

DISCRIMINATING GROUPS: A COMPREHENSIVE OVERVIEW

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ABSTRACT. Discriminating groups were introduced by G.Baumslag, A.Myasnikov and V.Remeslennikov as an outgrowth of their theory of algebraic geometry over groups. However they have taken on a life of their own and have been an object of a considerable amount of study. In this paper we survey the large array results concerning the class of discriminating groups that have been developed over the past decade.

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1. DISCRIMINATION AND DISCRIMINATING GROUPS

If G and H are groups then G **separates** H provided that to every nontrivial element h of H there is a homomorphism $\varphi_h : H \rightarrow G$ such that $\varphi_h(h) \neq 1$. G **discriminates** H if to every finite nonempty set S of nontrivial elements of H there is a homomorphism $\varphi_S : H \rightarrow G$ such that

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$\varphi_S(s) \neq 1$ for all $s \in S$. The group G is **discriminating** provided that it discriminates every group it separates.

Discriminating groups were invented by Baumslag, Myasnikov and Remeslennikov in [BMR 2] as a spin-off of their theory of algebraic geometry over groups [BMR 1]. Algebraic geometry over groups was created as a tool to attack the celebrated Tarski conjectures on the elementary theory of free groups. By analogy with classical algebraic geometry we may view the discrimination of H by G as an approximation to H much like the localization of a ring at a prime. (Think of a set of generators for H as a set of variables.)

The following is the main criterion for determining whether a group is discriminating.

Lemma 1.1. ([BMR 2]) *A group G is discriminating if and only if G discriminates $G \times G$.*

Since this is fundamental we will give the proof which is very easy:

Since G clearly separates $G \times G$ if it is discriminating it will discriminate $G \times G$.

Conversely suppose G discriminates $G \times G$. By induction then G will discriminate G^n . Suppose that G separates a group H ; we want to show that it discriminates H . Let h_1, \dots, h_n be nontrivial elements in H . Then for each there is a homomorphism $\phi_i : H \rightarrow G$ such that $\phi_i(h_i) \neq 1$. Taking $\phi = \phi_1 \times \dots \times \phi_n$ and using the assumption that G discriminates G^n gives us that G discriminates H .

It is clear from the Lemma that if $G \times G$ embeds in G , then G is discriminating. If this is the case we say that G is **trivially discriminating** which we abbreviate as **TD**. In [FGMS 1] the question was asked if there are finitely generated non-trivially discriminating groups. We will give some examples of these later in the survey.

1.1. Some Ties to Logic. We now present some ties between discriminating groups and logic.

A **universal sentence** is one of the form $\forall \mathbf{x} \varphi(\mathbf{x})$ where \mathbf{x} is a tuple of distinct variables and $\varphi(\mathbf{x})$ is a quantifier-free formula containing at most the variables in \mathbf{x} . Similarly, an existential sentence of L is one of the form $\exists \mathbf{x} \varphi(\mathbf{x})$ where \mathbf{x} and $\varphi(\mathbf{x})$ are as before. Vacuous quantifications are permitted and a quantifier-free sentence of L is considered a special case both of universal sentences and existential sentences. Observe that the negation of a universal sentence is logically equivalent to an existential sentence and vice-versa. If G is a group, then the **universal theory** of G , written $\text{Th}_{\forall}(G)$, is the set of all universal sentences of L true in G . Two groups G and H are **universally equivalent** or **have the same universal theory**,

written $G \equiv_{\forall} H$, provided $\text{Th}_{\forall}(G) = \text{Th}_{\forall}(H)$. Observe that two groups satisfy precisely the same universal sentences if and only if they satisfy precisely the same existential sentences. Notice that it is easy to prove that all finitely generated free groups have the same universal theory.

Theorem 1.1. (*[FGMS 1]*): *If G is discriminating, then $G \times G \equiv_{\forall} G$.*

In order to better capture the axiomatic properties of discriminating groups we define.

Definition 1.1. (*[FGMS 2]*): *The group G is **squarelike** provided $G \times G \equiv_{\forall} G$.*

Hence, every discriminating group is squarelike (but, as we shall see, **not** conversely).

1.1 Some Ties to the Algebraic Geometry of Groups. As mentioned discriminating groups arose out of the study of the algebraic geometry over groups. Here we present some of the ties.

A first order expression

$$\forall \mathbf{x} \wedge_i (w_i(\mathbf{x}) = 1) \rightarrow (u(\mathbf{x}) = \mathbf{1})$$

where $u(\mathbf{x})$ and the $w_i(\mathbf{x})$ are words on the variables and their formal inverses is called a **quasi-identity**

Observe that every identity $\forall \mathbf{x}(A(\mathbf{x}) = \mathbf{a}(\mathbf{x}))$ is equivalent to a quasi-identity $\forall \mathbf{x}((1 = 1) \rightarrow (A(\mathbf{x}) = \mathbf{a}(\mathbf{x})))$. So laws and the group axioms themselves may be viewed as instances of quasi-identities.

Definition 1.2. *The model class of a set of quasi-identities including the group axioms is a **quasivariety** of groups. If G is a group the quasivariety generated by G is denoted $\mathbf{qvar}(G)$*

Quasivarieties are examples of **axiomatic classes**. That is, they are model classes of sets of sentences of L , the language of group theory. Quasivarieties are closed under (unrestricted) direct products. They are also closed under subgroups since they have a set of universal axioms. Put another way, quasivarieties are axiomatic **prevarieties**. The class of all groups is a quasivariety of groups and the intersection of any family of quasivarieties is again a quasivariety. For that matter, the intersection of any family of universally axiomatizable model classes (i.e. axiomatic classes having a set of universal axioms) is again a universally axiomatizable model class. It follows that if \mathcal{X} is any class of groups there is a least quasivariety, denoted $\mathbf{qvar}(\mathcal{X})$, containing \mathcal{X} and a least universally axiomatizable model class, denoted $\mathbf{ucl}(\mathcal{X})$, containing \mathcal{X} . We call $\mathbf{qvar}(\mathcal{X})$ the quasivariety generated by \mathcal{X} and we call $\mathbf{ucl}(\mathcal{X})$ the universal closure of \mathcal{X} . If $\mathcal{X} = \{G\}$ is a singleton

we write $\text{qvar}(G)$ and $\text{ucl}(G)$ for $\text{qvar}(\mathcal{X})$ and $\text{ucl}(\mathcal{X})$ respectively. $\text{qvar}(G)$ is the model class of the quasi-identities satisfied by G and $\text{ucl}(G)$ is the model class of the universal sentences satisfied by G . In general $\text{ucl}(G)$ is a proper subclass of $\text{qvar}(G)$. (The more properties a structure is required to satisfy the fewer structures can satisfy them in general.)

One final concept and then the result.

Definition 1.4. *The group G is **equationally Noetherian** provided for every natural number $n \in \mathbb{N}$ and every system $S = 1$ of equations in n unknowns there is a finite subset $S_0 \subseteq S$ such that $V_G(S) = V_G(S_0)$.*

Examples of equationally Noetherian groups are groups linear over a commutative Noetherian ring with 1. In particular, every group linear over a field is equationally Noetherian.

Tying all of these together is:

Theorem 1.3 (FGMS 1). *Let G be a finitely generated equationally Noetherian group. Then G is discriminating if and only if $\text{qvar}(G) = \text{ucl}(G)$.*

1.3 Abelian Discriminating Groups. In the original paper by Baumslag, Myasnikov and Remeslennikov [BMR 2] the question was raised as to whether the abelian discriminating groups can be characterized. In [FGMS 1] this was extended to ask whether the abelian squarelike groups could be characterized. If there is any torsion this can be answered immediately.

Lemma 1.3.1. *Suppose the set of nontrivial elements of finite order in the group G is finite and nonempty. Then G is not squarelike. Hence, G is not discriminating.*

We immediately deduce that no nontrivial finitely generated nilpotent group with torsion can be squarelike. In particular, no nontrivial finitely generated abelian group with torsion and no nontrivial finite group can be squarelike. In the torsion-free case we always get discrimination.

Lemma 1.3.2. *Every torsion free abelian group is discriminating. Hence, every torsion free abelian group is squarelike.*

So, among finitely generated abelian groups, the discriminating groups and the squarelike groups are precisely the torsion-free ones. A moment's reflection produces nontrivial discriminating abelian groups (necessarily not finitely generated) with torsion. For example, every group free of infinite rank in the variety of abelian groups of exponent dividing n for any fixed integer $n > 1$ is trivially discriminating.

Baumslag, Myasnikov and Remeslennikov in [BMR 2] give a partial answer to the total characterization of abelian discriminating groups. We must introduce the **Szmielew invariants** of an abelian group. Given an

integer $m > 0$ and a family of elements (a_i) in an additively written abelian group A , (a_i) is **linearly independent modulo m** provided $\sum_i n_i a_i = 0$ implies $n_i \equiv 0 \pmod{m}$ for all i ; (a_i) is **linearly independent modulo m in the stronger sense** provided $\sum_i n_i a_i \in mA$ implies the coefficients $n_i \equiv 0 \pmod{m}$ for all i . For each prime p and positive integer k we define three ranks each of which is a nonnegative integer or the symbol ∞ .

- (1) $\rho^{(1)}[p, k](A)$ is the maximum number of elements of A of order p^k and linearly independent modulo p^k
- (2) $\rho^{(2)}[p, k](A)$ is the maximum number of elements of A linearly independent modulo p^k in the stronger sense.
- (3) $\rho^{(3)}[p, k](A)$ is the maximum number of elements of A of order p^k and linearly independent modulo p^k in the stronger sense.

Proposition 1.3.1. (*Szmielew [S]*) *Let A and B be abelian groups. Then A and B are elementarily equivalent if and only if the following two properties are satisfied.*

- (1) *A and B either both have finite exponent or both have infinite exponent.*
- (2) *For all primes p and positive integers k , $\rho^{(i)}[p, k](A) = \rho^{(i)}[p, k](B)$ for $i = 1, 2, 3$.*

Proposition 1.3.2. (*[BMR 2]*) *Let A be a torsion abelian group such that, for each prime p , the p -primary component of A modulo its maximal divisible subgroup contains no nontrivial elements of infinite p -height. Then A is discriminating if and only if, for each prime p , the following two properties are satisfied.*

- (1) *For all positive integers k , $\rho^{(1)}[p, k](A)$ is either 0 or ∞ .*
- (2) *The rank of the maximal divisible subgroup of the p -primary component of A is either 0 or infinite.*

Extending the ideas of the [BMR 2] paper, Fine, Gaglione, Mayasnikov and Spellman [FGMS 2] developed a complete characterization of abelian squarelike groups.

Proposition 1.3.3. (*[FGMS 2]*) *Let A be an abelian group. Then A is squarelike if and only if, for each prime p and positive integer k , $\rho^{(1)}[p, k](A)$ is either 0 or ∞ .*

2. POSITIVE EXAMPLES OF DISCRIMINATING GROUPS

In the theory as applied to the algebraic geometry of groups it is important to know both positive examples of discriminating groups and negative examples. The original positive examples of discriminating groups were either abelian or trivially discriminating. Recall that a group is trivially

discriminating if it embeds its direct square. In this section we present these positive examples. We will return later to examples of nontrivially discriminating groups.

Most of the early discovered positive examples were also “universal type groups”

(1) EXAMPLE 1: Abelian Groups:

Torsion-free abelian groups and the torsion abelian groups described above. ([BMR 2].)

(2) EXAMPLE 2: Higman’s universal group \mathcal{H} .

Higman constructed this group \mathcal{H} [Hi] which is a finitely presented group which embeds every other finitely presented group.

Since this group is finitely presented its direct square is also finitely presented. Therefore it embeds its direct square and hence is trivially discriminating.

(3) EXAMPLE 3: Thompson’s Group F .

This group consists of all orientation preserving piecewise linear homeomorphisms from the unit interval $[0,1]$ onto itself that are differentiable except at finitely many dyadic rational numbers and such that on intervals of differentiability the derivatives are powers of 2 is finitely presented. As in Higman’s group, this group embeds its direct square and hence is trivially discriminating. This is also finitely presented.

(4) EXAMPLE 4: The commutator subgroup of the Gupta-Sidki-Grigorchuk Group.

For each odd prime p , Gupta and Sidki and independently Grigorchuk constructed a group H_p which is a subgroup of the group of automorphisms of a rooted tree. H_p is a 2-generator infinite p -group. Consider the commutator subgroup H'_p of H_p . It can be shown that H'_p , while finitely generated, is not finitely presentable. It can be shown that H'_p is trivially discriminating.

(5) EXAMPLE 5: Groups which are isomorphic to their direct squares.

Examples of nontrivial finitely generated groups isomorphic to their direct squares were first constructed by Tyrer Jones [J] and subsequently by Hirshon and Meier [HM]. The question of whether or not there exists a finitely presented group isomorphic to its direct square remains open.

(6) EXAMPLE 6: The Grigorchuk group of intermediate growth. Grigorchuk developed a group of intermediate growth (see [FGMS 1]). It was

shown in that paper, due to conversations with Grigorchuk, that this group is discriminating. Whether it is trivially discriminating or not is open.

(7) EXAMPLE 7: It was shown in [BFGS 1] that

Lemma 2.1. *A simple discriminating group is trivially discriminating.*

Then: For each integer $n \geq 2$ and each integer $r \geq 1$, Higman defined a group $G_{n,r}$ (see [Hi 2]) as follows. Let \mathcal{V}_n be the variety of all algebras with one n -ary operation λ and n unary operations $\alpha_1, \dots, \alpha_n$ subject to the laws $\lambda(\alpha_1(x), \dots, \alpha_n(x)) = x$ and $\alpha_i(\lambda(x_1, \dots, x_n)) = x_i$, $i = 1, \dots, n$. If $V_{n,r}$ is an algebra free on r generators in \mathcal{V}_n , then we let $G_{n,r} = \text{Aut}(V_{n,r})$ be its group of automorphisms. Higman proved that the $G_{n,r}$ are finitely presented, when n is even $G_{n,r}$ is simple and when n is odd $G_{n,r}$ contains a simple subgroup $G_{n,r}^+$ of index 2. Setting $G_{n,r}^+ = G_{n,r}$ when n is even, he showed that, for fixed r , $G_{m,r}^+ \cong G_{n,r}^+$ implies $m = n$. Thus he found an infinite family of finitely presented infinite simple groups. The $G_{n,r}$ are all trivially discriminating.

(8) EXAMPLE 8: The group S_ω of all permutations of \mathbb{N} which move only finitely many integers and the subgroup A_ω of even permutations are each trivially discriminating

(9) EXAMPLE 9: The existence of a finitely generated but not finitely presentable trivially discriminating group G_1 (e.g. the commutator subgroup of a Gupta-Sidki-Grigorchuk group) together with the existence of a universal finitely presented group (Higman's Group G_2) allowed us to construct in [FGS 4] a group $G = G_1 \times G_2$ which is proven in that paper to be a finitely generated discriminating group which is neither finitely presented nor equationally Noetherian.

This led to the obvious question:

DOES THERE EXIST A FINITELY PRESENTED NONTRIVIALY DISCRIMINATING GROUP?

We will return to this shortly.

3. NEGATIVE EXAMPLES: EXAMPLES OF NONDISCRIMINATING GROUPS

In the application of algebraic geometry it is as important to know both whether a group is discriminating and whether it is not discriminating. This further raised general questions on nondiscrimination of various classes of groups in particular groups in various varieties.

The general idea to show nondiscrimination of a group is to show some universal property which is true in G but cannot be true in $G \times G$ or to find a number - dimension etc. - which is additive so cannot hold in $G \times G$ and in G .

Definition 3.1. *The group G is **commutative transitive** or **CT** provided the centralizer of every nontrivial element is abelian.*

We observe that the group G is CT if and only if it satisfies the universal sentence

$$\forall x, y, z((y \neq 1) \wedge (xy = yx) \wedge (yz = zy)) \rightarrow (xz = zx)).$$

If G is non-abelian then $G \times G$ cannot satisfy this sentence. Therefore the following lemma follows directly.

Lemma 3.1. *A nonabelian CT group is nondiscriminating.*

In particular we obtain the negative examples.

(1) Any torsion-free hyperbolic group and in particular any non-abelian free group is nondiscriminating

(2) Nonabelian free solvable groups and their nonabelian subgroups are nondiscriminating.

(3) The free product of two nondiscriminating groups is nondiscriminating.

In another direction, if we think of discrimination as having the direct square $G \times G$ almost embedding in the group G , then this would be impossible for a finite group. In particular it is impossible if the torsion elements form a finite subgroup.

Lemma 3.2. *If the torsion elements of a group G form a finite subgroup then G is nondiscriminating.*

This leads to the further negative examples.

(4) Any finitely generated nilpotent group with nontrivial torsion is nondiscriminating

(5) Any finite group is nondiscriminating.

One of the major questions raised in the early work on discriminating groups ([BMR 2], [FGMS]) was whether or not a nonabelian nilpotent group can be discriminating. The full result was:

Theorem 3.1. *A finitely generated nilpotent group is discriminating if and only if it is torsion-free abelian.*

There were several stages to obtaining the total final result (Theorem 3.1) although the final proof was rather direct. In [FGMS 1] it was shown that nonabelian finitely generated free nilpotent groups are nondiscriminating. To do this an extension of commutative transitivity was introduced. Myasnikov and Shumyatsky ([MS]) introduced the concept of centralizer dimensions (see [MS]) and showed that a group with finite centralizer dimension is nondiscriminating. As a consequence of this it follows that a finitely generated nilpotent group is discriminating if and only if it is torsion-free abelian. Their method yielded the following further result.

Theorem 3.2. *A finitely generated linear group is discriminating only if it is torsion-free abelian.*

Independently Baumslag, Fine, Gaglione and Spellman using vector space dimension proved that a nilpotent group is discriminating if and only if its Malcev completion is discriminating. Theorem 3.1 can then be recovered from this result.

Theorem 3.3. *([BFGS 2]) The Malcev completion of a finitely generated torsion-free nilpotent group is discriminating if and only if it is abelian.*

The nondiscrimination of nonabelian nilpotent groups leads to the question as to whether any nonabelian group in a variety must be nondiscriminating. There is some further evidence for this. Kassabov [K] proved it for solvable groups.

Theorem 3.4. *([K]) A finitely generated solvable group is discriminating only if it is torsion-free abelian.*

Further in this direction, Baumslag, Fine, Gaglione and Spellman [BFGS 1] proved some general results on nondiscrimination of relatively free groups.

Theorem 3.5 (BFGS 1). *Let F be a non-abelian free group and let R be a non-trivial, normal subgroup of F such that F/R is torsion-free and contains a free abelian group of rank two. Then $F/V(R)$ is not discriminating.*

Theorem 3.6 (BFGS 1). *Let \mathcal{V} and \mathcal{U} be two varieties. If the variety \mathcal{V} contains the infinite cyclic group, then the non-abelian free groups in the product variety $\mathcal{U}\mathcal{V}$ are not discriminating.*

Theorem 3.7 (BFGS 1). *For all sufficiently large primes p , the non-abelian free groups G in the variety of all groups of exponent p are not discriminating.*

We pose the following question:

DOES THERE EXIST A NONABELIAN, DISCRIMINATING, RELATIVELY FREE GROUP?

4. NONTRIVIALY DISCRIMINATING GROUPS

Recall the question of whether there exists a finitely generated nontrivially discriminating group. That is a finitely generated discriminating group which does not embed its direct square. One possibility (still open) was the Grigorchuk groups of intermediate growth. Here we give other examples - done with Peter Neumann and Gilbert Baumslag. We present two classes of examples. The groups in the first class are finitely presented. The second is a class of groups studied by B.H. Neumann which are non-trivially discriminating and finitely generated but probably not finitely presented.

EXAMPLE 1: To describe the first class of groups let X be a non-abelian, finitely generated, torsion free, nilpotent group and let Y be one of the infinite simple groups, $G_{n,r}$ as described by Higman. The following properties of X and Y are needed:

- (a) X and Y are finitely presented;
- (b) X is residually finite;
- (c) every finite group is embeddable in Y ;
- (d) X is not embeddable in Y ; in fact, a torsion free nilpotent subgroup of Y is abelian;
- (e) $Y \times Y \times Y$ is embeddable in Y . (This follows from the fact that $\mathcal{V}_{n,r} \cong \mathcal{V}_{n,s}$ in the notation of [GH2] if and only if $r \equiv s \pmod{n-1}$.)

Our class \mathcal{G}_1 consists of the groups $X \times Y$.

Theorem 4.1 (FGS 4). *Each group $G \in \mathcal{G}_1$ is a nonabelian, finitely presented non-trivially discriminating group.*

We present the proof to exhibit the techniques involved.

Proof. Clearly each group $G \in \mathcal{G}_1$ is non-abelian and finitely presented. Then suppose $G \in \mathcal{G}_1$. We must show that G is discriminating but that G does not embed its direct square.

Since $G = X \times Y$ with X and Y as described above to prove that G is discriminating it suffices to show that Y discriminates any group of the form $W \times Y \times Y$ where W is residually finite. This is sufficient for then $G = X \times Y$ will discriminate its direct square and hence be discriminating. For this purpose, let h_1, \dots, h_n be finitely many non-identity elements of $W \times Y \times Y$. Write $h_i = (a_i, b_i)$ where $a_i \in W$ and $b_i \in Y \times Y$. Without loss of generality, we may suppose that $a_i \neq 1$ for $1 \leq i \leq m$ and $a_i = 1$ for $m+1 \leq i \leq n$. Since W is residually finite, there exists a finite group V and a homomorphism $\alpha : W \rightarrow V$ such that $\alpha(a_i) \neq 1$ for $1 \leq i \leq m$. By (c) above we may embed V into Y and so we get a homomorphism $\beta : W \rightarrow Y$ such that $\beta(a_i) \neq 1$ for $1 \leq i \leq m$. Let γ be the homomorphism $\gamma : W \times Y \times Y \rightarrow Y \times Y \times Y$ defined by $(w, y, y) \xrightarrow{\gamma} (\beta(w), y, y)$. By (e) above let δ be an embedding of $Y \times Y \times Y$ into Y . If we put $\varphi = \delta \circ \gamma$, then clearly φ is a homomorphism such that $\varphi(h_i) \neq 1$ for all $i = 1, \dots, n$. Thus Y discriminates $W \times Y \times Y$ and therefore it follows that G discriminates $G \times G$. Thus G is discriminating.

To show that is nontrivially discriminating we must show that $G \times G$ is not embeddable into G . Assume to deduce a contradiction that $G \times G \hookrightarrow G$. Restricting the embedding we get an embedding $X \times X \hookrightarrow X \times Y$. Letting A be the projection of $X \times X$ into Y , then $X \times X \leq X \times A$, so $X' \times X' \leq X' \times A'$. But this tells us that if $h =$ the Hirsch length of the commutator subgroup X' , then $h \leq 2h \leq h$ since A' is finite. This contradicts the fact that X is non-abelian and torsion free nilpotent. Therefore $G \times G$ is not embeddable into G and hence is nontrivially discriminating. \square

EXAMPLE 2: The second class of non-trivially discriminating groups, which we will denote \mathcal{G}_2 was introduced by B. Neumann in a different context. Let n_1, n_2, \dots be a strictly increasing sequence of odd positive integers with $n_1 \geq 5$. Let $\Lambda_r = \{1, 2, \dots, n_r\}$ and let G_r be the alternating group on Λ_r . In G_r consider the two elements

$$x_r = (123), \quad y_r = (12\dots n_r).$$

In the unrestricted direct product $\prod_r G_r$ define

$$x = (x_1, x_2, \dots) \quad , \quad y = (y_1, y_2, \dots)$$

and let

$$G = \langle x, y \rangle \subseteq \prod_r G_r.$$

B.H. Neumann proved that the restricted direct product $D = G_1 \times G_2 \times \cdots$ is a subgroup of G . The class \mathcal{G}_2 consists of all the groups G constructed as above.

Theorem 4.2. *Each group $G \in \mathcal{G}_2$ is a finitely generated non-trivially discriminating group.*

The methods used in constructing these non-TD examples can be generalized via the use of ascending chains of subgroups.

Theorem 4.3 (BFGS 1). *Let U be the union of a properly ascending chain of subgroups*

$$U_1 < U_2 < \cdots < U_n < \cdots$$

and let P be the unrestricted direct product of the U_i :

$$P = \prod_{i=1}^{\infty} U_i$$

Furthermore let Q be the restricted direct product

$$Q = \prod_{i=1}^{\infty} U_i$$

Then every subgroup G of P containing Q is discriminating (where here we view Q as a subgroup of P).

Proof. The proof of this result is straightforward. To this end, let $D = G \times G$ and let $d_1 = (a_1, b_1), \dots, d_k = (a_k, b_k)$ be finitely many non-trivial elements of D . Each of the elements of G can be viewed as a sequence of elements whose n -th term is contained in U_n . Since G contains Q and the series of subgroups U_j is increasing, there exists an integer α such that all of $a_1, b_1, \dots, a_k, b_k \in U_\alpha$. We define now a homomorphism of D into G by projecting the first coordinates of the elements of D to U_α and the second coordinates to $U_{\alpha+1}$. The upshot of this is that we have defined a homomorphism θ of $G \times G$ into $U_\alpha \times U_{\alpha+1}$. If we view $U_\alpha \times U_{\alpha+1}$ as a subgroup of G , it follows that θ is a homomorphism of $G \times G$ into G mapping the given elements d_1, \dots, d_k non-trivially. This completes the proof. \square

This final theorem can be used to recover the proofs of the preceding examples and to obtain the fact that there exists continuously many 2-generator non-TD discriminating groups. In particular let A_i denote the alternating group of degree i and let G_r be the subgroup defined in the proof of Theorem 4.2. Then G_s contains the restricted direct product of the A_{n_i} and that

$$G_s \cong G_t \text{ if and only if } s = t$$

This implies that there are continuously many non-isomorphic 2-generator groups. Theorem 4.3 then applies to these groups. Further B.H. Neumann proved that the only finite normal subgroups of G_r are direct products of finitely many of the A_{n_i} . This implies, remembering that the alternating groups involved here are all simple, that none of the G_r are TD. We remark here that it has recently been proved that none of the groups G_s is finitely presented (G.Baumslag and C.F. Miller).

Corollary 4.1. *The groups G_r are discriminating and further are not TD. Thus there exist continuously many 2-generator nontrivially discriminating groups.*

5. SQUARELIKE GROUPS

In order to better capture the axiomatic properties of discriminating groups the class of squarelike groups were introduced. These are groups which share the same universal theory as their direct squares.

Definition 5.1. (*[FGMS 2]*): *The group G is squarelike provided $G \times G \equiv_{\forall} G$.*

Hence, every discriminating group is squarelike (but, as we shall see), **not** conversely).

Theorem 5.1. *The class of discriminating groups is a proper subclass of the class of squarelike groups.*

To prove this we explicitly construct a discriminating group which is provably non-squarelike. This can also be proved with an **axiomatic argument** done below.

EXAMPLE: Let H be the subgroup of the group of all permutations of the set \mathbb{Z} of integers generated by the 3-cycle $\xi = (012)$ and the translation $\eta(n) = n + 1$ for all $n \in \mathbb{Z}$. $H = \langle \xi, \eta \rangle$ can also be described as the semidirect product of the group M , of all even parity permutations within the group N of all permutations of the set \mathbb{Z} of integers which move only finitely many integers, by an infinite cyclic group $C = \langle c \rangle$ where the automorphism $\alpha(c) : M \rightarrow M$ acts by $\alpha(c)(\pi)(n) = \pi(n - 1) + 1$. (We say that $\alpha(c)$ acts by translation by 1.) Note that any bijection between \mathbb{N} and \mathbb{Z} induces an isomorphism between the infinite alternating group A_{ω} and M . H first appeared in print in the same paper of B.H. Neumann in which the uncountably many nontrivially discriminating groups $G_{\mathbf{n}}$ exhibited before were introduced. B.H. Neumann observed that (independent of \mathbf{n}) if K_0 is the restricted direct product of the family $(A_{n(r)})_{r \in \mathbb{N}}$ of alternating groups,

then the quotient of $G = G_{\mathbf{n}}$ modulo K_0 is isomorphic to H . It can be proved that H is nondiscriminating however H is squarelike.

In an earlier paper this was done differently. Recall that a class of groups is **axiomatic** if it is the model class of a set of first order sentences (axioms). It can be shown that a class of groups is axiomatic if it is closed under ultraproducts and elementary equivalence.

Theorem 5.2. (*[FGMS 2]*) *The class of squarelike groups is axiomatic but the class of discriminating groups is not axiomatic.*

To prove that the discriminating groups are nonaxiomatic a specific example of a nondiscriminating group (however squarelike) was constructed which was elementarily equivalent to a discriminating group. Hence the discriminating groups are not closed under elementary equivalence.

Originally the fact that the squarelike groups are axiomatic was proved by using the closure properties noted above. However subsequently V.H.Dyson [D] discovered an explicit axiom schema for the class of squarelike groups. To each ordered pair (\mathbf{w}, \mathbf{u}) of finite tuples of words on a fixed but arbitrary finite set $\{x_1, \dots, x_n\}$ of distinct variables and their formal inverses we assign the following sentence $\sigma(\mathbf{w}, \mathbf{u})$ of L .

$$\forall \mathbf{x} \left(\bigwedge_i (w_i(\mathbf{x}) = 1) \rightarrow \bigvee_j (u_j(\mathbf{x}) = 1) \right) \rightarrow \bigvee_j \forall \mathbf{x} \left(\bigwedge_i (w_i(\mathbf{x}) = 1) \rightarrow (u_j(\mathbf{x}) = 1) \right).$$

The contrapositive of $\sigma(\mathbf{w}, \mathbf{u})$ is (up to logical equivalence) the sentence $\tau(\mathbf{w}, \mathbf{u})$ asserting

$$\bigwedge_j \exists \mathbf{x} \left(\bigwedge_i (w_i(\mathbf{x}) = 1) \wedge (u_j(\mathbf{x}) \neq 1) \right) \rightarrow \exists \mathbf{x} \left(\bigwedge_i (w_i(\mathbf{x}) = 1) \wedge \bigwedge_i (u_j(\mathbf{x}) \neq 1) \right).$$

Theorem 5.3. (*Huber-Dyson [D]*) *The class of squarelike groups is the model class of the group axioms and the sentences $\sigma(\mathbf{w}, \mathbf{u})$. Hence, the class of squarelike groups is axiomatic.*

Although the squarelike groups properly contain the discriminating groups they are very close and in fact correspond in the presence of finite presentation. In particular:

Theorem 5.4. (*[FGMS 2]*) *Let G be a finitely presented group. Then G is discriminating if and only if G is squarelike.*

We give the proof because its very pretty

Proof. Suppose G is a finitely presented group. If it is discriminating, it is squarelike. Now we suppose that G is squarelike and we must show that it is discriminating.

Let

$$G = \langle x_1, \dots, x_n; R_1, \dots, R_m \rangle$$

be a finite presentation for G where $R_i = R_i(x_1, \dots, x_n)$ are words in x_1, \dots, x_n . To show that G is discriminating we must show that G discriminates $G \times G$.

A finite presentation for $G \times G$ is then given by

$$G \times G = \langle x_1, \dots, x_n, y_1, \dots, y_n; R_1(x_1, \dots, x_n) = 1, \dots, R_m(x_1, \dots, x_n) = 1,$$

$$R_1(y_1, \dots, y_n) = 1, \dots, R_m(y_1, \dots, y_n) = 1, [x_i, y_j] = 1, i, j = 1, \dots, n \rangle.$$

Now suppose W_1, \dots, W_k are non-trivial elements of $G \times G$. Then each $W_i, i = 1, \dots, k$ is given by $W_i = W_i(x_1, \dots, x_n, y_1, \dots, y_n)$, a word in the given generators of $G \times G$. Consider now the existential sentence

$$\begin{aligned} \exists x_1, \dots, x_n, y_1, \dots, y_n & \left(\left(\bigwedge_{i=1}^m R_i(x_1, \dots, x_n) = 1 \right) \wedge \left(\bigwedge_{i=1}^m R_i(y_1, \dots, y_n) = 1 \right) \right. \\ & \left. \wedge \left(\bigwedge_{i,j} [x_i, y_j] = 1 \right) \wedge \left(\bigwedge_{i=1}^k W_i(x_1, \dots, x_n, y_1, \dots, y_n) \neq 1 \right) \right) \end{aligned}$$

This existential sentence is clearly true in $G \times G$. Since G is squarelike, G and $G \times G$ have the same universal theory. Hence they have the same existential theory and therefore the above existential sentence is true in G . Therefore there exists elements $a_1, \dots, a_n, b_1, \dots, b_n$ in G such that $R_i(a_1, \dots, a_n) = 1$ for $i = 1, \dots, m$; $R_i(b_1, \dots, b_n) = 1$ for $i = 1, \dots, m$; $[a_i, b_j] = 1$ for $i, j = 1, \dots, n$ and $W_i(a_1, \dots, a_n, b_1, \dots, b_n) \neq 1$ for $i = 1, \dots, k$. The map from $G \times G$ to G given by mapping x_i to a_i and y_i to b_i for $i = 1, \dots, n$ defines a homomorphism for which the images of W_1, \dots, W_k are non-trivial. Hence G discriminates $G \times G$ and therefore G is discriminating. \square

Finally recall that

Theorem (FGMS 1). *Let G be a finitely generated equationally Noetherian group. Then G is discriminating if and only if $qvar(G) = ucl(G)$.*

For squarelike groups we do not need the equationally noetherian condition.

Theorem 5.5. *([FGMS 2]) Let G be a group. The following three conditions are pairwise equivalent.*

- G is squarelike.
- $ucl(G) = qvar(G)$.
- There is a discriminating group H such that $G \equiv_{\forall} H$.

Therefore the squarelike groups are the universal closure of the discriminating groups. In fact they are the axiomatic closure.

6. AXIOMATICS AND THE AXIOMATIC CLOSURE PROPERTY

The least axiomatic class containing the discriminating groups is the class of those groups G for which there exists a discriminating group H with $G \equiv H$. Here \equiv denotes elementary equivalence. We call this class the **axiomatic closure** of G .

In [FGS 2] it was established that the squarelike groups are precisely the axiomatic closure of the discriminating groups.

Theorem 6.1. (*[FGS 2]*) *Let G be a group. Then G is squarelike if and only if there is a discriminating group H with $G \equiv H$. Hence the class of squarelike groups is the axiomatic closure of the class of discriminating groups.*

Proof. We outline the proof which uses ultrapowers and ultralimits.

If $G \equiv H$, then $G \equiv_{\forall} H$; so, G elementarily equivalent to a discriminating group implies G is squarelike.

Assume now that G_0 is a squarelike group. Then $G_0 \forall (G_0 \times G_0)$ so there is an ultrapower $G_1 = G_0^I/D$ admitting an embedding $\varphi_1 : G_0 \times G_0 \rightarrow G_1$. Let $d_{0,1} : G_0 \rightarrow G_1$ be the canonical embedding and let $G_2 = G_1^I/D$. Then φ_1 induces $\varphi_2 : G_1 \times G_1 \rightarrow G_2$ and a diagram chase convinces one that the square

$$\begin{array}{ccccc}
 & & & \phi_1 & \\
 & & & \rightarrow & \\
 & G_0 \times G_0 & \rightarrow & G_1 & \\
 d_{0,1} \times d_{0,1} & \downarrow & & \downarrow & d_{1,2} \\
 & G_1 \times G_1 & \rightarrow & G_2 & \\
 & & & \phi_2 &
 \end{array}$$

is commutative where $d_{1,2} : G_1 \rightarrow G_2$ is the canonical embedding.

We may iterate a countable infinity of times. Taking G_ω to be the ultralimit of G_0 with respect to the constant sequence of ultrafilters (D, D, \dots, D, \dots) , we get an embedding $G_\omega \times G_\omega \rightarrow G_\omega$ so that G_ω is discriminating. But the limit map $d_{0,\omega} : G_0 \rightarrow G_\omega$ is elementary. Hence, $G_0 \equiv G_\omega$. \square

There is a great deal of further axiomatic information about both squarelike and discriminating groups (see [FGS 3]). As we have seen in the previous sections the class of squarelike groups is axiomatic, that is is given by a set of first -order axioms while the class of squarelike groups is not. Originally this was proved in [FGMS 2] using Malcev's condition for axiomability. Subsequently an explicit set of axioms was given by V.H. Dyson (see [FGS 3]). P. Schupp raised the question of whether the theory of squarelike groups was finitely axiomatizable, that is the axiom set can be taken to be finite. This was shown to be negative ([FGS 3]).

Theorem 6.2. (*[FGS 3]*) *The theory of squarelike groups is not finitely axiomatizable.*

Recall that if \mathcal{X} is a class of groups then the first-order theory of \mathcal{X} is **decidable** if there exists a recursive algorithm which, given a sentence ϕ of L_0 decides whether or not ϕ is true in every group in \mathcal{X} . As a consequence of the undecidability of the word problem (see [FGS 3]) it was shown that the theory of the squarelike groups is undecidable.

Theorem 6.3. (*[FGS 3]*) *The theory of squarelike groups is undecidable.*

7. VARIETAL DISCRIMINATION

There is an older notion of discrimination, which we call **varietal discrimination**, that was also introduced by G. Baumslag jointly with Bernard, Hanna and Peter Neumann (see [N]). The properties of this type of discrimination are described in the book of Hanna Neumann [N] and play a role in the structure of product varieties.

In [FGS 5] the relationship between these various notions of discrimination was examined. If we let \mathcal{D} denote the class of discriminating groups, \mathcal{S} the class of squarelike groups and \mathcal{VD} the class of varietally discriminating groups the main result is that

$$\mathcal{D} \subset \mathcal{S} \subset \mathcal{VD}$$

and all inclusions are proper.

In order to prove that $\mathcal{S} \subset \mathcal{VD}$ a further notion of discrimination called **q-discriminating** was introduced and then it was proved that a group is squarelike if and only if it is q-discriminating.

Definition 7.1. (*see [N]*) *Let \mathcal{V} be a variety of groups and $G \in \mathcal{V}$. G **discriminates** \mathcal{V} provided to every finite set $w_i(\bar{x})$ of words in the variables $X \cup X^{-1}$ with none of the equations $w_i(\bar{x}) = 1$ a law in \mathcal{V} there is a tuple \bar{g} of elements of G such that simultaneously $w_i(\bar{g}) \neq 1$ in G for all i . Further a group G is **varietally discriminating** if it discriminates the variety it generates.*

Theorem 7.1. *$\mathcal{D} \subset \mathcal{S} \subset \mathcal{VD}$ and all inclusions are proper.*

The proper inclusion $\mathcal{D} \subset \mathcal{S}$ was described already. The proper inclusion $\mathcal{D} \subset \mathcal{VD}$ can be proved independently of the middle. In order to prove the proper inclusion $\mathcal{S} \subset \mathcal{VD}$ we introduce a quasivarietal version of varietal discrimination.

Definition 2.1. Let \mathcal{Q} be a quasivariety and let $G \in \mathcal{Q}$. Then G **q-discriminates** \mathcal{Q} provided that given finitely many quasilaws

$$\forall \bar{x} ((\bigwedge_i u_i(\bar{x} = 1) \rightarrow (w_j(\bar{x}) = 1))$$

with the same antecedents and none of which holds in \mathcal{Q} there exists a tuple \bar{g} from G such that simultaneously $u_i(\bar{g}) = 1$ and $w_j(\bar{g}) \neq 1$ for all i, j . A group G is **q-discriminating** if G q-discriminates $\text{quar}(G)$.

This almost, except for having the same antecedents, appears to be the translation of varietal discrimination in terms of quasivarieties. The condition of having the same antecedents is necessary for the next theorem. Without this restriction the only q-discriminating group would be the trivial group. The above notion of q-discrimination characterizes squarelike groups. We introduce one more idea.

Definition 7.2. A group G is **q-algebraically closed** if and only if whenever a finite system

$$\begin{aligned} u_i(x_1, \dots, x_n) &= 1 \\ w_j(x_1, \dots, x_n) &\neq 1 \end{aligned}$$

of equations and inequations has a solution in some group $H \in \text{quar}(G)$ it also has a solution in G .

We can now characterize squarelike groups in terms of q-discrimination.

Theorem 7.3. ([FGS 5]) Let G be a group. The following are pairwise equivalent.

- (1) G is q-discriminating
- (2) G is q-algebraically closed
- (3) G is squarelike

G is q-discriminating

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