now prove in three cases that there must be an additional generator in the sequence $a_p \cdots a_h b_k \cdots b_j a_t \cdots a_{s+1} b_1 a_s \cdots a_1$ which is not one of the a_J generators guaranteed by Lemma 5.9, which does not increase the number of carets in the tree pair diagram.

1. If no $a_J = t$, then we have found two additional generators which do not increase the number of carets in the tree pair diagram, namely the two instances of the generator t guaranteed by Lemma 5.5.
2. If two t generators play the role of a_J, one between b_{j+1} and b_j, and the other between b_{m+1} and b_m, then at most one of b_j and b_m, say b_j, can be the rightmost generator in the sequence representing w. Then Lemma 5.9 guarantees one additional generator between b_{m+1} and b_m which does not increase the number of carets in the tree pair diagram.
3. If only one of the t generators in the expression is an a_J generator, then the other t generator guaranteed by Lemma 5.5 does not increase the number of carets in the tree pair diagram.

Thus we always have at least k generators in the expression $a_p \cdots a_h b_k \cdots b_j a_t \cdots a_{s+1} b_1 a_s \cdots a_1$ which do not increase the number of carets in the tree pair diagram.

We then obtain the following upper bound for the number of carets $N(w)$:

$$n + 1 \leq N(w) \leq 1 + 2k + (r - k - k) = r + 1 < n + 1$$

which is clearly a contradiction. Thus we are unable to express w using fewer generators than $n = |w|_F$. □

5.3. **Cyclic subgroups of T.** We can use the interpretation of elements of T as homeomorphisms of the circle to understand how F is contained in T and that the cyclic subgroups of T are quasi-isometrically embedded.

Theorem 5.10. *Every element in T has a power which is conjugate to an element in F.*

Proof. Ghys and Sergiescu [9] and Liousse [12] show that every element in T has rational rotation number as a homeomorphism of S^1. A standard result in dynamics is that a homeomorphism of the circle has rational rotation number if and only if it has a periodic orbit (see, for example Katok and Hasselblatt [11].) Thus, every element in T has a periodic orbit. That is, given $f \in T$, there exists a dyadic rational $x \in [0, 1]$ such that $f^n(x) = x$ for some n.

Given a dyadic rational x, we consider the rotation of the circle $t_x(p) = p + x$, where we view S^1 as the unit interval with the endpoints identified.

Then t_x is an element of T, and conjugation of $w \in T$ by t_x, for the appropriate choice of x yields an element with zero as a fixed point. Thus the conjugated element must lie in F. $\square$

We use Theorem 5.10 to show that cyclic subgroups of T are quasi-isometrically embedded. We again bound the distortion function, as in the proof of Corollary 5.2.

Theorem 5.11. *Let $x \in T$ be a non-torsion element. Then the cyclic subgroup $\langle x \rangle$ generated by x is quasi-isometrically embedded in T.*

Proof. If the element x lies in F, the result can be deduced from Corollary 5.2, since Burillo [3] shows that all cyclic subgroups in F are quasi-isometrically embedded, and F is quasi-isometrically embedded in T. If $x \notin F$, it has a power w^n which is conjugate to an element of F by some element $z \in T$. Since x is not a torsion element, this power is not conjugate to the identity element of F.

The cyclic subgroup generated by zw^nz^{-1} is quasi-isometrically embedded in T. Conjugation by z changes the number of carets by at most a factor of $2N(z)$, and thus we see that $\langle x \rangle$ must also be quasi-isometrically embedded in T. $\square$

6. Torsion elements

Although the group F is torsion free, both T and V contain torsion elements. It is easy to construct torsion elements in T or V by choosing any binary tree S and making any marked tree pair diagram with S as both source and target tree. If the labelling of the target tree is the same as the labelling of the source tree, we get an unreduced representative of the identity; otherwise, we get a non-trivial torsion element. If this is an element of T, the tree pair diagram has *pcq* factorization in which $q = p^{-1}$. In fact, any torsion element can be represented by such a tree pair diagram, though its reduced marked tree pair diagram may well not have the same source and target trees, corresponding to the fact that although it has a *pcq* word where $q = p^{-1}$, the normal form may well not have this special balanced appearance.

Proposition 6.1. *If $f \in F, T$ or V is a torsion element, then it can be represented by a (marked) tree pair diagram with the same source and target trees.*

Before proving Proposition 6.1, we establish some notation which links the analytic and geometric interpretations of these groups. For $f \in F$, T, or V, if (T_-, T_+) is a marked tree pair diagram representing f, then it is sometimes convenient to denote the tree T_+ by $f(T_-)$. The element f can be though of as mapping the leaves of T_- to the leaves of $f(T_-) = T_+$, where the marking defines this mapping of the leaves.

Given two rooted binary trees T and T', we say that T' is an *expansion* of T if T' can be obtained from T by attaching the roots of additional trees to some subset of the leaves of T. We observe that if $(T, f(T))$ is a marked tree pair diagram for f, and T' is an expansion of T, then there is always a tree pair diagram $(T', f(T'))$ for f, and $f(T')$ is an expansion of $f(T)$. Given two rooted binary trees S and T, by the *minimal common expansion* of S and T we mean the smallest rooted binary tee which is an expansion of both S and T. Using this language, if $(T, f(T))$ and $(S, g(S))$ are marked tree pair diagrams for f and g respectively, the process described in Section 2.3 for creating a tree pair diagram for the product gf could be summarized as follows. If E is the minimal common expansion of $f(T)$ and S, then there are tree pair diagrams $(f^{-1}(E), E)$ for f, $(E, g(E))$ for g, and $(f^{-1}(E), g(E))$ for gf.

Proof. Suppose that f is a torsion element, and that f cannot be represented by a tree pair diagram with the same source and target trees. Since there exists a positive integer m such that f^m is the identity, it follows that all tree pair diagrams for f^m have the same source and target trees. We reach a contradiction by constructing (marked) tree pair diagrams (A_n, B_n) for f^n such that $A_n \neq B_n$ for every $n \geq 1$. These tree pair diagrams are constructed inductively, viewing f^n as a product $(f^{n-1})(f)$. For $n = 1$, let (A_1, B_1) be the reduced marked tree pair diagram for f. If $k \geq 2$, suppose the marked tree pair diagram (A_{k-1}, B_{k-1}) for f^{k-1} has been constructed. Let E_{k-1} be the minimal common expansion of the trees A_1 and B_{k-1}. Then f^k has tree pair diagram $(f^{-(k-1)}(E_{k-1}), f(E_{k-1}))$, and we let $B_k = f(E_{k-1})$ and $A_k = f^{-(k-1)}(E_{k-1})$.

By construction, A_{k+1} is an expansion of A_k for all $k \geq 1$. We claim also that B_{k+1} is an expansion of B_k for all $k \geq 1$. For $k = 1$, E_1 is by definition an expansion of A_1, which implies that $B_2 = f(E_1)$ is an expansion of $B_1 = f(A_1)$. Suppose inductively that B_k is an expansion of B_{k-1}. Now E_k is an expansion of B_k and A_1, so E_k is an expansion of B_{k-1} and A_1. But E_{k-1} is the minimal common expansion of B_{k-1} and A_1, so E_k is an expansion of E_{k-1}, which implies that $B_{k+1} = f(E_k)$ is an expansion of $B_k = f(E_{k-1})$.

Now if $A_n = B_n$, then since A_n is an expansion of A_1, B_n is an expansion of A_1. But since E_{n-1} is the minimal common expansion of B_{n-1} and A_1, this implies that $B_n = f(E_{n-1})$ is an expansion of E_{n-1}. But they have the same number of carets, so in fact $f(E_n) = E_{n-1}$, which cannot be. Hence $A_n \neq B_n$ for all n, as claimed. □

Corollary 6.2. *An element of T is torsion if and only if it is a conjugate of some c_i^j.*

Proof. If an element is torsion, then it admits a diagram with two equal trees. The *pcq* factorization associated with this diagram has the form $pc_i^j p^{-1}$, where p is a positive element of F. □

A particularly natural torsion subgroup is the subgroup R of pure rotations, where by a pure rotation by $d = \frac{a}{2^n}$ (where a is not divisible by 2) we mean an element $g_d \in T$ which corresponds to the homeomorphism of S^1 given by

$$g_d(t) = \begin{cases} t + \frac{a}{2^n} & \text{if } 0 \leq t < 1 - \frac{a}{2^n} \\ t + \frac{a}{2^n} - 1 & \text{if } 1 - \frac{a}{2^n} \leq t < 1 \end{cases}$$

Such pure rotations were used in Section 5.3 to conjugate the fixed point of a homeomorphism to 0.

This subgroup is isomorphic to the group of dyadic rational numbers modulo 1, which has a 2-adic metric as follows: if $x = \frac{p}{2^l}$, $y = \frac{q}{2^m}$, and $z = |x - y| = \frac{r}{2^k}$, where p, q and r are odd, then $d(x, y) = 2^k$. With respect to this metric, the subgroup of rotations is quasi-isometrically embedded in T.

Proposition 6.3. *The subgroup R of the pure rotations, with the 2-adic metric, is quasi-isometrically embedded in T.*

Proof. We note that if $g \in T$ is the rotation by $\frac{a}{2^n}$ where a is not divisible by 2, then there are $2^n - 1$ carets in the reduced tree pair diagram representing g, so $N(g) = 2^n - 1$. Since we have shown that the word length of g in T is bi-Lipschitz equivalent to $N(g)$, the proposition follows. □

REFERENCES

[1] James Belk and Kenneth S. Brown. Forest diagrams for elements of Thompson's group F. Preprint.

[2] Kenneth S. Brown and Ross Geoghegan. An infinite-dimensional torsion-free FP_∞ group. *Inventiones mathematicae*, 77:367–381, 1984.

[3] José Burillo. Quasi-isometrically embedded subgroups of Thompson's group F. *J. Algebra*, 212(1):65–78, 1999.

[4] José Burillo, Sean Cleary, and Melanie Stein. Metrics and embeddings of generalizations of Thompson's group F. *Trans. Amer. Math. Soc.*, 353(4):1677–1689 (electronic), 2001.

[5] J. W. Cannon, W. J. Floyd, and W. R. Parry. Introductory notes on Richard Thompson's groups. *Enseign. Math. (2)*, 42(3-4):215–256, 1996.

[6] Sean Cleary and Jennifer Taback. Combinatorial properties of Thompson's group F. *Trans. Amer. Math. Soc.*, 356(7):2825–2849 (electronic), 2004.

[7] S. Blake Fordham. *Minimal Length Elements of Thompson's group F*. PhD thesis, Brigham Young Univ, 1995.

[8] S. Blake Fordham. Minimal length elements of Thompson's group F. *Geom. Dedicata*, 99:179–220, 2003.

[9] Étienne Ghys and Vlad Sergiescu. Sur un groupe remarquable de difféomorphismes du cercle. *Comment. Math. Helv.*, 62(2):185–239, 1987.

[10] Victor Guba. On the properties of the Cayley graph of Richard Thompson's group F. Preprint.

[11] Anatole Katok and Boris Hasselblatt. *Introduction to the modern theory of dynamical systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1995. With a supplementary chapter by Katok and Leonardo Mendoza.

[12] Isabelle Liousse. Nombre de rotation dan les groupes de thompson géneralisès et applications. Preprint.

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