PERIODIC SOLUTIONS OF SECOND-ORDER DIFFERENTIAL INCLUSIONS SYSTEMS WITH p–LAPLACIAN

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Abstract. Some existence results are obtained for periodic solutions of nonautonomous second-order differential inclusions systems with p–Laplacian.

1. Introduction

Consider the second order system

$$(1) \qquad \begin{aligned} \ddot{u}(t) &= \nabla F(t, u(t)) \text{ a.e. } t \in [0, T] \\ u(0) - u(T) &= \dot{u}(0) - \dot{u}(T) = 0 \end{aligned}$$

where $T > 0$ and $F : [0, T] \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumption:

(A) $F(t, x)$ is measurable in t for each $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in L^1(0, T; \mathbb{R}_+)$ such that

$$|F(t, x)| \leq a(\|x\|)b(t), \qquad \|\nabla F(t, x)\| \leq a(\|x\|)b(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$.

In the last years many authors starting with Mawhin and Willem (see [2]) proved the existence of solutions for problem (1) under suitable conditions on the potential F (see [6]-[17]). Also in a series of papers (see [3]-[5]) we have generalized some of these results for the case when the potential F is just locally Lipschitz in the second variable x not continuously differentiable.

The aim of this paper is to consider the problem (1) in a more general sense. More exactly our results represent the extensions to systems with p–Laplacian and also with discontinuity (we consider the generalized gradients unlike continuously gradient in classical results).

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Consider the second order differential inclusions system

$$\frac{d}{dt}\big(\|\dot u(t)\|^{p-2}\dot u(t)\big) \in \partial F(t,u(t)) \text{ a.e. } t\in[0,T], \qquad (2)$$
$$u(0)=u(T),\dot u(0)=\dot u(T),$$

where $p>1$, $T>0$, $F:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ and ∂ denotes the Clarke subdifferential.

We suppose that $F=F_1+F_2$ and F_1, F_2 satisfy the following assumption:

(A') F_1, F_2 are measurable in t for each $x\in\mathbb{R}^n$, at least F_1 or F_2 are strictly differentiable in x and there exist $k_1,k_2\in L^q(0,T;\mathbb{R})$ such that

$$|F_1(t,x_1)-F_1(t,x_2)|\le k_1(t)||x_1-x_2||,$$
$$|F_2(t,x_1)-F_2(t,x_2)|\le k_2(t)||x_1-x_2||,$$

for all $x_1,x_2\in\mathbb{R}^n$ and all $t\in[0,T]$.

The corresponding functional $\varphi:W_T^{1,p}\to\mathbb{R}$ is given by

$$\varphi(u)=\frac{1}{p}\int_0^T\|\dot u(t)\|^p dt+\int_0^T F(t,u(t))dt.$$

Definition 1. *A function $G:\mathbb{R}^n\to\mathbb{R}$ is called to be (λ,μ)–subconvex if*

$$G(\lambda(x+y))\le\mu(G(x)+G(y))$$

for some $\lambda,\mu>0$ and all $x,y\in\mathbb{R}^n$.

Remark 1. *When $\lambda=\mu=\frac{1}{2}$, a function $(\frac{1}{2},\frac{1}{2})$–subconvex is called convex.*
When $\lambda=\mu=1$, a function $(1,1)$–subconvex is called subadditive.
When $\lambda=1$, $\mu>0$, a function $(1,\mu)$–subconvex is called μ–subadditive.

2. Main results

Theorem 1. *Assume that $F=F_1+F_2$, where F_1, F_2 satisfy assumption (A') and the following conditions:*

(i) *$F_1(t,\cdot)$ is (λ,μ)–subconvex with $\lambda>1/2$ and $0<\mu<2^{p-1}\lambda^p$ for a.e. $t\in[0,T]$;*

(ii) *there exist $c_1,c_2>0$ and $\alpha\in[0,p-1)$ such that*

$$\zeta\in\partial F_2(t,x)\Rightarrow\|\zeta\|\le c_1\|x\|^\alpha+c_2$$

for all $x\in\mathbb{R}^n$ and a.e. $t\in[0,T]$;

(iii) *for $q=\frac{p}{p-1}$,*

$$\frac{1}{\|x\|^{q\alpha}}\Big[\frac{1}{\mu}\int_0^T F_1(t,\lambda x)dt+\int_0^T F_2(t,x)dt\Big]\to\infty,\ \textit{as } \|x\|\to\infty.$$

Then the problem (2) has at least one solution which minimizes φ on $W_T^{1,p}$.

Theorem 2. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A') and the following conditions:*

(iv) *$F_1(t, \cdot)$ is (λ, μ)-subconvex for a.e. $t \in [0, T]$, and there exists $\gamma \in L^1(0, T; \mathbb{R})$, $h \in L^1(0, T; \mathbb{R}^n)$ with $\int_0^T h(t)dt = 0$ such that*

$$F_1(t, x) \geq \langle h(t), x\rangle + \gamma(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

(v) *there exist $c_1 > 0$, $c_0 \in \mathbb{R}$ such that*

$$\zeta \in \partial F_2(t, x) \Rightarrow \|\zeta\| \ \leq c_1$$

for all $x \in \mathbb{R}^n$ and all $t \in [0, T]$, and

$$\int_0^T F_2(t, x)dt \geq c_0$$

for all $x \in \mathbb{R}^n$;

(vi)

$$\frac{1}{\mu}\int_0^T F_1(t, \lambda x)dt + \int_0^T F_2(t, x)dt \to \infty, \text{ as } \|x\| \to \infty.$$

Then the problem (2) has at least one solution which minimizes φ on $W_T^{1,p}$.

Theorem 3. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A') and the following conditions:*

(vii) *$F_1(t, \cdot)$ is (λ, μ)-subconvex for a.e. $t \in [0, T]$, and there exists $\gamma \in L^1(0, T; \mathbb{R})$, $h \in L^1(0, T; \mathbb{R}^n)$ with $\int_0^T h(t)dt = 0$ such that*

$$F_1(t, x) \geq \langle h(t), x\rangle + \gamma(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

(viii) *there exist $c_1, c_2 > 0$ and $\alpha \in [0, p - 1)$ such that*

$$\zeta \in \partial F_2(t, x) \Rightarrow \|\zeta\| \leq c_1\|x\|^\alpha + c_2$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0, T]$;

(ix) *for $q = \frac{p}{p-1}$,*

$$\frac{1}{\|x\|^{q\alpha}}\int_0^T F_2(t, x)dt \to \infty, \text{ as } \|x\| \to \infty.$$

Then the problem (2) has at least one solution which minimizes φ on $W_T^{1,p}$.

Remark 2. *Theorems 1, 2 and 3, generalizes the corresponding Theorems 2.1, 2.3 and 2.4 of* [5]*. In fact, it follows from these theorems letting $p = 2$.*

3. The preliminary results

We introduce some functional spaces. Let $[0,T]$ be a fixed real interval ($0 < T < \infty$) and $1 < p < \infty$. We denote by $W_T^{1,p}$ the Sobolev space of functions $u \in L^p(0,T;\mathbb{R}^n)$ having a weak derivative $\dot{u} \in L^p(0,T;\mathbb{R}^n)$. The norm over $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} = \Big(\int_0^T \|u(t)\|^p dt + \int_0^T \|\dot{u}(t)\|^p dt\Big)^{\frac{1}{p}}.$$

We recall that

$$\|u\|_{L^p} = \Big(\int_0^T \|u(t)\|^p dt\Big)^{\frac{1}{p}} \text{ and } \|u\|_\infty = \max_{t\in[0,T]} \|u(t)\|.$$

For our aims it is necessary to recall some very well know results (for proof and details see [2]).

Proposition 4. *If $u \in W_T^{1,p}$ then*

$$\|u\|_\infty \le c\|u\|_{W_T^{1,p}}.$$

If $u \in W_T^{1,p}$ and $\int_0^T u(t)dt = 0$ then

$$\|u\|_\infty \le c\|\dot{u}\|_{L^p}.$$

Proposition 5. *If the sequence $(u_k)_k$ converges weakly to u in $W_T^{1,p}$, then $(u_k)_k$ converges uniformly to u on $[0,T]$.*

Let X be a Banach space. Now follows [1], for each $x, v \in X$, we define the *generalized directional derivative* at x in the direction v of a given $f \in Lip_{loc}(X,\mathbb{R})$ as

$$f^0(x;v) = \limsup_{y\to x,\lambda\searrow 0} \frac{f(y+\lambda v) - f(y)}{\lambda}$$

and we denote by

$$\partial f(x) = \{x^* \in X^* : f^0(x;v) \ge \langle x^*, v\rangle, \text{ for all } v \in X\}$$

the *generalized gradient* of f at x (the Clarke subdifferential).

We recall the *Lebourg's mean value theorem* (see [1], Theorem 2.3.7).

Theorem 6. *Let x and y be points in X, and suppose that f is Lipschitz on open set containing the line segment [x,y]. Then there exists a point u in (x,y) such that*

$$f(y) - f(x) \in \langle \partial f(u), y - x\rangle.$$

Clarke consider in [1] the following abstract framework:

- let $(T,\mathcal{T},\,\mu)$ be a positive complete measure space with $\mu(T)<\infty$, and let Y be a separable Banach space;
- let Z be a closed subspace of $L^p(T;Y)$ (for some p in $[1,\infty)$), where $L^p(T;Y)$ is the space of p- integrable functions from T to Y;
- define a functional f on Z via

$$f(x)=\int_T f_t(x(t))\mu(dt),$$

where $f_t:Y\to\mathbb{R}$, $(t\in T)$ is a given family of functions;
- suppose that for each y in Y the function $t\to f_t(y)$ is measurable, and that x is a point at which $f(x)$ is defined (finitely).

Hypothesis 1: There is a function k in $L^q(T,\mathbb{R})$, $\left(\frac{1}{p}+\frac{1}{q}=1\right)$ such that, for all $t\in T$,

$$|f_t(y_1)-f_t(y_2)|\le k(t)\|y_1-y_2\|_Y \text{ for all } y_1,y_2\in Y$$

Hypothesis 2: Each function f_t is Lipschitz (of some rank) near each point of Y, and for some constant c, for all $t\in T$, $y\in Y$, one has

$$\zeta\in\partial f_t(y)\Rightarrow\|\zeta\|_{Y^*}\le c\{1+\|y\|_Y^{p-1}\}.$$

Under this conditions described above Clarke prove (see [1], Theorem 2.7.5):

Theorem 7. *Under the conditions described above, under either of Hypothesis 1 or 2, f is uniformly Lipschitz on bounded subsets of Z, and one has*

$$\partial f(x)\subset\int_T\partial f_t(x(t))\mu(dt).$$

Further, if each f_t is regular at $x(t)$ then f is regular at x and equality holds.

Remark 3. *f is globally Lipschitz on Z when Hypothesis 1 hold.*

Now we can prove the following result.

Theorem 8. *Let $F:[0,T]\times\mathbb{R}^n\to\mathbb{R}$ such that $F=F_1+F_2$ where F_1, F_2 are measurable in t for each $x\in\mathbb{R}^n$, and there exist $k_1\in L^q(0,T;\mathbb{R})$, $a\in C(\mathbb{R}^+,\mathbb{R}^+)$, $b\in L^1(0,T;\mathbb{R}^+)$, c_1, $c_2>0$ and $\alpha\in[0,p-1)$ such that*

$$|F_1(t,x_1)-F_1(t,x_2)|\le k_1(t)\|x_1-x_2\|$$

$$|F_2(t,x)|\le a(\|x\|)b(t)$$

$$\zeta\in\partial F_2(t,x)\Rightarrow\|\zeta\|\le c_1\|x\|^\alpha+c_2 \tag{3}$$

for all $t \in [0, T]$ *and all* x, x_1, $x_2 \in \mathbb{R}^n$. *We suppose that* $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$, *is given by* $L(t, x, y) = \frac{1}{p}\|y\|^p + F(t, x)$.
Then, the functional $f : Z \in \mathbb{R}$, *where*

$$Z = \left\{(u, v) \in L^p(0, T; Y) : u(t) = \int_0^t v(s)ds + c, c \in \mathbb{R}^n\right\}$$

given by $f(u, v) = \int_0^T L(t, u(t), v(t))dt$, *is uniformly Lipschitz on bounded subsets of* Z *and one has*

$$\partial f(u, v) \subset \int_0^T \{\partial F_1(t, u(t)) + \partial F_2(t, u(t))\} \times \{\|v(t)\|^{p-2}v(t)\}dt.$$

Proof. Let $L_1(t, x, y) = F_1(t, x)$, $L_2(t, x, y) = \frac{1}{p}\|y\|^p + F_2(t, x)$ and f_1,$f_2 : Z \to \mathbb{R}$ given by $f_1(u, v) = \int_0^T L_1(t, u(t), v(t))dt$, $f_2(u, v) = \int_0^T L_2(t, u(t), v(t))dt$. For f_1 we can apply Theorem 7 under Hypothesis 1, with the following cast of characters:

- $(T,\mathcal{T},\ \mu)$=[0,T] with Lebesgue measure, $Y = \mathbb{R}^n \times \mathbb{R}^n$ be the Hilbert product space (hence is separable);
- $p > 1$ and

$$Z = \left\{(u, v) \in L^p(0, T; Y) : u(t) = \int_0^t v(s)ds + c, c \in \mathbb{R}^n\right\}$$

  be a closed subspace of $L^p(0, T; Y)$;
- $f_t(x, y) = L_1(t, x, y) = F_1(t, x)$; in our assumptions it results that the integrand $L_1(t, x, y)$ is measurable in t for a given element (x, y) of Y and there exists $k \in L^q(0, T; \mathbb{R})$ such that

$$\begin{aligned} &|L_1(t, x_1, y_1) - L_1(t, x_2, y_2)| = |F_1(t, x_1) - F_1(t, x_2)| \le \\ &\le k_1(t)\|x_1 - x_2\| \le k_1(t)\big(\|x_1 - x_2\| + \|y_1 - y_2\|\big) = \\ &\qquad = k_1(t)\|(x_1, y_1) - (x_2, y_2)\|_Y \end{aligned} \tag{4}$$

  for all $t \in [0, T]$ and all (x_1, y_1), $(x_2, y_2) \in Y$. Hence f_1 is uniformly Lipschitz on bounded subsets of Z and one has

$$\partial f_1(u, v) \subset \int_0^T \partial L_1(t, u(t), v(t))dt.$$

For f_2 we can apply Theorem 7 under Hypothesis 2 with the same cast of characters, but now $f_t(x, y) = L_2(t, x, y) = \frac{1}{p}\|y\|^p + F_2(t, x)$. In our assumptions it results that the integrand $L_2(t, x, y)$ is measurable in t for a given element (x, y) of Y and locally Lipschitz in (x, y) for each $t \in [0, T]$.

Proposition 2.3.15 from [1] implies

$$\partial L_2(t, x, y) \subset \partial_x L_2(t, x, y) \times \partial_y L_2(t, x, y) = \partial F_2(t, x) \times \{\|y\|^{p-2}y\}.$$

Using (3) and (4), if $\zeta = (\zeta_1, \zeta_2) \in \partial L_2(t, x, y)$ it results $\zeta_1 \in \partial F_2(t, x)$ and $\zeta_2 = \|y\|^{p-2}y$, and hence

$$\|\zeta\| = \|\zeta_1\| + \|\zeta_2\| \leq c_1\|x\|^{\alpha} + c_2 + \|y\|^{p-1} \leq \tilde{c}\{1 + \|(x, y)\|^{p-1}\}$$

for each $t \in [0, T]$, since $\alpha < p-1$ and $p > 1$. Hence f_2 is uniformly Lipschitz on bounded subsets of Z and one has

$$\partial f_2(u, v) \subset \int_0^T \partial L_2(t, u(t), v(t))dt.$$

It follows that $f = f_1 + f_2$ is uniformly Lipschitz on the bounded subsets of Z.

Proposition 2.3.3 and Proposition 2.3.15 from [1] implies

$$\partial f(u, v) \subset \partial f_1(u, v) + \partial f_2(u, v) \subset$$

$$\subset \int_0^T \Big[\partial L_1(t, u(t), v(t)) + \partial L_2(t, u(t), v(t))\Big]dt \subset$$

$$\subset \int_0^T \Big[\Big(\partial_x L_1(t, u(t), v(t)) \times \partial_y L_1(t, u(t), v(t))\Big) +$$

$$+\Big(\partial_x L_2(t, u(t), v(t)) \times \partial_y L_2(t, u(t), v(t))\Big)\Big]dt \subset$$

$$\subset \int_0^T \Big[\Big(\partial_x L_1(t, u(t), v(t)) + \partial_x L_2(t, u(t), v(t))\Big) \times$$

$$\times \Big(\partial_y L_1(t, u(t), v(t)) + \partial_y L_2(t, u(t), v(t))\Big)\Big]dt =$$

$$= \int_0^T \Big(\partial F_1(t, u(t)) + \partial F_2(t, u(t))\Big) \times \{\|v(t)\|^{p-2}v(t)\}dt.$$

Moreover, Corollary 1 of Proposition 2.3.3 from from [1] imply that, if at least of the functions F_1, F_2 is strictly differentiable in x for all $t \in [0, T]$ then

$$\partial f(u, v) \subset \int_0^T \partial F(t, u(t)) \times \{\|v(t)\|^{p-2}v(t)\}dt. \tag{5}$$

□

Remark 4. *The interpretation of expression (5) is as follows: if (u_0, v_0) is an element of Z (so that $v_0 = \dot{u}_0$) and if $\zeta \in \partial f(u_0, v_0)$, we deduce the existence of a measurable function $(q(t), p(t))$ such that*

(6) $\quad q(t) \in \partial F(t, u_0(t))$ *and* $p(t) = \|v_0(t)\|^{p-2}v_0(t)$ *a.e. on* $[0, T]$

and for any (u,v) *in* Z, *one has*

$$\langle \zeta, (u,v)\rangle = \int_0^T \{\langle q(t), u(t)\rangle + \langle p(t), v(t)\rangle\} dt.$$

In particular, if $\zeta = 0$ *(so that* u_0 *is critical point for* $\varphi(u) = \int_0^T \Big[\frac{1}{p}\|\dot u(t)\|^p + F(t,u(t))\Big] dt$*), it then follows easily that* $q(t) = \dot p(t)$ *a.e., or taking into account (6)*

$$\frac{d}{dt}\big(\|\dot{u_0}(t)\|^{p-2}\dot{u_0}(t)\big) \in \partial F(t, u_0(t)) \text{ a.e. on } [0,T],$$

so that u_0 *satisfies the inclusions system (2).*

Remark 5. *Of course if* $p = 2$ *and* F *is continuously differentiable in* x, *then the system (2) becomes system (1).*

4. Proofs of the Theorems

Proof of Theorem 1. From (A') it follows immediately there exist $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b \in L^1(0,T;\mathbb{R}_+)$ such that

$$|F_1(t,x)| \le a(\|x\|)b(t),$$

for all $x \in \mathbb{R}^n$ and all $t \in [0,T]$. Like in [11] we obtain

$$F_1(t,x) \le (2\mu\|x\|^\beta + 1)a_0 b(t)$$

for all $x \in \mathbb{R}^n$ and all $t \in [0,T]$, where $\beta < p$ and $a_0 = \max_{0\le s\le 1} a(s)$.
For $u \in W_T^{1,p}$, let $\bar u = \frac{1}{T}\int_0^T u(t)dt$ and $\tilde u = u - \bar u$. From Lebourg's mean value theorem it follows that for each $t \in [0,T]$ there exist $z(t)$ in $(\bar u, u(t))$ and $\zeta \in \partial F_2(t, z(t))$ such that $F_2(t,u(t)) - F_2(t,\bar u) = \langle \zeta, \tilde u(t)\rangle$. It follows from *(ii)* and Hölder's inequality that

$$\Big|\int_0^T [F_2(t,u(t)) - F_2(t,\bar u)]dt\Big| \le \int_0^T |F_2(t,u(t)) - F_2(t,\bar u)|dt \le$$

$$\le \int_0^T \|\zeta\|\|\tilde u(t)\|dt \le \int_0^T \Big[2c_1(\|\bar u\|^\alpha + \|\tilde u(t)\|^\alpha) + c_2\Big]\|\tilde u(t)\|dt \le$$

$$\le C_1\|\tilde u\|_\infty^{\alpha+1} + C_2\|\tilde u\|_\infty\|\bar u\|^\alpha + C_3\|\tilde u\|_\infty \le$$

$$\le C_4\|\dot u\|_{L^p}^{\alpha+1} + \frac{1}{2p}\|\dot u\|_{L^p}^p + C_5\|\dot u\|_{L^p} + C_6\|\bar u\|^{q\alpha}$$

for all $u \in W_T^{1,p}$ and some positive constants C_4, C_5 and C_6. Hence we have

$$\varphi(u) \ge \frac{1}{p}\int_0^T \|\dot u(t)\|^p dt + \frac{1}{\mu}\int_0^T F_1(t, \lambda\bar u)dt - \int_0^T F_1(t, -\tilde u(t))dt +$$

$$+\int_0^T F_2(t,\bar{u})dt + \int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]dt \geq$$

$$\geq \frac{1}{2p}\|\dot{u}\|_{L^p}^p - C_4\|\dot{u}\|_{L^p}^{\alpha+1} - C_5\|\dot{u}\|_{L^p} - C_6\|\bar{u}\|^{q\alpha} - (2\mu\|\tilde{u}\|_\infty^\beta + 1)\int_0^T a_0 b(t)dt +$$

$$+\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})dt + \int_0^T F_2(t,\bar{u})dt \geq \frac{1}{2p}\|\dot{u}\|_{L^p}^p - C_4\|\dot{u}\|_{L^p}^{\alpha+1} - C_5\|\dot{u}\|_{L^p} -$$

$$-C_7\|\dot{u}\|_{L^p}^\beta - C_8 + \|\bar{u}\|^{q\alpha}\Big\{\frac{1}{\|\bar{u}\|^{q\alpha}}\Big[\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u})dt + \int_0^T F_2(t,\bar{u})dt\Big] - C_6\Big\}$$

for all $u \in W_T^{1,p}$, which implies that $\varphi(u) \to \infty$ as $\|u\| \to \infty$ by *(iii)* because $\alpha < p-1$, $\beta < p$, and the norm $\|u\| = (\|\bar{u}\|^p + \|\dot{u}\|_{L^p}^p)^{\frac{1}{p}}$ is an equivalent norm on $W_T^{1,p}$. Now we write $\varphi(u) = \varphi_1(u) + \varphi_2(u)$ where

$$\varphi_1(u) = \frac{1}{p}\int_0^T \|\dot{u}(t)\|^p dt \text{ and } \varphi_2(u) = \int_0^T F(t,u(t))dt.$$

The function φ_1 is weakly lower semi-continuous (w.l.s.c.) on $W_T^{1,p}$. From *(i), (ii)* and Theorem 7, taking to account Remark 3 and Proposition 5, it follows that φ_2 is w.l.s.c. on $W_T^{1,p}$. By Theorem 1.1 in [2] it follows that φ has a minimum u_0 on $W_T^{1,p}$. Evidently $Z \simeq W_T^{1,p}$ and $\varphi(u) = f(u,v)$ for all $(u,v) \in Z$. From Theorem 8, it results that f is uniformly Lipschitz on bounded subsets of Z, and therefore φ possesses the same properties relative to $W_T^{1,p}$. Proposition 2.3.2 in [1] implies that $0 \in \partial\varphi(u_0)$ (so that u_0 is critical point for φ). Now from Theorem 8 and Remark 4 it follows that the problem (2) has at least one solution $u \in W_T^{1,p}$. ∎

Proof of Theorem 2. Let (u_k) be a minimizing sequence of φ. It follows from *(iv), (v)*, Lebourg's mean value theorem and Proposition 4 that

$$\varphi(u_k) \geq \frac{1}{p}\|\dot{u}_k\|_{L^p}^p + \int_0^T \langle h(t), u_k(t)\rangle dt + \int_0^T \gamma(t)dt +$$

$$+\int_0^T F_2(t,\bar{u}_k)dt - \int_0^T \|\zeta\|\|\tilde{u}_k(t)\|dt \geq \frac{1}{p}\|\dot{u}_k\|_{L^p}^p -$$

$$-\|\tilde{u}_k\|_\infty \int_0^T \|h(t)\|dt + \int_0^T \gamma(t)dt - c_1\|\tilde{u}_k\|_\infty + c_0 \geq$$

$$\geq \frac{1}{p}\|\dot{u}_k\|_{L^p}^p - c_2\|\dot{u}_k\|_{L^p} - c_3$$

for all k and some constants c_2, c_3, which implies that $(\tilde{u}_k)$ is bounded. On the other hand, in a way similar to the proof of Theorem 1, one has

$$\Big|\int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]dt\Big| \le \frac{1}{2p}\|\dot{u}\|_{L^p}^p + C_1\|\dot{u}\|_{L^p}$$

for all k and some positive constant C_1, which implies that

$$\varphi(u_k) \ge \frac{1}{p}\|\dot{u}_k\|_{L^p}^p + \frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u}_k)dt - \int_0^T F_1(t,-\tilde{u}_k(t))dt +$$

$$+\int_0^T F_2(t,\bar{u}_k)dt + \int_0^T [F_2(t,u(t)) - F_2(t,\bar{u}_k)]dt \ge$$

$$\ge \frac{1}{2p}\|\dot{u}_k\|_{L^p}^p - a(\|\tilde{u}_k\|_\infty)\int_0^T b(t)dt - C_1\|\dot{u}_k\|_{L^p} +$$

$$+\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u}_k)dt + \int_0^T F_2(t,\bar{u}_k)dt$$

for all k and some positive constant C_1. It follows from *(vi)* and the boundedness of $(\tilde{u}_k)$ that $(\bar{u}_k)$ is bounded. Hence φ has a bounded minimizing sequence (u_k). Now Theorem 2 follows like Theorem 1. ∎

Proof of Theorem 3. From *(vii)*, *(viii)* and Proposition 4 it follows that

$$\varphi(u) \ge \frac{1}{p}\|\dot{u}\|_{L^p}^p + \int_0^T \langle h(t), u(t)\rangle dt +$$

$$+\int_0^T \gamma(t)dt + \int_0^T F_2(t,\bar{u})dt + \int_0^T [F_2(t,u(t)) - F_2(t,\bar{u})]dt \ge$$

$$\ge \frac{1}{2p}\|\dot{u}\|_{L^p}^p - \|\tilde{u}\|_\infty \int_0^T \|h(t)\|dt +$$

$$+\int_0^T \gamma(t)dt - C_1\|\dot{u}\|_{L^p}^{\alpha+1} - C_2\|\dot{u}\|_{L^p} + \int_0^T F_2(t,\bar{u})dt - C_3\|\bar{u}\|^{q\alpha} \ge$$

$$\ge \frac{1}{2p}\|\dot{u}\|_{L^p}^p - C_1\|\dot{u}\|_{L^p}^{\alpha+1} - C_4(\|\dot{u}\|_{L^p}+1) + \|\bar{u}\|^{q\alpha}\Big[\frac{1}{\|\bar{u}\|^{q\alpha}}\int_0^T F_2(t,\bar{u})dt - C_3\Big]$$

for all $u \in W_T^{1,p}$ and some positive constants C_1, C_3 and C_4. Now follows like in the proof of Theorem 1 that φ is coercive by *(ix)*, which completes the proof. ∎

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