# PARABOLIC AND QUASIPARABOLIC SUBGROUPS OF FREE PARTIALLY COMMUTATIVE GROUPS

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Abstract. Let $\Gamma$ be a finite graph and $G$ be the corresponding free partially commutative group. In this paper we study subgroups generated by vertices of the graph $\Gamma$, which we call canonical parabolic subgroups. A natural extension of the definition leads to canonical quasiparabolic subgroups. It is shown that the centralisers of subsets of $G$ are the conjugates of canonical quasiparabolic centralisers satisfying certain graph theoretic conditions.

## 1. Preliminaries

Free partially commutative groups arise naturally in many branches of mathematics and computer science and consequently have many aliases: they are known as semifree groups [1, 2], graph groups [14, 22, 24, 26, 31, 33], right-angled Artin groups [4, 5, 6, 8, 10, 23, 30, 35], trace groups [13, 29], locally free groups [9, 28, 34] and of course (free) partially commutative groups [3, 7, 11, 12, 15, 17, 18, 21, 25, 27, 32]. we refer the reader to [5, 21, 13, 22] for further references, more comprehensive surveys, introductory material and discussion of the various manifestations of these groups.

In this section we give a brief overview of some definitions and results from [21, 20]. We begin with the basic notions of the theory of free partially commutative groups: which, for the sake of brevity we refer to simply as partially commutative groups. Let $\Gamma$ be a finite, undirected, simple graph. Let $X = V(\Gamma) = \{x_1, \dots, x_n\}$ be the set of vertices of $\Gamma$ and let $F(X)$ be the free group on $X$. Let

$$R=\{[x_i, x_j]\in F(X) \mid x_i, x_j \in X \text{ and there is an edge of } \Gamma \text{ joining } x_i \text{ to } x_j\}.$$

We define the *partially commutative group* with (*commutation*) *graph* $\Gamma$ to be the group $G(\Gamma)$ with presentation $\langle X \mid R\rangle$. When the underlying graph is clear from the context we write simply $G$.

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Denote by $l(g)$ the minimum of the lengths words that represents the element $g$. If $w$ is a word representing $g$ and $w$ has length $l(g)$ we call $w$ a *minimal form* for $g$. When the meaning is clear we shall say that $w$ is a minimal element of $G$ when we mean that $w$ is a minimal form of an element of $G$. We say that $w \in G$ is *cyclically minimal* if and only if

$$l(g^{-1}wg) \geq l(w)$$

for every $g \in G$. We write $u \circ w$ to express the fact that $l(uw) = l(u) + l(w)$, where $u, w \in G$. We will need the notions of a divisor and the greatest divisor of a word $w$ with respect to a subset $Y \subseteq X$, defined in [21]. Let $u$ and $w$ be elements of $G$. We say that $u$ is a *left* (*right*) *divisor* of $w$ if there exists $v \in G$ such that $w = u \circ v$ ($w = v \circ u$). We order the set of all left (right) divisors of a word $w$ as follows. We say that $u_2$ is greater than $u_1$ if and only if $u_1$ left (right) divides $u_2$. It is shown in [21] that, for any $w \in G$ and $Y \subseteq X$, there exists a unique maximal left divisor of $w$ which belongs to the subgroup $G(Y) < G$ which is called the *greatest left divisor* $\mathrm{gd}^l_Y(w)$ of $w$ in $Y$. The *greatest right divisor* of $w$ in $Y$ is defined analogously. We omit the indices when no ambiguity occurs.

The *non-commutation* graph of the partially commutative group $G(\Gamma)$ is the graph $\Delta$, dual to $\Gamma$, with vertex set $V(\Delta) = X$ and an edge connecting $x_i$ and $x_j$ if and only if $[x_i, x_j] \neq 1$. The graph $\Delta$ is a union of its connected components $\Delta_1, \ldots, \Delta_k$ and words that depend on letters from distinct components commute. For any graph $\Gamma$, if $S$ is a subset of $V(\Gamma)$ we shall write $\Gamma(S)$ for the full subgraph of $\Gamma$ with vertices $S$. Now, if the vertex set of $\Delta_k$ is $I_k$ and $\Gamma_k = \Gamma(I_k)$ then $G = G(\Gamma_1) \times \cdots \times G(\Gamma_k)$. For $g \in G$ let $\alpha(g)$ be the set of elements $x$ of $X$ such that $x^{\pm 1}$ occurs in a minimal word $w$ representing $g$. It is shown in [21] that $\alpha(g)$ is well-defined. Now suppose that the full subgraph $\Delta(\alpha(w))$ of $\Delta$ with vertices $\alpha(w)$ has connected components $\Delta_1, \ldots, \Delta_l$ and let the vertex set of $\Delta_j$ be $I_j$. Then, since $[I_j, I_k] = 1$, we can split $w$ into the product of commuting words, $w = w_1 \circ \cdots \circ w_l$, where $w_j \in G(\Gamma(I_j))$, so $[w_j, w_k] = 1$ for all $j, k$. If $w$ is cyclically minimal then we call this expression for $w$ a *block decomposition* of $w$ and say $w_j$ a *block* of $w$, for $j = 1, \ldots, l$. Thus $w$ itself is a block if and only if $\Delta(\alpha(w))$ is connected. In general let $v$ be an element of $G$ which is not necessarily cyclically minimal. We may write $v = u^{-1} \circ w \circ u$, where $w$ is cyclically minimal and then $w$ has a block decomposition $w = w_1 \cdots w_l$, say. Then $w_j^u = u^{-1} \circ w_j \circ u$ and we call the expression $v = w_1^u \cdots w_l^u$ the *block decomposition* of $v$ and say that $w_j^u$ is a *block* of $v$, for $j = 1, \ldots, l$. Note that this definition is slightly different from that given in [21].

Let $Y$ and $Z$ be subsets of $X$. As in [20] we define the *orthogonal complement* of $Y$ *in* $Z$ to be

$$\mathcal{O}^Z(Y) = \{u \in Z | d(u, y) \le 1, \text{ for all } y \in Y\}.$$

By convention we set $\mathcal{O}^Z(\emptyset) = Z$. If $Z = X$ we call $\mathcal{O}^X(Y)$ the orthogonal complement of $Y$, and if no ambiguity arises then we write $Y^\perp$ instead of $\mathcal{O}^X(Y)$ and $x^\perp$ for $\{x\}^\perp$. Let $\mathrm{CS}(\Gamma)$ be the set of all subsets $Z$ of $X$ of the form $Y^\perp$ for some $Y \subseteq X$. The set $\mathrm{CS}(\Gamma)$ is shown in [20] to be a lattice, the *lattice of closed sets* of $\Gamma$.

The *centraliser* of a subset $S$ of $G$ is

$$C(S) = C_G(S) = \{g \in G : gs = sg, \text{ for all } s \in S\}.$$

The set $\mathrm{C}(G)$ of centralisers of a group is a lattice. An element $g \in G$ is called a *root element* if $g$ is not a proper power of any element of $G$. If $h = g^n$, where $g$ is a root element and $n \ge 1$, then $g$ is said to be a *root* of $h$. As shown in [16] every element of the partially commutative group $G$ has a unique root, which we denote $r(g)$. If $w \in G$ define $A(w) = \langle Y \rangle = G(Y)$, where $Y = \alpha(w)^\perp \setminus \alpha(w)$. Let $w$ be a cyclically minimal element of $G$ with block decomposition $w = w_1 \cdots w_k$ and let $v_i = r(w_i)$. Then, from [16, Theorem 3.10],

$$C(w) = \langle v_1 \rangle \times \cdots \times \langle v_k \rangle \times A(w). \tag{1.1}$$

We shall use [19, Corollary 2.4] several times in what follows, so for ease of reference we state it here: first recalling the necessary notation. It follows from [19, Lemma 2.2] that if $g$ is a cyclically minimal element of $G$ and $g = u \circ v$ then $vu$ is cyclically minimal. For a cyclically minimal element $g \in G$ we define $\tilde{g} = \{h \in G | h = vu, \text{ for some } u, v \text{ such that } g = u \circ v\}$. (We allow $u = 1$, $v = g$ so that $g \in \tilde{g}$.)

**Lemma 1.1.** [19, Corollary 2.4] *Let $w, g$ be (minimal forms of) elements of $G$ and $w = u^{-1} \circ v \circ u$, where $v$ is cyclically minimal. Then there exist minimal forms $a$, $b$, $c$, $d_1$, $d_2$ and $e$ such that $g = a \circ b \circ c \circ d_2$, $u = d_1 \circ a^{-1}$, $d = d_1 \circ d_2$, $w^g = d^{-1} \circ e \circ d$, $\tilde{e} = \tilde{v}$, $e = v^b$, $\alpha(b) \subseteq \alpha(v)$ and $[\alpha(b \circ c), \alpha(d_1)] = [\alpha(c), \alpha(v)] = 1$.*

Figure 1 expresses the conclusion of Lemma 1.1 as a Van Kampen diagram. In this diagram we have assumed that $v = b \circ f$ and so $e = f \circ b$. The regions labelled $B$ are tessellated using relators corresponding to the relation $[\alpha(b \circ c), \alpha(d_1)] = 1$ and the region labelled $A$ with relators corresponding to $[\alpha(c), \alpha(v)] = 1$. Reading anticlockwise from the vertex labelled 0 the boundary label of the exterior region is $g^{-1}wg$ and the label of the interior region (not labelled $A$ or $B$) is $e^d$.

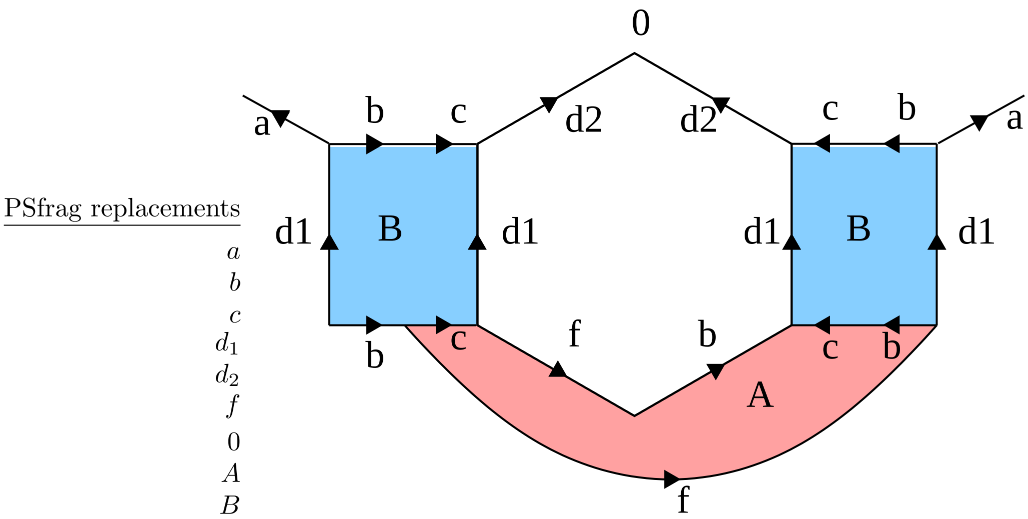


FIGURE 1.1. A Van Kampen diagram for Lemma 1.1

## 2. Parabolic Subgroups

2.1. **Parabolic and Block-Homogeneous Subgroups.** As usual let $\Gamma$ be a graph with vertices $X$ and $G = G(\Gamma)$. If $Y$ is a subset of $X$ denote by $\Gamma(Y)$ the full subgraph of $\Gamma$ with vertices $Y$. Then $G(\Gamma(Y))$ is the free partially commutative group with graph $\Gamma(Y)$. As Baudisch [1] observed $G(\Gamma(Y))$ is the subgroup $\langle Y \rangle$ of $G(\Gamma)$ generated by $Y$. We call $G(\Gamma(Y))$ a *canonical parabolic* subgroup of $G(\Gamma)$ (in keeping with the terminology for analogous subgroups of Coxeter groups) and, when no ambiguity arises, denote it $G(Y)$. Note that such subgroups are called *graphical* in [31], *full* in [23] and *special* in [5]. The elements of $Y$ are termed the *canonical* generators of $G(Y)$.

**Definition 2.1.** *A subgroup $P$ of $G$ is called* parabolic *if it is conjugate to a canonical parabolic subgroup $G(Y)$ for some $Y \subseteq X$. The* rank *of $P$ is the cardinality $|Y|$ and $Y$ is called a set of* canonical generators *for $P$.*

To see that the definition of rank of a parabolic subgroup is well defined note that if $Y, Z \subseteq X$ and $G(Y) = G(Z)^g$, for some $g \in G$, then we have $y = g^{-1} w_y g$, for some $w_y \in G(Z)$, for all $y \in Y$. It follows, from [21, Lemma 2.5], by counting the exponent sums of letters in a geodesic word representing $g^{-1}wg$, that $y \in \alpha(w_y)$, so $y \in Z$. Hence $Y \subseteq Z$ and similarly $Z \subseteq Y$ so $Y = Z$.

**Definition 2.2.** *A subgroup $H$ is called* block-homogeneous *if, for all $h \in H$, if $h$ has block decomposition $h = w_1 w_2 \dots w_k$ then $w_i \in H$, for $i = 1, \dots, k$.*

**Lemma 2.3.** *An intersection of block-homogeneous subgroups is again a block-homogeneous subgroup. If $H$ is block-homogeneous and $g \in G$ then $H^g$ is block-homogeneous. In particular parabolic subgroups are block-homogeneous.*

*Proof.* The first statement follows directly from the definition. Let $H$ be block-homogeneous and $g \in G$ and let $w^g \in H^g$, where $w \in H$. Write $w = u^{-1} \circ v \circ u$, where $v$ is cyclically reduced and has block-decomposition $v = v_1 \cdots v_k$. Then the blocks of $w$ are $v_j^u$, so $v_j^u \in H$, for $j = 1, \dots, k$. From Lemma 1.1 there exist $a, b, c, d_1, d_2, e$ such that $g = a \circ b \circ c \circ d_2$, $u = d_1 \circ a^{-1}$, $d = d_1 \circ d_2$, $w^g = d^{-1} \circ e \circ d$, $\tilde{e} = \tilde{v}$, $e = v^b$, $\alpha(b) \subseteq \alpha(v)$ and $[\alpha(b \circ c), \alpha(d_1)] = [\alpha(c), \alpha(v)] = 1$. As $\tilde{e} = \tilde{v}$ it follows that $\Delta(\alpha(e)) = \Delta(\alpha(v))$ so $e$ has block-decomposition $e = e_1 \cdots e_k$, where $\tilde{e_j} = \tilde{v_j}$. Therefore $w^g$ has block-decomposition $w^g = e^d = e_1^d \cdots e_k^d$. Moreover $e = v^b$ so $e_j = v_j^b$. Thus

$$e_j^d = e_j^{cd} = v_j^{bcd} = v_j^{d_1 bcd_2} = v_j^{ug} \in H^g,$$

which implies that $H^g$ is block-homogeneous. It follows from [21, Lemma 2.5] that any canonical parabolic subgroup is block-homogeneous and this gives the final statement. □

2.2. **Intersections of parabolic subgroups.** In this section we show that an intersection of parabolic subgroups is again a parabolic subgroup. To begin with we establish some preliminary results.

**Lemma 2.4.** *Let $Y, Z \subseteq X$, let $w \in G(Y)$ and let $g \in G(X)$ be such that*

$$\mathrm{gd}\,^l_Y(g) = \mathrm{gd}\,^r_Z(g) = 1.$$

*(1) If $w^g \in G(Z)$ then $g \in A(w)$ and $w \in G(Y) \cap G(Z) = G(Y \cap Z)$.*
*(2) If $Y = Z$ and $g \in C(w)$ then $g \in A(w)$.*

*Proof.* For 1, in the notation of Lemma 1.1 we have $w = u^{-1} \circ v \circ u$, $w^g = d_2^{-1} \circ d_1^{-1} \circ e \circ d_1 \circ d_2$ and $g = a \circ b \circ c \circ d_2$. Applying the conditions of this Lemma we obtain $a = b = d_2 = 1$, $u = d_1$ and $e = v$ so $w^g = w$ and $g = c$. Moreover, from Lemma 1.1 again we obtain $[\alpha(g), \alpha(w)] = 1$. If $x \in \alpha(w) \cap \alpha(g)$ this means that $g = x \circ g'$, with $x \in Y$, contradicting the hypothesis on $g$. Hence $\alpha(w) \cap \alpha(g) = \emptyset$ and $g \in A(w)$. Statement 2 follows from 1. □

**Corollary 2.5.** *Let $Y, Z \subseteq X$ and $g \in G$. If $G(Y)^g \subseteq G(Z)$ and $\mathrm{gd}^l_{Y^\perp}(g) = 1$ then $Y \subseteq Z$ and $\alpha(g) \subseteq Z$.*

*Proof.* Assume first that $\mathrm{gd}^l_Y(g) = \mathrm{gd}^r_Z(g) = 1$. Let $y \in Y$ and $w = y$ in Lemma 2.4; so $y^g \in G(Z)$ implies that $g \in A(y)$ and $y \in Z$. This holds for all $y \in Y$ so we have $Y \subseteq Z$ and $g \in A(Y)$. Hence, in this case, $g = 1$. Now suppose that $g = g_1 \circ d$, where $g_1 = \mathrm{gd}^l_Y(g)$. Then $\mathrm{gd}^l_{Y^\perp}(g) = 1$ implies that $\mathrm{gd}^l_{Y^\perp}(d) = 1$. Now write $d = e \circ g_2$, where $g_2 = \mathrm{gd}^r_Z(d)$. Then $G(Y)^g = G(Y)^d$ and $G(Y)^d = G(Z)$ implies $G(Y)^e = G(Z)$. As $\mathrm{gd}^l_{Y^\perp}(d) = 1$ the same is true of $e$ and from the above we conclude that $e = 1$ and that $Y \subseteq Z$. Now $g = g_1 \circ g_2$, where $\alpha(g_1) \subseteq Y \subseteq Z$ and $\alpha(g_2) \subseteq Z$. Thus $\alpha(g) \subseteq Z$, as required. □

**Proposition 2.6.** *Let $P_1$ and $P_2$ be parabolic subgroups. Then $P = P_1 \cap P_2$ is a parabolic subgroup. If $P_1 \not\subseteq P_2$ then the rank of $P$ is strictly smaller than the rank of $P_1$.*

This lemma follows easily from the next more technical result.

**Lemma 2.7.** *Let $Y, Z \subset X$ and $g \in G$. Then*

$$G(Y) \cap G(Z)^g = G(Y \cap Z \cap T)^{g_2},$$

*where $g = g_1 \circ d \circ g_2$, $\mathrm{gd}^l_Z(d) = \mathrm{gd}^r_Y(d) = 1$, $g_1 \in G(Z)$, $g_2 \in G(Y)$ and $T = \alpha(d)^\perp$.*

*Derivation of Proposition 2.6 from Lemma 2.7.* Let $P_1 = G(Y)^a$ and $P_2 = G(Z)^b$, for some $a, b \in G$. Then $P = \left(G(Y) \cap G(Z)^{ba^{-1}}\right)^a$, which is parabolic since Lemma 2.7 implies that $G(Y) \cap G(Z)^{ba^{-1}}$ is parabolic. Assume that the rank of $P$ is greater than or equal to the rank of $P_1$. Let $g = ba^{-1}$. The rank of $P$ is equal to the rank of $G(Y) \cap G(Z)^g$ and, in the notation of Lemma 2.7, $G(Y) \cap G(Z)^g = G(Y \cap Z \cap T)^{g_2}$, where $g = g_1 \circ d \circ g_2$, with $T = \alpha(d)^\perp$, $g_2 \in G(Y)$ and $g_1 \in G(Z)$. Therefore $Y \subseteq Y \cap Z \cap T$ which implies $Y \subseteq Z \cap T$. Thus $G(Y) \subseteq G(Z \cap T) = G(Z \cap T)^d$ so $G(Y) = G(Y)^{g_2} \subseteq G(Z \cap T)^{dg_2} \subseteq G(Z)^{dg_2} = G(Z)^g$. Hence $P_1 \subseteq P_2$. □

*Proof of Lemma 2.7.* Let $g_1 = \mathrm{gd}^l_Z(g)$ and write $g = g_1 \circ g'$. Let $g_2 = \mathrm{gd}^r_Y(g')$ and write $g' = d \circ g_2$. Then $g_1, g_2$ and $d$ satisfy the conditions of the lemma. Set $T = \alpha(d)^\perp$. As $G(Y) \cap G(Z)^g = G(Y) \cap G(Z)^{dg_2} = G(Y)^{g_2} \cap G(Z)^{dg_2} = \left(G(Y) \cap G(Z)^d\right)^{g_2}$ it suffices to show that $G(Y) \cap G(Z)^d = G(Y \cap Z \cap T)$. If $d = 1$ then $T = X$ and $G(Y) \cap G(Z) = G(Y \cap Z)$, so the result holds. Assume then that $d \neq 1$. Let $p = w^d \in G(Y) \cap G(Z)^d$, with $w \in G(Z)$. Applying Lemma 2.4 to $w^d \in G(Y)$ we have $d \in A(w)$ and $w \in G(Z) \cap G(Y) = G(Y \cap Z)$. Thus $w \in \alpha(d)^\perp = T$ and so $w \in G(Y \cap Z \cap T)$. This shows that $G(Y) \cap G(Z)^d \subseteq G(Y \cap Z \cap T)$ and as the reverse inclusion follows easily the proof is complete. □

**Proposition 2.8.** *The intersection of parabolic subgroups is a parabolic subgroup and can be obtained as an intersection of a finite number of subgroups from the initial set.*

*Proof.* In the case of two parabolic subgroups the result follows from Proposition 2.6. Consequently, the statement also holds for a finite family of parabolic subgroups. For the general case we use Proposition 2.6 again, noting that a proper intersection of two parabolic subgroups is a parabolic subgroup of lower rank, and the result follows. □

As a consequence of this Proposition we obtain: given two parabolic subgroups $P$ and $Q$ the intersection $R$ of all parabolic subgroups containing $P$ and $Q$ is the unique minimal parabolic subgroup containing both $P$ and $Q$. Define $P \vee Q = R$ and $P \wedge Q = P \cap Q$.

**Corollary 2.9.** *The parabolic subgroups of $G$ with the operations $\vee$ and $\wedge$ above form a lattice.*

2.3. **The Lattice of Parabolic Centralisers.** Let $Z \subseteq X$. Then the subgroup $C_G(Z)^g$ is called a parabolic centraliser. As shown in [20, Lemma 2.3] every parabolic centraliser is a parabolic subgroup: in fact $C_G(Z)^g = G(Z^\perp)^g$. The converse also holds as the following proposition shows.

**Proposition 2.10.** *A parabolic subgroup $G(Y)^g$, $Y \subseteq X$ is a centraliser if and only if there exists $Z \subseteq X$ so that $Z^\perp = Y$. In this case $G(Y)^g = C_G(Z^g)$.*

*Proof.* It suffices to prove the proposition for $g = 1$ only. Suppose that there exists such a $Z$. It is then clear that $G(Y) \subseteq C_G(Z)$. If $w \in G$, $w$ is a reduced word and $\alpha(w) \nsubseteq Y$ then there exists $x \in \alpha(w)$ and $z \in Z$ so that $[x, z] \neq 1$ and consequently, by [21, Lemma 2.4], $[w, z] \neq 1$. Assume further that $G(Y)$ is a centraliser of a set of elements $w_1, \dots, w_k$ written in a reduced form. Since for any $y \in Y$ holds $[y, w_i] = 1$ then again, by [21, Lemma 2.4], $[y, \alpha(w_i)] = 1$. Denote $Z = \bigcup\limits_{i=1}^{k} \alpha(w_i)$. We have $[y, z] = 1$ for all $z \in Z$ and consequently $Y \subseteq Z^\perp$. Conversely if $x \in Z^\perp$ then $x \in C_G(w_1, \dots, w_k)$ so $x \in Y$. □

We now introduce the structure of a lattice on the set of all parabolic centralisers. As we have shown above the intersection of two parabolic subgroups is a parabolic subgroup. So, we set $P_1 \wedge P_2 = P_1 \cap P_2$. The most obvious way to define $P_1 \vee P_2$ would be to set $P_1 \vee P_2 = \langle P_1, P_2 \rangle$. However, in this case $P_1 \vee P_2$ is not necessarily a centraliser, though it is a parabolic subgroup. For any $S \subseteq G$ we define the $\overline{S} = \cap\{P : P$ is a parabolic centraliser and $S \subseteq P\}$. Then $\overline{S}$ is the minimal parabolic centraliser containing $S$; since intersections of centralisers are centralisers and intersections of parabolic subgroups are parabolic subgroups. We now define $P_1 \vee P_2 = \overline{\langle P_1, P_2 \rangle}$.

## 3. Quasiparabolic subgroups

3.1. **Preliminaries.** As before let $\Gamma$ be a finite graph with vertex set $X$ and $G = G(\Gamma)$ be the corresponding partially commutative group.

**Definition 3.1.** *Let $w$ be a cyclically minimal root element of $G$ with block decomposition $w = w_1 \cdots w_k$ and let $Z$ be a subset of $X$ such that $Z \subseteq \alpha(w)^\perp$. Then the subgroup $Q = Q(w, Z) = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Z)$ is called a* canonical quasiparabolic subgroup *of $G$.*

Note that we may choose $w = 1$ so that canonical parabolic subgroups are canonical quasiparabolic subgroups. Given a canonical quasiparabolic subgroup $Q(w, Z)$, with $w$ and $Z$ as above, we may reorder the $w_i$ so that $l(w_i) \geq 2$, for $i = 1, \ldots, s$ and $l(w_i) = 1$, for $i = s + 1, \ldots, k$. Then setting $w' = w_1 \cdots w_s$ and $Z' = Z \cup \{w_{s+1}, \ldots, w_k\}$ we have $Z' \subseteq \alpha(w)^\perp$ and $Q(w, Z) = Q(w', Z')$. This prompts the following definition.

**Definition 3.2.** *We say that a canonical quasiparabolic subgroup $Q = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Z)$ is written in standard form if $|\alpha(w_i)| \geq 2$, $i = 1, \ldots, k$, or $w = 1$.*

There are two obvious advantages to the standard form which we record in the following lemma.

**Lemma 3.3.** *The standard form of a canonical quasiparabolic subgroup $Q$ is unique, up to reordering of blocks of $w$. If $Q(w, Z)$ is the standard form of $Q$ then $Z \subseteq \alpha(w)^\perp \setminus \alpha(w)$.*

*Proof.* That the standard form is unique follows from uniqueness of roots of elements in partially commutative groups. The second statement follows directly from the definitions. □

**Definition 3.4.** *A subgroup $H$ of $G$ is called quasiparabolic if it is conjugate to a canonical quasiparabolic subgroup.*

Let $H = Q^g$ be a quasiparabolic subgroup of $G$, where $Q$ is the canonical quasiparabolic subgroup of $G$ in standard form

$$Q = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Z).$$

We call $(|Z|, k)$ the *rank* of $H$. We use the left lexicographical order on ranks of quasiparabolic subgroups: if $H$ and $K$ are quasiparabolic subgroups of ranks $(|Z_H|, k_H)$ and $(|Z_K|, k_K)$, respectively, then $\operatorname{rank}(H) < \operatorname{rank}(K)$ if $(|Z_H|, k_H)$ precedes $(|Z_K|, k_K)$ in left lexicographical order.

The centraliser $C_G(g)$ of an element $g \in G$ is a typical example of a quasiparabolic subgroup [16]. We shall see below (Theorem 3.12) that the centraliser of any set of elements of the group $G$ is a quasiparabolic subgroup.

**Lemma 3.5.** *A quasiparabolic subgroup is a block-homogeneous subgroup and consequently any intersection of quasiparabolic subgroups is again block-homogeneous.*

*Proof.* Let $Q(w, Z)$ be a canonical quasiparabolic subgroup. Since $w$ is a cyclically minimal root element it follows that $Q(w, Z)$ is block-homogeneous. An application of Lemma 2.3 then implies $Q(w, Z)^g$ is also block-homogeneous. □

We shall need the following lemma in Section 4.

**Lemma 3.6.** *Let $Q_1 = Q(u, Y)$ and $Q_2 = Q(v, Z)$ be canonical quasiparabolic subgroups in standard form and let $g \in G$. If $Q_2^g \subseteq Q_1$, $g \in G(Z^\perp)$ and $\mathrm{gd}^r_Y(g) = 1$ then $Q_2^g$ is a canonical quasiparabolic subgroup.*

*Proof.* Let $u$ and $v$ have block decompositions $u = u_1 \cdots u_k$ and $v = v_1 \cdots v_l$, respectively. As $g \in G(Z^\perp)$ we have

$$Q_2^g = \langle v_1^g \rangle \times \cdots \times \langle v_l^g \rangle \times G(Z).$$

Therefore, for $j = 1, \ldots, l$, either $v_j^g = u_i$ for some $i = 1, \ldots, k$, or $v_j^g \in G(Y)$. If $v_j^g = u_i$ then $v_j^g$ is a cyclically minimal root element. If, on the other hand, $v_j^g \in G(Y)$ then, from Lemma 1.1, there exist elements $b, c, d$ and $e$ such that $g = b \circ c \circ d$, $v_j^g = d^{-1} \circ e \circ d$ and $e = v_j^b$ is a cyclically minimal root element. As $v_j^g \in G(Y)$ and $\mathrm{gd}^r_Y(g) = 1$ we have $d = 1$ and so $v_j^g = e$ and is a cyclically minimal root element. Therefore $Q_2^g$ is a canonical quasiparabolic subgroup. □

3.2. **Intersections of Quasiparabolic Subgroups.** The main result of this section is the following

**Theorem 3.7.** *An intersection of quasiparabolic subgroups is a quasiparabolic subgroup.*

We shall make use of the following results.

**Lemma 3.8.** *Let $A = A_1 \times \cdots \times A_l$ and $B = B_1 \times \cdots \times B_k$, $A_i$, $B_j$, $i = 1, \ldots, l$, $j = 1, \ldots, k$ be block-homogeneous subgroups of $G$ and $C = A \cap B$. Then*

$$C = \prod_{\substack{i = 1, \ldots, l;\\ j = 1, \ldots, k}} (A_i \cap B_j).$$

*Proof.* If $C = 1$ then the result is straightforward. Assume then that $C \neq 1$, $w \in C$ and $w \neq 1$ and let $w = w_1 \dots w_t$ be the block decomposition of $w$. Since $C$ is a block-homogeneous subgroup, $w_i \in C$, $i = 1, \dots, t$. As $w_i$ is a block element we have $w_i \in A_r$ and $w_i \in B_s$ and consequently $w_i$ lies in $\prod_{i,j}(A_i \cap B_j)$. As it is clear that $C \geq \prod_{i,j}(A_i \cap B_j)$ this proves the lemma. $\square$

**Lemma 3.9.** *Let $Z \subseteq X$, $w \in G(Z)$, $g \in G$. Suppose that $u = g^{-1}wg$ is cyclically minimal and $\mathrm{gd}^l_{\alpha(w)}(g) = 1$, then $g$ and $w$ commute.*

*Proof.* Let $g = d \circ g_1$, where $d = \mathrm{gd}^l_{\alpha(w)^\perp}(g)$. If $g_1 = 1$ then $g \in C(w)$. Suppose $g_1 \neq 1$. Then $\mathrm{gd}^l_{\alpha(w)}(g_1) = 1$ so we write $g_1 = x \circ g_2$, where $x \in (X \cup X^{-1}) \setminus (\alpha(w) \cup \alpha(w)^\perp)$ and thus $u = g_2^{-1}x^{-1}wxg_2$ is written in geodesic form. This is a contradiction for $l(w) < l(u)$. $\square$

**Lemma 3.10.** *Let*

$$Q_1 = \langle u_1\rangle \times \cdots \times \langle u_l\rangle \times G(Y) \text{ and } Q_2 = \langle v_1\rangle \times \cdots \times \langle v_k\rangle \times G(Z)$$

*be canonical quasiparabolic subgroups in standard form and let $g \in G$ such that $\mathrm{gd}^l_Z(g) = 1$. Write $g = d \circ h$, where $h = \mathrm{gd}^r_Y(g)$ and set $T = \alpha(d)^\perp$. Then, after reordering the $u_i$ and $v_j$ if necessary, there exist $m, s, t$ such that*

$$Q_1 \cap Q_2^g = \left( \prod_{i=1}^{s} \langle v_i\rangle \times \prod_{i=s+1}^{t} \langle v_i\rangle \times \prod_{j=s+1}^{m} \langle u_i\rangle \times G(Y \cap Z \cap T) \right)^g \tag{3.1}$$

*and*

*(i)* $\langle u_i\rangle = \langle v_i\rangle^g$, *for* $i = 1, \dots, s$;
*(ii)* $\langle v_i\rangle^g \subseteq G(Y)$, *for* $i = s+1, \dots, t$; *and*
*(iii)* $\langle u_i\rangle \subseteq G(Z)$, *for* $i = s+1, \dots, m$.

*Proof.* As $Q_1 \cap Q_2^g = (Q_1 \cap Q_2^d)^h$ we may assume that $h = 1$ and $d = g$, so $\mathrm{gd}^r_Y(g) = 1$. As $Q_1$ and $Q_2^g$ are block-homogeneous we may apply Lemma 3.8 to compute their intersection. Therefore we consider the various possible intersections of factors of $Q_1$ and $Q_2^g$.

(i) If $\langle u_i\rangle \cap \langle v_j\rangle^g \neq 1$ then, as $u_i$ and $v_j$ are root elements, $\langle u_i\rangle = \langle v_j\rangle^g$. Suppose that this is the case for $u_1, \dots u_s$ and $v_1, \dots, v_s$ and that $\langle u_i\rangle \cap \langle v_j\rangle^g = 1$, if $i > s$ or $j > s$.
(ii) If $\langle v_j\rangle^g \cap G(Y) \neq 1$ then, since $v_j$ is cyclically minimal, $\langle v_j\rangle^g \subset G(Y)$. This cannot happen if $j \leq s$ so suppose it is the case for $v_{s+1}, \dots, v_t$, and that $\langle v_j\rangle^g \cap G(Y) = 1$, for $j > t$.

(iii) If $\langle u_i\rangle \cap G(Z)^g \neq 1$ then $u_i = w^g$, $w \in G(Z)$ and by Lemma 3.9, $w$ and $g$ commute so does $u_i = w = u_i^g$. This cannot happen if $i \leq s$ so suppose that it's the case for $u_{s+1}, \dots, u_m$, and not for $i > m$.

(iv) Finally, using Lemma 2.7 and the assumption that $\mathrm{gd}^r_Y(g) = \mathrm{gd}^l_Z(g) = 1$, we have $G(Y) \cap G(Z)^g = G(Y \cap Z \cap T) = G(Y \cap Z \cap T)^g$, where $T = \alpha(g)^\perp$.

Combining these intersections (3.1) follows from Lemma 3.8. □

**Corollary 3.11.** *Let $H_1$ and $H_2$ be quasiparabolic subgroups of $G$ then $H_1 \cap H_2$ is quasiparabolic and* $\mathrm{rank}(H_1 \cap H_2) \leq \min\{\mathrm{rank}(H_1), \mathrm{rank}(H_2)\}$.

*Proof.* Let $H_1 = Q_1^f$ and $H_2 = Q_2^g$, where $Q_1 = Q(u, Y)$ and $Q_2 = Q(v, Z)$ are quasiparabolic subgroups in standard form, as in Lemma 3.10. As in the proof of Proposition 2.6 we may assume that $f = 1$ and $\mathrm{gd}^l_Z(g) = 1$ and so Lemma 3.10 implies $H_1 \cap H_2$ is quasiparabolic. If $\mathrm{rank}(H_1 \cap H_2) \geq \mathrm{rank}(H_1)$ then $|Y| \leq |Y \cap Z \cap T|$ so $Y \subseteq Z \cap T$. In this case (ii) of Lemma 3.10 cannot occur. Therefore, in the notation of Lemma 3.10, $\mathrm{rank}(H_1 \cap H_2) = s + m$. If $\mathrm{rank}(H_1 \cap H_2) \geq \mathrm{rank}(H_1)$ then $s + m \geq l$ which implies $m = l - s$ and so $u_i \in G(Z)^g$, for $i = s + 1, \dots, l$. As $u_i = v_i^g$, for $i = 1, \dots, s$ it follows that $H_1 \subseteq H_2$. □

*Proof of Theorem 3.7.* Given Corollary 3.11 the intersection of an infinite collection of quasiparabolic subgroups is equal to the intersection of a finite sub-collection. From Corollary 3.11 again such an intersection is quasiparabolic and the result follows. □

### 3.3. A Criterion for a Subgroup to be a Centraliser.

**Theorem 3.12.** *A subgroup $H$ of $G$ is a centraliser if and only if the two following conditions hold.*

*(1) $H$ is conjugate to some canonical quasiparabolic subgroup $Q$.*
*(2) If $Q$ is written in standard form*

$$Q = \langle w_1\rangle \times \cdots \times \langle w_k\rangle \times G(Y),$$

*where $w = w_1 \dots w_k$ is the block decomposition of a cyclically minimal element $w$, $w_i$ is a root element and $|\alpha(w_i)| \geq 2$, $i = 1, \dots, k$, then*

$Y \in \mathrm{CS}(\Gamma)$ *and* $Y \in \mathrm{CS}(\Gamma_w)$ *where* $\Gamma_w = \Gamma(\alpha(w)^\perp \setminus \alpha(w))$.

*Proof.* Let $H = C(u_1, \dots, u_l)$. Then $H = \bigcap_{i=1}^{k} C(u_i)$ and we may assume that each $u_i$ is a block root element. Since $C(u_i)$ is a quasiparabolic subgroup, then by Theorem 3.7, $H$ is also a quasiparabolic subgroup and is conjugate to a canonical quasiparabolic subgroup $Q = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Y)$ written in standard form. Thus condition 1 is satisfied.

Then $H = Q^g$ and, after conjugating all the $u_i$'s by $g^{-1}$ we have a centraliser $H^{g^{-1}} = Q$. Thus we may assume that $H = Q$. Let $w = w_1 \cdots w_k$, set $Z = \alpha(w)^\perp \setminus \alpha(w)$ and $T = \bigcup_{i=1}^{l} \alpha(u_i)$. As $w$ has block decomposition $w = w_1 \cdots w_k$ we have $C(w) = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Z)$. For all $y \in Y$ we have $y \in C(u_i)$ so and thus $y \in C(\alpha(u_i))$ and $Y \subseteq T^\perp$. Conversely if $y \in T^\perp$ then $y \in C(u_i)$ so $y \in Q$ and, by definition of standard form, $y \in Y$. Therefore $Y = T^\perp$. It follows that $Y \in \mathrm{CS}(\Gamma)$ and since by Lemma 3.3 we have $Y \cap \alpha(w) = \emptyset$ we also have $Y \subseteq Z$.

It remains to prove that $Y \in \mathrm{CS}(\Gamma_w) = \mathrm{CS}(\Gamma(Z))$. Set $W = \alpha(w)$. We show that $T \cup Z \subseteq W \cup Z$. Take $t \in T = \bigcup_{i=1}^{l} \alpha(u_i)$, $t \notin W$ and suppose that $t \in \alpha(u_m)$. Since $w \in C(u_i)$, we have $u_m \in C(w) = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Z)$. Now $u_m$ is a root block element and $C(w)$ is a block-homogeneous subgroup so if $u_m = w_j^{\pm 1}$ for some $j$ then $t \in W = \alpha(w)$, contrary to the choice of $t$. Therefore $u_m \in G(Z)$, so $t \in Z$ and $T \cup Z \subseteq W \cup Z$, as claimed.

Assume now that $Y \notin \mathrm{CS}(\Gamma(Z))$. In this case there exists an element $z \in Z \setminus Y$ such that $z \in \mathrm{cl}_Z(Y)$. Since $z \notin Y = T^\perp$, there exists $u_m$ such that $[u_m, z] \neq 1$ and so there exists $t \in \alpha(u_m)$ such that $[t, z] \neq 1$. As $[z, W] = 1$, we have $t \notin W$ and since $W \cup Z \supseteq T \cup Z$, we get $t \in Z$. This together with $t \in \alpha(u_m) \subseteq Y^\perp$ implies that $t \in \mathcal{O}^Z(Y)$. Since $[z, t] \neq 1$, we obtain $z \notin \mathrm{cl}_Z(Y)$, in contradiction to the choice of $z$. Hence $\mathrm{cl}_Z(Y) = Y$ and $Y \in \mathrm{CS}(\Gamma(Z))$.

Conversely, let $Q = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Y)$ be a canonical quasiparabolic subgroup written in the standard form, $Y \in \mathrm{CS}(X)$ and $Y \in \mathrm{CS}(\Gamma(Z))$, where $Z = \alpha(w)^\perp \setminus \alpha(w)$. We shall prove that $Q = C(w, z_1, \dots, z_l)$, where $z_1, \dots, z_l$ are some elements of $Z$. If $Y = Z$ then $Q = C(w)$. If $Y \subsetneq Z$ then, since $Y = \mathrm{cl}_Z(Y)$, there exist $z_1, u \in Z$ so that $z_1 \in \mathcal{O}^Z(Y)$ and $[z_1, u] \neq 1$. In which case $C(w, z_1) = \langle w_1 \rangle \times \cdots \times \langle w_k \rangle \times G(Y_1)$, $Y \subseteq Y_1 \subsetneq Z$ (the latter inclusion is strict for $u \notin Y_1$). If $Y_1 = Y$ then $Q = C(w, z_1)$, otherwise iterating the procedure above, the statement follows. □

A centraliser which is equal to a canonical quasiparabolic subgroup is called a *canonical quasiparabolic centraliser*.

## 4. Height of the Centraliser Lattice

In this section we will give a new shorter proof of the main theorem of [19].

**Theorem 4.1.** *Let $G = G(\Gamma)$ be a free partially commutative group, let $\mathrm{C}(G)$ be its centraliser lattice and let $L = \mathrm{CS}(\Gamma)$ be the lattice of closed sets of $\Gamma$. Then the height $h(\mathrm{C}(G)) = m$ equals the height $h(L)$ of the lattice of closed sets $L$.*

In order to prove this theorem we introduce some notation for the various parts of canonical quasiparabolic subgroups.

**Definition 4.2.** *Let $Q = \langle w_1\rangle \times \cdots \times \langle w_k\rangle \times G(Z)$ be a quasiparabolic subgroup in standard form. Define the* block set *of $Q$ to be $\mathcal{B}(Q) = \{w_1 \dots, w_k\}$ and the* parabolic part *of $Q$ to be $\mathcal{P}(Q) = G(Z)$. Let $Q'$ be a quasiparabolic subgroup with block set $\langle v_1\rangle \times \cdots \times \langle v_l\rangle$ and parabolic part $G(Y)$. Define the* block difference *of $Q$ and $Q'$ to be $b(Q, Q') = |\mathcal{B}(Q) \setminus \mathcal{B}(Q')|$, that is the number of blocks occurring in the block set of $Q$ but not $Q'$. Define the* parabolic difference *of $Q$ and $Q'$ to be $p(Q, Q') = |Z \setminus Y|$.*

The following lemma is the key to the proof of the theorem above.

**Lemma 4.3.** *Let $C$ and $D$ be canonical quasiparabolic centralisers such that $C > D$ and $b = b(D, C) > 0$. Then $p(C, D) > 0$ and there exists a strictly descending chain of canonical parabolic centralisers*

$$\mathcal{P}(C) > C_b > \cdots > C_1 > \mathcal{P}(D) \tag{4.1}$$

*of length $b + 1$.*

*Proof.* Let $C$ and $D$ have parabolic parts $\mathcal{P}(C) = G(Y)$ and $\mathcal{P}(D) = G(Z)$, for closed subsets $Y$ and $Z$ in $\mathrm{CS}(\Gamma)$. Let the block sets of $C$ and $D$ be $\mathcal{B}(C) = \{u_1, \dots, u_k\}$ and $\mathcal{B}(D) = \{v_1, \dots, v_l\}$. Fix $i$ with $1 \le i \le l$. As $D < C$, either $\langle v_i\rangle = \langle u_j\rangle$, for some $j$, or $\langle v_i\rangle \subseteq G(Y)$. As $b(D, C) > 0$ there exists $i$ such that $\langle v_i\rangle \subseteq G(Y)$. Moreover, for such $i$, we have $\alpha(v_i) \subseteq Y \setminus Z$, so $p(C, D) > 0$.

Assume that, after relabelling if necessary, $\langle v_i\rangle = \langle u_i\rangle$, for $i = 1, \dots, s$, and that $\langle v_{s+1}\rangle, \dots, \langle v_l\rangle \subseteq G(Y)$, so $b = l - s$. Choose $t_i \in \alpha(v_{s+i})$ and let $Y_i = \mathrm{cl}(Z \cup \{t_1, \dots, t_i\})$, for $i = 1, \dots, l-s = b$. Let $C_i = G(Y_i) = C_G(Y_i^\perp)$, so $C_i$ is a canonical parabolic centraliser. We claim that the chain (4.1) is strictly descending. To begin with, as $t_1 \in Y_1 \setminus Z$ we have $G(Z) < C_1$. Now fix $i$ and $n$ such that $1 \le i < n \le b$. If $a \in \alpha(v_{s+n})$ then $a \in Z^\perp$ and $a \in \alpha(v_{s+j})^\perp \subseteq t_j^\perp$, for $1 \le j < n$, by definition of the standard form of quasiparabolic subgroups. Hence $a \in (Z \cup \{t_1, \dots, t_i\})^\perp$. Thus $[a, b] = 1$, for all $b \in Y_i$. This holds for all $a \in \alpha(v_{s+n})$ so $Y_i \subseteq \alpha(v_{s+n})^\perp$. As $v_{s+n}$ is a block of length at least 2 we have $\alpha(v_{s+n})\cap\alpha(v_{s+n})^{\perp}=\emptyset$, so $Y_i\cap\alpha(v_{s+n})=\emptyset$. Hence $t_n\notin Y_i$ and it follows that $C_i<C_{i+1}$, $i=1,\dots,b-1$. Now choose $c\in\alpha(v_{s+1})$ such that $[c,t_1]\neq 1$. Then $c\in Y$, as $\alpha(v_{s+1})\subseteq Y$, however $c\notin Y_b$, since $t_1\in Z^{\perp}\cap t_1^{\perp}\cap\cdots\cap t_b^{\perp}=Y_b^{\perp}$ and $Y_b=Y_b^{\perp\perp}$. As $D<C$ we have $Z\subseteq Y$ so $C_b=G(Y_b)<G(Y)$. □

We can use this lemma to prove the following about chains of canonical quasiparabolic subgroups.

**Lemma 4.4.** *Let $C_0>\cdots>C_d$ be a strictly descending chain of canonical quasiparabolic centralisers such that $C_0$ and $C_d$ are canonical parabolic centralisers. Then there exists a strictly descending chain $C_0>P_1>\cdots>P_{d-1}>C_d$, of canonical parabolic centralisers.*

*Proof.* First we divide the given centraliser chain into types depending on block differences. Then we replace the chain with a chain of canonical parabolic centralisers, using Lemma 4.3. A simple counting argument shows that the new chain has length at least as great as the old one. In detail let $I=\{0,\dots,d-1\}$ and

$$\begin{aligned} I_+ &= \{i\in I : b(C_{i+1},C_i)>0\},\\ I_0 &= \{i\in I : b(C_{i+1},C_i)=0 \text{ and } p(C_i,C_{i+1})>0\} \text{ and}\\ I_- &= \{i\in I : b(C_{i+1},C_i)=p(C_i,C_{i+1})=0\}.\end{aligned}$$

Then $I=I_+\sqcup I_0\sqcup I_-$. For $i\in I_+$ let $\Delta_i$ be the strictly descending chain of canonical parabolic centralisers of length $b(C_{i+1},C_i)+1$ from $\mathcal{P}(C_i)$ to $\mathcal{P}(C_{i+1})$, constructed in Lemma 4.3. For $i\in I_0$ let $\Delta_i$ be the length one chain $\mathcal{P}(C_i)>\mathcal{P}(C_{i+1})$ and for $i\in I_-$ let $\Delta_i$ be the length zero chain $\mathcal{P}(C_i)=\mathcal{P}(C_{i+1})$. This associates a chain $\Delta_i$ of canonical parabolic centralisers to each $i\in I$ and we write $l_i$ for the length of $\Delta_i$. If $\Delta_i=P_0>\cdots>P_{l_i}$ and $\Delta_{i+1}=P'_0>\cdots>P'_{l_{i+1}}$ then by definition $P_{l_i}=P'_0$, for $i=1,\dots,d-1$. We may therefore concatenate $\Delta_i$ and $\Delta_{i+1}$ to give a chain of canonical parabolic centralisers

$$P_0>\cdots>P_{l_i}=P'_0>\cdots>P'_{l_{i+1}}$$

of length $l_i+l_{i+1}$. Concatenating $\Delta_1,\dots,\Delta_{d-1}$ in this way we obtain a strictly descending chain of canonical parabolic centralisers of length $l=\sum_{i=0}^{d-1}l_i$. Moreover

$$l=\sum_{i\in I_+}b(C_{i+1},C_i)+|I_+|+|I_0|,$$

since $l_i = b(C_{i+1}, C_i) + 1$, for all $i \in I_+$, $l_i = 1$, for all $i \in I_0$ and $l_i = 0$, for all $l_i \in I_-$. As $|I| = d$ we have now

$$l - d = \sum_{i \in I_+} b(C_{i+1}, C_i) - |I_-|.$$

To complete the argument we shall show that

$$\sum_{i \in I_+} b(C_{i+1}, C_i) = |\cup_{i=0}^{d} \mathcal{B}(C_i)| \geq |I_-|.$$

As $\mathcal{B}(C_0) = \emptyset$ we have $b(C_1, C_0) = |\mathcal{B}(C_0) \cup \mathcal{B}(C_1)|$. Assume inductively that

$$\sum_{i=0}^{k} b(C_{i+1}, C_i) = |\cup_{i=0}^{k+1} \mathcal{B}(C_i)|,$$

for some $k \geq 0$. Then

$$\sum_{i=0}^{k+1} b(C_{i+1}, C_i) = |\cup_{i=0}^{k+1} \mathcal{B}(C_i)| + |\mathcal{B}(C_{k+2}) \setminus \mathcal{B}(C_{k+1})|.$$

Moreover, if $w \in \mathcal{B}(C_{k+2}) \setminus \mathcal{B}(C_{k+1})$ then $w \in \mathcal{P}(C_j)$, for all $j \leq k+1$, so $w \notin \mathcal{B}(C_j)$, for $j = 0, \ldots, k+1$. Hence

$$\mathcal{B}(C_{k+2}) \setminus \mathcal{B}(C_{k+1}) = \mathcal{B}(C_{k+2}) \setminus \cup_{i=0}^{k+1} \mathcal{B}(C_i)$$

and it follows that

$$\sum_{i=0}^{k+1} b(C_{i+1}, C_i) = |\cup_{i=0}^{k+2} \mathcal{B}(C_i)|.$$

As $b(C_{i+1}, C_i) = 0$ if $i \notin I_+$ it follows that

$$\sum_{i \in I_+} b(C_{i+1}, C_i) = |\cup_{i=0}^{d} \mathcal{B}(C_i)|,$$

as required. If $i \in I_-$ then $b(C_{i+1}, C_i) = p(C_i, C_{i+1}) = 0$, so $b(C_i, C_{i+1}) > 0$. Therefore there is at least one element $w \in \mathcal{B}(C_i) \setminus \mathcal{B}(C_{i+1})$. It follows that $w \notin \mathcal{B}(C_j)$, for all $j \geq i+1$ and so $I_- \leq |\cup_{i=0}^{d} \mathcal{B}(C_i)|$. Therefore $l - d \geq 0$ and the proof is complete. □

*Proof of Theorem 4.1.* Let

$$G = C_0 > \cdots > C_d = Z(G)$$

be a maximal descending chain of centralisers of $G$. By Theorem 3.12, each of the $C_i$'s is a quasiparabolic subgroup. If each $C_i$ is canonical then, since $G$ and $Z(G)$ are both canonical parabolic centralisers the result follows from Lemma 4.4.

Suppose that now $C_1, \dots, C_s$ are canonical quasiparabolic and $C_{s+1}$ is not: say $C_{s+1} = Q^g$, where $Q$ is a canonical quasiparabolic subgroup. Let $C_s = Q(u, Y)$ and $Q = Q(v, Z)$ both in standard form. Write $g = f \circ h$, where $f = \mathrm{gd}^l_{Z^\perp}(g)$ and let $f = e \circ d$, where $d = \mathrm{gd}^r_Y(f)$, so $d \in G(Y \cap Z^\perp)$. Then $G(Z)^h = G(Z)^{dh} = G(Z)^g \subseteq G(Y)$ and $\alpha(h) \subseteq Y$, from Corollary 2.5. Hence $\alpha(d \circ h) \subseteq Y$ which implies that $\alpha(d \circ h) \subseteq \mathcal{P}(C_s) \subseteq \cdots \subseteq \mathcal{P}(C_0)$. It follows that $C_r^{dh} = C_r$, for $r = 0, \dots, s$. Therefore conjugating $C_0 > C_1 > \cdots > C_d$ by $(dh)^{-1}$ we obtain a chain in which $C_0, \dots, C_s$ are unchanged and $C_{s+1} = Q^e = \langle v_1 \rangle^e \times \langle v_l \rangle^e \times G(Z)$, with $\mathrm{gd}^l_Z(e) = \mathrm{gd}^r_Y(e) = 1$, $e \in G(Z^\perp)$. As Lemma 3.6 implies that $Q^e$ is a canonical quasiparabolic subgroup we now have a chain in which $C_0, \dots, C_{s+1}$ are canonical quasiparabolic. Continuing this way we eventually obtain a chain, of length $d$, of canonical quasiparabolic centralisers to which the first part of the proof may be applied. □

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