A GENERAL CONSTRUCTION OF INTERNAL SHEAVES IN ALGEBRAIC SET THEORY

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Abstract. We present a solution to the problem of defining a counterpart in Algebraic Set Theory of the construction of internal sheaves in Topos Theory. Our approach is general in that we consider sheaves as determined by Lawvere-Tierney coverages, rather than by Grothendieck coverages, and assume only a weakening of the axioms for small maps originally introduced by Joyal and Moerdijk, thus subsuming the existing topos-theoretic results.

Introduction

Algebraic Set Theory provides a general framework for the study of category-theoretic models of set theories [16]. The fundamental objects of interest are pairs $(\mathcal{E}, \mathcal{S})$ consisting of a category $\mathcal{E}$ equipped with a distinguished family of maps $\mathcal{S}$, whose elements are referred to as small maps. The category $\mathcal{E}$ is thought of as a category of classes, and $\mathcal{S}$ as the family of functions between classes whose fibers are sets. The research in the area has been following two general directions: the first is concerned with isolating axioms for the pair $(\mathcal{E}, \mathcal{S})$ that guarantee the existence in $\mathcal{E}$ of a model for a given set theory; the second is concerned with the study of constructions, such as that of internal sheaves, that allow us to obtain new pairs $(\mathcal{E}, \mathcal{S})$ from given ones, in analogy with the existing development of Topos Theory [17, Chapter 5]. The combination of these developments is intented to give general methods that subsume the known techniques to define sheaf and realizability models for classical, intuitionistic, and constructive set theories [8, 9, 11, 14, 18, 20, 23].

Our aim here is to contribute to the study of the construction of internal sheaves in Algebraic Set Theory. The starting point of our development is the notion of a *Lawvere-Tierney coverage*. If our ambient category $\mathcal{E}$ were an elementary topos, Lawvere-Tierney coverages would be in bijective correspondence with Lawvere-Tierney local operators on the subobject classifier of the topos. However, since $\mathcal{E}$ is assumed here to be only a Heyting

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pretopos, we work with the more general Lawvere-Tierney coverages. As we will see, when $\mathcal{E}$ is a category of internal presheaves, these correspond bijectively to the Grothendieck coverages considered in [12]. Therefore, our development gives as a special case a treatment of the construction of internal sheaves relative to those Grothendieck sites, extending the results in [12]. Given a Lawvere-Tierney coverage, we can define an associated universal closure operator on the subobjects of $\mathcal{E}$, which allows us to define the notion of a sheaf as in the standard topos-theoretic context.

Our main result asserts that the category of internal sheaves for a Lawvere-Tierney coverage is a Heyting pretopos and that it can be equipped with a family of small maps satisfying the same axioms that we assumed on the small maps $\mathcal{S}$ in the ambient category $\mathcal{E}$. The first part of this result involves the definition of an associated sheaf functor, a finite-limit preserving left adjoint to the inclusion of sheaves into $\mathcal{E}$. For this, we adapt the topos-theoretic argument due to Lawvere [17, §V.3]. Since the argument involves the construction of power-objects, which in our setting classify indexed families of small subobjects, our proof requires a preliminary analysis of *locally small maps*, which form the family of small maps between sheaves. We will apply this analysis also to prove the second part of our main result, which involves the verification that locally small maps between sheaves satisfy the axioms for a family of small maps.

In recent years, substantial work has been devoted to isolating axioms on $(\mathcal{E}, \mathcal{S})$ that provide a basic setting for both directions of research mentioned above. Let us briefly consider two such possible settings. The first, to which we shall refer as the *exact setting*, involves assuming that $\mathcal{E}$ is a Heyting pretopos, and that $\mathcal{S}$ satisfies a a weakening of the axioms for small maps introduced in [16]. The second, to which we shall refer as the *bounded exact setting*, involves assuming that $\mathcal{E}$ is a Heyting category, that $\mathcal{S}$ satisfies not only the axioms for small maps of the exact setting, but also the axiom asserting that for every object $X \in \mathcal{E}$, the diagonal $\Delta_X : X \to X \times X$ is a small map, that universal quantification along small maps preserves smallness of monomorphisms, and finally that $\mathcal{E}$ has quotients of bounded equivalence relations, that is to say equivalence relations given by small monomorphisms [4]. Categories of ideals provide examples of the bounded exact setting [2]. The exact completion and the bounded exact completion of syntactic categories of classes arising from constructive set theories provide other examples of the exact setting and of the bounded exact setting, respectively [4, Proposition 2.10]. Neither setting is included in the other, and they are somehow incompatible. Indeed, if we wish to avoid the assumption that every equivalence relation is given by a small monomorphism, which is necessary to include constructive set theories [1] within the general development, it is not possible to assume both that $\mathcal{E}$ is exact and that every object has a small diagonal. Each setting has specific advantages. On the one hand, the assumption of exactness of $\mathcal{E}$ is useful to define an internal version of the associated sheaf functor [12]. On the other hand, the assumption that diagonals are small has been applied in the coalgebra construction for cartesian comonads [25] and to establish results on W-types [4, Proposition 6.16].

The choice of developing our theory within the exact setting is motivated by the desire for the theory to be appropriately general. Even for the special case of Grothendieck sites, the assumption that the ambient category is exact seems to be essential in order to define the associated sheaf functor without additional assumptions on the site [12]. In the bounded exact setting, Benno van den Berg and Ieke Moerdijk have recently announced a result concerning internal sheaves on a site [5, Theorem 6.1], building on previous work of Ieke Moerdijk and Erik Palmgren [21]. Apart from the axioms for small maps that are part of the bounded exact setting, this result assumes a further axiom for small maps, the Exponentiation Axiom, and the additional hypothesis that the Grothendieck site has a basis. We prefer to avoid these assumptions since the Grothendieck site that provides a category-theoretic version of the double-negation translation cannot be shown to have a basis without assuming additional axioms for small maps, which do not hold in categories of classes arising from constructive set theories [11, 14].

1. Algebraic Set Theory in Heyting pretoposes

1.1. **Preliminaries.** We begin by stating precisely the axioms for small maps that we are going to work with. As in [16], we assume that $\mathcal{E}$ is a Heyting pretopos, and that $\mathcal{S}$ is a family of maps in $\mathcal{E}$ satisfying the axioms (A1)-(A7) stated below.

(A1) The family $\mathcal{S}$ contains isomorphisms and is closed under composition.

(A2) For every pullback square of the form

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r, "h"'] & A \end{tikzcd} \tag{1}$$

if $f : X \to A$ is in $\mathcal{S}$, then so is $g : Y \to B$.

(A3) For every pullback square as (1), if $h : B \to A$ is an epimorphism and $g : Y \to B$ is in $\mathcal{S}$, then $f : X \to A$ is in $\mathcal{S}$.

(A4) The maps $0 \to 1$ and $1 + 1 \to 1$ are in $\mathcal{S}$.

(A5) If $f : X \to A$ and $g : Y \to B$ are in $\mathcal{S}$, then $f + g : X + Y \to A + B$ is in $\mathcal{S}$.

(A6) For every commutative triangle of the form

$$\begin{tikzcd} X \arrow[rr, two heads, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & A & \end{tikzcd}$$

where $h : X \twoheadrightarrow Y$ is an epimorphism, if $f : X \to A$ is in $\mathcal{S}$, then $g : Y \to A$ is in $\mathcal{S}$.

(A7) For map $f : X \to A$ in $\mathcal{S}$ and every epimorphism $p : P \twoheadrightarrow X$, there exists a quasi-pullback diagram of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & P \arrow[r, two heads, "p"] & X \arrow[d, "f"] \\ B \arrow[rr, two heads, "h"'] & & A \end{tikzcd}$$

where $g : Y \to B$ is in $\mathcal{S}$ and $h : B \twoheadrightarrow A$ is an epimorphism.

We refer to (A2) as the Pullback Stability axiom, to (A3) as the Descent axiom, to (A6) as the Quotients axiom, to (A7) as the Collection axiom.

1.2. **Power objects.** Our basic axiomatisation of small maps involves one more axiom. In order to state it, we need some terminology. Given a family of maps $\mathcal{S}$ satisfying (A1)-(A7), an $\mathcal{S}$-*object* is an object X such that the unique map $X \to 1$ is in $\mathcal{S}$. For a fixed object $A \in \mathcal{E}$, an A-*indexed family of* $\mathcal{S}$-*subobjects* is a subobject $S \rightarrowtail A \times X$ such that its composite with the first projection $A \times X \to A$ is in $\mathcal{S}$. We abbreviate this by saying that the diagram $S \rightarrowtail A \times X \to A$ is an indexed family of $\mathcal{S}$-subobjects. Recall that, writing $\Gamma(f) : X \rightarrowtail A \times X \to A$ for the evident indexed family of subobjects consisting of the graph of a map $f : X \to A$, it holds that f is in $\mathcal{S}$ if and only if $\Gamma(f)$ is an indexed family of $\mathcal{S}$-subobjects. Axiom (P1), stated below, expresses that indexed families of $\mathcal{S}$-subobjects can be classified.

(P1) For each object X of $\mathcal{E}$ there exists an object $\mathcal{P}(X)$ of $\mathcal{E}$, called the *power object* of X, and an indexed family of $\mathcal{S}$-subobjects of X

$$\ni_X \rightarrowtail \mathcal{P}(X) \times X \to \mathcal{P}(X)\,,$$

called the *membership relation* on X, such that for any indexed family of $\mathcal{S}$-subobjects $S \rightarrowtail A \times X \to A$ of X, there exists a unique map $\chi_S : A \to \mathcal{P}(X)$ fitting in a double pullback diagram of the form

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail] & \ni_X \arrow[d, tail] \\
A \times X \arrow[r] \arrow[d] & \mathcal{P}(X) \times X \arrow[d] \\
A \arrow[r, "\chi_S"'] & \mathcal{P}(X)\,.
\end{tikzcd}$$

Writing $\mathrm{Sub}_{\mathcal{S}}(X)(A)$ for the lattice of A-indexed families of $\mathcal{S}$-subobjects of X, axiom (P1) can be expressed equivalently by saying that for every map $\chi : A \to \mathcal{P}(X)$, the functions

$$\mathrm{Hom}(A, \mathcal{P}(X)) \to \mathrm{Sub}_{\mathcal{S}}(X)(A)\,,$$

defined by pulling back $\ni_X \rightarrowtail \mathcal{P}(X) \times X \to \mathcal{P}(X)$, are a family of bijections, natural in A. In the following, we often omit the subscript in the membership relation.

Our basic axiomatisation of small maps involves only axioms (A1)-(A7) and (P1). Therefore, when we speak of a *family of small maps* without further specification, we mean a family $\mathcal{S}$ satisfying axioms (A1)-(A7) and (P1). In this case, elements of $\mathcal{S}$ will be referred to as small maps, and we speak of small objects and indexed families of small subobjects rather than $\mathcal{S}$-objects and indexed families of $\mathcal{S}$-subobjects, respectively.

1.3. **Exponentiability and Weak Representabily.** One can also consider an alternative axiomatisation of small maps by requiring, in place of (P1), the axioms of Exponentiability (S1) and Weak Representability (S2), stated below.

(S1) If $f : X \to A$ is in $\mathcal{S}$, then the pullback functor $f^* : \mathcal{E}/A \to \mathcal{E}/X$ has a right adjoint, which we write $\Pi_f : \mathcal{E}/X \to \mathcal{E}/A$.

(S2) There exists a map $u : E \to U$ in $\mathcal{S}$ such that every map $f : X \to A$ in $\mathcal{S}$ fits in a diagram of form

$$\begin{tikzcd}
X \arrow[d, "f"'] & Y \arrow[l] \arrow[r] \arrow[d] & E \arrow[d, "u"] \\
A & B \arrow[l, two heads, "h"] \arrow[r] & U
\end{tikzcd} \tag{2}$$

where $h : B \twoheadrightarrow A$ is an epimorphism, the square on the left-hand side is a quasi-pullback and the square on the right-hand side is a pullback.

The axiomatisation of small maps with (A1)-(A7) and (S1)-(S2) is a slight variant of the one introduced in [16]. The only difference concerns the formulation of the Weak Representability axiom, which is a weakening of the Representability axiom in [16, Definition 1.1]. The weakening involves having a quasi-pullback rather than a genuine pullback in the left-hand side square of the diagram in (2). Example 1.5.2 and Example 1.5.3 illustrate how the weaker form of representability in (S2) is the most appropriate to consider when working within exact categories without assuming additional axioms for small maps. See [2, 24] for other forms of representability.

This axiomatisation is a strengthening of the one consisting of (A1)-(A7) and (P1). On the one hand, the combination of (S1) and (S2) implies (P1), since the proof in [16, §I.3] carries over when Weak Representability is assumed instead of Representability [5]. On the other hand, Example 1.5.1 shows that there are examples satisfying (P1) but not (S2). Let us also recall that (P1) implies (S1) by an argument similar to the usual construction of exponentials from power objects in a topos [2, Proposition 5.17].

1.4. **Internal language.** We will make extensive use of the internal language of Heyting pretoposes [19] This is a form of many-sorted first-order intuitionistic logic which allows us to manipulate objects and maps of $\mathcal{E}$ syntactically. As an illustration of the internal language, let us recall that for any map $f : X \to Y$, we have a *direct image* map $f_! : \mathcal{P}(X) \to \mathcal{P}(Y)$. Assuming $f : X \to Y$ to be small, there is also an *inverse image* map $f^* : \mathcal{P}(Y) \to \mathcal{P}(X)$, which is related to $f_! : \mathcal{P}(X) \to \mathcal{P}(Y)$ by the internal adjointness expressed in the internal language as follows:

$$(\forall s : \mathcal{P}(X))(\forall t : \mathcal{P}(Y)) \; \big(f_!(s) \subseteq t \Leftrightarrow s \subseteq f^*(t)\big) \, . \tag{3}$$

The internal language allow us also to give a characterisation of small maps. Indeed, a map $f : X \to A$ is small if and only if the following sentence is valid:

$$(\forall a : A)(\exists s : \mathcal{P}(X))(\forall x : X)\big(f(x) = a \Leftrightarrow x \in s\big) \, . \tag{4}$$

The sentence in (4) can be understood informally as expressing that the fibers of $f : X \to A$ are small. Formulation of some of the axioms for small in the internal language can be found in [3]. For $s : \mathcal{P}(X)$ and a formula $\phi(x)$ where $x : X$ is a free variable, we define the restricted quantifiers by letting

$$\begin{aligned} (\forall x \in s)\phi(x) \quad &=_{\mathrm{def}} \quad (\forall x : X)\big(x \in s \Rightarrow \phi(x)\big) \, , \\ (\exists x \in s)\phi(x) \quad &=_{\mathrm{def}} \quad (\exists x : X)\big(x \in s \wedge \phi(x)\big) \, . \end{aligned}$$

We denote anonymous variables of sort X by writing $\_ : X$.

1.5. **Examples.** We end this section with some examples of Heyting pretoposes equipped with families of small maps. Example 1.5.1 shows that our development of internal sheaves applies to elementary toposes [17, Chapter V], while Example 1.5.2 and Example 1.5.3 show that it includes important examples for which the ambient category $\mathcal{E}$ is not an elementary topos. Note that neither Example 1.5.2 nor Example 1.5.3 satisfies the Representability axiom of [16, Definition 1.1], but only the Weak Representabily Axiom, as stated in (S2) above. Furthermore, neither of these examples satisfies the additional axiom that every object X of $\mathcal{E}$ has a small diagonal map $\Delta_X : X \to X \times X$.

1.5.1. *Example.* Consider an elementary topos $\mathcal{E}$ and let $\mathcal{S}$ consist of all maps in $\mathcal{E}$. It is evident that the axioms (A1)-(A7) and (P1) are verified, while (S2) is not.

1.5.2. *Example.* Consider Constructive Zermelo-Fraenkel set theory (CZF), presented in [1]. We take $\mathcal{E}$ to be the exact completion [6, 7] of the corresponding category of classes [3, 10], considered as a regular category. By the general theory of exact completions, the category $\mathcal{E}$ is a Heyting pretopos and the category of classes of CZF embeds faithfully in it [6, 7]. We write the objects of $\mathcal{E}$ as X/r_X, where X is a class and $r_X \subseteq X \times X$ is an equivalence relation on it. A map $f : X/r_X \to A/r_A$ in $\mathcal{E}$ is a relation $f \subseteq X \times A$ that is functional and preserves the equivalence relation, in the sense made precise in [6]. We declare a map $f : X/r_X \to A/r_A$ to be small if it fits into a quasi-pullback of the form

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "g"'] & X/r_X \arrow[d, "f"] \\ B \arrow[r, two heads, "h"'] & A/r_A \end{tikzcd}$$

where $h : B \to A$ is an epimorphism and $g : Y \to B$ is a function of classes whose fibers are sets. This family satisfies the axioms (A1)-(A7) and (S1)-(S2) by a combination of the results on the category of classes of CZF in [3, 10] with those on small maps in exact completions in [4, Section 4].

1.5.3. *Example.* Consider Martin-Löf's constructive type theory with rules for all the standard forms of dependent types and for a type universe reflecting them [22, 13]. We take $\mathcal{E}$ to be the corresponding category of setoids, which has been shown to be a Heyting pretopos in [21, Theorem 12.1]. We declare a map $f : X \to A$ in $\mathcal{E}$ to be small if it fits into a quasi-pullback of the form

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r, two heads, "h"'] & A \end{tikzcd}$$

where $h : B \to A$ is an epimorphism and $g : Y \to B$ is a map such that for every $b \in B$ the setoid $g^{-1}(b)$, as defined in [21, Section 12], is isomorphic to setoid whose carrier and equivalence relation are given by elements of the type universe. This family satisfies the axioms (A1)-(A7) and (S1)-(S2) by combining the results on display maps in [4, Section 4] with those on setoids in [21, Section 12].

2. Lawvere-Tierney sheaves

2.1. **Lawvere-Tierney coverages.** Let $\mathcal{E}$ be a Heyting pretopos equipped with a family of small maps $\mathcal{S}$ satisfying axioms (A1)-(A7) and (P1). We define the object of *small truth values* Ω by letting $\Omega =_{\mathrm{def}} \mathcal{P}(1)$. By the universal property of Ω, it is immediate to see that we have a global element $\top : 1 \to \Omega$. To simplify notation, we write p instead of $p = \top$, for $p : \Omega$. For example, this allows us to write $\{p : \Omega \mid p\}$ instead of $\{p : \Omega \mid p = \top\}$. Note, then, that the monomorphism $\{p : \Omega \mid p\} \rightarrowtail \Omega$ is the map $\top : 1 \to \Omega$. Similarly, $p \Rightarrow \phi$ is equivalent to $(\forall_{\_} \in p)\phi$, for every $p : \Omega$ and every formula ϕ of the internal language. The internal language is used in Definition 2.1.1 to specify what will be our starting point to introduce a notion of sheaf.

2.1.1. **Definition.** Let $(\mathcal{E}, \mathcal{S})$ be a Heyting pretopos with a family of small maps. A *Lawvere-Tierney coverage* in $\mathcal{E}$ is a subobject $J \rightarrowtail \Omega$ making the following sentences valid in $\mathcal{E}$

(C1) $J(\top)$,

(C2) $(\forall p : \Omega)(\forall q : \Omega)\Big[\big(p \Rightarrow J(q)\big) \Rightarrow \big(J(p) \Rightarrow J(q)\big)\Big]$.

2.1.2. *Remark.* Our development of internal sheaves relative to a Lawvere-Tierney coverage generalises the existing theory of internal sheaves relative to a Lawvere-Tierney local operator in an elementary topos [17]. Indeed, when $\mathcal{E}$ is an elementary topos and $\mathcal{S}$ consists of all maps in $\mathcal{E}$, as in Example 1.5.1, Lawvere-Tierney coverages are in bijective correspondence with Lawvere-Tierney local operators [15, A.4.4.1]. The correspondence is given by the universal property of Ω, which is the subobject classifier of $\mathcal{E}$, via a pullback square of the form

$$\begin{tikzcd} J \arrow[r] \arrow[d, tail] & 1 \arrow[d, "\top"] \\ \Omega \arrow[r, "j"'] & \Omega \end{tikzcd}$$

The verification of the correspondence between the axioms for a Lawvere-Tierney coverage and those for a Lawvere-Tierney local operator is a simple calculation. Since in the general the monomorphism $J \rightarrowtail \Omega$ may fail to be small, we focus on Lawvere-Tierney coverages.

From now on, we will work with a fixed Lawvere-Tierney coverage as in Definition 2.1.1. Our first step towards defining sheaves is to construct a universal closure operator on subobjects, that is to say a natural family of functions

$$\mathbf{C}_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X)\,,$$

for $X \in \mathcal{E}$, satisfying the familiar mononotonicity, inflationarity, and idempotency properties [15, A.4.3]. Note that we do not need to require meet-stability, since this follows from the other properties by the assumption that the operator is natural [15, Lemma A.4.3.3]. Naturality of the operator means that for $f : X \to Y$, the diagram below commutes

$$\begin{tikzcd} \mathrm{Sub}(Y) \arrow[rr, "\mathbf{C}_Y"] \arrow[d, "f^*"'] & & \mathrm{Sub}(Y) \arrow[d, "f^*"] \\ \mathrm{Sub}(X) \arrow[rr, "\mathbf{C}_X"'] & & \mathrm{Sub}(X) \end{tikzcd}$$

We define the universal closure operator associated to the Lawvere-Tierney coverage by letting, for $S \rightarrowtail X$

$$\mathbf{C}_X(S) =_{\mathrm{def}} \big\{ x : X \mid (\exists p : \Omega)\big(J(p) \wedge \big(p \Rightarrow S(x)\big)\big)\big\}\,. \tag{5}$$

2.1.3. **Proposition.** *The family* $\mathbf{C}_X : \mathrm{Sub}(X) \to \mathrm{Sub}(X)$*, for* $X \in \mathcal{E}$*, associated to a Lawvere-Tierney operator is a universal closure operator.*

Proof. First, we verify that the operator is natural. This is immediate, since for a subobject $T \rightarrowtail Y$ we have

$$\begin{aligned} \mathbf{C}_X(f^*T) &= \big\{ x : X \mid (\exists p : \Omega)\big(J(p) \wedge \big(p \Rightarrow f^*T(x)\big)\big)\big\} \\ &= f^*\{ y : Y \mid (\exists p : \Omega)\big(J(p) \wedge \big(p \Rightarrow T(y)\big)\big)\big\} \\ &= f^*(\mathbf{C}_Y(T))\,. \end{aligned}$$

For inflationarity, let $S \rightarrowtail X$, $x : X$ and assume that $S(x)$ holds. Then, define $p : \Omega$ by letting $p =_{\mathrm{def}} \top$. We have that $J(p)$ holds by (C1), and that

$p \Rightarrow S(x)$ holds by assumption. Monotonicity is immediate by the definition in (5). Idempotence is the only part that is not straightforward, since it makes use of the Collection Axiom for small maps. For $S \rightarrowtail X$, we need to show that $\mathbf{C}^2(S) \subseteq \mathbf{C}(S)$. Let $x : X$ and assume that there exists $p : \Omega$ such that $J(p)$ and $p \Rightarrow \mathbf{C}_X(S)(x)$ hold. For $\_ : 1$ and $q : \Omega$, let us define

$$\phi(\_, q) =_{\mathrm{def}} J(q) \wedge \big(q \Rightarrow S(x)\big) \,.$$

By the definition in (5), $p \Rightarrow \mathbf{C}_X(S)(x)$ implies

$$(\forall \_ \in p)(\exists q : \Omega)\phi(\_, q) \,.$$

We can apply Collection and derive the existence of $u : \mathcal{P}(\Omega)$ such that

$$(\forall \_ \in p)(\exists q \in u)\phi(\_, q) \wedge (\forall q \in u)(\exists \_ \in p)\phi(\_, q) \,.$$

Define $r : \Omega$ by $r =_{\mathrm{def}} \bigcup u$. We wish to show that $J(r)$ and $r \Rightarrow S(x)$ hold, which will allow us to conclude $\mathbf{C}S_X(x)$, as required. To prove that $J(r)$ holds, we observe that

$$\begin{aligned} p \quad &\Rightarrow \quad (\exists q \in u)\, J(q) \\ &\Rightarrow \quad (\exists q : \Omega)\big(J(q) \wedge q \subseteq r\big) \\ &\Rightarrow \quad J(r) \,. \end{aligned}$$

Therefore $p \Rightarrow J(r)$. Since we have $J(p)$ by hypothesis, Axiom (C2) for a Lawvere-Tierney coverage allows to derive $J(r)$, as required. By definition of $r : \Omega$, we note that $r \Rightarrow S(x)$ holds if and only if for every $q \in u$ we have $q \Rightarrow S(x)$. But since $q \in u$ implies $q \Rightarrow S(x)$, we obtain $r \Rightarrow S(x)$, as desired. □

2.1.4. *Remark.* Given a universal closure operator on $\mathcal{E}$, we can define a Lawvere-Tierney coverage $J \rightarrowtail \Omega$ by taking J to be the closure of $\{p : \Omega \mid p\}$. The closure operator induced by J coincides with the given one if and only if the latter satisfies the equation in (5). Therefore, we are considering here only a special class of universal closure operators. Some restriction on the universal closure operators seems necessary since it does not seem possible to develop a treatment of sheaves for arbitrary universal closure operators without assuming additional axioms for small maps, such as that asserting that every monomorphism is small. Focusing on the class of universal closure operations determined by Lawvere-Tierney coverages captures an appropriate level of generality. First, as we will see in Section 2.3, they correspond precisely to the Grothendieck sites considered in [12]. Secondly, as we will see in Section 4 and Section 5, they allow us to develop a treatment of internal sheaves.

We shall be particularly interested in the closure of the membership relation $\ni_X \rightarrowtail \mathcal{P}(X) \times X \to X$, which we are going to write as

$$\overline{\ni}_X \rightarrowtail \mathcal{P}(X) \times X \to \mathcal{P}(X)\,.$$

For $x : X$ and $s : \mathcal{P}(X)$, the definition in (5) implies

$$x \,\overline{\in}\, s \Leftrightarrow (\exists p : \Omega)\big(J(p) \wedge \big(p \Rightarrow x \in s\big)\big)\,,$$

2.2. **Sheaves.** Having defined a universal closure operator on $\mathcal{E}$, we can define a notion of sheaf and introduce the family of maps that we will consider as small maps between sheaves. In order to do this, we define a monomorphism $m : B \rightarrowtail A$ to be *dense* if it holds that $\mathbf{C}_A(B) = A$.

2.2.1. **Definition.** An object X of $\mathcal{E}$ is said to be a *separated* if for every dense monomorphism $m : B \rightarrowtail A$ the function

$$\mathrm{Hom}(m, X) : \mathrm{Hom}(A, X) \to \mathrm{Hom}(B, X)\,, \tag{6}$$

induced by composition with m, is injective. Equivalently, X is a separated if and only if for every map $v : B \to X$ there exists at most one extension $u : A \to X$ making the following diagram commute

$$\begin{tikzcd} B \arrow[r, "v"] \arrow[d, tail, "m"'] & X \\ A \arrow[ur, bend right, "u"'] & \end{tikzcd} \tag{7}$$

We say that X is a *sheaf* if the function in (6) is bijective. Equivalently, X is a sheaf if and only if every map $v : B \to X$ has a unique extension $u : A \to X$ as in (7).

Definition 2.2.2 defines what it means for a map in $\mathcal{E}$ to be *locally small*. As explained further in Section 3, the idea underlying this notion is that a map is locally small if and only if each of its fibers contains a small dense subobject. By their very definition, locally small maps are stable under pullback, since the properties of the defining diagrams all are.

2.2.2. **Definition.** A map $f : X \to A$ in $\mathcal{E}$ is said to be *locally small* if its graph $\Gamma(f) : X \rightarrowtail A \times X \to A$ fits in a diagram of the form

$$\begin{tikzcd} T \arrow[r] \arrow[d, tail] & X \arrow[d, tail] \\ B \times X \arrow[r] \arrow[d, "\pi_B"'] & A \times X \arrow[d, "\pi_A"] \\ B \arrow[r, two heads, "h"'] & A \end{tikzcd}$$

where $h : B \twoheadrightarrow A$ is an epimorphism, $T \rightarrowtail B \times X \to B$ is an indexed family of small subobjects of X, and the canonical monomorphism $T \rightarrowtail B \times_A X$ is dense. An indexed family of subobjects $S \rightarrowtail A \times X \to X$ is said to be an *indexed family of locally small subobjects* if the composite map $S \to A$ is locally small.

We write $\mathcal{E}_J$ for the full subcategory of $\mathcal{E}$ whose objects are sheaves, and $\mathcal{S}_J$ for the family of locally small maps in $\mathcal{E}_J$. The aim of the remainder of the paper is to prove the following result. As fixed in Section 1.2, a family of small maps is required to satisfy only axioms (A1)-(A7) and (P1).

2.2.3. **Theorem.** *Let $(\mathcal{E}, \mathcal{S})$ be a Heyting pretopos with a family of small maps. For every Lawvere-Tierney coverage J in $\mathcal{E}$, $(\mathcal{E}_J, \mathcal{S}_J)$ is a Heyting pretopos equipped with a family of small maps. Furthermore, if $\mathcal{S}$ satisfies the Exponentiability and Weak Representability axioms, so does $\mathcal{S}_J$.*

2.3. **Grothendieck sheaves.** Before developing the theory required to prove Theorem 2.2.3, we explain how this result subsumes the treatment of sheaves for a Grothendieck site. Let $\mathbb{C}$ be a small internal category in $\mathcal{E}$. Smallness of $\mathbb{C}$ means that both its objects $\mathbb{C}_0$ and its arrows $\mathbb{C}_1$ are given by small objects in $\mathcal{E}$. We write category $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ of internal presheaves over $\mathbb{C}$. It is well-known that $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ is a Heyting pretopos. When $\mathbb{C}_0$ and $\mathbb{C}_1$ have small diagonals, so that both the equality of objects and that of arrows is given by a small monomorphism, it is possible to equip $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ with a family of small maps, consisting of the internal natural transformations that are pointwise small maps in $\mathcal{E}$ [21, 25].

Lawvere-Tierney coverages in $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ are in bijective correspondence with Grothendieck coverages with small covers on $\mathbb{C}$, as defined in [12]. To explain this correspondence, we need to recall some terminology and notation. A *sieve* P on $a \in \mathbb{C}$ is a subobject $P \to \mathbf{y}(a)$, where we write $\mathbf{y}(a)$ for the Yoneda embedding of a. Such a sieve can be identified with a family of arrows with codomain a that is closed under composition, in the sense that for every pair of composable maps $\phi : b \to a$ and $\psi : c \to b$ in $\mathbb{C}$, $\phi \in P$ implies $\phi\,\psi \in P$. For a sieve P on a and arrow $\phi : b \to a$, we write $P \cdot \phi$ for the sieve on b defined by letting

$$P \cdot \phi =_{\mathrm{def}} \{\psi : c \to b \mid \phi\,\psi \in P\}\,. \tag{8}$$

Recall from [12] that a *Grothendieck coverage with small covers* on $\mathbb{C}$ consists of a family $(\mathrm{Cov}(a) \mid a \in \mathbb{C})$ such that elements of $\mathrm{Cov}(a)$ are small sieves, and the conditions of Maximality (M), Local Character (L), and Transitivity (T) hold:

(M) $M_a \in \mathrm{Cov}(a)$.
(L) If $\phi : b \to a$ and $S \in \mathrm{Cov}(a)$, then $S \cdot \phi \in \mathrm{Cov}(b)$.

(T) If $S \in \mathrm{Cov}(a)$, T is a small sieve on a, and for all $\phi : b \to a \in S$ we have $T \cdot \phi \in \mathrm{Cov}(b)$, then $T \in \mathrm{Cov}(a)$.

The object Ω in $\mathrm{Psh}_{\mathcal{E}}(\mathbb{C})$ is given by

$$\Omega(a) =_{\mathrm{def}} \{S \rightarrowtail \mathbf{y}(a) \mid S \text{ small}\} .$$

Therefore, a Grothendieck coverage with small covers can be identified with a family of subobjects $\mathrm{Cov}(a) \rightarrowtail \Omega(a)$. Condition (L) means that this family is a subpresheaf of Ω, while conditions (M) and (T) for a Grothendieck coverage are the rewriting of conditions (C1) and (C2) for a Lawvere-Tierney coverage. By instanciating the general definitions of Section 2.2 we obtain a notion of sheaf, which can be shown to be equivalent to the familiar notion of a sheaf for a Grothendieck coverage, and a notion of small map. Writing $\mathrm{Sh}_{\mathcal{E}}(\mathbb{C}, \mathrm{Cov})$ for the category of internal sheaves, and $\mathcal{S}(\mathbb{C}, \mathrm{Cov})$ for the corresponding family of small maps, Theorem 2.2.3 implies the following result.

2.3.1. **Corollary.** *Let $(\mathcal{E}, \mathcal{S})$ be a Heyting pretopos with a family of small maps. Let $(\mathbb{C}, \mathrm{Cov})$ be a small category with small diagonals equipped with a Grothendieck coverage with small covers. Then $\big(\mathrm{Sh}_{\mathcal{E}}(\mathbb{C}, \mathrm{Cov}), \mathcal{S}(\mathbb{C}, \mathrm{Cov})\big)$ is a Heyting pretopos with a family of small maps. Furthermore, if $\mathcal{S}$ satisfies the Exponentiability and Weak Representability axioms, so does $\mathcal{S}_J$.*

3. Classification of locally small subobjects

3.1. **Locally small maps.** We begin by characterising locally small maps in the internal language, analogously to how small maps are characterised in (4). For each object X, define an equivalence relation $R \rightarrowtail \mathcal{P}(X) \times \mathcal{P}(X)$ by letting

$$R =_{\mathrm{def}} \{(s,t) : \mathcal{P}(X) \times \mathcal{P}(X) \mid (\forall x : X)\big(x \mathrel{\overline{\in}} s \Leftrightarrow x \mathrel{\overline{\in}} t\big)\} .$$

Informally, $R(s,t)$ holds whenever s and t have the same closure. Using the exactness of the Heyting pretopos $\mathcal{E}$, we define $\mathcal{P}_J(X)$ as the quotient of $\mathcal{P}(X)$ by R, fitting into an exact diagram of the form

$$R \underset{\pi_2}{\overset{\pi_1}{\rightrightarrows}} \mathcal{P}(X) \xrightarrow{[\,\cdot\,]} \mathcal{P}_J(X) . \tag{9}$$

The quotient map $\mathcal{P}(X) \twoheadrightarrow \mathcal{P}_J(X)$ is to be interpreted as performing the closure of a small subobject of X.

3.1.1. *Remark.* The exactness of the Heyting pretopos $\mathcal{E}$ is exploited here in a crucial way to define $\mathcal{P}_J(X)$. In particular, without further assumptions on the Lawvere-Tierney coverage, the equivalence relation in (9) cannot be shown to be given by a small monomorphism.

We define a new indexed family of subobjects of X, $\ni_J \rightarrowtail \mathcal{P}_J(X) \times X \to \mathcal{P}_J(X)$, by letting, for $x : X$ and $p : \mathcal{P}_J(X)$,

$$x \in_J p \Leftrightarrow (\exists s : \mathcal{P}X)\big(p = [s] \wedge x \mathrel{\overline{\in}} s\big)\,.$$

In particular, for $x : X$ and $s : \mathcal{P}X$, this gives $x \in_J [s] \Leftrightarrow x \mathrel{\overline{\in}} s$, which is to say that the following squares are pullbacks

$$\begin{tikzcd}
\overline{\ni} \arrow[r] \arrow[d, tail] & \ni_J \arrow[d, tail] \\
\mathcal{P}(X) \times X \arrow[r] \arrow[d] & \mathcal{P}_J(X) \times X \arrow[d] \\
\mathcal{P}(X) \arrow[r, "{[\,\cdot\,]}"'] & \mathcal{P}_J(X)
\end{tikzcd} \tag{10}$$

This definition of $\in_J$ and of the relation R imply that $\mathcal{P}_J(X)$ satisfies a form of extensionality, in the sense that for $p, q : \mathcal{P}_J(X)$ it holds that

$$p = q \Leftrightarrow (\forall x : X)(x \in_J p \Leftrightarrow x \in_J q)\,.$$

From diagram (10), we also see that $\in_J$ is closed in $X \times \mathcal{P}_J X$, since it is closed when pulled back along an epimorphism. Given a map $f : X \to A$, it is convenient to define, for $a : A$, $s : \mathcal{P}(X)$,

$$s \approx f^{-1}(a) =_{\mathrm{def}} (\forall x : X)\,[(x \in s \Rightarrow f(x) = a) \wedge (f(x) = a \Rightarrow x \mathrel{\overline{\in}} s)]\,.$$

3.1.2. **Lemma.** *A map $f : X \to A$ is locally small if and only if the following sentence is valid:*

$$(\forall a : A)(\exists s : \mathcal{P}X) s \approx f^{-1}(a)\,. \tag{11}$$

Proof. First, let us assume that $f : X \to A$ is locally small. By Definition 2.2.2 there is a diagram

$$\begin{tikzcd}
T \arrow[r] \arrow[d, tail] & X \arrow[d, tail] \\
B \times X \arrow[r] \arrow[d] & A \times X \arrow[d] \\
B \arrow[r, two heads, "h"'] & A
\end{tikzcd}$$

where $T \rightarrowtail B \times X \to B$ is an indexed family of small subobjects of X, $h : B \twoheadrightarrow A$ is an epimorphism, and the monomorphism $T \rightarrowtail B \times_A X$ is dense.

There is a classifying map $\chi_T : B \to \mathcal{P}(X)$ such that $x \in \chi_T(b) \Leftrightarrow T(b,x)$, and the commutativity of the diagram implies

$$x \in \chi_T(b) \Rightarrow f(x) = h(b)\,,$$

while the density of $T \rightarrowtail B \times_A X$ implies

$$f(x) = h(b) \Rightarrow x \mathrel{\overline{\in}} \chi_T(b)\,.$$

Given $a : A$, there exists $b : B$ such that $h(b) = a$ and so, defining $s =_{\mathrm{def}} \chi_T(b)$, we obtain the data required to prove the statement. For the converse implication, assume (11). We define

$$B =_{\mathrm{def}} \{(a,s) : A \times \mathcal{P}(X) \mid s \approx f^{-1}(a)\}$$

and

$$T =_{\mathrm{def}} \{\big((a,s),x\big) : B \times X \mid x \in s\}\,.$$

We have that $T \rightarrowtail B \times X \to B$ is small by construction and that the projection $h : B \to A$ is an epimorphism by hypothesis. Since

$$B \times_A X \cong \{\big((a,s),x\big) : B \times X \mid f(x) = a\}\,,$$

we have that $T \rightarrowtail B \times_A X$ is dense by the definition of B. □

3.1.3. *Remark.* We could have considered maps $f : X \to A$ satisfying the condition

$$(\forall a : A)(\exists s : \mathcal{P}X)(\forall x : X.) f(x) = a \Leftrightarrow x \mathrel{\overline{\in}} s\,.$$

This amounts to saying that each fiber of f is the closure of a small subobject. Let us call such maps *closed-small*. For maps with codomain a separtated object, and so for maps with codomain a sheaf, these definitions coincide, so either would give our desired class of small maps on $\mathcal{E}_J$. However, considered on the whole of $\mathcal{E}$, they generally give different classes of maps, each retaining different properties. For instance, closed-small subobjects of X are classified by $\mathcal{P}_J(X)$, while locally small subobjects may not be classified. On the other hand, locally small maps satisfy axioms (A1)-(A7) in $\mathcal{E}$, whereas the closed-small maps may not, since the identity map on a non-separated object will not be closed-small. Thus, the choice of either as the extension of local smallness from $\mathcal{E}_J$ to $\mathcal{E}$ is a matter of convenience.

3.2. **Universal property of $\mathcal{P}_J(X)$.** We conclude this section by showing how locally small closed subobjects can be classified. This is needed in Section 4 for the proof of the associated sheaf functor theorem.

3.2.1. **Lemma.** *An indexed family of subobjects $S \rightarrowtail A \times X \to A$ is an indexed family of locally small subobjects if and only if there exists a diagram* *of the form*

$$\begin{tikzcd}
T \arrow[r] \arrow[d, tail] & S \arrow[d, tail] \\
B \times X \arrow[r] \arrow[d, "\pi_B"'] & A \times X \arrow[d, "\pi_A"] \\
B \arrow[r, two heads, "h"'] & A
\end{tikzcd}$$

where $h : B \twoheadrightarrow A$ is an epimorphism, $T \rightarrowtail B \times X \to S$ is an indexed family of small subobjects of X, and the canonical monomorphism $T \rightarrowtail B \times_A S$ is dense.

Proof. Assume $S \rightarrowtail A \times X \to A$ to be an indexed family of locally small subobjects. By Definition 2.2.2, this means that the composite $S \to A$ is a locally small map, which in turn implies that there exists a diagram of the form

$$\begin{tikzcd}
T \arrow[r] \arrow[d, tail] & S \arrow[d, tail] \\
B \times S \arrow[r] \arrow[d, "\pi_B"'] & A \times S \arrow[d, "\pi_A"] \\
B \arrow[r, two heads, "h"'] & A
\end{tikzcd} \tag{12}$$

where $h : B \twoheadrightarrow A$ is an epimorphism, $T \rightarrowtail B \times S \to S$ is an indexed family of small subobjects of S, and the canonical monomorphism $T \rightarrowtail B \times_A S$ is dense. The composite

$$T \rightarrowtail B \times S \rightarrowtail B \times A \times X \longrightarrow B \times X$$

can be shown to be a monomorphism using the commutativity of the diagram in (12). We obtain an indexed family of small subobjects $T \rightarrowtail B \times X \to B$, which clearly satisfies the required property. $\square$

Observe that $\ni_J \to \mathcal{P}_J(X) \times X \to \mathcal{P}_J(X)$ is a family of locally small subobjects of X, as witnessed by the diagram

$$\begin{tikzcd}
\ni \arrow[r] \arrow[d, tail] & \ni_J \arrow[d, tail] \\
\mathcal{P}(X) \times X \arrow[r] \arrow[d] & \mathcal{P}_J(X) \times X \arrow[d] \\
\mathcal{P}(X) \arrow[r, two heads, "{[\,\cdot\,]}"'] & \mathcal{P}_J(X)
\end{tikzcd}$$

The pullback of $\ni_J \rightarrowtail \mathcal{P}_J(X) \times X \to \mathcal{P}_J(X)$ along a map $\chi : A \to \mathcal{P}_J(X)$ is therefore an A-indexed family of locally small closed subobjects of X. We write $\mathrm{Sub}_{\mathcal{S}_J}(X)(A)$ for the lattice of such subobjects.

3.2.2. **Proposition.** *For every object X, the object $\mathcal{P}_J(X)$ classifies indexed families of locally small closed subobjects of X, which is to say that for every such family $S \rightarrowtail A \times X \to A$ there exists a unique map $\phi_S : A \to \mathcal{P}_J(X)$ such that both squares in the diagram*

$$\begin{tikzcd}
S \arrow[r] \arrow[d, tail] & \ni_J \arrow[d, tail] \\
A \times X \arrow[r, "\phi_S \times 1_X"] \arrow[d, "\pi_A"'] & \mathcal{P}_J(X) \times X \arrow[d, "\pi_{\mathcal{P}_J(X)}"] \\
A \arrow[r, "\phi_S"'] & \mathcal{P}_J(X)
\end{tikzcd}$$

are pullbacks. Equivalently, the functions

$$\mathrm{Hom}(A, \mathcal{P}_J(X)) \to \mathrm{Sub}_{\mathcal{S}_J}(X)(A)$$

given by pulling back $\ni_J \rightarrowtail \mathcal{P}_J(X) \times X \to \mathcal{P}_J(X)$ are a family of bijections, natural in A.

Proof. Let an indexed family as in the statement be given, together with data as in Lemma 3.2.1. By the universal property of $\mathcal{P}(X)$, we get a classifying map $\chi_T : B \to \mathcal{P}(X)$. Using the naturality of the closure operation, we obtain

$$(\chi_T \times 1_X)^* \mathbf{C}(\ni_X) = \mathbf{C}\big((\chi_T \times 1_X)^*(\ni_X)\big) = \mathbf{C}(T) = B \times_A S\,,$$

where the last equality is a consequence of the density of T in $B \times_A S$. Therefore, we have a sequence of pullbacks of the form

$$
\begin{tikzcd}
\ni_J \arrow[d, tail] & \overline{\ni} \arrow[l] \arrow[d, tail] & B \times_A S \arrow[l] \arrow[r] \arrow[d, tail] & S \arrow[d, tail] \\
\mathcal{P}_J(X) \times X \arrow[d] & \mathcal{P}(X) \times X \arrow[l] \arrow[d] & B \times X \arrow[l] \arrow[r] \arrow[d] & A \times X \arrow[d] \\
\mathcal{P}_J(X) & \mathcal{P}(X) \arrow[l, two heads, "{[\,\cdot\,]}"'] & B \arrow[l, "\chi_T"'] \arrow[r, two heads, "h"'] & A
\end{tikzcd}
$$

This, combined with the epimorphism $h : B \to A$, allows us to show that

$$(\forall a : A)(\exists p : \mathcal{P}_J(X))(\forall x : X)\,\big(S(a, x) \Leftrightarrow x \in_J p\big)\,.$$

But the definition of $\mathcal{P}_J(X)$ as a quotient ensures that for a given $a : A$, there is a unique $p : \mathcal{P}_J(X)$ satisfying $S(a, x) \Leftrightarrow x \in_J p$ for all $x : X$. Hence, by functional completeness we obtain the existence of a map $\phi_S : A \to \mathcal{P}_J(X)$ such that

$$(\forall a : A)(\forall x : X)\, S(a, x) \Leftrightarrow x \in_J \phi_S(x)\,,$$

as required. $\square$

4. The associated sheaf functor theorem

4.1. **The associated sheaf functor.** We are now ready to define the associated sheaf functor $\mathbf{a} : \mathcal{E} \to \mathcal{E}_J$, the left adjoint to the inclusion $\mathbf{i} : \mathcal{E}_J \to \mathcal{E}$ of the full subcategory of sheaves into $\mathcal{E}$. Given $X \in \mathcal{E}$, define $\sigma : X \to \mathcal{P}_J(X)$ to be the composite

$$X \xrightarrow{\{\,\cdot\,\}} \mathcal{P}(X) \xrightarrow{[\,\cdot\,]} \mathcal{P}_J(X)\,.$$

To define the associated sheaf functor, first factor $\sigma : X \to \mathcal{P}_J(X)$ as an epimorphism $X \twoheadrightarrow X'$ followed by a monomorphism $X' \rightarrowtail \mathcal{P}_J(X)$. Then, define $\mathbf{a}(X)$ to be the closure of the subobject X' in $\mathcal{P}_J(X)$

$$\mathbf{a}(X) =_{\mathrm{def}} \mathbf{C}(X')\,.$$

The unit of the adjunction $\eta_X : X \to \mathbf{a}(X)$ is then defined as the composite of $X \twoheadrightarrow X'$ with the inclusion $X' \rightarrowtail \mathbf{C}(X')$. We need to show that $\mathbf{a}(X)$ is indeed a sheaf and that it satisfies the appropriate universal property. The proof of the former involves the verification that $\mathcal{P}_J(X)$ is a sheaf. This, in turn, requires further analysis of the notion of locally small map, which we carry out in Section 4.2 below.

4.2. **Characterisation of locally small maps.** We say that a map is *dense* if it factors as an epimorphism followed by a dense monomorphism. For monomorphisms this definition agrees with the definition of dense monomorphism given in Section 2.2. It is immediate to see that the pullback of a dense map is again dense, and a direct calculation shows that dense maps are closed under composition. It will be convenient to introduce some additional terminology: we refer to a commutative square of the form

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \arrow[d] \\ B \arrow[r] & A \end{tikzcd}$$

such that the canonical map $Y \to B \times_A X$ is dense as a *local quasi-pullback*. Note that every dense map $h : B \to A$ fits into a local quasi-pullback of the form

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, tail, "m"'] & A \arrow[d, "1_A"] \\ B' \arrow[r, two heads, "p"'] & A \end{tikzcd}$$

where $m : B \rightarrowtail B'$ and $p : B' \twoheadrightarrow A$ are respectively the dense monomorphism and the epimorphism forming the factorisation of h. The diagram is a local quasi-pullback because the map $B \to B' \times_A A$ is m itself. Let us also observe that if both squares in a diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[d] \\ C \arrow[r] & B \arrow[r] & A \end{tikzcd}$$

are local quasi-pullbacks, then the whole rectangle is also a local quasi-pullback. This implies that any finite pasting of local quasi-pullbacks is again a local quasi-pullback. We establish a very useful factorisation for dense mononomorphisms.

4.2.1. **Lemma.** *Every dense monomorphism $h : B \rightarrowtail A$ can be factored as*

$$\begin{tikzcd} B \arrow[rr, tail, "m"] \arrow[dr, tail, "h"'] & & B' \arrow[dl, two heads, "p"] \\ & A & \end{tikzcd}$$

where $m : B \rightarrowtail B'$ is a small dense monomorphism and $p : B' \twoheadrightarrow A$ is an epimorphism.

Proof. Let us define $B' =_{\mathrm{def}} \{(p, a) : \Omega \times A \mid p \Rightarrow B(a)\}$. By the definition of closure in (5) and the assumption that h is dense, the projection $\pi_2 : B' \to A$ is an epimorphism. Furthermore, there is a monomorphism $m : B \to B'$ defined by mapping $b : B$ into $(\top, b) : B'$. Diagrammatically, we have a pullback of the form

$$\begin{tikzcd} B \arrow[r] \arrow[d, tail, "m"'] & 1 \arrow[d, tail, "\top"] \\ B' \arrow[r, "\pi_1"'] & J \end{tikzcd}$$

The map $\top : 1 \to J$ is small and dense. It is small because it is the pullback of the map $\top : 1 \to \Omega$, which is small by the definition of Ω, along the inclusion $J \rightarrowtail \Omega$. It is dense by the very definition of closure in (5). Therefore, preservation of smallness and density along pullbacks implies that $m : B \to B'$ is small and dense, as required. $\square$

4.2.2. *Remark.* Lemma 4.2.1 exploits in a crucial way the fact that the closure operation, and hence the notion of density, are determined by a Lawvere-Tierney coverage. Indeed, it does not seem possible to prove an analogue of its statement for arbitary closure operations without assuming further axioms for small maps.

4.2.3. **Lemma.** *A map $f : X \to A$ is locally small if and only if it fits into a local quasi-pullback square of the form*

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r, two heads, "h"'] & A \end{tikzcd} \tag{13}$$

where $h : B \twoheadrightarrow A$ is an epimorphism and $g : Y \to B$ is small.

Proof. Assuming that $f : X \to A$ is locally small, the required diagram is already given by Definition 2.2.2. For the converse implication, consider a diagram as in (13). Consider the factorisation of the canonical map $Y \to B \times_A X$ as an epimorphism followed by a monomorphism, say $Y \twoheadrightarrow T \rightarrowtail$

$B \times_A X$. We have a diagram of form

$$\begin{tikzcd}
Y \arrow[r, two heads] \arrow[d, tail] & T \arrow[r] \arrow[d, tail] & X \arrow[d, tail] \\
B \times Y \arrow[r] \arrow[d, "\pi_B"'] & B \times X \arrow[r] \arrow[d, "\pi_B"] & A \times X \arrow[d, "\pi_A"] \\
B \arrow[r, "1_B"'] & B \arrow[r, two heads, "h"'] & A
\end{tikzcd}$$

By the Quotients Axiom for small maps $T \rightarrowtail B \times X \to B$ is a family of small subobjects. Furthermore, $T \rightarrowtail B \times_A X$ is dense by construction. □

Note that the map $k : Y \to X$ in Lemma 4.2.3 is also dense, since it is the composition of the pullback of an epimorphism with a dense map.

4.2.4. **Proposition.** *A map $f : X \to A$ is locally small if and only if it fits into a pullback diagram of the form*

$$\begin{tikzcd}
Y \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\
B \arrow[r, "h"'] & A
\end{tikzcd} \tag{14}$$

where $h : B \to A$ is dense and $g : Y \to B$ is locally small.

Proof. Assume to be given a diagram as (14). By the factorisation of dense maps as epimorphisms followed by dense monomorphisms and Lemma 4.2.3, it is sufficient to prove the statement when $h : B \to A$ is a dense monomorphism. We construct a diagram of the form

$$\begin{tikzcd}
B' \times_C Z \arrow[r] \arrow[d] \arrow[dr, phantom, "(5)"] & Z \arrow[r] \arrow[d] \arrow[dr, phantom, "(3)"] & Y \arrow[r, "k"] \arrow[d, "g"'] \arrow[dr, phantom, "(1)"] & X \arrow[d, "f"] \\
B' \arrow[r] \arrow[d, "n"'] \arrow[drr, phantom, "(4)"] & C \arrow[r, two heads] & B \arrow[r, "h"] \arrow[d, "m"'] \arrow[dr, phantom, "(2)"] & A \arrow[d, "1_A"] \\
A'' \arrow[rr, two heads, "p'"'] & & A' \arrow[r, two heads, "p"'] & A
\end{tikzcd}$$

The given diagram is in (1). First, factor $h : B \to A$ using Lemma 4.2.1 as a small dense monomorphism $m : B \rightarrowtail A'$ followed by an epimorphism $p : A' \to A$. The resulting commutative square in (2) is a local quasi-pullback since m is dense. Next, apply Lemma 4.2.3 to the locally small map $g : Y \to B$ so as to obtain the local quasi-pullback in (3) with $Z \to C$

a small map and $C \twoheadrightarrow B$ an epimorphism. Next, we apply the Collection Axiom to the small map $m : B \to A'$ and the epimorphism $C \to B$, so as to obtain the quasi-pullback in (4) where $n : B' \to A''$ is a small map and $p' : A'' \twoheadrightarrow A'$ is an epimorphism. Finally, we construct the pullback square in (5) to obtain another small map $B' \times_C Z \to B'$. The whole diagram is a local quasi-pullback, and we can apply Lemma 4.2.3 to deduce that $f : X \to A$ is locally small. The converse implication is immediate since locally small maps are stable under pullback. □

Corollary 4.2.5 shows that the assumption of being an epimorphism for the map $h : B \to A$ in Lemma 4.2.3 can be weakened.

4.2.5. **Corollary.** *If a map $f : X \to A$ fits into a local quasi-pullback diagram of the form*

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r, "h"'] & A \end{tikzcd} \tag{15}$$

where $g : Y \to B$ is small and $h : B \to A$ is dense, then $f : X \to A$ is locally small.

Proof. Given the diagram in (15), we construct the following one

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & B \times_A X \arrow[r] \arrow[d, "g'"] & X \arrow[d, "f"] \\ B \arrow[r, "1_B"'] & B \arrow[r, "h"'] & A \end{tikzcd}$$

The left-hand side square is a local quasi-pullback by assumption. So, by Lemma 4.2.3, the map $g' : B \times_A X \to B$ is locally small. The right-hand side square is a pullback by definition. So, by Proposition 4.2.4, the map $f : X \to A$ is locally small, as desired. □

4.3. **Proof of the associated sheaf functor theorem.** We exploit our characterisation of locally small maps in the proof of the following proposition.

4.3.1. **Proposition.** *For every object X of $\mathcal{E}$, $\mathcal{P}_J(X)$ is a sheaf.*

Proof. By Proposition 3.2.2, it suffices to show that every dense monomorphism $m : B \rightarrowtail A$ induces by pullback an isomorphism

$$m^* : \mathrm{Sub}_{\mathcal{S}_J}(X)(A) \to \mathrm{Sub}_{\mathcal{S}_J}(X)(B) .$$

Let us define a proposed inverse $m_\sharp$ as follows. For a family of locally small closed subobjects $T \rightarrowtail B \times X \to B$, we define

$$m_\sharp(T) = \mathbf{C}_{A \times X}(T)\,.$$

Here, we view T as a subobject of $A \times X$ via composition with the evident monomorphism $B \times X \rightarrowtail A \times X$. We need to show that the result is a locally small family, and that $m_\sharp$ and m^* are mutually inverse. First, note that m^* is just intersection with $B \times X$. Therefore, for a locally small closed family $T \rightarrowtail B \times X \to B$ we have

$$\begin{aligned} m^* m_\sharp(T) &= m^* \mathbf{C}_{A \times X}(T) \\ &= \mathbf{C}_{B \times X}(m^* T) \\ &= \mathbf{C}_{B \times X}(T) \\ &= T\,, \end{aligned}$$

where we used the naturality of the closure operation, that $T \leq A \times X$, and the assumption that T is closed. This also shows that $m_\sharp(T)$ becomes locally small over B when pulled back along the dense monomorphism $m : B \rightarrowtail A$. By Proposition 4.2.4, it is locally small over A, as required. Finally, for a locally small closed family $S \rightarrowtail A \times X \to A$, we have

$$\begin{aligned} m_\sharp m^*(S) &= m_\sharp(S \cap (B \times X)) \\ &= \mathbf{C}_{A \times X}(S \cap (B \times X)) \\ &= \mathbf{C}_{A \times X}(S) \cap \mathbf{C}_{A \times X}(B \times X) \\ &= S \cap (A \times X) \\ &= S\,, \end{aligned}$$

as desired. ☐

4.3.2. **Lemma.** *A subobject of a sheaf is a sheaf if and only if it is closed.*

Proof. Let $i : S \rightarrowtail X$ be a monomorphism, and assume that X is a sheaf. We begin by proving that if S is closed, then it is a sheaf. For this, assume given a dense monomorphism $m : B \rightarrowtail A$ and a map $f : B \to S$. Define $g : B \to X$ to be $i\,f : B \to X$. By the assumption that X is a sheaf, there exists a unique $\bar{g} : A \to X$ making the following diagram commute

$$\begin{tikzcd} B \arrow[r, "f"] \arrow[d, tail, "m"'] & S \arrow[r, tail, "i"] & X \\ A \arrow[urr, dotted, bend right, "\bar{g}"'] & & \end{tikzcd}$$

We have

$$\begin{array}{rcl} \mathrm{Im}(\bar{g}) & = & \bar{g}_!(A) \\ & = & \bar{g}_!(\mathbf{C}_A(B)) \\ & \leq & \mathbf{C}_X(\bar{g}_!(B)) \\ & = & \mathbf{C}_X(\mathrm{Im}(g)) \\ & \leq & \mathbf{C}_X(S) \\ & = & S\,, \end{array}$$

where the first inequality follows from adjointness. Therefore, $\bar{g} : A \to X$ factors through $\bar{f} : A \to S$, extending $f : B \to S$ as required. Uniqueness of this extension follows by the uniqueness of $\bar{g} : A \to X$ and the assumption that $i : S \to X$ is a monomorphism.

For the converse implication, we need to show that if S is a sheaf then it is closed. The monomorphism $m : S \rightarrowtail \mathbf{C}_X(S)$ is dense and therefore the identity $1_S : S \to S$ has an extension $n : \mathbf{C}_X(S) \to S$ with $n\,m = 1_S$. But by the preceding part, we know that $\mathbf{C}_X(S)$ is a sheaf and that $m\,n\,m = n = 1_{\mathbf{C}_X(S)} \cdot m$. Therefore, both $n \cdot m$ and $1_{\mathbf{C}(S)}$ are extensions of $m : S \rightarrowtail \mathbf{C}_X(S)$ to $\mathbf{C}_X(S)$. Since S is a sheaf, they must be equal. Thus m and n are mutually inverse, and so $S \cong \mathbf{C}_X(S)$ as desired. □

Lemma 4.3.1 and Lemma 4.3.2 imply that the object $\mathbf{a}(X)$ is a sheaf for every $X \in \mathcal{E}$, since $\mathbf{a}(X)$ is a closed subobject of the sheaf $\mathcal{P}_J(X)$. In order to show that we have indeed defined a left adjoint to the inclusion, we need the following Lemma 4.3.3, concerning the map $\eta_X : X \to \mathbf{a}(X)$, also defined in Section 4.1 using the map $\sigma_X : X \to \mathcal{P}_J(X)$ taking $x : X$ into $[\{x\}] : \mathcal{P}_J(X)$.

4.3.3. **Lemma.** *For every $X \in \mathcal{E}$, we have*

$$\mathrm{Ker}(\eta_X) = \mathrm{Ker}(\sigma_X) = \mathbf{C}_{X \times X}(\Delta_X)$$

in $\mathrm{Sub}(X \times X)$.

Proof. For $x, y : X$, we have

$$\begin{array}{rcl} \sigma(x) = \sigma(y) & \Leftrightarrow & \mathbf{C}(\{x\}) = \mathbf{C}(\{y\}) \\ & \Leftrightarrow & (\exists p : \Omega)\big(J(p) \wedge \big(p \Rightarrow x \in \{y\}\big)\big) \\ & \Leftrightarrow & (x, y) \in \mathbf{C}_{X \times X}(\Delta_X)\,, \end{array}$$

as required. □

Generally, a map $f : X \to Y$ with $\mathrm{Ker}(f) \leq \mathbf{C}(\Delta_X)$ is called *codense*.

4.3.4. **Theorem.** *For every $X \in \mathcal{E}$, $\mathbf{a}(X) \in \mathcal{E}_J$ is the associated sheaf of X, in the sense that for every map $f : X \to Y$ into a sheaf Y, there exists a unique $\bar{f} : \mathbf{a}(X) \to Y$ making the following diagram commute*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, bend right, "f"'] & \mathbf{a}(X) \arrow[d, "\bar{f}"] \\ & Y \end{tikzcd}$$

The resulting left adjoint $\mathbf{a} : \mathcal{E} \to \mathcal{E}_J$ preserves finite limits.

Proof. Given $f : X \to Y$, $\mathrm{Ker}(f)$ is a pullback of $Y \rightarrowtail Y \times Y$. Since $Y \times Y$ is a sheaf, Lemma 4.3.2 implies that Y is closed, and hence $\mathrm{Ker}(f)$ is closed in $X \times X$. But certainly $\Delta_X \leq \mathrm{Ker}(f)$ and therefore

$$\mathrm{Ker}(\sigma_X) = \mathbf{C}(\Delta_X) \leq \mathrm{Ker}(f)\,.$$

Since any epimorphism is the coequaliser of its kernel pair, f factors uniquely through the codense epimorphism $X \twoheadrightarrow \mathrm{Im}(\sigma_X)$. Since Y is a sheaf, the map from $\mathrm{Im}(\sigma_X)$ to Y extends uniquely along the dense monomorphism $\mathrm{Im}(\sigma_X) \rightarrowtail \mathbf{a}(X)$, giving a unique factorisation of f through η_X as desired.

To show that the associated sheaf functor preserves finite limits, we may proceed exactly as in [17, §V.3], since the argument there uses only the structure of a Heyting category on $\mathcal{E}$ and the fact that the associated sheaf functor is defined by embedding each object X by a codense map into a sheaf. In particular, injectivity of the sheaf in which X is embedded is not required to carry over the proof. □

Theorem 4.3.4 allows us to deduce Proposition 4.3.5, which contains the first part of Theorem 2.2.3. After stating it, we discuss in some detail the structure of the category $\mathcal{E}_J$.

4.3.5. **Proposition.** *The category $\mathcal{E}_J$ is a Heyting pretopos.*

Proof. This is an immediate consequence of Theorem 4.3.4. □

4.4. **The Heyting pretopos structure of sheaves.** We discuss the relationship between the structure of $\mathcal{E}$ and that of $\mathcal{E}_J$ is some detail. Here and subsequently, operations in $\mathcal{E}$, will be denoted without subscript, such as Im, while their counterparts in $\mathcal{E}_J$ will be denoted with subscript, such as Im_J. Limits in $\mathcal{E}_J$ are just limits in $\mathcal{E}$, since sheaves are closed under limits. Colimits are the sheafifications of colimits taken in $\mathcal{E}$, since the associated sheaf functor, being a left adjoint, preserves colimits. The lattice $\mathrm{Sub}_J(X)$ is the sub-lattice of closed elements of $\mathrm{Sub}(X)$. Here, the closure operation is a reflection and therefore meets in $\mathrm{Sub}_J(X)$ are meets in $\mathrm{Sub}(X)$, and joins in $\mathrm{Sub}_J(X)$ are closures of joins in $\mathrm{Sub}(X)$. Moreover, for $S, T \in \mathrm{Sub}(X)$, we have

$$\begin{array}{rcl} S \wedge \mathbf{C}(S \Rightarrow T) & \leq & \mathbf{C}(S) \wedge \mathbf{C}(S \Rightarrow T) \\ & = & \mathbf{C}(S \wedge (S \Rightarrow T)) \\ & \leq & \mathbf{C}(T) \\ & = & T\,. \end{array}$$

Therefore $\mathbf{C}(S \Rightarrow T) \leq (S \Rightarrow T)$ and so $S \Rightarrow T$ is closed. Thus, the implication of $\mathrm{Sub}(X)$ restricts so as to give implication in $\mathrm{Sub}_J(X)$. Since limits in $\mathcal{E}_J$ agree with limits in $\mathcal{E}$, the inverse image functors in $\mathcal{E}_J$ are just the restrictions of the inverse image functors of $\mathcal{E}$.

It is clear that the sheafification of a dense monomorphism $m : B \rightarrowtail A$ is an isomorphism, since for any sheaf X we have

$$\mathcal{E}_J\big(\mathbf{a}(A), X\big) \cong \mathcal{E}(A, X) \cong \mathcal{E}(B, X) \cong \mathcal{E}_J\big(\mathbf{a}(B), X\big)\,.$$

Therefore, epimorphisms in $\mathcal{E}_J$ are precisely the dense maps in $\mathcal{E}$, since coequalisers in $\mathcal{E}_J$ are sheafifications of coequalisers in $\mathcal{E}$, and dense maps are the sheafifications of epimorphisms of $\mathcal{E}$. We have seen before that dense maps are stable under pullback. Consequently, quasi-pullbacks in $\mathcal{E}_J$ are exactly those squares that are local quasi-pullbacks in $\mathcal{E}$.

If $(r_1, r_2) : R \rightarrowtail X \times X$ is an equivalence relation in $\mathcal{E}_J$, it is also an equivalence relation in $\mathcal{E}$, so has an effective quotient $q : X \twoheadrightarrow Q$ in $\mathcal{E}$. Its sheafification $\mathbf{a}(q) = \eta_Q \cdot q : X \to \mathbf{a}(Q)$ is then a coequaliser for (r_1, r_2) in sheaves; but this is an effective quotient for R, since

$$\mathrm{Ker}(\eta_Q \cdot q) = q^*\mathbf{C}(\Delta_Q) = \mathbf{C}(\mathrm{Ker}\, q) = \mathbf{C}(R)\,,$$

and R is closed in $X \times X$, as it is a subsheaf.

Images in $\mathcal{E}_J$ are the closures of images in $\mathcal{E}$. For $f : X \to Y$, $\mathrm{Im}_J(f) \rightarrowtail Y$ is the least subsheaf of Y throught which f factors, so the least closed subobject of Y containing $\mathrm{Im}(f)$, which can be readily identified with $\mathbf{C}_Y(\mathrm{Im}\, f)$. Consequently, the corresponding forward image functor

$$(\exists_J)_f : \mathrm{Sub}_J(X) \to \mathrm{Sub}_J(Y)$$

is given by

$$(\exists_J)_f(S) = \mathbf{C}_Y(\exists_f S)\,.$$

Dual image functors in $\mathcal{E}_J$ are again just the restrictions of the dual images functors of $\mathcal{E}$, since if $S \rightarrowtail X$ is closed so is $\forall_f(S) \rightarrowtail Y$, and therefore $\mathbf{C}(\forall_f S) \leq \forall_f(\mathbf{C}S)$. These give us an immediate translation from the internal logic of $\mathcal{E}_J$ into that of $\mathcal{E}$.

5. Small maps in sheaves

5.1. **Preliminaries.** Within this section, we study locally small maps between sheaves, completing the proof of Theorem 2.2.3. The associated sheaf functor followed by the inclusion of sheaves into $\mathcal{E}$ preserves dense maps. Indeed, avoiding explicit mention of the inclusion functor, if $h : B \to A$ is dense then we have

$$\mathrm{Im}(\eta_A\, h) = \mathrm{Im}(\mathbf{a}(h)\, \eta_B) \le \mathrm{Im}(\mathbf{a}(h))\,.$$

But $\eta_A\, h : B \to \mathbf{a}(A)$ is a dense map, and so its image is dense in $\mathbf{a}(A)$. Also, if X is a sheaf then $\eta_X : X \to \mathbf{a}(X)$ is easily seen to be an isomorphism.

5.1.1. **Lemma.** *For every $f : X \to Y$ in $\mathcal{E}$, the naturality square*

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "f"'] & \mathbf{a}(X) \arrow[d, "\mathbf{a}(f)"] \\ Y \arrow[r, "\eta_Y"'] & \mathbf{a}(Y) \end{tikzcd}$$

is a local quasi-pullback.

Proof. The canonical map $X \to \mathbf{a}X \times_{\mathbf{a}Y} Y$, which we wish to show to be dense, factors through $X \times_{\mathbf{a}Y} Y$ as in the diagram below:

$$\begin{tikzcd} X \arrow[dr] \arrow[drr, bend left] \arrow[drrr, bend left, "f"] \arrow[ddr, bend right, "1_X"'] & & & \\ & X \times_{\mathbf{a}(Y)} Y \arrow[r] \arrow[d] & \mathbf{a}(X) \times_{\mathbf{a}(Y)} Y \arrow[r] \arrow[d] & Y \arrow[d, "\eta_Y"] \\ & X \arrow[r, "\eta_X"'] & \mathbf{a}(X) \arrow[r, "\mathbf{a}(f)"'] & \mathbf{a}(Y) \end{tikzcd}$$

The map $X \times_{\mathbf{a}Y} Y \to \mathbf{a}X \times_{\mathbf{a}Y} Y$ is a pullback of η_X and therefore it is dense. We just need to show that $X \to X \times_{\mathbf{a}Y} Y$ is dense. For this, let us consider the following diagram, in which each square is a pullback

$$\begin{tikzcd}
X \arrow[r, "\cong"] \arrow[dr, bend right, "{(f,f)}"'] & \Gamma_f \arrow[r] \arrow[d] & X \times_{\mathbf{a}(Y)} Y \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& \Delta_Y \arrow[r] \arrow[dr, bend right, "\cong"'] & \mathbf{C}(\Delta_Y) \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & Y \arrow[d, "\eta_Y"] \\
& & Y \arrow[r, "\eta_Y"'] & \mathbf{a}(Y)
\end{tikzcd}$$

We then have the following chain of isomorphisms of subobjects of $X \times Y$

$$\begin{aligned}
X \times_{\mathbf{a}Y} Y &\cong X \times_Y \mathrm{Ker}(\eta_Y) \\
&\cong (f, 1_Y)^* \mathbf{C}_{Y \times Y}(\Delta_Y) \\
&\cong \mathbf{C}_{X \times Y}\big((f, 1_Y)^* \Delta_Y\big) \\
&\cong \mathbf{C}_{X \times Y}(\Gamma_f)\,.
\end{aligned}$$

Therefore, the map $X \to X \times_{\mathbf{a}Y} Y$ can be identified as the map $\Gamma_f \to \mathbf{C}(\Gamma_f)$, which is dense. □

5.1.2. **Corollary.**

(i) If $f : X \to Y$ is small, then $\mathbf{a}(f) : \mathbf{a}(X) \to \mathbf{a}(Y)$ is locally small.

(ii) If $f : X \to A$ is locally small, then there is some small map $g : Y \to B$ and dense $h : \mathbf{a}(B) \to A$ such that the following diagram is a pullback

$$\begin{tikzcd}
\mathbf{a}(Y) \arrow[r] \arrow[d, "\mathbf{a}(g)"'] & X \arrow[d, "f"] \\
\mathbf{a}(B) \arrow[r, "h"'] & A
\end{tikzcd}$$

Proof. Since the components of the unit of the adjunction are dense maps, part (i) follows from Corollary 4.2.5 and Lemma 5.1.1. For part (ii), use the definition of locally small map to get a dense map $h : B \to A$ and an indexed family of subobjects $Y \rightarrowtail B \times X \to B$, together with a diagram of form

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, tail] & X \arrow[d, tail] \\
B \times X \arrow[r] \arrow[d] & A \times X \arrow[d] \\
B \arrow[r, "h"'] & A
\end{tikzcd}$$

such that $Y \rightarrowtail B \times_A X$ is dense. Then, let $g : Y \to B$ be the evident map, which is small. The required square can be readily obtained by recalling that $\mathbf{a} : \mathcal{E} \to \mathcal{E}_J$ preserves pullbacks, sends dense monomorphisms to isomorphisms, and preserves dense maps. □

Corollary 5.1.2 allows us to regard the family of locally small maps as the smallest family of maps in $\mathcal{E}_J$ containing the sheafifications of the small maps in $\mathcal{E}$ and closed under descent along dense maps.

5.1.3. **Lemma.**

(i) Identity maps are locally small.
(ii) Composites of locally small maps are locally small.

Proof. All identities are trivially locally small. For composition, suppose $f : X \to A$ and $f' : X' \to X$ are locally small. We construct the diagram

$$\begin{tikzcd}
Y' \times_C Z \arrow[r] \arrow[d] \arrow[dr, phantom, "(5)"] & Z \arrow[r] \arrow[d, "h"] \arrow[dr, phantom, "(3)"] & Y \times_X X' \arrow[r] \arrow[d, "f''"] \arrow[dr, phantom, "(2)"] & X' \arrow[d, "f'"] \\
Y' \arrow[r] \arrow[d] & C \arrow[r, two heads] \arrow[d, phantom, "(4)"] & Y \arrow[r] \arrow[d, "g"] \arrow[dr, phantom, "(1)"] & X \arrow[d, "f"] \\
B' \arrow[rr, two heads] & & B \arrow[r, two heads, "p"'] & A
\end{tikzcd}$$

as follows. We begin by applying Lemma 4.2.3 to $f : X \to A$ so as to obtain the local quasi-pullback (1), where $g : Y \to B$ is a small map and $p : B \to A$ is an epimorphism. Then, we construct the pullback (2) and obtain the locally small map $f'' : Y \times_X X' \to Y$. We can apply Lemma 4.2.3 to it so as to obtain the local quasi-pullback in (3), where $h : Z \to C$ is a small map and $C \twoheadrightarrow Y$ is an epimorphism. Next, we apply the Collection Axiom to the small map $Y \to B$ and the epimorphism $C \twoheadrightarrow Y$, so as to obtain (4), where $Y' \to B'$ is a small map and $B' \twoheadrightarrow B$ is an epimorphism. Finally, (5) is a pullback and therefore $Y' \times_C Z \to Y'$ is small since $h : Z \to C$ is so. To conclude that the composite of $f' : X' \to X$ and $f : X \to A$ is locally small, it is sufficient to apply Lemma 4.2.3 to the whole diagram, which is a local quasi-pullback since it is obtained as the pasting of local quasi-pullbacks. □

5.1.4. **Lemma.** *For every commutative diagram of the form*

$$\begin{tikzcd}
X \arrow[rr, "d"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\
& A &
\end{tikzcd}$$

if $f : X \to A$ is locally small and $d : X \to X'$ is dense, then $f' : X' \to A$ is locally small.

Proof. By Lemma 4.2.3 we have a local quasi-pullback

$$\begin{tikzcd} Y \arrow[r, "k"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r, two heads, "h"'] & A \end{tikzcd}$$

where $g : Y \to B$ is a small map and $h : B \twoheadrightarrow A$ is an epimorphism. Since $f : X \to A$ is $f' \, d : X \to A$, we can expand the diagram above as follows

$$\begin{tikzcd} Y \arrow[dr] \arrow[drr, bend left, "k"] \arrow[ddr, bend right] \arrow[dddr, bend right=60, "g"'] & & \\ & B \times_A X \arrow[r] \arrow[d] & X \arrow[d, "d"] \\ & B \times_A X' \arrow[r] \arrow[d] & X' \arrow[d, "f'"] \\ & B \arrow[r, "h"'] & A \end{tikzcd}$$

The map $Y \to B \times_A X'$ is dense since it is the composite of the dense map $Y \to B \times_A X$ with the pullback of the dense map $d : X \to X'$. Hence, the commutative square

$$\begin{tikzcd} Y \arrow[r, "d\,k"] \arrow[d, "g"'] & X' \arrow[d, "f'"] \\ B \arrow[r, "h"'] & A \end{tikzcd}$$

is a local quasi-pullback, and so $f' : X' \to A$ is locally small. □

5.1.5. **Lemma.** *Let X, A, P be sheaves. For every locally small map $f : X \to A$ and every dense map $P \to X$, there exists a local quasi-pullback of the form*

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "g"'] & P \arrow[r] & X \arrow[d, "f"] \\ B \arrow[rr, "h"'] & & A \end{tikzcd}$$

where $g : Y \to B$ is locally small and $h : B \to A$ is dense.

Proof. Given such a pair, we construct the diagram

$$\begin{tikzcd}
 & & & & Y \times_X P \arrow[r] \arrow[dd, "e"] \arrow[dr, phantom, "(2)"] & P \arrow[dd, "d"] \\
Z' \times_C Z \arrow[r] \arrow[d] \arrow[dr, phantom, "(6)"] & Z \arrow[r] \arrow[d] \arrow[drr, phantom, "(4)"] & Y \times_X P \arrow[r] \arrow[urr, bend left, "1_{Y \times_X P}"] & Y' \arrow[d] \arrow[r, phantom, "(3)"] & & \\
Z' \arrow[r] \arrow[d] \arrow[drrrr, phantom, "(5)"] & C \arrow[rr] & & B' \arrow[r] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "(1)"] & X \arrow[d, "f"] \\
C' \arrow[rrrr] & & & & B \arrow[r] & A
\end{tikzcd}$$

as follows. The commutative square (1) is obtained by applying Lemma 4.2.3 to $f : X \to A$. In particular, it is a local quasi-pullback. Diagram (2) is a pullback. To construct (3), first we factor the dense map $e : Y \times_X P \to Y$ first as an epimorphism $Y \times_X P \to Y'$ followed by a dense monomorphism $Y' \to Y$, and then we factor the dense monomorphism as a small dense monomorphism $Y' \to B'$ followed by an epimorphism $B' \to Y$ by Lemma 4.2.1. We can then apply the Collection Axiom in $\mathcal{E}$ to construct a diagram in (4) which is a quasi-pullback. By the definition of quasi-pullback and the fact that $Y' \to B'$ is dense, it follows that $Z \to C$ is a dense map since it is the composition of a dense map with an epimorphism. We apply again the Collection Axiom to construct (5), and finally (6) is obtained by another pullback.

Since the pasting of (1) and (5) is a local quasi-pullback and the map $Z' \times_C Z \to Z'$ is dense, the resulting diagram

$$\begin{tikzcd}
Z' \times_C Z \arrow[r] \arrow[d] & P \arrow[r, "d"] & X \arrow[d, "f"] \\
C' \arrow[rr] & & A
\end{tikzcd}$$

is a local quasi-pullback as well. Note that some of the objects in the diagram above need not be sheaves, since they have been obtained by applying the Collection Axiom in $\mathcal{E}$. To complete the proof, it suffices to apply the associated sheaf functor, so as to obtain the diagram

$$\begin{tikzcd}
\mathbf{a}(Z' \times_C Z) \arrow[r] \arrow[d] & P \arrow[r, "d"] & X \arrow[d, "f"] \\
\mathbf{a}(C') \arrow[rr] & & A
\end{tikzcd}$$

This provides the required diagram, since the associated sheaf functor preserves dense maps and pullbacks and sends small maps into locally small maps. □

5.1.6. **Proposition.** *The family $\mathcal{S}_J$ of locally small maps in $\mathcal{E}_J$ satisfies the axioms for a family of small maps.*

Proof. Lemma 5.1.3 proves Axiom (A1). Axiom (A2), asserting stability under pullbacks, holds by the very definition of locally small map, as observed before Definition 2.2.2. Axiom (A3) follows by Proposition 4.2.4. Axioms (A4) and (A5) follow from the corresponding axioms in $\mathcal{E}$, using the fact that sheafification of small maps is locally small and the fact that the initial object and the coproducts of $\mathcal{E}_J$ are the sheafification of the initial object and of coproducts in $\mathcal{E}$, respectively. Lemma 5.1.4 proves Axiom (A6), and Lemma 5.1.5 proves Axiom (A7). Axiom (P1) follows by Proposition 3.2.2 and Lemma 4.3.2. □

5.1.7. *Remark.* The theorem asserting that the category internal sheaves over a Lawvere-Tierney local operator in an elementary topos is again an elementary topos [17, §V.2] can be seen as following from a special case of Proposition 4.3.5 and Proposition 5.1.6. Indeed, elementary toposes are examples of our setting, as explained in Example 1.5.1, and Lawvere-Tierney coverages are equivalent to Lawvere-Tierney local operators, as explained in Remark 2.1.2. By Proposition 4.3.5, we know that the category of sheaves is a Heyting pretopos, and by Proposition 5.1.6 we know that it has a subobject classifier.

5.1.8. **Proposition.** *If the family of small maps $\mathcal{S}$ in $\mathcal{E}$ satisfies the Exponentiability and Weak Representability Axiom, so does the family of locally small maps $\mathcal{S}_J$ in $\mathcal{E}_J$.*

Proof. Since we know by Proposition 5.1.6 that axiom (P1) holds for $\mathcal{S}_J$ and (P1) implies (S1), as recalled in Section 1.3, we have that (S1) holds for $\mathcal{S}_J$. For (S2), we need to show that there exists a locally small map $u_J : E_J \to U_J$ in $\mathcal{E}_J$ such that every locally small map $f : X \to A$ fits into a diagram of form

$$\begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l] \arrow[r] \arrow[d] & E_J \arrow[d, "u_J"] \\ A & B \arrow[l, two heads, "h"] \arrow[r] & U_J \end{tikzcd}$$

where the left-hand square is a quasi-pullback, the right-hand side square is a pullback and $h : B \twoheadrightarrow A$ is an epimorphism. Let $u : E \to U$ the weakly representing map for small maps in $\mathcal{E}$, which exists by our assumption that

$\mathcal{S}$ satisfies (S2). We define $u_J : E_J \to U_J$ to be its sheafification, $\mathbf{a}(u) : \mathbf{a}(E) \to \mathbf{a}(U)$. Given $f : X \to A$ locally small, let us consider a diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r, "h"'] & A \end{tikzcd} \tag{16}$$

as in part *(ii)* of Corollary 5.1.2. In particular, $g : Y \to B$ is small in $\mathcal{E}$. By the Weak Representability Axiom for small maps applied to $g : Y \to B$, we obtain a diagram of form

$$\begin{tikzcd} X \arrow[d, "f"'] & \mathbf{a}(Y) \arrow[l] \arrow[d] & \mathbf{a}(Z) \arrow[l] \arrow[r] \arrow[d] & \mathbf{a}(E) \arrow[d, "\mathbf{a}(u)"] \\ A & \mathbf{a}(B) \arrow[l] & \mathbf{a}(C) \arrow[l] \arrow[r] & \mathbf{a}(U) \end{tikzcd}$$

This is obtained by applying the associated sheaf functor to the diagram expressing that $g : Y \to B$ is weakly classified by $u : E \to U$ and to the one in (16), recalling that $X \cong \mathbf{a}(X)$ if X is a sheaf. The desired conclusion follows by recalling that the associated sheaf functor preserves dense maps and pullbacks. □

The combination of Proposition 4.3.5, Proposition 5.1.6, and Proposition 5.1.8 provides a proof of Theorem 2.2.3.

5.1.9. *Remark.* We may note that for a locally small sheaf X, witnessed by an epimorphism $h : B \twoheadrightarrow 1$ and an indexed family of small subobjects $S \rightarrowtail B \times X \to B$, fitting in a diagram

$$\begin{tikzcd} S \arrow[r, "k"] \arrow[d, tail] & X \arrow[d, tail] \\ B \times X \arrow[r] \arrow[d] & 1 \times X \arrow[d] \\ B \arrow[r, two heads, "h"'] & 1 \end{tikzcd}$$

with $S \rightarrowtail B \times X$ dense, an exponential of the form Y^X, where Y is a sheaf, can be constructed explicitly as the quotient of

$$\sum_{b:B} Y^{S_b}$$

by the equivalence relation defined by

$$(b, f) \sim (b', f') \Leftrightarrow (\forall s : S_b)(\forall s' : S_{b'}) \, [k(s) = k(s') \Rightarrow \; f(s) = f'(s')] \; .$$

5.1.10. *Remark.* Of course, if $\mathcal{E}$ has more structure than we have assumed so far, or satisfies stronger axioms, we would like these to be preserved by the construction of internal sheaves. For example, it is immediate to see that if $\mathcal{E}$ has small diagonals, so does $\mathcal{E}_J$, and that existence of a universal object in $\mathcal{E}$ in the sense of [24] implies the existence of a universal object in $\mathcal{E}_J$.

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