On the growth rate of minor-closed classes of graphs

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Abstract

A minor-closed class of graphs is a set of labelled graphs which is closed under isomorphism and under taking minors. For a minor-closed class $\mathcal{G}$, we let g_n be the number of graphs in $\mathcal{G}$ which have n vertices. A recent result of Norine *et al.* [13] shows that for all minor-closed class $\mathcal{G}$, there is a constant c such that $g_n \leq c^n n!$. Our main results show that the growth rate of g_n is far from arbitrary. For example, no minor-closed class $\mathcal{G}$ has $g_n = c^{n+o(n)} n!$ with $0 < c < 1$ or $1 < c < \xi \approx 1.76$.

1 Introduction

In 1994, Scheinerman and Zito [15] introduced the study of the possible growth rates of hereditary classes of graphs (that is, sets of graphs which are closed under isomorphism and induced subgraphs). Here we study the same problem for classes which are closed under taking minors. Clearly, being minor-closed is a much stronger property than to be hereditary. However, many of the more structured hereditary classes such as graphs embeddable in a fixed surface or graphs of tree width bounded by a fixed constant are minor-closed and the possible growth rates attainable are of independent interest.

A broad classification of possible growth rates for hereditary classes given by Scheinermann and Zito [15] is into four categories, namely constant, polynomial, exponential and factorial. This has been considerably extended in a series of papers by Balogh, Bollobas and Weinrich [2, 3, 4] who use the term *speed* for what we call growth rate.

A first and important point to note is that if a class of graphs is minor-closed then it is hereditary. Hence, in what follows we are working within the confines described by the existing classifications of growth rates of hereditary classes. Working in this more restricted context, we obtain simpler characterization of the different categories of growth rate and simpler proofs. This is done in Section 2. In Section 3, we establish some results about the possible behaviour about classes in the most interesting range of growth rates, namely the factorial range. We conclude by listing some open questions in Section 4.

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A significant difference between hereditary and minor-closed classes is due to the following recent result by Norine *et al.* A class is proper if it does not contain all graphs.

Theorem 1 (Norine et al. [13]). *If $\mathcal{G}$ is a proper minor-closed class of graphs then $g_n \leq c^n n!$ for some constant c.*

Remark. In contrast, a hereditary class such as the set of bipartite graphs can have growth rate of order 2^{cn^2} with $c > 0$.

We close this introduction with some definitions and notations. We consider simple labelled graphs. The *size* of a graph is the number of vertices; graphs of size n are labelled with vertex set $\{1, 2, \ldots, n\}$. A *class* of graphs is a family of labelled graphs closed under isomorphism. For a class of graphs $\mathcal{G}$, we let $\mathcal{G}_n$ be the graphs in $\mathcal{G}$ with n vertices, and we let $g_n = |\mathcal{G}_n|$. The (exponential) *generating function* associated to a class $\mathcal{G}$ is $G(z) = \sum_{n\geq 0} \frac{g_n}{n!} z^n$.

The relation $H < G$ between graphs means H *is a minor of* G. A family $\mathcal{G}$ is *minor-closed* if $G \in \mathcal{G}$ and $H < G$ implies $H \in \mathcal{G}$. A class is *proper* if it does not contain all graphs. A graph H is a (minimal) *excluded minor* for a minor-closed family $\mathcal{G}$ if $H \notin \mathcal{G}$ but every proper minor of H is in $\mathcal{G}$. We write $\mathcal{G} = \mathrm{Ex}(H_1, H_2, \cdots)$ if $H_1, H_2, \ldots$ are the excluded minors of $\mathcal{G}$. By the theory of graph minors developed by Robertson and Seymour [14], the number of excluded minors is always finite.

2 A classification theorem

Our classification theorem for the possible growth rate of minor-closed classes of graphs involves the following classes; it is easy to check that they are all minor-closed.

- $\mathcal{P}$ is the class of *path forests*: graphs whose connected components are paths.
- $\mathcal{S}$ is the class of *star forests*: graphs whose connected components are stars (this includes isolated vertices).
- $\mathcal{M}$ is the class of *matchings*: graphs whose connected components are edges and isolated vertices.
- $\mathcal{X}$ is the class of *stars*: graphs made of one star and some isolated vertices.

Theorem 2. *Let $\mathcal{G}$ be a proper minor-closed family and let g_n be the number of graphs in $\mathcal{G}$ with n vertices.*

1. *If $\mathcal{G}$ contains all the paths, then g_n has* factorial growth*, that is,*
   $n! \leq g_n \leq c^n n!$ *for some $c > 1$;*

2. *else, if $\mathcal{G}$ contains all the star forests, then g_n has* almost-factorial growth*, that is,*
   $B(n) \leq g_n \leq \epsilon^n n!$ *for all $\epsilon > 0$, where $B(n)$ is the n^{th} Bell number;*

3. *else, if $\mathcal{G}$ contains all the matchings, then g_n has* semi-factorial growth*, that is,*
   $a^n n^{(1-1/k)n} \leq g_n \leq b^n n^{(1-1/k)n}$ *for some integer $k \geq 2$ and some $a, b > 0$;*

4. *else, if $\mathcal{G}$ contains all the stars, then g_n has* exponential growth*, that is,* $2^{n-1} \leq g_n \leq c^n$ *for some $c > 2$;*

5. *else, if $\mathcal{G}$ contains all the graphs with a single edge, then g_n has* polynomial growth*, that is,* $g_n = P(n)$ *for some polynomial $P(n)$ of degree at least 2 and n sufficiently large*;

6. *else, g_n is* constant*, namely g_n is equal to 0 or 1 for n sufficiently large.*

Remark. As mentioned in the introduction, some of the results given by Theorem 2 follow from the previous work on hereditary classes. In particular, the classification of growth between *pseudo factorial* (this includes our categories factorial, almost-factorial and semi-factorial), *exponential*, *polynomial* and *constant* was proved by Scheinerman and Zito in [15]. A refined description of the exponential growth category was also proved in this paper (we have not included this refinement in our statement of the classification Theorem 2 since we found no shorter proof of this result in the context of minor-closed classes). The refined descriptions of the semi-factorial and polynomial growth categories stated in Theorem 2 were established in [2]. Finally, the *jump* between the semi-factorial growth category and the almost-factorial growth category was established in [4].

The rest of this section is devoted to the proof of Theorem 2. This proof is self-contained and does not use the results from [15, 2, 3, 4]. We begin by the following easy estimates.

Lemma 3. *1. The number of path forests of size n satisfies $|\mathcal{P}_n| \geq n!$.*
2. The number of star forests of size n satisfies $|\mathcal{S}_n| \geq B(n)$.
3. The number of matchings of size n satisfies $|\mathcal{M}_n| \geq n!! = n(n-2)(n-4)\ldots$.
4. The number of stars of size n satisfies $|\mathcal{X}_n| \geq 2^{n-1}$.

We recall that $\log(n!) = n\log(n) + O(n)$, $\log B(n) = n\log(n) - n\log(\log(n)) + O(n)$ and $\log(n!!) = n\log(n)/2 + O(n)$.

Proof. 1. The number of path forests of size $n \geq 2$ made of a single path is $n!/2$; the number of path forests of size $n \geq 2$ made of an isolated vertex and a path is $n!/2$.
2. A star-forest defines a partition of $[n] := \{1, 2, \ldots, n\}$ (together with some marked vertices: the centers of the stars) and the partitions of $[n]$ are counted by the Bell numbers $B(n)$.
3. The vertex n of a matching of size n can be isolated or joined to any of the $(n-1)$ other vertices, hence $|\mathcal{M}_n| \geq |\mathcal{M}_{n-1}| + n|\mathcal{M}_{n-2}|$. The property $|\mathcal{M}_n| \geq n!!$ follows by induction.
4. The number of stars for which 1 is the center of the star is 2^{n-1}. $\square$

Proof of Theorem 2

• The lower bound for classes of graphs containing all paths follows from Lemma 3 while the upper bound follows from Theorem 1.

• The lower bound for classes of graphs containing all the star forests but not all the paths follows from Lemma 3. The upper bound is given by the following Claim (and the observation that if a class $\mathcal{G}$ does not contain a given path P, then $\mathcal{G} \subseteq \mathrm{Ex}(P)$).

Claim 4. *For any path P, the growth rate of $Ex(P)$ is bounded by $\epsilon^n n^n$ for all $\epsilon > 0$.*

The proof of Claim 4 use the notion of *depth-first search spanning tree* (or *DFS tree* for short) of a graph. A DFS tree of a connected graph G is a rooted spanning tree obtained by a *depth-first search algorithm* on G (see, for instance, [5]). If G is not connected, a choice of a DFS tree on each component of G is a *DFS spanning forest*. We recall that if T is a DFS spanning forest of G, every edge of G which is not in T joins a vertex of T to one of its ancestors (see [5]).

Proof. Let P be the path of size k. Let G be a graph in $\mathrm{Ex}(P)$ and let T be a DFS spanning forest of G. We wish to bound the number of pairs (G, T) of this kind.
• First, the height of T is at most $k-1$ (otherwise G contains P). The number of (rooted labelled) forests of bounded height is at most $\epsilon^n n^n$ for all $\epsilon > 0$; this is because the associated exponential generating function is analytic everywhere and hence has infinite radius of convergence (see Section III.8.2 in [7]).
• Second, since T is a DFS spanning forest, any edge in G which is not in T joins a vertex of T to one of its ancestors. Since the height of T is at most $k-1$, each vertex has at most k ancestors, so can be joined to its ancestors in at most 2^k different ways. This means that, given T, the graph G can be chosen in at most 2^{kn} ways, and so the upper bound $\epsilon^n n^n$ for all $\epsilon > 0$ holds for the number of pairs (G, T). $\square$

• We now consider minor-closed classes which do not contain all the paths nor all the star forests. Given two sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$, we write $f_n \asymp_{\exp} g_n$ if there exist $a, b > 0$ such that $f_n \leq a^n g_n$ and $g_n \leq b^n f_n$. Observe that if $\mathcal{G}$ contains all the matchings, then $g_n \geq n!! \asymp_{\exp} n^{n/2}$ by Lemma 3. We prove the following more precise result.

Claim 5. *Let $\mathcal{G}$ be a minor-closed class containing all matchings but not containing all the paths nor all the star forests. Then, there exists an integer $k \geq 2$ such that $g_n \asymp_{exp} n^{(1-1/k)n}$.*

Remark. For any integer $k \geq 2$, there exists a minor-closed class of graphs $\mathcal{G}$ such that $g_n \asymp_{\exp} n^{(1-1/k)n}$. For instance, the class $\mathcal{G}$ in which the connected components have no more than k vertices satisfies this property (see Lemma 7 below).

Proof. Let $\mathcal{G}$ be a minor-closed class $\mathcal{G}$ containing all matchings but not a given path P nor a given star forest S. We denote by p and s the size of P and S respectively. Let $\mathcal{F}$ be set of graphs in $\mathcal{G}$ such that every vertex has degree at most s. The following lemma compares the growth rate of $\mathcal{F}$ and $\mathcal{G}$.

Lemma 6. *The number f_n of graphs of size n in $\mathcal{F}$ satisfies $f_n \asymp_{exp} g_n$.*

Proof. Clearly $f_n \leq g_n$ so we only have to prove that there exists $b > 0$ such that $g_n \leq b^n f_n$. Let c be the number of stars in the star forest S and let $s_1, \ldots, s_c$ be the respective number of edges of these stars (so that $s = c + s_1 + \ldots + s_c$).

• We first prove that *any graph in $\mathcal{G}$ has less than c vertices of degree greater than s.* We suppose that a graph $G \in \mathcal{G}$ has c vertices $v_1, \ldots, v_c$ of degree at least s and we want to prove that G contains the forest S as a subgraph (hence as a minor; which is impossible). For $i = 1 \ldots, n$, let V_i be the set of vertices distinct from $v_1, \ldots, v_c$ which are adjacent to v_i. In order to prove that G contains the forest S as a subgraph it suffices to show that there exist disjoint subsets $S_1 \subseteq V_1, \ldots, S_c \subseteq V_c$ of respective size $s_1, \ldots, s_c$. Suppose, by induction, that for a given $k \leq c$ there exist disjoint subsets $S_1 \subseteq V_1, \ldots, S_{k-1} \subseteq V_{k-1}$ of respective size $s_1, \ldots, s_{k-1}$. The set $R_k = V_k - \bigcup_{i \leq k} S_i$ has size at least $s - c - \sum_{i<k} s_i \geq s_k$, hence there is a subset $S_k \subseteq V_k$ distinct from the S_i, $i < k$ of size s_k. The induction follows.
• We now prove that $g_n \leq \binom{n}{c} 2^{cn} f_n$. For any graph in $\mathcal{G}$ one obtains a graph in $\mathcal{F}$ by deleting all the edges incident to the vertices of degree greater than s. Therefore, any graph of $\mathcal{G}_n$ can be obtained from a graph of $\mathcal{F}_n$ by choosing c vertices and adding some edges incident to these vertices. There are at most $\binom{n}{c} 2^{cn} f_n$ graphs obtained in this way. □

It remains to prove that $f_n \asymp_{\exp} n^{(1-1/k)n}$ for some integer $k \geq 2$. Let G be a graph in $\mathcal{F}$ and let T be a tree spanning of one of its connected components. The tree T has height less than p (otherwise G contains the path P as a minor) and vertex degree at most s. Hence, T has at most $1 + s + \ldots + s^{p-1} \leq s^p$ vertices. Thus the connected components of the graphs in $\mathcal{F}$ have at most s^p vertices. For a connected graph G, we denote by $m(G)$ the maximum r such that $\mathcal{F}$ contains the graph consisting of r disjoint copies of G. We say that G has *unbounded multiplicity* if $m(G)$ is not bounded. Note that the graph consisting of 1 edge has unbounded multiplicity since $\mathcal{G}$ contains all matchings.

Lemma 7. *Let k be the size of the largest connected graph in $\mathcal{F}$ having unbounded multiplicity. Then, $f_n \asymp_{exp} n^{(1-1/k)n}$.*

Proof. • Let G be a connected graph in $\mathcal{F}$ of size k having unbounded multiplicity. The class of graphs consisting of disjoint copies of G and isolated vertices (these are included in order to avoid parity conditions) is contained in $\mathcal{F}$ and has exponential generating function $\exp(z + z^k/a(G))$, where $a(G)$ is the number of automorphisms of G. Hence f_n is of order at least $n^{(1-1/k)n}$, up to an exponential factor (see Corollary VIII.2 in [7]).
• Let $\mathcal{L}$ be the class of graphs in which every connected component C appears at most $m(C)$ times. Then clearly $\mathcal{F} \subseteq \mathcal{L}$. The exponential generating function for $\mathcal{L}$ is $P(z)\exp(Q(z))$, where $P(z)$ collects the connected graphs with bounded multiplicity, and $Q(z)$ those with unbounded multiplicity. Since $Q(z)$ has degree k, we have an upper bound of order $n^{(1-1/k)n}$. □

This finishes the proof of Claim 5. □

• We now consider the classes of graphs containing all the stars but not all the matchings. The lower bound for these classes follows from Lemma 3 while the upper bound is given by the following claim.

Claim 8. *Let M_k be a perfect matching on $2k$ vertices. The growth rate of $Ex(M_k)$ is at most exponential.*

Proof. Let G be a graph of size n in $\text{Ex}(M_k)$ and let M be a maximal matching of G. The matching M has no more than $2k-2$ vertices (otherwise, $M_k < G$). Moreover, the remaining vertices form an independent set (otherwise, M is not maximal). Hence G is a subgraph of the sum H_n of the complete graph K_{2k-2} and $n-(2k-2)$ independent vertices. There are $\binom{n}{2k-2}$ ways of labeling the graph H_n and $2^{e(H_n)}$ ways of taking a subgraph, where $e(H_n) = \binom{2k-2}{2} + (2k-2)(n-2k+2)$ is the number of edges of H_n. Since $\binom{n}{2k-2}$ is polynomial and $e(H_n)$ is linear, the number of graphs of size n in $\text{Ex}(M_k)$ is bounded by an exponential. $\square$

• We now consider consider classes of graphs $\mathcal{G}$ containing neither all the matchings nor all the stars. If $\mathcal{G}$ does not contain all the graphs with a single edge, then either $\mathcal{G}$ contains all the graphs without edges and $g_n = 1$ for n large enough or $g_n = 0$ for n large enough. Observe that if $\mathcal{G}$ contains the graphs with a single edge, then $g_n \geq \frac{n(n-1)}{2}$. It only remains to prove the following claim:

Claim 9. *Let $\mathcal{G}$ be a minor-closed class containing neither all the matching nor all the stars. Then, there exists an integer N and a polynomial P such that $g_n = P(n)$ for all $n \geq N$.*

Remark. For any integer $k \geq 2$, there exists a minor-closed class of graphs $\mathcal{G}$ such that $g_n = P(n)$ where P is a polynomial of degree k. Indeed, we let the reader check that the class $\mathcal{G}$ of graphs made of one star of size at most k plus some isolated vertices satisfies this property.

Proof. Since $\mathcal{G}$ does not contain all matchings, one of the minimal excluded minors of $\mathcal{G}$ is a graph M which is made of a set of k independent edges plus l isolated vertices. Moreover, $\mathcal{G}$ does not contain all the stars, thus one of the minimal excluded minors of $\mathcal{G}$ is a graph S made of one star on s vertices plus r isolated vertices.
• We first prove that *for every graph G in $\mathcal{G}$ having $n \geq \max(s+r, 2k+l)$ vertices, the number of isolated vertices is at least $n - 2ks$.* Observe that for every graph G in $\mathcal{G}$ having at least $s+r$ vertices, the degree of the vertices is less than s (otherwise, G contains the star S as a minor). Suppose now that a graph G in $\mathcal{G}$ has $n \geq \max(s+r, 2k+l)$ vertices from which at least $2ks$ are not isolated. Then, one can perform a greedy algorithm in order to find k independent edges. In this case, G contains the graph M as a minor, which is impossible.
• Let $M, S, H_1, \ldots, H_h$ be the minimal excluded minors of $\mathcal{G}$ and let $M', S', H'_1, \ldots, H'_h$ be the same graphs after deletion of their isolated vertices. We prove that *there exists $N \in \mathbb{N}$ such that $\mathcal{G}_n = \mathcal{F}_n$ for all $n \geq N$, where $\mathcal{F} = Ex(H'_1, \ldots, H'_h)$.* Let m be the maximal number of isolated vertices in the excluded minors $M, S, H_1, \ldots, H_h$ and let $N = \max(s+r, 2k+l, 2ks+m)$. If G has at least N vertices, then G has at least m isolated vertices, hence G is in $\mathcal{G}$ if and only if it is in $\mathcal{F}$.
• We now prove that there exists a polynomial P with rational coefficients such that $f_n \equiv |\mathcal{F}_n| = P(n)$. Let $\mathcal{C}$ be the set of graphs in $\mathcal{F}$ without isolated vertices; by convention we consider the graph of size 0 as being in $\mathcal{C}$. The graphs in $\mathcal{C}$ have at most $\max(s+r, 2k+l, 2ks)$ vertices, hence C is a finite set. We say that a graph in G *follows the pattern* of a graph $C \in \mathcal{C}$ if C is the graph obtained from G by deleting the isolated vertices of G and reassigning the labels in

$\{1,\ldots,r\}$ respecting the order of the labels in G. By the preceding points, any graph in $\mathcal{F}$ follows the pattern of a graph in $\mathcal{C}$ and, conversely, any graph following the pattern of a graph in $\mathcal{C}$ is in $\mathcal{F}$ (since the excluded minors $M', S', H'_1, \ldots, H'_h$ of $\mathcal{F}$ have no isolated vertices). The number of graphs of size n following the pattern of a given graph $C \in \mathcal{C}$ is $\binom{n}{|C|}$, where $|C|$ is the number of vertices of C. Thus, $f_n = \sum_{C\in\mathcal{C}} \binom{n}{|C|}$ which is a polynomial. $\square$

This conclude the proof of Theorem 2.

3 Growth constants

We say that class $\mathcal{G}$ *has growth constant* γ if $\lim_{n\to\infty} (g_n/n!)^{1/n} = \gamma$, and we write $\gamma(\mathcal{G}) = \gamma$.

Proposition 10. *Let $\mathcal{G}$ be a minor-closed class such that all the excluded minors of $\mathcal{G}$ are 2-connected. Then, $\gamma(\mathcal{G})$ exists.*

Proof. In the terminology of [12], the class $\mathcal{G}$ is small (because of Theorem 1), and it is addable because of the assumption on the forbidden minors. Hence, Theorem 3.3 from [12] applies and there exists a growth constant. $\square$

We know state a theorem about the set Γ of growth constants of minor-closed classes. In what follows we denote by $\xi \approx 1.76$ the inverse of the unique positive root of $x\exp(x) = 1$.

Theorem 11. *Let Γ be the set of real numbers which are growth constants of minor-closed classes of graphs.*

1. *The values $0,\ 1,\ \xi$ and e are in Γ.*
2. *If $\gamma \in \Gamma$ then $2\gamma \in \Gamma$.*
3. *There is no $\gamma \in \Gamma$ with $0 < \gamma < 1$.*
4. *There is no $\gamma \in \Gamma$ with $1 < \gamma < \xi$.*

Remarks. • The property 1 of Theorem 11 can be extended with the growth constants of the minor-closed classes listed in table by table 1.
• The properties 2, 3 and 4 of Theorem 11 remain valid if one replaces Γ by the set $\Gamma' = \{\gamma' = \limsup \left(\frac{g_n}{n!}\right)^{1/n} / \mathcal{G}$ minor-closed$\}$.

Before the proof of Theorem 11, we make the following remark. Let $\mathcal{G}$ be a minor-closed class, let $\mathcal{C}$ be the family of all connected members of $\mathcal{G}$, and let $G(z)$ and $C(z)$ be the corresponding generating functions. Then if $\mathcal{C}$ has growth constant γ, so does $\mathcal{G}$. This is because the generating functions $G(z)$ is bounded by $\exp(C(z))$ (they are equal if the forbidden minors for $\mathcal{G}$ are all connected), and both functions have the same dominant singularity.

Class of graphs	Growth constant	Reference
$\mathrm{Ex}(P_k)$	0	This paper
Path forests	1	Standard
Caterpillar forests	$\xi \approx 1.76$	This paper
Forests = $\mathrm{Ex}(K_3)$	$e \approx 2.71$	Standard
$\mathrm{Ex}(C_4)$	3.63	[10]
$\mathrm{Ex}(K_4 - e)$	4.18	[10]
$\mathrm{Ex}(C_5)$	4.60	[10]
Outerplanar = $\mathrm{Ex}(K_4, K_{2,3})$	7.320	[1]
$\mathrm{Ex}(K_{2,3})$	7.327	[1]
Series parallel = $\mathrm{Ex}(K_4)$	9.07	[1]
$\mathrm{Ex}(W_4)$	11.54	[10]
$\mathrm{Ex}(K_5 - e)$	12.96	[10]
$\mathrm{Ex}(K_2 \times K_3)$	14.13	[10]
Planar	27.226	[9]
Embeddable in a fixed surface	27.226	[11]
$\mathrm{Ex}(K_{3,3})$	27.229	[8]

Table 1: A table of some known growth constants.

Proof. 1) • All classes whose growth is not at least factorial have growth constant 0. In particular, $\gamma(\mathrm{Ex}(P)) = 0$ for any path P.
• The number of labelled paths is $n!/2$. Hence, by the remark made before the proof, the growth constant of the class of path forests is 1.
• A *caterpillar* is a tree consisting of a path and vertices directly adjacent to (i.e. one edge away from) that path. Let $\mathcal{C}$ be the class of graphs whose connected components are caterpillars, which is clearly minor-closed. A rooted caterpillar can be considered as an ordered sequence of stars. Hence the associated generating function is $1/(1 - ze^z)$. The dominant singularity is the smallest positive root of $1 - ze^z = 0$, and $\gamma(\mathcal{C})$ is the inverse ξ of this value.
• The growth constant of the class of acyclic graphs (forests) is the number e. This is because the number of labelled trees is n^{n-2} which, up to a sub-exponential factor, is asymptotic to $\sim e^n n!$.

2) This property follows from an idea by Colin McDiarmid. Suppose $\gamma(\mathcal{G}) = \gamma$, and let $\mathcal{AG}$ be family of graphs G having a vertex v such that $G - v$ is in $\mathcal{G}$; in this case we say that v is an apex of G. It is easy to check that if $\mathcal{G}$ is minor-closed, so is $\mathcal{AG}$. Now we have

$$2^n|\mathcal{G}_n| \le |\mathcal{AG}_{n+1}| \le (n+1)2^n|\mathcal{G}_n|.$$

The lower bound is obtained by taking a graph $G \in \mathcal{G}$ with vertices $[n]$, adding $n+1$ as a new vertex, and making $n+1$ adjacent to any subset of $[n]$. The upper bound follows the same argument by considering which of the vertices $1, 2, \dots, n+1$ acts as an apex. Dividing by $n!$ and taking n-th roots, we see that $\gamma(\mathcal{AG}) = 2\gamma(\mathcal{G})$.

3) This has been already shown during the proof of Theorem 2. Indeed, if a minor-closed class $\mathcal{G}$ contains all paths, then $|\mathcal{G}_n| \geq n!/2$ and the growth constant is at least 1. Otherwise $g_n < \epsilon^n n^n$ for all $\epsilon > 0$ and $\gamma(\mathcal{G}) = 0$.

4) We consider the graphs Cat_l and Ap_l represented in Figure 1.

Figure 1: The graph Cat_l (left) and the graph Ap_l (right).

If a minor-closed class $\mathcal{G}$ contains the graphs Cat_l for all l, then $\mathcal{G}$ contains all the caterpillars hence $\gamma(\mathcal{G}) \geq \xi \approx 1.76$. If $\mathcal{G}$ contains the graphs Ap_l for all l, then $\mathcal{G}$ contains the apex class of path forests and $\gamma(\mathcal{G}) \geq 2$. Now, if $\mathcal{G}$ contains neither Cat_k nor Ap_l for some k, l, then $\mathcal{G} \subseteq \mathrm{Ex}(\mathrm{Cat}_l, \mathrm{Ap}_l)$. Therefore, it is sufficient to prove the following claim.

Claim 12. *The growth constant of the class* $\mathrm{Ex}(\mathrm{Cat}_k, \mathrm{Ap}_l)$ *is 1 for all* $k > 2, l > 1$.

Remark. Claim 12 gives in fact a characterization of the minor-closed classes with growth constant 1. These are the classes containing all the paths but neither all the caterpillars nor all the graphs in the apex class of the path forests. For instance, the class of trees not containing a given caterpillar (as a minor) and the class of graphs not containing a given star (as a minor) both have growth constant 1.

Proof. Observe that the class $\mathrm{Ex}(\mathrm{Cat}_k, \mathrm{Ap}_l)$ contains all paths as soon as $k > 2$ and $l > 1$. Hence, $\gamma(\mathrm{Ex}(\mathrm{Cat}_k, \mathrm{Ap}_l)) \geq 1$ (by Lemma 3) and we only need to prove that $\gamma(\mathrm{Ex}(\mathrm{Cat}_k, \mathrm{Ap}_l)) \leq 1$. We first prove a result about the simple paths of the graphs in $\mathrm{Ex}(\mathrm{Cat}_k, \mathrm{Ap}_l)$.

Lemma 13. *Let G be a graph in* $\mathrm{Ex}(\mathrm{Cat}_k, \mathrm{Ap}_l)$ *and let P be a simple path in G. Then, there are less than $kl + 4k^3 l$ vertices in P of degree greater than 2.*

Proof. • We first prove that *any vertex not in P is adjacent to less than l vertices of P* and *any vertex in P is adjacent to less than $2l$ vertices of P*. Clearly, if G contains a vertex v not in P and adjacent to l vertices P, then G contains Ap_l as a minor. Suppose now that there is a vertex v in P adjacent to $2l$ other vertices of P. In this case, v is adjacent to at least l vertices in one of the simple paths P_1, P_2 obtained by removing the vertex v from the path P. Hence G contains Ap_l as a minor.

• We now prove that *there are less than kl vertices in P adjacent to at least one vertex not in P.* We suppose the contrary and we prove that there exist k independent edges $e_i = (u_i, v_i)$, $i = 1 \ldots k$ such that u_i is in P and v_i is not in P (thereby implying that Cat_k is a minor of G). Let $r < k$ and let $e_i = (u_i, v_i)$, $i \leq r$ be independent edges with $u_i \in P$ and $v_i \notin P$. The set of vertices in P adjacent to some vertices not in P but to none of the vertices $v_i, i \leq r$ has size at least $kl - rl > 0$

(this is because each of the vertex v_i is adjacent to less than l vertices of P). Thus, there exists an edge $e_{r+1} = (u_{r+1}, v_{r+1})$ independent of the edges $e_i, i \leq r$ with $u_{r+1} \in P$ and $v_{r+1} \notin P$. Thus, any set of $r < k$ independent edges with one endpoint in P and one endpoint not in P can be increased.

• We now prove that *there are no more than $4k^3l$ vertices in P adjacent to another vertex in P beside its 2 neighbors in P*. We suppose the contrary and we prove that either Cat_k or Ap_l is a minor of G. Let E_P be the set of edges not in the path P but joining 2 vertices of P. We say that two independent edges $e = (u, v)$ and $e' = (u', v')$ of E_P *cross* if the vertices u, u', v, v' appear in this order along the path P; this situation is represented in Figure 2 (a).

- We first show that *there is a subset $E'_P \subseteq E_P$ of k^3 independent edges.* Let S be any set of $r < k^3$ edges in E_P. The number of edges in E_P sharing a vertex with one of the edges in S is at most $2r \times 2l < 4k^3l$ (this is because any vertex in P is adjacent to less than $2l$ vertices in P). Since $|E_P| \geq 4k^3l$, any set of independent edges in E_P of size less than k^3 can be increased.

- We now show that *for any edge e in E'_P there are at most k edges of E'_P crossing e.* Suppose that there is a set $S \subseteq E'_P$ of k edges crossing e. Let P' be the path obtained from $P \cup e$ by deleting the edges of P that are between the endpoints of e. The graph made of P' and the set of edges S contains the graph Cat_l as a minor which is impossible.

- We now show that *there exists a subset $E''_P \subseteq E'_P$ of k^2 non-crossing edges.* Let S be any set of $r < k^2$ edges in E'_P. By the preceding point, the number of edges in E'_P crossing one of the edges in S is less than $rk < k^3$. Since $|E'_P| \geq k^3$, any set of non-crossing edges in E'_P of size less than k^2 can be increased.

- Lastly, we show that *the graph Cat_k is a minor of G.* We say that an edge $e = (u, v)$ of E''_P is *inside* another edge $e' = (u', v')$ if u', u, v, v' appear in this order along the path P; this situation is represented in Figure 2 (b). We define the *height* of the edges in E''_P as follows: the height of an edge e is 1 plus the maximum height of edges of E''_P which are inside e (the height is 1 if there is no edge inside e). The height of edges have been indicated in Figure 2 (c). Suppose that there is an edge of height k in E''_P. Then there is a set S of k edges $e_1 = (u_1, v_1), \ldots, e_k = (u_k, v_k)$ such that the vertices $u_1, u_2, \ldots, u_k, v_k, v_{k-1}, \ldots, v_1$ appear in this order along P. In this case, the subgraph made of S and the subpath of P between u_1 and u_k contains Cat_k as a minor. Suppose now that there is no edge of height k. Since there are k^2 edges in E''_P, there is a integer $i < k$ such that the number of edges of height i is greater than k. Thus, there is a set S of k edges $e_1 = (u_1, v_1), \ldots, e_k = (u_k, v_k)$ such that the vertices $u_1, v_1, u_2, v_2, \ldots, u_k, v_k$ appear in this order along P. In this case, the subgraph obtained from $P \cup \{e_1, \ldots, e_k\}$ by deleting an edge of P between u_i and v_i for all i contains Cat_k as a minor. □

For any integer N, we denote by $\mathcal{G}_T^N$ the set of pairs (G, T) where G is a graph and T is a DFS spanning forest on G having height at most N (the definition of *DFS spanning forest* was given just after Claim 4).

Lemma 14. *For any graph G in $Ex(Cat_k, Ap_l)$, there exists a pair (G', T') in $\mathcal{G}_T^{kl+4k^3l}$ such that G is obtained from G' by subdividing some edges of T.*

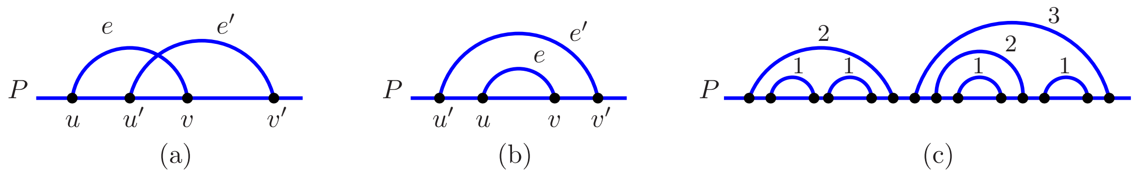

Figure 2: (a) Two crossing edges. (b) An edge inside another. (c) A set of non-crossing edges.

Proof. Let G be a graph in $\mathrm{Ex}(\mathrm{Cat}_k, \mathrm{Ap}_l)$, let T be a DFS spanning forest of G and let R be the set of roots of T (one root for each connected components of G). One *contracts* a vertex v of degree 2 by deleting v and joining its two neighbors by an edge. Let G' and T' be the graphs and trees obtained from G and T by contracting the vertices $v \notin R$ of degree 2 which are incident to 2 edges of T. We want to prove that (G', T') is in $\mathcal{G}_T^{kl+4k^3l}$.
• Since T is a DFS spanning forest of G, every edge of G which is not in T connects a vertex to one of its ancestors [5]. This property characterize the DFS spanning forests and is preserved by the contraction of the vertices of degree 2. Hence, T' is a DFS spanning forest of G'.
• By Lemma 13, the number of vertices which are not of degree 2 along a path of T from a root to a leaf is less than $kl + 4k^3l$. Thus, the height of T' is at most $kl + 4k^3l$. □

We have already shown in the proof of Claim 4 that the radius of convergence of the generating function $G_T^N(z)$ of the set $\mathcal{G}_T^N$ is infinite. Moreover, the generating function of the set of graphs that can be obtained from pairs (G', T') in $\mathcal{G}_T^N$ by subdividing the tree T' is bounded (coefficient by coefficient) by $G_T^N(\frac{z}{1-z})$ (since a forest T' on a graph G' of size n has at most $n-1$ edges to be subdivided). Thus, Lemma 14 implies that the generating function of $\mathrm{Ex}(\mathrm{Cat}_k, \mathrm{Ap}_l)$ is bounded by $G_T^{kl+4k^3l}(\frac{z}{1-z})$ which has radius of convergence 1. Hence, the growth constant $\gamma(\mathrm{Ex}(\mathrm{Cat}_k, \mathrm{Ap}_l))$ is at most 1. □

This concludes the proof of Claim 12 and Theorem 11. □

We now investigate the topological properties of the set Γ and in particular its limit points. First note that Γ is countable (as a consequence of the Minor Theorem of Robertson and Seymour [14]).

Lemma 15. *Let $H_1, H_2, \dots H_k$ be a family of 2-connected graphs, and let $\mathcal{H} = \mathrm{Ex}(H_1, H_2, \dots H_k)$. If G is a 2-connected graph in $\mathcal{H}$, then $\gamma(\mathcal{H} \cap Ex(G)) < \gamma(\mathcal{H})$.*

Proof. The condition on 2-connectivity guarantees that the growth constants exist. By Theorem 4.1 from [12], the probability that a random graph in $\mathcal{H}_n$ contains G as a subgraph is a least $1 - e^{-\alpha n}$ for some $\alpha > 0$. Hence the probability that a random graph in $\mathcal{H}_n$ does not contain G as a minor is at most $e^{-\alpha n}$. If we denote $\mathcal{G} = \mathcal{H} \cap \mathrm{Ex}(G)$, then we have

$$\frac{|\mathcal{G}_n|}{|\mathcal{H}_n|} = \frac{|\mathcal{G}_n|}{n!} \frac{n!}{|\mathcal{H}_n|} \leq e^{-\alpha n}.$$

Taking limits, this implies

$$\frac{\gamma(\mathcal{G})}{\gamma(\mathcal{H})} \leq \lim \left(e^{-\alpha n}\right)^{1/n} = e^{-\alpha} < 1.$$

□

We recall that given a set A of real numbers, a is a *limit point* of A if for every $\epsilon > 0$ there exists $x \in A - \{a\}$ such that $|a - x| < \epsilon$.

Theorem 16. *Let $H_1, \ldots, H_k$ be 2-connected graphs which are not cycles. Then, $\gamma = \gamma(Ex(H_1, \ldots, H_k))$ is a limit point of Γ.*

Proof. For $k \geq 3$, let $\mathcal{G}_k = \mathcal{G} \cap \mathrm{Ex}(C_k)$, where C_k is the cycle of size k. Because of Proposition 10, the class $\mathcal{G}_k$ has a growth constant γ_k, and because of Lemma 15 the γ_k are strictly increasing and $\gamma_k < \gamma$ for all k. It follows that $\gamma' = \lim_{k\to\infty} \gamma_k$ exists and $\gamma' \leq \gamma$. In order to show equality we proceed as follows.

Let $g_n = |\mathcal{G}_n|$ and let $g_{k,n} = |(\mathcal{G}_k)_n|$. Since $\gamma = \lim_{n\to\infty}(g_n/n!)^{1/n}$, for all $\epsilon > 0$ there exists N such that for $n > N$ we have

$$(g_n/n!)^{1/n} \geq \gamma - \epsilon.$$

Now define $f_n = \dfrac{g_n}{e^2 n!}$ and $f_{k,n} = \dfrac{g_{k,n}}{e^2 n!}$. From [12, Theorem 3], the sequence f_n is supermultiplicative and $\gamma = \lim\limits_{n\to\infty} (f_n)^{1/n} = \lim\limits_{n\to\infty} (g_n/n!)^{1/n}$ exists and equals $\sup_n (f_n)^{1/n}$. Similarly, $\gamma_k = \lim_{n\to\infty} (f_{k,n})^{1/n} = \sup_n (f_{k,n})^{1/n}$.

But since a graph on less than k vertices cannot contain C_k as a minor, we have $g_{k,n} = g_n$ for $k > n$. Equivalently, $f_{k,n} = f_n$ for $k > n$. Combining all this, we have

$$\gamma_k \geq (f_{k,n})^{1/n} \geq (f_n)^{1/n} \geq \gamma - \epsilon$$

for $k > N$. This implies $\gamma' = \lim \gamma_k \geq \gamma$. □

Notice that Theorem 16 applies to all the classes in Table 1 starting at the class of outerplanar graphs. However, it does not apply to the classes of of forests. In this case we offer an independent proof based on generating functions.

Lemma 17. *The number e is a limit point of Γ.*

Proof. Let $\mathcal{F}_k$ be the class of forests whose trees are made of a path and rooted trees of height at most k attached to vertices of the path. Observe that the classes $\mathcal{F}_k$ are minor-closed, that $\mathcal{F}_k \subset \mathcal{F}_{k+1}$, and that $\cup_k \mathcal{F}_k = \mathcal{F}$, where $\mathcal{F}$ is the class of forests. We prove that $\gamma(\mathcal{F}_k)$ is a strictly increasing sequence tending to $e = \gamma(\mathcal{F})$.

Recall that the class $\mathcal{F}_k$ and the class $\mathcal{T}_k$ of its connected members have the same growth constant. Moreover, the class $\vec{\mathcal{T}}_k$ of trees with a distinguished oriented path to which rooted trees of height at most k are attached has the same growth constant as $\mathcal{T}_k$ (this is because there are only $n(n-1)$ of distinguishing and orienting a path in a tree of size n). The generating function associated to $\vec{\mathcal{T}}_k$ is $1/(1-F_k(z))$, where $F_k(z)$ of is the generating function of rooted trees of height at most k. Hence, $\gamma(\mathcal{F}_k) = \gamma(\vec{\mathcal{T}}_k)$ is the inverse of the unique positive root ρ_k of $F_k(\rho_k) = 1$.

Recall that the generating functions F_k are obtained as follows; see Section III.8.2 in [7]).

$$F_0(z) = z; \qquad F_{k+1}(z) = ze^{F_k(z)} \quad \text{for } k > 0.$$

It is easy to check that the roots ρ_k of $F_k(\rho_k) = 1$ are strictly decreasing. Recall that the generating function $F(z)$ of rooted trees has a singularity at $1/e$ and that $F(1/e) = 1$ (see [7]). Moreover, for all n, $0 \leq [z^n]F_k(z) \leq [z^n]F(z)$ and $\lim_{k\to\infty}[z^n]F_k(z) = [z^n]F(z)$, thus $\lim_{k\to\infty} F_k(1/e) = F(1/e) = 1$. Furthermore, the functions $F_k(z)$ are convex and $F_k'(1/e) \geq 1$ (since the coefficients of F_k are positive and $[z^1]F_k(z) = 1$). Thus, $F_k(z) > F_k(1/e) + (z - 1/e)$ which implies $1/e \leq \rho_k \leq 1/e + (F_k(1/e) - F(1/e))$. Thus, the sequence ρ_k tends to $1/e$ and the growth constants $\gamma(\mathcal{F}_k) = 1/\rho_k$ tend to e. □

Remark. The number $\nu \approx 2.24$, which is the inverse of the smallest positive root of $z\exp(z/(1-z)) = 1$, can be shown to be a limit point of Γ by similar methods. It is the smallest number which we know to be a limit point of Γ. It is the growth constant of the family whose connected components are made of a path P and any number of paths of any length attached to the vertices of P.

Remark. All our examples of limit points in Γ come from strictly increasing sequences of growth constants that converge to another growth constant. Is it possible to have an infinite strictly decreasing sequences of growth constants? As we see now, this is related to a classical problem. A quasi-ordering is a reflexive and transitive relation. A quasi-ordering $\leq$ in X is a *well-quasi ordering* if for every infinite sequence $x_1, x_2, \ldots$ in X there exist $i < j$ such that $x_i \leq x_j$. Now consider the set X of minor-closed classes of graphs ordered by inclusion. It is an open problem whether this is a well-quasi ordering [6]. Assuming this is the case, it is clear that an infinite decreasing sequence $\gamma_1 > \gamma_2 > \cdots$ of growth constants cannot exist. For consider the corresponding sequence of graph classes $\mathcal{G}_1, \mathcal{G}_2, \ldots$. For some $i < j$ we must have $\mathcal{G}_i \subseteq \mathcal{G}_j$, but this implies $\gamma_i \leq \gamma_j$.

4 Conclusion: some open problems

We close by listing some of the open questions which have arisen in this work.

1) We know that a class $\mathcal{G}$ has a growth constant provided that all its excluded minor are 2-connected. The condition that the excluded-minors are 2-connected is certainly not necessary as is seen by noting that the apex family of any class which has a growth constant also has a growth constant. It is also easy to see that such an apex family is also minor-closed and that at least one of its excluded minors is disconnected.

Thus our first conjecture is that every minor-closed family has a growth constant, that is, $\lim \left(\frac{g_n}{n!}\right)^{1/n}$ exists for every minor-closed class $\mathcal{G}$.

2) A minor-closed class is *smooth* if $\lim \frac{g_n}{ng_{n-1}}$ exists. It follows that this limit must be the growth constant and that a random member of $\mathcal{G}$ will have expected number of isolated vertices converging to $1/\gamma$. Our second conjecture is that if every excluded minor of a minor-closed class is 2-connected then the class is smooth.

If true, then it would follow that a random member of the class would qualitatively exhibit all the Poisson type behaviour exhibited by the random planar graph. However proving smoothness for a class seems to be very difficult and the only cases which we know to be smooth are when the exponential generating function has been determined exactly.

3) We have shown that the intervals $(0, 1)$ and $(1, \xi)$ are "gaps" which contain no growth constant. We know of no other gap, though if there is no infinite decreasing sequence of growth constants they must exist. One particular question which we have been unable to settle is whether $(\xi, 2)$ is also a gap.

4) We have shown that for each nonnegative integer k, 2^k is a growth constant. A natural question is whether any other integer is a growth constant. More generally, is there any algebraic number in Γ besides the powers of 2?

5) All our results concern labelled graphs. In unlabelled setting, the most important question to settle is whether there is an analogue of the theorem of Norine *et al.* More precisely, suppose $\mathcal{G}$ is a minor-closed class of graphs and that u_n denotes the number of unlabelled members of $\mathcal{G}_n$. Does there exist a finite d such that u_n is bounded above by d^n?

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