SUBGROUPS OF SMALL INDEX IN $Aut(F_n)$ AND KAZHDAN'S PROPERTY (T)

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ABSTRACT. We give a series of interesting subgroups of finite index in $Aut(F_n)$. One of them has index 42 in $Aut(F_3)$ and infinite abelianization. This implies that $Aut(F_3)$ does not have Kazhdan's property (T) (see [3] and [6] for another proofs). We proved also that every subgroup of finite index in $Aut(F_n)$, $n \geqslant 3$, which contains the subgroup of IA-automorphisms, has a finite abelianization.

1. DEFINITIONS, PROBLEMS AND MOTIVATIONS

The Kazhdan property (T) was introduced by D. Kazhdan in 1967 in his investigations on Lie groups.

Let H be a Hilbert space and $U(H)$ be the group of unitary transformations of H.

Definition 1.1. *A discrete group G has Kazhdan's property (T) if there exists an $\varepsilon > 0$ and a finite subset $S \subset G$ such that for all nontrivial irreducible representations $\rho : G \to U(H)$ and for all vectors $v \in H$ with $||v|| = 1$ we have $||\rho(s)v - v|| > \varepsilon$ for some $s \in S$.*

It is known, that in this case S generates G.

Definition 1.2. *A group G has Serre's property (FA) if acting simplicially and without inversions of edges on any simplicial tree, G has a global fixed point.*

Theorem 1.3. *(J.-P. Serre; 1974). A finitely generated group G has the property (FA) if and only if the following two statements hold:*

(1) *G is not a nontrivial amalgamated product, that is $G \ncong A *_B C$ with $B \neq A$ and $B \neq C$,*

(2) *G does not have a quotient which is isomorphic to $\mathbb{Z}$,*

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Theorem 1.4. *(Y. Watatani; 1982). Let G be a countable group. If G has the property (T) of Kazhdan, then it has the property (FA) of Serre.*

In the following table we summarize known facts on (T) and (FA) properties for $SL_n(\mathbb{Z})$ and $(S)Aut(F_n)$, $n \geqslant 3$.

	(T) $\Rightarrow$	(FA)
$SL_n(\mathbb{Z}), n \geqslant 3$	$+$ $(Kazhdan, 1967)$	$+$ $(Serre, 1974)$
$(S)Aut(F_3)$	$-$ $(McCool, 1989)$	$+$ $(Bogopolski, 1987)$
$(S)Aut(F_n), n \geqslant 4$	?	$+$ $(Bogopolski, 1987)$

Remark 1.5. *The property (T) is preserved under taking subgroups of finite index, the property (FA) is not preserved. Groups with the property (T) have no subgroups of finite index, which can be mapped onto $\mathbb{Z}$.*

Problems 1.6. 1) *Does every finite index subgroup of $SAut(F_n)$, $n \geqslant 4$, have the (FA) property?*

2) *Characterize (in terms of actions on trees or algebraically) those finitely generated groups, whose subgroups of finite index have the (FA) property.*

3) *Does every group $SAut(F_n)$, $n \geqslant 4$, have Kazhdan's property (T)?*

In [10, Problems 15 and 14], K. Vogtmann formulated the Out-versions of Problems 1) and 3).

Now we explain how the Kazhdan property (T) can be used to construct an infinite series of ε-expanders.

Definition 1.7. *A finite graph Γ is called an ε-expander, if for each subset $B \subset \Gamma^0$ with $|B| \leqslant |\Gamma^0|/2$ we have $\partial B \geqslant \varepsilon |B|$, where*

$$\partial B = \{v \in \Gamma^0 \,|\, v \notin B, \ \operatorname{dist}(v, B) = 1\}.$$

Definition 1.8. (*Graph $\Gamma_n(G)$*) *Fix a natural number n and a finite group G, which can be generated by n elements. The vertices of the graph $\Gamma_n(G)$ are all tuples $(g) = (g_1, \dots, g_n)$ such that $\langle g_1, \dots, g_n \rangle = G$. Two tuples (g) and (g') are connected by an edge if (g') can be obtained from (g) by applying one of the following replacement operations:*

$$R^{\pm}_{i,j} : (g_1, \dots, g_i, \dots, g_n) \to (g_1, \dots, g_i \cdot g_j^{\pm}, \dots, g_n),$$

$$L^{\pm}_{i,j} : (g_1, \dots, g_i, \dots, g_n) \to (g_1, \dots, g_j^{\pm} \cdot g_i, \dots, g_n).$$

Clearly the graph $\Gamma_n(G)$ is $4n(n-1)$-regular.

Theorem 1.9. *(A. Lubotzky, I. Pak; 2001). If $Aut(F_n)$ (or $SAut(F_n)$) has the property (T), then there exists an $\varepsilon > 0$, such that the graph $\Gamma_n(G)$ (if connected) is an ε-expander for any n-generated finite group G.*

However, the following two theorems state that $Aut(F_3)$ does not have the Kazhdan property (T). In this paper we suggest some approach towards the solving Problem 1.6.3) for rank $n \geqslant 4$.

Theorem 1.10. *(J. McCool; 1989) There is a subgroup of finite index in $Out(F_3)$, which can be approximated by torsion-free nilpotent groups. In particular, this subgroup can be mapped onto $\mathbb{Z}$. Therefore $Out(F_3)$ and $Aut(F_3)$ do not have the Kazhdan property (T).*

Theorem 1.11. [1] *(F. Grunewald and A. Lubotzky; 2006) There exists a subgroup of index 168 in $Aut(F_3)$ which can be mapped onto F_2. In particular, $Aut(F_3)$ has no Kazhdan's property (T).*

In the following section we give a sketch of the proof of this theorem. In Section 5 we use a part of this proof to construct some interesting subgroups in $Aut(F_3)$. One of them, $C(3)$ has index 42 in $Aut(F_3)$ and it has infinite abelianization. In Section 6 we consider some generalizations of these groups for ranks $n \geqslant 4$. We are especially interested in the subgroup $K(4)$ of $Aut(F_4)$. It has index 80640 and we conjecture that it has infinite abelianization. In Section 4 we explain, why we should avoid the so called IA-automorphisms in constructing subgroups of finite index in $Aut(F_n)$ with infinite abelianizations.

2. A sketch of the proof of F. Grunewald and A. Lubotzky that $Aut(F_3)$ has no Kazhdan's property (T)

Let $F_3(a, b, c)$ be the free group on free generators a, b, c. There exist exactly 7 nontrivial homomorphisms $F_3(a, b, c) \to \mathbb{Z}_2$. Therefore there exist exactly 7 subgroups of index 2 in F_3. Denote them by $F_5^{(1)}, \dots, F_5^{(7)}$. Clearly, every such subgroup has rank 5. We will work with one of them

[1] The proof of this theorem existed a long time in a folklore form, that is it was not published. We know this proof from private talks with A. Casson (2000) and M. Bridson (2004).

$F_5^{(1)} = \langle a, b, c^2, c^{-1}ac, c^{-1}bc\rangle$. It is easy to check that $Aut(F_3)$ acts transitively on the set of these subgroups. Therefore the index of $St(F_5^{(1)})$ in $Aut(F_3)$ is 7, where we use the notation

$$St(G) = \langle \alpha \in Aut(F_n) \mid \alpha(G) = G\rangle$$

for $G \leqslant F_n$. Now we introduce an important inner automorphism $\tau : x \mapsto c^{-1}xc$, $x \in F_3$. We use the following notation for the commutator: $[x, y] = x^{-1}y^{-1}xy$.

The following claim can be verified straightforward.
Claim.
1) $\tau|_{F_5^{(1)}} \notin Inn(F_5^{(1)})$,
2) $[\tau, \varphi]|_{F_5^{(1)}} \in Inn(F_5^{(1)})$ for every $\varphi \in St(F_5^{(1)})$.

Consider the canonical homomorphism $\Psi : St(F_5^{(1)}) \to GL_5(\mathbb{Z})$, which sends an automorphism of $F_5^{(1)}$ to the automorphism induced on the abelianization of $F_5^{(1)}$ (we identify the last automorphism with a matrix, using the prescribed basis of $F_5^{(1)}$). Note, that $Inn(F_5^{(1)})$ lies in the kernel of Ψ.

One can easily compute, that

$$\Psi(\tau) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Since $(\Psi(\tau))^2 = \mathrm{Id}$, we have $\mathbb{Z}^5 \supset V_+ \oplus V_-$, where

$$V_+ = \mathrm{Ker}(\Psi(\tau) + \mathrm{Id}), \qquad V_- = \mathrm{Ker}(\Psi(\tau) - \mathrm{Id}).$$

The $\mathbb{Z}$-module V_+ has the basis $\{(1, 0, 0, -1, 0)^T, (0, 1, 0, 0, -1)^T\}$. By the statement 2) of the above Claim, V_+ is $\Psi(\varphi)$-invariant for every $\varphi \in St(F_5^{(1)})$. Thus, there exists the natural homomorphism

$$\theta : St(F_5^{(1)}) \to GL(V_+) \cong GL_2(\mathbb{Z}).$$

One can easily verify that this homomorphism is onto.

Notice that $GL_2(\mathbb{Z}) \cong D_4 *_{D_2} D_6$, where D_m denotes the dihedral group of order $2m$. Therefore there exists an epimorphism $\mu : GL_2(\mathbb{Z}) \to D_{12}$. The kernel of μ is a free group of rank 2, denote it by F_2. Thus we have the following chain of embeddings and epimorphisms:

$$Aut(F_3) \geqslant St(F_5^{(1)}) \xrightarrow{\theta} GL_2(\mathbb{Z}) \xrightarrow{\mu} D_{12}. \tag{1}$$

Let $H_1 = \mathrm{Ker}(\theta\mu)$. Then H_1 has index 24 in $St(F_5^{(1)})$. Hence H_1 has index 168 in $Aut(F_3)$. Moreover, $\theta(H_1) = \mathrm{Ker}(\mu) = F_2$. In particular, H_1 can be homomorphically mapped onto $\mathbb{Z}$. Hence $Aut(F_3)$ does not have the Kazhdan property (T).

REMARK. The group H_1 is not normal in $Aut(F_3)$.

The above construction can not be generalized for $Aut(F_n)$ with $n \geqslant 4$, since in this case $GL_{n-1}(\mathbb{Z})$ (contrary to $GL_2(\mathbb{Z})$) does not contain a subgroup of finite index with infinite abelianization.

3. Some notations and useful automorphisms

Let F_n be the free group on free generators $x_1, x_2, \dots, x_n$. First we define some automorphisms of F_n. For easy we will write the image of x_i only if it differs from x_i. Recall that the commutator of two elements x, y is defined to be $[x, y] = x^{-1}y^{-1}xy$.

1) For any $i, j, k \in \{1, 2, \dots, n\}$, where $i \neq k$ and $j \neq k$, we define the automorphsm

$$\alpha_{ijk} : x_i \to x_i[x_j, x_k].$$

In particular,

$$\alpha_{iik} : x_i \to x_k^{-1}x_i, x_k.$$

Note that $\alpha_{ijk} = \alpha_{ikj}^{-1}$ for distinct i, j, k.

2) For any $i, j \in \{1, 2, \dots, n\}$, where $i \neq j$, we define the automorphism

$$E_{ij} : x_i \to x_i x_j.$$

3) For any $i \in \{1, 2, \dots, n\}$ we define the automorphism

$$n_i : x_i \to x_i^{-1}.$$

The kernel of the canonical epimorphism $Aut(F_n) \to GL_n(\mathbb{Z})$ is denoted by $IA(F_n)$. It is known, that $IA(F_2) = Inn(F_2)$ and $IA(F_n)$ is strictly larger than $Inn(F_n)$ for $n \geqslant 3$. N. Nielsen (for $n = 3$) and W. Magnus (for $n \geqslant 3$) proved that $IA(F_n)$ is generated by all automorphisms α_{ijk}.

4. Finite index subgroups of $Aut(F_n)$ containing $IA(F_n)$

Theorem 4.1. *Let $n \geqslant 3$. Any subgroup of finite index in $Aut(F_n)$, containing $IA(F_n)$, has a finite abelianization.*

To prove this theorem we need to introduce more automorphisms of $Aut(F_n)$ and to formulate a theorem of B. Sury and T.N. Venkataramana on generators of congruence subgroups of $SL_n(\mathbb{Z})$.

4) For any $i \in \{1, 2, \dots, n\}$ we define the automorphism

$$T_i : x_i \mapsto x_i^{-1},\ x_{i+1} \mapsto x_{i+1}^{-1} x_i.$$

5) For any $i, j \in \{1, 2, \dots, n\}$, where $i \neq j$, we define the automorphism

$$T_{ij} : x_i \mapsto x_j,\ x_j \mapsto x_i^{-1}.$$

Denote

$$T = \{T_k \,|\, k = 1, \dots, n-1\} \cup \{T_{ij} \,|\, i, j = 1, \dots, n;\ i \neq j\} \cup \{\mathrm{Id}\}.$$

Let $^{-} : Aut(F_n) \to GL_n(\mathbb{Z})$ be the canonical epimorphism. For any $\alpha \in Aut(F_n)$ we denote by $\overline{\alpha}$ its canonical image in $GL_n(\mathbb{Z})$.

Let m be a natural number. The kernel of the canonical epimorphism $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}_m)$ is denoted by $SL_n(\mathbb{Z}, m\mathbb{Z})$ and is called the congruence subgroup of $SL_n(\mathbb{Z})$ modulo m. Of course, $SL_n(\mathbb{Z}, m\mathbb{Z})$ is normal and has a finite index in $SL_n(\mathbb{Z})$. The following famous theorem is called the congruence subgroup theorem. It was proved by H. Bass, M. Lazard, and J.-P. Serre in [1] and by J. Mennicke in [7]

Theorem 4.2. *Let $n \geqslant 3$ be a natural number. Any subgroup of finite index in $SL_n(\mathbb{Z})$ contains a congruence subgroup $SL_n(\mathbb{Z}, m\mathbb{Z})$ for some m.*

Theorem 4.3. *(B. Sury and T.N. Venkataramana; 1994) Let $n \geqslant 3$, $m \geqslant 2$. The congruence subgroup $SL_n(\mathbb{Z}, m\mathbb{Z})$ is generated by the following set of matrices*

$$\{(\overline{\alpha})(\overline{E}_{ij})^m(\overline{\alpha})^{-1} \,|\, \alpha \in T;\ i, j \in \{1, \dots, n\};\ i \neq j\}.$$

Proof of Theorem 4.1. Let G be a subgroup of finite index in $Aut(F_n)$ containing $IA(F_n)$. Let $\overline{G}$ be the image of G is $GL_n(\mathbb{Z})$. By the congruence subgroup theorem, there exists an $m \geqslant 2$, such that $SL_n(\mathbb{Z}, m\mathbb{Z}) \leqslant \overline{G}$. Let H be the preimage of $SL_n(\mathbb{Z}, m\mathbb{Z})$ in G. Since H has a finite index in G, it is sufficient to show that H/H' is finite. Since H contains $IA(F_n)$, the results of Nielsen – Magnus and Sury – Venkataramana imply that H is generated by the union of two sets:

$$\{\alpha_{ijk} \,|\, i, j, k \in \{1, \dots, n\};\ i, j \neq k\},$$
$$\{\alpha E_{ij}^m \alpha^{-1} \,|\, \alpha \in T;\ i, j \in \{1, \dots, n\};\ i \neq j\}.$$

It is sufficient to prove that the m-th power of each of these generators lies in $[H, H]$. But this follows from the next formulas (they can be checked using the formulas in Appendix).

1) $[\alpha_{iij}, E_{jk}^m] = \alpha_{iik}^m$ for distinct $i, j, k \in \{1, \dots, n\}$;

2) $[\alpha_{iij}, E_{ik}^m] = (\alpha_{ikj}\alpha_{jjk}^{-1})^{m-1}\alpha_{ikj}\alpha_{jjk}^{m-1} \equiv \alpha_{ikj}^m \pmod{[IA(F_n), IA(F_n)]}$ for distinct $i, j, k \in \{1, \dots, n\}$;

3) $E_{ik}^{m^2} = \left(\prod_{s=0}^{m-2}[\alpha_{iij}^s, E_{ik}^{m^2-(s+1)m}][E_{ik}^{m^2-(s+1)m}, \alpha_{iij}^{s+1}]\right)[E_{ij}^m, E_{jk}^m]$. □

Definition 4.4. *Consider the chain of canonical epimorphisms*

$$SAut(F_n) \to SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}_2).$$

Let $H(n)$ be the kernel of the composition of these epimorphisms. This is an analogue of the congruence subgroup of $SL_n(\mathbb{Z})$ modulo 2.

Proposition 4.5. 1) *$H(n)$ is a normal subgroup in $SAut(F_n)$ and even in $Aut(F_n)$.*

2) *$H(n)$ contains $IA(F_n)$. In particular, $H(n)$ has a finite abelianization.*

3) *$H(n)$ lies in $St(N)$ for every subgroup N of index 2 in F_n.*

Our aim is to find a subgroup $K(n)$ of index 2 in $H(n)$, which has an infinite abelianization. By Theorem 4.1, $K(n)$ can not contain $IA(F_n)$.

5. New subgroups of small finite index in $Aut(F_3)$ with infinite abelianization

Note that $H(3)$ has index $2 \cdot 168$ in $Aut(F_3)$ and a finite abelianization. In this section we will define subgroups $K(3), A(3), B(3)$, and $C(3)$ of $Aut(F_3)$, which have indexes $4 \cdot 168$, 168, 84, and 42, and infinite abelianization.

Definition 5.1. *By Proposition 4.5.3), the group $H(3)$ lies in $St(F_5^{(1)})$. Using the chain of homomorphisms (1), one can verify that $H(3)\,\theta\mu \cong D_6$. Let $K(3)$ be the subgroup of index 2 in $H(3)$, such that $K(3)\,\theta\mu \cong D_3$.*

Using the Reidemeister–Schreier method and GAP, we proved the following theorem.

Theorem 5.2. *The group $K(3)$ is generated by the following 16 automorphisms:*

(1) $\alpha_{112}, \alpha_{221}, \alpha_{331}, \alpha_{332}, \alpha_{123}, \alpha_{213}, \alpha_{312}$,

(2) $E_{12}^2, E_{13}^2, E_{21}^2, E_{23}^2, E_{31}^2, E_{32}^2$,

(3) $n_1 n_2$, $\alpha_{113}\alpha_{223}^{-1}$, $\alpha_{113} n_2 n_3$.

Theorem 5.3. 1) *$K(3)$ has index $4 \cdot 168$ in $Aut(F_3)$.*

2) *$K(3) \cap IA(F_3)$ has index 2 in $IA(F_3)$.*

3) *$K(3)/K(3)' \cong \mathbb{Z}_2^{14} \times \mathbb{Z} \times \mathbb{Z}$.*

Theorem 5.4. *The* 16 *automorphisms from Theorem* 5.2 *form a minimal generating set of* $K(3)$. *All of them, except of* $\alpha_{123}, \alpha_{213}, E_{12}^2, E_{21}^2$ *have finite order in* $K(3)/K(3)'$. *These four automorphisms have infinite order in* $K(3)/K(3)'$.

Moreover, $E_{12}^{-4} \equiv \alpha_{123}^2 (\bmod K(3)')$ *and* $E_{21}^{-4} \equiv \alpha_{213}^2 (\bmod K(3)')$. *In particular, the image of the group* $\langle E_{12}^2, E_{21}^2 \rangle$ *in* $K(3)/K(3)'$ *is isomorphic to* $\mathbb{Z} \times \mathbb{Z}$. □

Still, $K(3)$ has the large index in $Aut(F_3)$. We will enlarge $K(3)$ by adding special generators. In this way we construct the following chain of subgroups:

$$Aut(F_3) \geqslant C(3) \geqslant B(3) \geqslant A(3) \geqslant K(3),$$

where

$$\begin{aligned} A(3) &= \langle K(3), E_{31}, E_{32} \rangle, \\ B(3) &= \langle K(3), E_{31}, E_{32}, E_{21} \rangle, \\ C(3) &= \langle K(3), E_{31}, E_{32}, E_{21}, n_3 \rangle. \end{aligned}$$

Theorem 5.5. 1) $A(3)$ *has index* 168 *in* $Aut(F_3)$.

2) $A(3)/A(3)' \cong \mathbb{Z}_2^7 \times \mathbb{Z} \times \mathbb{Z}$.

3) $A(3)$ *has the following minimal set of generators:*

$$\alpha_{112}, \alpha_{221}, \alpha_{123}, \alpha_{213}, E_{13}^2, E_{23}^2, E_{31}, E_{32}, n_1 n_2.$$

Theorem 5.6. 1) $B(3)$ *has index* 84 *in* $Aut(F_3)$.

2) $B(3)/B(3)' \cong \mathbb{Z}_2^4 \times \mathbb{Z}$.

Remark: we do not know a minimal generating set of $B(3)$.

Theorem 5.7. 1) $C(3)$ *has index* 42 *in* $Aut(F_3)$.

2) $C(3)/C(3)' \cong \mathbb{Z}_2^3 \times \mathbb{Z}_4 \times \mathbb{Z}$.

3) $C(3)$ *has the following minimal set of generators:*

$$\alpha_{123}, E_{13}^2, E_{21}, E_{32}, n_3.$$

Corollary 5.8. *The image of* $C(3)$ *in* $Out(F_3)$ *has index* 21 *and it can be mapped onto* $\mathbb{Z}$.

Conjecture 5.9. *No one subgroup of* $Aut(F_3)$ *of index smaller than* 42 *can be mapped onto* $\mathbb{Z}$. *No one subgroup of* $Out(F_3)$ *of index smaller than* 21 *can be mapped onto* $\mathbb{Z}$.

6. A conjecture for $Aut(F_n)$, $n \geqslant 4$

Recall that we want to define a subgroup $K(n)$ of index 2 in $H(n)$ with infinite abelianization. For $n = 3$ it was done in Definition 5.1 with the help of the formula (1). But for $n \geqslant 4$ an analogue of this formula does not exist. Therefore we will define $K(n)$ for $n \geqslant 3$ in another way, which (magically) gives us the same group for $n = 3$.

Let $F^{(1)}_{2n-1}$ be a subgroup of index 2 in $F_n = F(x_1, \dots, x_n)$. Consider its stabilizer and the natural homomorphism:

$$Aut(F_n) \geqslant St(F^{(1)}_{2n-1}) \xrightarrow{\Psi} GL_{2n-1}(\mathbb{Z}).$$

Notice, that $H(n) \leqslant St(F^{(1)}_{2n-1})$ and $\alpha_{11n} \in H(n)$. One can compute, that $\det(\Psi(\alpha_{11n})) = -1$. Therefore the full preimage (respective to Ψ) of $SL_{2n-1}(\mathbb{Z})$ in $H(n)$ has index 2 in $H(n)$. Denote this preimage by $K(n)$.

Claim. $K(3)$ as defined in this section coincides with $K(3)$ as defined in Section 5.

Theorem 6.1. *The subgroup $K(4)$ has index $2 \cdot |SL_4(2)| = 80640$ in $Aut(F_4)$ and is generated by the following 38 automorphisms:*

(1) α_{ijk}, *where $i, j, k \in \{1, 2, 3, 4\}$ are all distinct and $j < k$,*
(12 elements)

(2) α_{iij}, *where $i, j \in \{1, 2, 3, 4\}$ are distinct,*
and the automorphisms $\alpha_{114}, \alpha_{224}, \alpha_{334}$ must be excluded,
(9 elements)

(3) E^2_{ij}, *where $i, j \in \{1, 2, 3, 4\}$ are distinct,*
(12 elements)

(4) *exceptional automorphisms*
$\alpha_{224}\alpha_{114}$, $\alpha_{334}\alpha_{114}$, $\alpha_{114}n_1n_4$, n_1n_2, n_1n_3.
(5 elements)

At present we can not verify, whether the group $K(4)$ has infinite abelianization. The obstacle is that the index of $K(4)$ in $Aut(F_4)$ is too large. Let us try to enlarge $K(4)$ (that will decrease the index) by adding special generators. In this way we construct the following chain of subgroups:

$$Aut(F_4) \geqslant D(4) \geqslant C(4) \geqslant B(4) \geqslant A(4) \geqslant K(4),$$

where

$$\begin{aligned} A(4) = \quad & \langle K(4), E_{41}, E_{42}, E_{43}\rangle, \\ B(4) = \quad & \langle K(4), E_{41}, E_{42}, E_{43}, E_{31}, E_{32}\rangle, \\ C(4) = \quad & \langle K(4), E_{41}, E_{42}, E_{43}, E_{31}, E_{32}, E_{21}\rangle. \\ D(4) = \quad & \langle K(4), E_{41}, E_{42}, E_{43}, E_{31}, E_{32}, E_{21}, n_4\rangle. \end{aligned}$$

Theorem 6.2. *The indexes of the subgroups $A(4)$, $B(4)$, $C(4)$, $D(4)$ in $Aut(F_4)$ are equal to $10080, 2520, 1260$ and 630 respectively. These subgroups have finite abelianizations.*

Conjecture 6.3. *$K(n)/K(n)'$ is infinite for $n \geqslant 4$.*

If this conjecture is true, then $Aut(F_n)$ has no Kazhdan property (T) for $n \geqslant 4$.

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Appendix

$$
\begin{aligned}
E_{12}\alpha_{112}E_{12}^{-1} &= \alpha_{112},\\
E_{12}\alpha_{113}E_{12}^{-1} &= \alpha_{132}\alpha_{113},\\
E_{12}\alpha_{221}E_{12}^{-1} &= \alpha_{221}\alpha_{112}^{-1},\\
E_{12}\alpha_{223}E_{12}^{-1} &= \alpha_{223}\alpha_{123},\\
E_{12}\alpha_{331}E_{12}^{-1} &= \alpha_{332}^{-1}\alpha_{331},\\
E_{12}\alpha_{332}E_{12}^{-1} &= \alpha_{332},\\
E_{12}\alpha_{123}E_{12}^{-1} &= \alpha_{332}^{-1}\alpha_{123}\alpha_{332},\\
E_{12}\alpha_{213}E_{12}^{-1} &= \alpha_{113}\alpha_{213}\alpha_{113}^{-1}\alpha_{223}^{-1}\alpha_{221}\alpha_{332}\alpha_{123}^{-1}\alpha_{332}^{-1}\alpha_{112}^{-1}\alpha_{113}\alpha_{112}\alpha_{221}^{-1}\\
&\equiv \alpha_{213}\alpha_{223}^{-1}\alpha_{123}^{-1}\alpha_{113} \,(\bmod\, (IA(F_n)')),\\
E_{12}\alpha_{312}E_{12}^{-1} &= \alpha_{312}\alpha_{112}^{-1},
\end{aligned}
$$

$$
\begin{aligned}
E_{12}n_{12}E_{12}^{-1} &= \alpha_{112}n_{12},\\
E_{13}n_{12}E_{13}^{-1} &= E_{13}^{2}\alpha_{113}^{-1}n_{12},\\
E_{21}n_{12}E_{21}^{-1} &= \alpha_{221}n_{12},\\
E_{23}n_{12}E_{23}^{-1} &= E_{23}^{2}\alpha_{223}^{-1}n_{12},\\
E_{31}n_{12}E_{31}^{-1} &= E_{31}^{2}n_{12},\\
E_{32}n_{12}E_{32}^{-1} &= E_{32}^{2}n_{12},
\end{aligned}
$$

$$
\begin{aligned}
n_{12}\alpha_{112}n_{12}^{-1} &= \alpha_{112}^{-1},\\
n_{12}\alpha_{113}n_{12}^{-1} &= \alpha_{113},\\
n_{12}\alpha_{221}n_{12}^{-1} &= \alpha_{221}^{-1},\\
n_{12}\alpha_{223}n_{12}^{-1} &= \alpha_{223},\\
n_{12}\alpha_{331}n_{12}^{-1} &= \alpha_{331}^{-1},\\
n_{12}\alpha_{332}n_{12}^{-1} &= \alpha_{332}^{-1},\\
n_{12}\alpha_{123}n_{12}^{-1} &= \alpha_{123}^{-1}\alpha_{113}\alpha_{112}^{-1}\alpha_{113}^{-1}\alpha_{112}\alpha_{332}\alpha_{123}\alpha_{332}^{-1}\alpha_{123}\\
&\equiv \alpha_{123} \,(\bmod\, (IA(F_n))'),\\
n_{12}\alpha_{213}n_{12}^{-1} &= \alpha_{213}^{-1}\alpha_{223}\alpha_{221}^{-1}\alpha_{223}^{-1}\alpha_{221}\alpha_{331}\alpha_{213}\alpha_{331}^{-1}\alpha_{213}\\
&\equiv \alpha_{213} \,(\bmod\, (IA(F_n))'),\\
n_{12}\alpha_{312}n_{12}^{-1} &= \alpha_{332}^{-1}\alpha_{331}^{-1}\alpha_{332}\alpha_{331}\alpha_{312}^{-1}\alpha_{331}^{-1}\alpha_{332}^{-1}\alpha_{331}\alpha_{332}\alpha_{221}\alpha_{112}\alpha_{312}^{2}\\
&\equiv \alpha_{221}\alpha_{112}\alpha_{312} \,(\bmod\, (IA(F_n))').
\end{aligned}
$$