HOMOTOPY EXPONENTS FOR LARGE H-SPACES

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ABSTRACT. We show that H-spaces with finitely generated cohomology, as an algebra or as an algebra over the Steenrod algebra, have homotopy exponents at all primes. This provides a positive answer to a question of Stanley.

INTRODUCTION

A simply connected space is elliptic if both its rational homotopy and rational homology are finite. Moore's conjecture, see for example [9], predicts that elliptic complexes have an exponent at any prime p, meaning that there is a bound on the p-torsion in the graded group of all homotopy groups. Any finite H-space is known to be elliptic as it is rationally equivalent to a finite product of (odd dimensional) spheres. Relying on results by James [6] and Toda [11] about the homotopy groups of spheres, the fourth author (re)proved in [10] Long's result that finite H-spaces have an exponent at any prime [7]. He proved in fact a stronger result which holds for example for H-spaces for which the mod p cohomology is finite. He also asked whether this would hold for finitely generated cohomology rings. The aim of this note is to give a positive answer to this question and provide a way larger class of H-spaces which have homotopy exponents.

Theorem 0.1. *1.2 Let X be a connected and p-complete H-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Then X has an exponent at p.*

This class of H-spaces is optimal in the sense that H-spaces with a larger mod p cohomology, such as an infinite product of Eilenberg-Mac Lane spaces

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2000 *Mathematics Subject Classification.* Primary 55P45; Secondary 55Q99, 55S10, 55U20.

The first author is supported in part by NSF grant DMS-0296117, Vetenskapsrådet grant 2001-4296, and Göran Gustafssons Stiftelse. The second and third authors are partially supported by FEDER/MEC grant MTM2007-61545. The second author is supported by the program Ramón y Cajal, MEC, Spain.

$K(\mathbb{Z}/p^n, n)$, will not have in general an exponent at p. As a corollary, we obtain the desired result. In fact we obtain the following global theorem.

Theorem 0.2. *1.4 Let X be a connected H-space such that $H^*(X;\mathbb{Z})$ is finitely generated as an algebra. Then X has an exponent at each prime p.*

The methods we use are based on the deconstruction techniques of the third author in his joint work with Castellana and Crespo, [3]. Our results on homotopy exponents should also be compared with the computations of homological exponents done with Clément, [4]. Whereas such H-spaces always have homotopy exponents, they almost never have homological exponents. The only simply connected H-spaces for which the 2-torsion in $H_*(X;\mathbb{Z})$ has a bound are products of mod 2 finite H-spaces with copies of the infinite complex projective space $\mathbb{C}P^\infty$ and $K(\mathbb{Z}, 3)$.

Acknowledgments

This project originated during a workshop at the CRM, Barcelona, held in the emphasis year on algebraic topology (2007-08). We would like to thank the organizers for making it possible to meet in such a pleasant atmosphere.

1. Homotopy exponents

Our starting point is the fact that mod p finite H-spaces have always homotopy exponents. The following is a variant of Stanley's [10, Corollary 2.9]. Whereas he focused on spaces localized at a prime, we will stick to p-completion in the sense of Bousfield and Kan, [2]. Since the p-localization map $X \to X_{(p)}$ is a mod p homology equivalence, his result implies the following.

Proposition 1.1 (Stanley)**.** *Let p be a prime and X be a p-complete and connected H-space such that $H^*(X;\mathbb{F}_p)$ is finite. Then X has an exponent at p.*

We will not repeat the proof, but let us sketch the main steps. Let us consider a decomposition of X by p-complete cells, i.e. X is obtained by attaching cones along maps from $(S^n)^\wedge_p$. The natural map $X \to \Omega\Sigma X$ factors then through the loop spaces on a wedge W of a finite numbers of such p-completed spheres, up to multiplying by some integer N: the composite $X \to \Omega\Sigma X \xrightarrow{N} \Omega\Sigma X$ is homotopic to $X \to \Omega W \to \Omega\Sigma X$. The proof goes by induction on the number of p-complete cells and the key ingredient here is Hilton's description of the loop space on a wedge of spheres, [5]. Note that the suspension of a map between spheres is torsion except for the multiples of the identity. This idea to "split off" all the cells of X up to multiplication by some integer is dual to Arlettaz' way to split off Eilenberg-Mac Lane spaces in H-spaces with finite order k-invariants, [1, Section 7]. The final step relies on the classical results by James, [6], and Toda, [11], that spheres do have homotopy exponents at all primes.

Theorem 1.2. *Let X be a connected and p-complete H-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra. Then X has an exponent at p.*

Proof. A connected H-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra over the Steenrod algebra can always be seen as the total space of an H-fibration $F \to X \to Y$ where Y is an H-space with finite mod p cohomology and F is a p-torsion Postnikov piece whose homotopy groups are finite direct sums of copies of cyclic groups $\mathbb{Z}/p^r$ and Prüfer groups $\mathbb{Z}_{p^\infty}$, [3, Theorem 7.3]. This is a fibration of H-spaces and H-maps, so that we obtain another fibration $F^\wedge_p \to X \to Y^\wedge_p$ by p-completing it. The base space $Y^\wedge_p$ now satisfies the assumptions of Proposition 1.1. It has therefore an exponent at p. The homotopy groups of the fiber $F^\wedge_p$ are finite direct sums of cyclic groups $\mathbb{Z}/p^n$ and copies of the p-adic integers $\mathbb{Z}^\wedge_p$. Thus $F^\wedge_p$ has an exponent at p as well. The homotopy long exact sequence of the fibration allows us to conclude. □

We see here how the p-completeness assumption plays an important role. The space $K(\mathbb{Z}_{p^\infty}, 1)$ for example has obviously no exponent at p, but its p-completion is $K(\mathbb{Z}^\wedge_p, 2) = (\mathbb{C}P^\infty)^\wedge_p$, which is a torsion free space. The mod p cohomology of $K(\mathbb{Z}_{p^\infty}, 1)$ is a polynomial ring on one generator in degree 2, we must thus also work with p-complete spaces to give an answer to Stanley's question [10, Question 2.10].

Corollary 1.3. *Let X be a connected and p-complete H-space such that $H^*(X;\mathbb{F}_p)$ is finitely generated as an algebra. Then X has an exponent at p.*

In fact, when the mod p cohomology is finitely generated, the fiber F in the fibration described in the proof of Theorem 1.2 is a single Eilenberg-Mac Lane space $K(P, 1)$. Thus the typical example of an H-space with finitely generated mod p cohomology is the 3-connected cover of a simply connected finite H-space (P is $\mathbb{Z}_{p^\infty}$ in this case). Likewise, the typical example in Theorem 1.2 are highly connected covers of finite H-spaces. This explains why such spaces have homotopy exponents!

If one does not wish to work at one prime at a time and prefers to find a global condition which permits to conclude that a certain class of spaces have exponents at all primes, one must replace mod p cohomology by integral cohomology.

Theorem 1.4. *Let X be a connected H-space such that $H^*(X;\mathbb{Z})$ is finitely generated as an algebra. Then X has an exponent at each prime p.*

Proof. Since the integral cohomology groups are finitely generated it follows from the universal coefficient exact sequence (see [8]) that the integral homology groups are also finitely generated. Since X is an H-space we may use a standard Serre class argument to conclude that so are the homotopy groups. Therefore the p-completion map $X \to X_p^\wedge$ induces an isomorphim on the p-torsion at the level of homotopy groups. The theorem is now a direct consequence of the next lemma. □

Lemma 1.5. *Let X be a connected space. If $H^*(X;\mathbb{Z})$ is finitely generated as an algebra, then so is $H^*(X;\mathbb{F}_p)$.*

Proof. Let $u_1, \dots, u_r$ generate $H^*(X;\mathbb{Z})$ as an algebra. Consider the universal coefficients short exact sequences

$$0 \to H^n(X;\mathbb{Z}) \otimes \mathbb{Z}/p \longrightarrow H^n(X;\mathbb{F}_p) \xrightarrow{\partial} \mathrm{Tor}(H^{n+1}(X;\mathbb{Z});\mathbb{Z}/p) \to 0\,.$$

Since $H^*(X;\mathbb{Z})$ is finitely generated as an algebra it is degree-wise finitely generated as a group and therefore $\mathrm{Tor}(H^*(X;\mathbb{Z});\mathbb{Z}/p)$ can be identified with the ideal of elements of order p in $H^*(X;\mathbb{Z})$. This ideal must be finitely generated since $H^*(X;\mathbb{Z})$ is Noetherian. Choose generators $a_1, \dots, a_s$. Each a_i corresponds to a pair $\alpha_i, \beta\alpha_i$ in $H^*(X;\mathbb{F}_p)$, where β denotes the Bockstein.

We claim that the elements $\alpha_1, \dots, \alpha_s$ together with the mod p reduction of the algebra generators, denoted by $\bar{u}_1, \dots, \bar{u}_r$, generate $H^*(X;\mathbb{F}_p)$ as an algebra. Let $x \in H^*(X;\mathbb{F}_p)$ and write its image $\partial(x) = \sum \lambda_j a_j$ with $\lambda_j = \lambda_j(u)$ a polynomial in the u_i's. Define now $\bar{\lambda}_j = \lambda_j(\bar{u}) \in H^*(X;\mathbb{F}_p)$ to be the corresponding polynomial in the $\bar{u}_i$'s. As the action of $H^*(X;\mathbb{Z})$ on the ideal $\mathrm{Tor}(H^*(X;\mathbb{Z});\mathbb{Z}/p)$ factors through the mod p reduction map $H^*(X;\mathbb{Z}) \to H^*(X;\mathbb{F}_p)$, the element $x - \sum \bar{\lambda}_j \alpha_j$ belongs to the kernel of ∂, i.e. it lives in the image of the mod p reduction. It can be written therefore as a polynomial $\bar{\mu}$ in the $\bar{u}_i$'s. Thus $x = \bar{\mu} + \sum \bar{\lambda}_j \alpha_j$. □

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