PERIODIC SOLUTIONS FOR NONAUTONOMOUS SECOND ORDER DIFFERENTIAL INCLUSIONS SYSTEMS WITH p–LAPLACIAN

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ABSTRACT. Using the nonsmooth variant of minimax point theorems, some existence results are obtained for periodic solutions of nonautonomous second-order differential inclusions systems with p–Laplacian.

1. INTRODUCTION

Consider the second order system

$$\begin{aligned} \ddot{u}(t) &= \nabla F(t, u(t)) \text{ a.e. } t \in [0, T], \\ u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0 \end{aligned} \tag{1}$$

where $T > 0$, $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for each $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in L^1(0, T; \mathbb{R}_+)$ such that

$$|F(t, x)| + \|\nabla F(t, x)\| \leq a(\|x\|)b(t),$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$.

Suppose that the nonlinearity $\nabla F(t, x)$ is bounded, that is, there exists $g \in L^1(0, T; \mathbb{R}_+)$ such that

$$\|\nabla F(t, x)\| \leq g(t),$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$. In [3] the authors proved the existence of solutions for problem (1) under the condition that

$$\int_0^T F(t, x)dt \to +\infty \text{ as } \|x\| \to \infty,$$

or that

$$\int_0^T F(t, x)dt \to -\infty \text{ as } \|x\| \to \infty.$$

Tang in [5] proved the existence of solutions for problem (1) under more general conditions. He supposes that assumption (A) holds, that

$$\|\nabla F(t, x)\| \leq f(t)\|x\|^\alpha + g(t),$$

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for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$, where $f, g \in L^1(0,T;\mathbb{R}_+)$, $\alpha \in [0,1)$ and

$$\|x\|^{-2\alpha} \int_0^T F(t,x)dt \to +\infty \text{ as } \|x\| \to \infty,$$

or that

$$\|x\|^{-2\alpha} \int_0^T F(t,x)dt \to -\infty \text{ as } \|x\| \to \infty.$$

In order to prove the above results, Mawhin-Willem and Tang apply the classical (smooth) variant of minimax methods. In [4] we have considered the following problem which is a generalization of problem (1)

$$\begin{aligned} &\ddot{u}(t) \in \partial F(t,u(t)) \text{ a.e. } t \in [0,T], \\ &u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0 \end{aligned} \tag{2}$$

where $T > 0$, $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, ∂ denotes the Clarke subdifferential (see [2]) and $F(t,x)$ is measurable in t for each $x \in \mathbb{R}^n$, and locally Lipschitz and regular (see [2]) in x for each $t \in [0,T]$. Under some additional assumptions (see [4]) on F and ∂F we proved the existence of solutions for problem (2).

The aim of this paper is to consider the problem (2) in a more general sense. More exactly our results represent the extensions to systems with p–Laplacian.

Consider the second order differential inclusions system

$$\begin{aligned} &\frac{d}{dt}\Big(|\dot{u}(t)|^{p-2}\dot{u}(t)\Big) \in \partial F(t,u(t)) \text{ a.e. } t \in [0,T], \\ &u(0) = u(T), \dot{u}(0) = \dot{u}(T), \end{aligned} \tag{3}$$

where $p > 1$, $T > 0$, $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, and ∂ denotes the Clarke subdifferential.

The corresponding functional $\varphi(u) : W_T^{1,p} \to \mathbb{R}$ is given by

$$\varphi(u) = \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt + \int_0^T F(t,u(t))dt.$$

2. Main results

Theorem 1. *Let $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that $F(t,x)$ is measurable in t for each $x \in \mathbb{R}^n$ and regular in x for each $t \in [0,T]$. We suppose that exist $k \in L^q(0,T;\mathbb{R})$ such that*

$$|F(t,x_1) - F(t,x_2)| \le k(t)\|x_1 - x_2\| \tag{4}$$

for all $t \in [0,T]$ and all $x_1, x_2 \in \mathbb{R}^n$. If there exist $c_1, c_2 > 0$ and $\alpha \in [0,1)$ such that

$$\zeta_1 \in \partial F(t,x) \Rightarrow \|\zeta_1\| \le c_1\|x\|^\alpha + c_2 \tag{5}$$

for all $t \in [0,T]$ and all $x \in \mathbb{R}^n$, and if for $q = \frac{p}{p-1}$

$$\|x\|^{-q\alpha} \int_0^T F(t,x)dt \to +\infty \text{ as } \|x\| \to \infty \tag{6}$$

then problem (3) has at least one solution which minimizes the functional φ on $W_T^{1,p}$.

Theorem 2. *Let $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ such that $F(t,x)$ is measurable in t for each $x \in \mathbb{R}^n$ and locally Lipschitz and regular in x for each $t \in [0,T]$. We suppose that exist $a \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $b \in L^1(0,T;\mathbb{R}_+)$ such that*

$$\|F(t,x)\| \leq a(\|x\|)b(t) \tag{7}$$

for all $t \in [0,T]$ and all $x \in \mathbb{R}^n$. If there exist $c_1, c_2 > 0$ and $\alpha \in [0,1)$ such that

$$\zeta_1 \in \partial F(t,x) \Rightarrow \|\zeta_1\| \leq c_1\|x\|^\alpha + c_2$$

for all $t \in [0,T]$ and all $x \in \mathbb{R}^n$, and if for $q = \frac{p}{p-1}$

$$\|x\|^{-q\alpha} \int_0^T F(t,x)dt \to -\infty \text{ as } \|x\| \to \infty \tag{8}$$

then problem (3) has at least one solution on $W_T^{1,p}$.

Remark 1. *Theorems 1 and 2 generalizes the corresponding Theorems 1 and 2 of* [4]*. In fact, it follows from these theorems letting $p = 2$.*

3. The preliminary results

We introduce some functional spaces. Let T a positive real number and $1 < p < \infty$. We denote by $W_T^{1,p}$ the Sobolev space of functions $u \in L^p(0,T;\mathbb{R}^n)$ having a weak derivative $\dot{u} \in L^p(0,T;\mathbb{R}^n)$. The norm over X is defined by

$$\|u\|_{W_T^{1,p}} = \Big(\int_0^T \|u(t)\|^p dt + \int_0^T \|\dot{u}(t)\|^p dt\Big)^{\frac{1}{p}}.$$

We recall that

$$\|u\|_{L^p} = \Big(\int_0^T \|u(t)\|^p dt\Big)^{\frac{1}{p}} \text{ and } \|u\|_\infty = \max_{t\in[0,T]} \|u(t)\|.$$

For our aims it is necessary to recall some very well know results (for proof and details see [3]).

Proposition 3. *If $u \in W_T^{1,p}$ then*

$$\|u\|_\infty \leq c\|u\|_{W_T^{1,p}} \ .$$

If $u \in W_T^{1,p}$ and $\int_0^T u(t)dt = 0$ then

$$\|u\|_\infty \leq c\|\dot{u}\|_{L^p} \text{ (Sobolev inequality)},$$

$$\|u\|_{L^p} \leq c\|\dot{u}\|_{L^p} \text{ (Wirtinger's inequality)}.$$

Proposition 4. *If the sequence $(u_k)_k$ converges weakly to u in $W_T^{1,p}$, then $(u_k)_k$ converges uniformly to u on $[0,T]$.*

Let X be a Banach space. Now follows [2], for each $x, v \in X$, we define the *generalized directional derivative* at x in the direction v of a given $f \in Lip_{loc}(X,\mathbb{R})$ as

$$f^0(x;v) = \limsup_{y\to x, \lambda\searrow 0} \frac{f(y+\lambda v) - f(y)}{\lambda}$$

and we denote by

$$\partial f(x) = \{x^* \in X^* : f^0(x; v) \geq \langle x^*, v\rangle, \text{ for all } v \in X\}$$

the *generalized gradient* of f at x (the Clarke subdifferential).

We recall the *Lebourg's mean value theorem* (see [2], Theorem 2.3.7).

Theorem 5. *Let x and y be points in X, and suppose that f is Lipschitz on open set containing the line segment [x,y]. Then there exists a point u in (x,y) such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x\rangle.$$

Clarke consider in [2] the following abstract framework:

- let $(T,\mathcal{T}, \mu)$ be a positive complete measure space with $\mu(T) < \infty$, and let Y be a separable Banach space;
- let Z be a closed subspace of $L^p(T; Y)$ (for some p in $[1, \infty)$), where $L^p(T; Y)$ is the space of p- integrable functions from T to Y;
- define a functional f on Z via
$$f(x) = \int_T f_t(x(t))\mu(dt),$$
where $f_t : Y \to \mathbb{R}$, $(t \in T)$ is a given family of functions;
- suppose that for each y in Y the function $t \to f_t(y)$ is measurable, and that x is a point at which $f(x)$ is defined (finitely).

Hypothesis 1: There is a function k in $L^q(T, \mathbb{R})$, $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ such that, for all $t \in T$,

$$|f_t(y_1) - f_t(y_2)| \leq k(t)\|y_1 - y_2\|_Y \text{ for all } y_1, y_2 \in Y$$

Hypothesis 2: Each function f_t is Lipschitz (of some rank) near each point of Y, and for some constant c, for all $t \in T$, $y \in Y$, one has

$$\zeta \in \partial f_t(y) \Rightarrow \|\zeta\|_{Y^*} \leq c\{1 + \|y\|_Y^{p-1}\}.$$

Under this conditions described above Clarke prove (see [2], Theorem 2.7.5):

Theorem 6. *Under the conditions described above, under either of Hypothesis 1 or 2, f is uniformly Lipschitz on bounded subsets of Z, and one has*

$$\partial f(x) \subset \int_T \partial f_t(x(t))\mu(dt).$$

Further, if each f_t is regular at $x(t)$ then f is regular at x and equality holds.

Remark 2. *f is globally Lipschitz on Z when Hypothesis 1 hold.*

Now we can prove the following result.

Theorem 7. *Let $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ such that $F(t, x)$ is measurable in t for each $x \in \mathbb{R}^n$, and locally Lipschitz and regular in x for each $t \in [0, T]$, and there exist $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in L^1(0, T; \mathbb{R}_+)$, c_1, $c_2 > 0$ and $\alpha \in [0, p - 1)$ such that*

$$|F(t, x)| \leq a(\|x\|)b(t), \tag{9}$$

$$\zeta_1 \in \partial F(t, x) \Rightarrow \|\zeta_1\| \leq c_1\|x\|^\alpha + c_2, \tag{10}$$

for all $t \in [0,T]$ *and all* $x \in \mathbb{R}^n$. *We suppose that* $L : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, *is given by* $L(t,x,y) = \frac{1}{p}\|y\|^p - F(t,x)$.
Then, the functional $f : Z \in \mathbb{R}$, *where*

$$Z = \left\{(u,v) \in L^p(0,T;Y) : u(t) = \int_0^t v(s)ds + c, c \in \mathbb{R}^n\right\}$$

given by $f(u,v) = \int_0^T L(t,u(t),v(t))dt$, *is uniformly Lipschitz on bounded subsets of* Z *and one has*

$$\partial f(u,v) \subset \int_0^T \partial L(t,u(t),v(t))dt. \tag{11}$$

Proof. We can apply Theorem 6 under Hypothesis 2, with the following cast of characters:

- $(T,\mathcal{T},\mu) = [0,T]$ with Lebesgue measure, $Y = \mathbb{R}^n \times \mathbb{R}^n$ be the Hilbert product space (hence is separable);
- $p > 1$ and

  $$Z = \left\{(u,v) \in L^p(0,T;Y) : u(t) = \int_0^t v(s)ds + c, c \in \mathbb{R}^n\right\}$$

  be a closed subspace of $L^p(0,T;Y)$;
- $f_t(x,y) = L(t,x,y) = \frac{1}{p}\|y\|^p + F(t,x)$; in our assumptions it results that the integrand $L(t,x,y)$ is measurable in t for a given element (x,y) of Y, locally Lipschitz in (x,y) for each $t \in [0,T]$.

Proposition 2.3.15 from [2] implies

$$\partial L(t,x,y) \subset \partial_x L(t,x,y) \times \partial_y L(t,x,y) = \partial\{F(t,x)\} \times \{\|y\|^{p-2}y\}.$$

Using (3) and (4), if $\zeta = (\zeta_1,\zeta_2) \in \partial L(t,x,y)$ it results $\zeta_1 \in \partial\{F(t,x)\}$ and $\zeta_2 = \|y\|^{p-2}y$, and hence

$$\|\zeta\| = \|\zeta_1\| + \|\zeta_2\| \leq c_1\|x\|^\alpha + c_2 + \|y\|^{p-1} \leq \tilde{c}\{1 + \|(x,y)\|^{p-1}\}$$

for each $t \in [0,T]$, since $\alpha < p-1$ and $p > 1$. The hypotheses of Theorem 6 are satisfied, therefore f is uniformly Lipschitz on the bounded subsets of Z and one has (11). □

Remark 3. *The interpretation of expression (11) is as follows: if* (u_0,v_0) *is an element of* Z *(so that* $v_0 = \dot{u}_0$*) and if* $\zeta \in \partial f(u_0,v_0)$, *we deduce the existence of a measurable function* $(q(t),p(t))$ *such that*

$$q(t) \in \partial\{F(t,u_0(t))\} \text{ and } p(t) = \|v_0(t)\|^{p-2}v_0(t) \text{ a.e. on } [0,T] \tag{12}$$

and for any (u,v) *in* Z, *one has*

$$\langle\zeta,(u,v)\rangle = \int_0^T \{\langle q(t),u(t)\rangle + \langle p(t),v(t)\rangle\}dt.$$

In particular, if $\zeta = 0$ *(so that* u_0 *is critical point for* $\varphi(u) = \int_0^T \Big[\frac{1}{p}\|\dot{u}(t)\|^p +$

$F(t,u(t))\Big]dt$), it then follows easily that $q(t)=\dot p(t)$ a.e., or taking into account (12)

$$\frac{d}{dt}\Big(\|\dot u_0(t)\|^{p-2}\dot u_0(t)\Big)\in\partial F(t,u_0(t)) \ a.e. \ on\ [0,T],$$

so that u_0 satisfies the inclusions system (3).

Remark 4. *If $p=2$ then the system (3) becomes system (2). If in addition F is continuously differentiable in x, then the system (3) becomes system (1).*

In proving Theorem 2 we will invoke the following nonsmooth variant of the Rabinowitz's saddle point theorem (see [1], Theorem 3.3):

Theorem 8. *Let X be a real Banach space, and let f be a locally Lipschitz function defined on X satisfies (PS) condition. Suppose $X=X_1\oplus X_2$ with a finite-dimensional subspace X_1, and there exist constants $b_1<b_2$ and a bounded neighborhood N of θ in X_1 such that*

$$f\mid_{X_2}\geq b_2,\qquad f\mid_{\partial N}\leq b_1,$$

then f has a critical point.

The definitions of a critical point and the Palais-Smale condition are now recalled.

Definition 1. *A point $u\in X$ is said to be a critical point of $f\in Lip_{loc}(X,\mathbb{R})$ if $\theta\in\partial f(u)$, namely $f^0(u,v)\geq 0$ for every $v\in X$. A real number c is called a critical value of f if there is a critical point $u\in X$ such that $f(u)=c$.*

Definition 2. *If $f\in Lip_{loc}(X,\mathbb{R})$, we say that f satisfies the Palais-Smale condition (in short (PS)) if each sequence (x_n) in X such that $(f(x_n))$ is bounded and $\lim_{n\to\infty}\lambda(x_n)=0$ has a convergent subsequence. We denote $\lambda(x)=\min_{x^*\in\partial f(x)}\|x^*\|$.*

4. Proof of the Theorems

4.1. **Proof of Theorem 1.** For $u\in W_T^{1,p}$, let $\bar u=\frac{1}{T}\int_0^T u(t)dt$ and $\tilde u=u-\bar u$. From Lebourg's mean value theorem it follows that for each $t\in[0,T]$ there exist $z(t)$ in $(\bar u,u(t))$ and $\zeta\in\partial F(t,z(t))$ such that $F(t,u(t))-F(t,\bar u)=\langle\zeta,\tilde u(t)\rangle$. It follows from (5) and Hölder's inequality that

$$\begin{aligned}\Big|\int_0^T[F(t,u(t))-F(t,\bar u)]dt\Big| &\leq \int_0^T|F(t,u(t))-F(t,\bar u)|dt\leq\\ \leq\int_0^T|\zeta||\tilde u(t)|dt &\leq \int_0^T\Big[2c_1(|\bar u|^\alpha+|\tilde u(t)|^\alpha)+c_2\Big]|\tilde u(t)|dt\leq\\ &\leq C_1\|\tilde u\|_\infty^{\alpha+1}+C_2\|\tilde u\|_\infty\|\bar u\|^\alpha+C_3\|\tilde u\|_\infty\leq\\ &\leq C_4\|\dot u\|_{L^p}^{\alpha+1}+\frac{1}{2p}\|\dot u\|_{L^p}^p+C_5\|\dot u\|_{L^p}+C_6\|\bar u\|^{q\alpha}\end{aligned}$$

for all $u\in W_T^{1,p}$ and some positive constants C_4, C_5 and C_6. Hence we have

$$\varphi(u)\geq\frac{1}{p}\int_0^T|\dot u(t)|^p dt+\int_0^T F(t,\bar u)dt+\int_0^T[F(t,u(t))-F(t,\bar u)]dt\geq$$

$$\geq \frac{1}{2p}\|\dot{u}\|^p_{L^p} - C_4\|\dot{u}\|^{\alpha+1}_{L^p} - C_5\|\dot{u}\|_{L^p} - C_6\|\bar{u}\|^{q\alpha} + \int_0^T F(t,\bar{u})dt \geq$$

$$\geq \frac{1}{2p}\|\dot{u}\|^p_{L^p} - C_4\|\dot{u}\|^{\alpha+1}_{L^p} - C_5\|\dot{u}\|_{L^p} + \|\bar{u}\|^{q\alpha}\Big\{\frac{1}{\|\bar{u}\|^{q\alpha}}\int_0^T F(t,\bar{u})dt - C_6\Big\}$$

for all $u \in W^{1,p}_T$, which implies that $\varphi(u) \to \infty$ as $\|u\| \to \infty$ by (6) because $\alpha < p-1$, and the norm $\|u\| = (\|\bar{u}\|^p + \|\dot{u}\|^p_{L^p})^{\frac{1}{p}}$ is an equivalent norm on $W^{1,p}_T$. Now we write $\varphi(u) = \varphi_1(u) + \varphi_2(u)$ where

$$\varphi_1(u) = \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt \text{ and } \varphi_2(u) = \int_0^T F(t,u(t))dt.$$

The function φ_1 is weakly lower semi-continuous (w.l.s.c.) on $W^{1,p}_T$. From (4), (5) and Theorem 7, taking to account Remark 2 and Proposition 4, it follows that φ_2 is w.l.s.c. on $W^{1,p}_T$. By Theorem 1.1 in [3] it follows that φ has a minimum u_0 on $W^{1,p}_T$. Evidently $Z \simeq W^{1,p}_T$ and $\varphi(u) = f(u,v)$ for all $(u,v) \in Z$. From Theorem 7, it results that f is uniformly Lipschitz on bounded subsets of Z, and therefore φ possesses the same properties relative to $W^{1,p}_T$. Proposition 2.3.2 in [2] implies that $0 \in \partial\varphi(u_0)$ (so that u_0 is critical point for φ). Now from Theorem 7 and Remark 3 it follows that the problem (3) has at least one solution $u \in W^{1,p}_T$. ∎

Remark 5. *Evidently if $p = 2$ then we obtain the existence of solutions of problem (2). If in addition F is continuously differentiable in x, then we obtain the existence of solutions of problem (2).*

4.2. **Proof of Theorem 2.** We will see that the functional

$$\varphi(u) : W^{1,p}_T \to \mathbb{R}, \quad \varphi(u) = \frac{1}{p}\int_0^T |\dot{u}(t)|^p dt + \int_0^T F(t,u(t))dt.$$

verify the assumptions of Theorem 8. Evidently $Z \simeq W^{1,p}_T$ and $\varphi(u) = f(u,v)$ for all $(u,v) \in Z$. From Theorem 7, it results that f is uniformly Lipschitz on bounded subsets of Z and regular at each $(u,v) \in Z$, and therefore φ possesses the same properties relative to $W^{1,p}_T$. The functional φ is neither bounded from below, nor from above. Indeed, if $w \in W^{1,p}_T$ is a constant function, then

$$\varphi(w) = \int_0^T F(t,w)dt = \|w\|^{q\alpha}\Big(\|w\|^{-q\alpha}\int_0^T F(t,w)dt\Big) \to -\infty \text{ as } \|w\| \to \infty$$

and, if $v \in W^{1,p}_T$ has mean zero, by the proof of Theorem 1 one has

$$\varphi(v) = \frac{1}{p}\int_0^T |\dot{v}(t)|^p dt + \int_0^T F(t,0)dt + \int_0^T [F(t,v(t)) - F(t,0)]dt =$$

$$= \frac{1}{p}\int_0^T |\dot{v}(t)|^p dt + \int_0^T F(t,0)dt + \int_0^T \langle \zeta_1, v(t)\rangle dt \geq$$

$$\geq \frac{1}{2p}\|\dot{u}\|^p_{L^p} - C_4\|\dot{u}\|^{\alpha+1}_{L^p} - C_5\|\dot{u}\|_{L^p} + \int_0^T F(t,0)dt$$

where we applied the Lebourg's mean value theorem and Sobolev inequality, and where C_1 and C_2 are positive constants, so that φ is not bounded from above. We denote

$$X_1 = \{w \in W_T^{1,p} : w = \text{constant}\}$$

and

$$X_2 = \Big\{v \in W_T^{1,p} : \int_0^T v(t) = 0\Big\}.$$

Evidently $W_T^{1,p} = X_1 \oplus X_2$ with $\dim X_1 < \infty$. From the above observations, we see that there exists $R > 0$ such that

$$\sup_{S_R} \varphi < \inf_{X_2} \varphi$$

where $S_R = \{w \in X_1 : \|w\|_{W_T^{1,p}} = R\}$.

We shall show that φ satisfies the (PS) condition. Let (u_k) be a sequence in $W_T^{1,p}$ such that $\varphi(u_k)$ is bounded and $\lambda(u_k) \to 0$ as $k \to \infty$. Writing $u_k(t) = \tilde{u}_k(t) + \bar{u}_k$ with $\bar{u}_k = \frac{1}{T}\int_0^T u_k(t)dt$, and using the definition of $\lambda(u_k)$ it results that there is some k_0 such that for each $k \geq k_0$ there exist $u_k^* \in \partial\varphi(u_k)$ with

$$|\langle u_k^*, h\rangle| \leq \|h\|_{W_T^{1,p}}, \quad \text{for all } h \in W_T^{1,p}.$$

From Theorem 7, if $u_k^* \in \partial\varphi(u_k)$ it results that there exist $q_k(t) \in \partial F(t, u_k(t))$ such that

$$|\langle u_k^*, \tilde{u}_k\rangle| = \Big|\int_0^T \Big[\|\dot{u}_k(t)\|^p + \langle q_k(t), \tilde{u}_k(t)\rangle\Big]dt\Big| \leq \|\tilde{u}_k\|_{W_T^{1,p}}, \quad \text{for all } k \geq k_0.$$

In similar way to the proof of Theorem 1, we have

$$\Big|\int_0^T \langle q_k(t), \tilde{u}_k(t)\rangle dt\Big| \leq \frac{1}{2p}\|\dot{u}_k\|_{L^p}^p + C_4\|\dot{u}_k\|_{L^p}^{\alpha+1} + C_5\|\dot{u}_k\|_{L^p} + C_6\|\bar{u}_k\|^{q\alpha}$$

for all k. Hence one has

$$\|\tilde{u}_k\|_{W_T^{1,p}} \geq \langle u_k^*, \tilde{u}_k\rangle = \int_0^T \Big[\|\dot{u}_k(t)\|^p + \langle q_k(t), \tilde{u}_k(t)\rangle\Big]dt \geq$$
$$\geq \frac{2p-1}{2p}\|\dot{u}_k\|_{L^p}^p - C_4\|\dot{u}_k\|_{L^p}^{\alpha+1} - C_5\|\dot{u}_k\|_{L^p} - C_6\|\bar{u}_k\|^{q\alpha}$$

for $k \geq k_0$. It follows from Wirtinger's inequality that

$$\|\tilde{u}_k\|_{W_T^{1,p}} \leq (1+c)^{\frac{1}{p}}\|\dot{\tilde{u}}_k\|_{L^p}$$

for all k. Hence we obtain

$$(1+c)^{\frac{1}{p}}\|\dot{\tilde{u}}_k\|_{L^p} \geq \frac{2p-1}{2p}\|\dot{\tilde{u}}_k\|_{L^p}^p - C_4\|\dot{\tilde{u}}_k\|_{L^p}^{\alpha+1} - C_5\|\dot{\tilde{u}}_k\|_{L^p} - C_6\|\bar{u}_k\|^{q\alpha}$$

for $k \geq k_0$, and it follows that

$$C_6\|\bar{u}_k\|^{q\alpha} \geq \frac{2p-1}{2p}\|\dot{\tilde{u}}_k\|_{L^p}^p - C_4\|\dot{\tilde{u}}_k\|_{L^p}^{\alpha+1} - \Big[(1+c)^{\frac{1}{p}} + C_5\Big]\|\dot{\tilde{u}}_k\|_{L^p}$$

or

$$C_7\|\bar{u}_k\|^{q\alpha} \geq \|\dot{\tilde{u}}_k\|_{L^p}^p \tag{13}$$

for some $C_7 > 0$ and for $k \geq k_0$. By the proof of Theorem 1 we have

$$\left| \int_0^T [F(t, u_k(t)) - F(t, \bar{u}_k)]dt \right| \leq \frac{1}{2p} \|\dot{u}_k\|_{L^p}^p + C_4 \|\dot{u}_k\|_{L^p}^{\alpha+1} + C_5 \|\dot{u}_k\|_{L^p} + C_6 \|\bar{u}_k\|^{q\alpha}$$

for all k. It follows from the boundedness of $\big(\varphi(u_k)\big)$, (13) and the above inequality that

$$C_8 \leq \varphi(u_k) = \frac{1}{p} \int_0^T |\dot{u}_k(t)|^p dt + \int_0^T [F(t, u_k(t)) - F(t, \bar{u}_k)]dt + \int_0^T F(t, \bar{u}_k)dt \leq$$

$$\leq \frac{2p-1}{2p} \|\dot{u}_k\|_{L^p}^p + C_4 \|\dot{u}_k\|_{L^p}^{\alpha+1} + C_5 \|\dot{u}_k\|_{L^p} + C_6 \|\bar{u}_k\|^{q\alpha} + \int_0^T F(t, \bar{u}_k)dt \leq$$

$$\leq \|\bar{u}_k\|^{q\alpha} \Big(\|\bar{u}_k\|^{-q\alpha} \int_0^T F(t, \bar{u}_k)dt + C_9 \Big)$$

for $k \geq k_0$ and some positive constants C_8 and C_9. The above inequality and (8) implies that $(\|\bar{u}_k\|)$ is bounded. Hence (u_k) is bounded by (13). Thus (u_k) is bounded in $W_T^{1,p}$ and hence contains a subsequence, relabeled (u_k), which converge to some $u \in W_T^{1,p}$, weakly in $W_T^{1,p}$ and strongly in $C([0,T];\mathbb{R}^n)$ (see Proposition 4). Therefore we have for $u_k^* \in \partial\varphi(u_k)$ and $u^* \in \partial\varphi(u)$

$$\langle u_k^* - u^*, u_k - u \rangle \to 0 \text{ as } k \to \infty.$$

But

$$\langle u_k^* - u^*, u_k - u \rangle = \int_0^T \Big[\langle q_k(t) - q(t), u_k(t) - u(t) \rangle + \|\dot{u}_k(t) - \dot{u}(t)\|^p \Big] dt =$$

$$= \|\dot{u}_k - \dot{u}\|_{L^p}^p + \int_0^T \langle q_k(t) - q(t), u_k(t) - u(t) \rangle dt$$

where $q_k(t) \in \partial F(t, u_k(t))$ and $q(t) \in \partial F(t, u(t))$. It is easy to verify, that $\|\dot{u}_k - \dot{u}\|_{L^p} \to 0$ as $k \to \infty$, and hence $u_k \to u$ in $W_T^{1,p}$. We conclude that (PS) is satisfied and from Theorem 8, φ admits a critical point. Now from Theorem 7 and Remark 3 it follows that the problem (3) has at least one solution $u \in W_T^{1,p}$. ∎

Remark 6. *Evidently if $p = 2$ then we obtain the existence of solutions of problem (2). If in addition F is continuously differentiable in x, then we obtain the existence of solutions of problem (2).*

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