THE EXISTENCE OF PERIODIC SOLUTIONS OF A TWO DIMENSIONAL LATTICE

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Abstract. We consider a two dimensional lattice coupled with nearest neighbor interaction potential of power type. The existence of infinite many periodic solutions is shown by using minimax methods.

1. Introduction and main results

In this paper we consider a two dimensional forced lattice coupled with nearest neighbor interaction potential of power type $(1 < p < \infty)$, i.e.,

$$\Box u^l = |u^{l+1} - u^l|^{p-1}(u^{l+1} - u^l) - |u^l - u^{l-1}|^{p-1}(u^l - u^{l-1}) + g^l(x,t) \tag{1.1}$$

for $x \in (0, 2\pi), t \in \mathbb{R}, l \in \mathfrak{N}$, satisfying the periodic-Dirichlet conditions

$$u^l(0,t) = 0 = u^l(\pi, t) \quad \forall t \in \mathbb{R}, l \in \mathfrak{N}, \tag{1.2}$$

$$u^l(x,t) = u^l(x, t + 2\pi) \quad \forall x \in (0, \pi), t \in \mathbb{R}, l \in \mathfrak{N}, \tag{1.3}$$

where $\Box u^l = (u^l_{tt} - u^l_{xx})$, $\mathfrak{N} = \{1, 2, \ldots, N\}$, $N \in \mathbb{N}$, $u^{N+1} = u^1, u^0 = u^N$ and $g^l(x, t+2\pi) = g^l(x,t)$ for $l \in \mathfrak{N}$. We are looking for the solutions which are periodic in time.

In last years a considerable effort has been devoted to mathematical study of two dimensional lattice constituted by coupled flexible or elastic elements as strings beams, membranes or plates, etc.. These systems are known as multi-link or multi-body structures, which may generate new, unexpected phenomena. Their practical relevance is huge. However, the mathematical models describing their motions are quite complex. They can be view as systems of partial differential equations on networks or graphs. For the control problems of linear systems, wide information may be found in [5].

On the other hand, a one dimensional lattice takes the form as (1.1)-(1.3) by replacing the operator $\Box$ with d^2/dt^2. Its completely integrability and existence of periodic solutions are well established by [1][6][7][18]. In particular, the classical Toda lattice was shown to be completely integrable, with

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explicit periodic and soliton solutions in [18]. It is well known by the KAM (Kolmogorov-Arnold-Moser) theory [1] that periodic and quasi-periodic solutions of Toda lattice persist under small perturbations. A surprising result, the existence of soliton was got for the power type interaction in [6]. In [7] travelling wave solutions (periodic oscillation and heteroclinic solutions) were constructed on a lattice, corresponding to mass particles interacting nonlinearly with their nearest neighbour (the Fermi-Pasta-Ulam model). It had wide application in many physical systems and biology models.

For simplicity, here we only consider the simplified version of those models to study a lattice coupled with nearest neighbor interaction on one line. This may give us the way to later address more complex situations. To the best of our knowledge, the nonlinear problem of 2-dimensional Toda lattice like (1.1)-(1.3) was firstly introduced by Mikhailov in [10], which showed its integrability by using inverse problem method. Then the reduction problem about the two-dimensional generalized Toda lattice was considered in [11], all involving exponential potentials. The existence of periodic solutions for the 2-dim Toda lattice has been explored in [9] by nonlinear analysis methods. Here we will use minimax methods to find the critical points which correspond to the solutions of (1.1)-(1.3) but with power type interaction. Our main goals are to prove that

Theorem 1.1. *If $g^l \equiv 0$ for all $l \in \mathfrak{N}$, then lattice (1.1)-(1.3) has infinite many solutions.*

Theorem 1.2. *Let $A = \partial_{tt} - \partial_{xx}$ and $\Omega = (0,\pi) \times (0, 2\pi)$. If $g^l \in L^{\alpha/(\alpha-1)}(\Omega)$ satisfies*

$$\int_\Omega g^l \zeta \, dxdt = 0 \quad \text{for all } \zeta \in L^\infty(\Omega) \cap kerA, \;\; l \in \mathfrak{N} = \{1, 2, \ldots, N\},$$

then lattice (1.1)-(1.3) has infinite many solutions, where $\{2, p\} < \alpha < p+1$.

Remark. If $g^l \in \ker A$ satisfies $\sum_l g^l \neq 0$, then the problem (1.1)-(1.3) has no periodic solution.

Since the null of the operator $\Box u = (u^l_{tt} - u^l_{xx})_{l=1,\ldots,N}$ is infinite dimensional and the embedding operator from vector space of p-integrable functions to this null is not compact, we turn to study the functional $I(\varphi)$ (see (2.8)) as Tanaka did in [17]. Firstly we shall apply linking theorem of Benci and Rabinowtz in [4] to obtain Theorem 1.1, which is in section 2. Theorem 1.2 is established from section 4 to section 7.

Theorem 1.2 will be proved with the aid of standard argument for a perturbation from Z_2 symmetry. Here our steps follow Tanaka's framework in [17]. Firstly we introduce a functional I of the problem (1.1)-(1.3), (see

(2.8))

$$I(\varphi) = \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - Q(\varphi),$$

where $\|\cdot\|$ is the norm of the space E (see (2.5)), $Q(\varphi) := \min_{\psi\in E^0}\int_\Omega F_{\delta,g}(\varphi+\psi)dxdt$ and

$$F_{\delta,g}(u) = \frac{1}{p+1}\sum_{l=1}^{N}|u^{l+1}-u^l|^{p+1} - (g,u) + \frac{\delta}{p+1}\sum_{l=1}^{N}|u^l|^{p+1},$$

for $\varphi = \varphi^+ + \varphi^- \in E := E^+ \oplus E^-$, which is strictly convex functional (the term $\delta\sum_{l=1}^N |u^l|$ makes it strictly convex). Note that critical points of I and weak solutions of (1.1)-(1.3) possess one-to-one correspondence after taking limit $\delta \to 0$ and that $Q'(\varphi)$ is compact in E. Since there is the force term $g \not\equiv 0$, the method, which is applicable to treat I, is to make a simple modification: (see (4.1) in section 4),

$$J(\varphi) = \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - Q_0(\varphi) - \tilde{\chi}(\varphi)(Q(\varphi) - Q_0(\varphi)).$$

Secondly in section 5 we apply the methods of Rabinowitz [14] to $I(\varphi^+ + \varphi^-)$ and obtain the existence of infinitely many solutions of (1.1)-(1.3) under some assumption

$$b_{q_j} \geq C_1 q_j^{(p+1)/(p-1)-\epsilon} - C_2 \qquad \text{for a large number } q_j \in \mathbb{N}, \tag{1.4}$$

where $b_q = \inf_{\gamma\in\Gamma_q}\sup_{\varphi\in D_q} J(\gamma(\varphi))$ (see (5.2)). This assumption will tell us that the critical points, which we find by minimax methods, have large critical values. To prove the assumption (1.4), we follow Tanaka's steps which were introduced by Bahri-Berestycki to find comparison value σ_q such that $\sigma_q \leq b_q + C_2$ and

$$\sigma_{q_j} \geq C_1 q_j^{(p+1)/(p-1)-\epsilon}, \qquad \text{for a large number } q_j \in \mathbb{N}, \tag{1.5}$$

where $\sigma_q^n = \sup_{\sigma\in A_q^n}\min_{y\in S^{n-q}} K(\sigma(y))$ (see (5.7)). In the proof of (1.5), a Morse index estimate and the spectral estimate play an essential role as Tanaka in [17], which are in section 6 and finally we shall complete the proof of Theorem 1.2 in section 7.

2. Preliminaries

For convenience, we use the following standard notations. Let E be a Hilbert space, $\langle\cdot,\cdot\rangle$ be inner product in Hilbert space E or the dual bracket between Hilbert space E and its dual space E^*. $(\cdot,\cdot)$ be inner product in $\mathbb{R}^N$.

Let $\Omega = (0,\pi)\times(0,2\pi)$. We firstly recall basic properties of the operator $A = \partial_{tt} - \partial_{xx}$ acting on integrable functions which are $2\pi-$periodic and satisfy Dirichlet boundary conditions. It is well known that $A\zeta = 0$ (in the weak sense) if and only if

$$\zeta(x,t) = \gamma(x+t) - \gamma(x-t)$$

for some $\gamma \in L^1_{loc}(\mathbb{R})$, 2π-periodic and such that $\int_0^{2\pi} \gamma(s)\, ds = 0$. Also, for a given integrable f_0, we have that if

$$\int_\Omega f_0 \zeta \, dxdt = 0 \quad \text{for all } \zeta \in L^\infty(\Omega) \cap \ker A,$$

then there is a unique 2π-periodic continuous function w^0 satisfying

$$w^0_{tt} - w^0_{xx} = f_0 \tag{2.1}$$

and the boundary condition $w^0(0,t) = w^0(\pi,t) = 0$ for all t. From here, it is clear that if $f_0 \in \ker A$, then there is no solution for the equation (2.1).

Denote $u(x,t) = (u^1(x,t), \dots, u^N(x,t))$ for $(x,t) \in \Omega$. For $p \in [1,\infty)$ set by L^q the vector of 2π-time periodic functions of t whose p-th powers are integrable, i.e.,

$$\|u\|_p = (\int_\Omega \sum_{l=1}^N |u^l(x,t)|^p \, dxdt)^{1/p} < \infty.$$

The vector of smooth functions satisfying (1.2),(1.3) has a Fourier expansion of the form

$$u = \sum_{l=1}^N \sum_{j=1}^\infty \sum_{k=-\infty}^\infty a^l_{jk} \sin jx e^{ikt} e_l, \quad a^l_{j,-k} = \bar{a}^l_{jk}, \tag{2.2}$$

where $\{e_1, \cdots, e_N\}$ is the usual orthogonal basis in $\mathbb{R}^N$. We define

$$\langle u, v\rangle_W = \frac{1}{4}|\Omega| \sum_{l,j,k} |k^2 - j^2| a^l_{jk} \bar{b}^l_{jk}, \quad \|u\|^2_W = \langle u, u\rangle,$$

for $u = \sum_{l=1}^N \sum_{j=1}^\infty \sum_{k=-\infty}^\infty a^l_{jk} \sin jx e^{ikt} e_l$ and $v = \sum_{l=1}^N \sum_{j=1}^\infty \sum_{k=-\infty}^\infty b^l_{jk} \sin jx e^{ikt} e_l$. We observe that $\|\cdot\|_W$ is a norm on the set $\{u | \, a^l_{jk} = 0 \text{ if } j = |k|\}$. Let

$$W^+ = \overline{span}\{(\sin jx e^{ikt} e_l) | \, j < |k|\},$$
$$W^- = \overline{span}\{(\sin jx e^{ikt} e_l) | \, j > |k|\},$$
$$W = W^+ \oplus W^-,$$

where the closures are taken under the norm $\|\cdot\|_W$ and $(\sin jx e^{ikt} e_l)$ is the vector $(\sin jx e^{ikt} e_1, \cdots, \sin jx e^{ikt} e_l, \cdots, \sin jx e^{ikt} e_N)$. Note that $(W, \langle, \rangle)$

is a Hilbert space. Further set

$$W^0 = L^{p+1} - \text{closure of } span\{(sinjxe^{\pm ijt}e_l)|\, j \in \mathbb{N}, l \in \mathfrak{N}\}.$$

Then W^+, W^-, W^0 are complementary subspaces of the vector space of functions satisfying (1.2),(1.3). Moreover, the space W has the following property (see Lemma 2.1):

(2.3) $\|u\|_p \le C_p \|u\|_W$ for all $u \in W$ and $p \in [1, \infty)$,

(2.4) the embedding $W \to L^p$ is compact for all $p \in [1, \infty)$,

where C_p is positive constant. However the embedding $W^0 :\to L^p$ is not compact.

Now we are going to look for a subspace of W, which will be our workspace, called E. For this purpose, we need some facts in $\mathbb{R}^N$. Given $\xi \in \mathbb{R}^N$ let $[\xi] = \max\{\xi^1 - \xi^N, \xi^2 - \xi^1, \cdots, \xi^N - \xi^{N-1}\}$. Denote $1\!\!1 := (1, 1, \cdots, 1) \in \mathbb{R}^N$ and its orthogonal space by $\{1\!\!1\}^\perp$. It known from [9] that

$$[\xi] \ge \frac{2}{N-1}|\xi|_\infty \ge \frac{2}{\sqrt{N}(N-1)}|\xi| \quad \text{for } \xi \in 1\!\!1^\perp.$$

Let

$$F_{\delta,0}(\xi) = \frac{1}{p+1}\sum_{l=1}^{N} |\xi^{l+1} - \xi^l|^{p+1} + \frac{\delta}{p+1}\sum_{l=1}^{N} |\xi^l|^{p+1}, \quad \delta \in (0, 1),$$

where $\xi^{N+1} = \xi^1, \xi^0 = \xi^N$. Then we have that qualitative properties of $F_{\delta,0}$ are as follows (see Lemma 2.2):
(F1) $F_{\delta,0} \in C^2(\mathbb{R} \cap \{1\!\!1\}^\perp, \mathbb{R})$ is strictly convex and coercive.
(F2) There exists $\alpha \in (2, p+1)$ such that $(\xi, \nabla F_{\delta,0}(\xi)) \ge \alpha F_{\delta,0}(\xi) > 0$ for $\xi \in \mathbb{R}^N \cap \{1\!\!1\}^\perp$.

Let $\overline{w}$ be a function such that $(\overline{w}(x,t)1\!\!1) := (\overline{w}(x,t), \overline{w}(x,t), \cdots, \overline{w}(x,t)) \in W \oplus W^0$. Denote $X_1 = \{\overline{w}1\!\!1 \in W \oplus W^0\}$ and its orthogonal space by $X_2 = X_1^\perp$. It is clear that $X_2 = \{(u^0, \cdots, u^N)|\ \sum_{l=0}^N u^l = 0\}$. Define

$$E^+ = W^+ \cap X_2, \quad E^- = W^- \cap X_2,$$
$$E = E^+ \oplus E^-, \quad E^0 = W^0 \cap X_2. \tag{2.5}$$

Note that $W \oplus W^0 = [(W \oplus W^0) \cap X_1] \oplus (E \oplus E^0)$.

On the other hand, it is easy to see that the solutions of problem (1.1) satisfying conditions (1.2),(1.3) are corresponded to a critical points of the functional

$$G_0(u) := \frac{1}{2}\int_\Omega (|\partial_t u|^2 - |\partial_x u|^2)\, dxdt - \int_\Omega F_{0,g}(u)\, dxdt, \tag{2.6}$$

where

$$F_{0,g}(u)=\frac{1}{p+1}\sum_{l=1}^{N}|u^{l+1}-u^l|^{p+1}-(g,u), u^{N+1}=u^1, u^0=u^N,\ u\in W\oplus W^0.$$

We can divide $g=\bar{g}\mathbb{1}+\hat{g}$ such that $(\bar{g}\mathbb{1})\in(W\oplus W^0)\cap X_1, \hat{g}\in E\oplus E^0=(W\oplus W^0)\cap X_2$. Similarly we have $w=\overline{w}\mathbb{1}+\hat{w}$. It is easy to see that the problem (1.1)-(1.3) in $(W\oplus W^0)\cap X_1$ becomes a scalar equation

$$\overline{w}_{tt}-\overline{w}_{xx}=\bar{g},$$

where $\overline{w}$ be a 2π periodic function satisfying Dirichlet condition and $\bar{g}\in(W\oplus W^0)\cap X_1$. If $g^l\in L^{\alpha/(\alpha-1)}(\Omega)$ $(\alpha>1)$satisfies

$$\int_\Omega g^l\zeta\,dxdt=0\quad\text{for all }\zeta\in L^\infty(\Omega)\cap\ker A,\ \ l\in\mathfrak{N},$$

then we know from (2.1) that there is a unique solution $\overline{w}$. Moreover, note that if u is a critical point of $G_0(u)$ in E, $u+w$ is a solution. To see that, we know that for $\xi\in(W\oplus W^0)\cap X_1$ and $\zeta\in E\oplus E^0$,

$$\begin{aligned}&\langle(G_0)'(u+w),(\zeta+\xi)\rangle=\\&\quad=\int_\Omega\sum_{l=0}^{N}\Big[u_t^l\zeta_t^l+w_t^l\xi_t^l-u_t^l\zeta_x^l-w_x^l\xi_t^l-(F_{0,0}(u),\zeta)+\hat{g}^l\zeta^l+\tilde{g}^l\xi^l\Big]\,dt\\&\quad=\langle(G_0|_{g\in E\oplus E^0})'(u),\zeta\rangle.\end{aligned}$$

Without loss of generality, we may assume $\sum_{i=0}^N g^l=0$, that is, the critical points of G_0 in $E\oplus E^0$ are solutions for lattice (1.1)-(1.3).

To find the critical points of G_0, we introduce the aid of a modified functional to treat the loss of compactness of the embedding $E^0\to L^{p+1}$ as follows:

$$G_{\delta,g}(u)\ :=\ \frac{1}{2}\int_\Omega|\partial_t u|^2-|\partial_x u|^2\,dxdt-\int_\Omega F_{\delta,g}(u)\,dxdt,$$

for $u\in E\oplus E^0$, where

$$F_{\delta,g}(u)=F_{0,g}(u)+\frac{\delta}{p+1}\sum_{l=1}^{N}|u^l|^{p+1}\quad\text{for}\quad\delta\in(0,1),\tag{2.7}$$

which is strictly convex. The wave form is positive definitely, negative definitely and null on E^+,E^-,E^0 respectively.

From the above properties of $F_{\delta,0}$, we know that for $\varphi = \varphi^+ + \varphi^- \in E$ and $\psi \in E^0$,

$$\begin{aligned} G_{\delta,g}(u) &= G_{\delta,g}(\varphi^+ + \varphi^- + \psi) = \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 \\ &\quad - \int_\Omega F_{\delta,g}(\varphi^+ + \varphi^- + \psi)\, dxdt \in C^2(E^+ \oplus E^- \oplus E^0, \mathbb{R}). \end{aligned}$$

Observe also that for fixed φ^+, φ^-, the functional $G_{\delta,g}(\varphi^+ + \varphi^- + \psi)$ is a strictly concave function of $\psi \in E^0$. So there is one to one correspondence between every critical point of $G_{\delta,g}$ and that of I. Here $I : E \to \mathbb{R}$ is a functional for $\varphi = \varphi^+ + \varphi^- \in E$, defined by

$$I(\varphi) = \max_{\psi \in E^0} G_{\delta,g}(\varphi + \psi) = \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - Q(\varphi) \tag{2.8}$$

where

$$Q(\varphi) := \min_{\psi \in E^0} \int_\Omega F_{\delta,g}(\varphi + \psi)\, dxdt \text{ for } \varphi \in E. \tag{2.9}$$

We will treat the problem (1.1)-(1.3) by finding the critical points of the functional $I(\varphi)$ in E and taking limit $\delta \to 0$. Note that $Q(\varphi)$ can be also defined for all $\varphi \in L^{p+1}$ by (2.9). Now we give the proof of some lemmas that we mentioned above and also study the space E and treat $Q(\varphi)$ as a functional from L^{p+1} to $\mathbb{R}$ at the end of this section.

Lemma 2.1. *There is a positive constant C_q such that*

$$\begin{aligned} &\|u\|_q \le C_q \|u\|_W \text{ for all } u \in W \text{ and } q \in [1, \infty), \\ &\text{the embedding } W \to L^q \text{ is compact for all } q \in [1, \infty). \end{aligned}$$

In particular, the Hilbert space E has the same compact embedding properties of the space W.

Proof. Here we follow the steps in [16]. It is easy to see the case $q = 2$, so we can set $q > 2$. Note that $u \in W$ have the form $(j \ne |k|)$:

$$u = \sum_{l=1}^{N} \sum_{j=1}^{\infty} \sum_{k=-\infty}^{\infty} a_{jk}^l \sin jx e^{ikt} e_l, \quad a_{j,-k}^l = \bar{a}_{jk}^l.$$

Denote $a_{jk} = (a_{jk}^1, \cdots, a_{jk}^N)$ with the modulus $|a_{jk}|$. We first note that for $p > 1$

$$\begin{aligned} \sum_{j \ne |k|} |j^2 - k^2|^{-p} &\le 4 \sum_{j<|k|} (k^2 - j^2)^{-p} + \sum_{j \ge 1} j^{-2p} = \sum_{j,l \ge 1} l^{-p}(2j + l)^{-p} + \sum_{j \ge 1} j^{-2p} \\ &\le 2 \sum_{j,l \ge 1} l^{-p} j^{-p} + \sum_{j \ge 1} j^{-2p} \le 3 \sum_{j \ge 1} j^{-2p} \le 3(\frac{p}{p-1})^2. \end{aligned}$$

Then for $q > 2$, by F. Riesz's Theorem in [19], Hölder's inequality with $p = q/(q-2)$,

$$\|u\|_q \leq \pi^{2/q}\Big(\sum_{j\neq|k|}|a_{jk}|^{q/(q-1)}\Big)^{(q-1)/q}$$
$$\leq \pi^{2/q}\Big(\sum_{j\neq|k|}|j^2-k^2||a_{jk}|^2\Big)^{1/2}\Big(\sum_{j\neq|k|}|j^2-k^2|^{-q/(q-2)}\Big)^{(q-2)/2q}$$
$$\leq C_q\|u\|_E.$$

The compactness of the embedding is a standard result. It is enough to prove that there exist constants $0 < \mu, \nu < 1/2$, $C > 0$ such that

$$|u(x,t) - u(x',t')| \leq C(|x-x'|^\mu + |t-t'|^\nu).$$

Firstly we claim

$$|\sin jxe^{ikt} - \sin jx'e^{ikt'}| \leq c_\mu|x-x'|^\mu + c_\nu|t-t'|^\nu.$$

In fact,

$$|e^{ikt} - e^{ikt'}| = |e^{ik(t-t')} - 1| \leq c_\mu|t-t'|^\mu \quad \text{for } 0 \leq \mu \leq 1.$$

Moreover, if $j|x-x'| \leq 1$ then

$$|\sin jx - \sin jx'| \leq j|x-x'| \leq j^\nu|x-x'|^\nu;$$

if $j|x-x'| > 1$, then

$$|\sin jx - \sin jx'| \leq 2 \leq 2j^\nu|x-x'|^\nu.$$

Therefore, using

$$|\sin jxe^{ikt} - \sin jx'e^{ikt'}| \leq 2|\sin jx - \sin jx'| + |e^{ikt} - e^{ikt'}|,$$

we can get the claim. By the form of (2.2), we have

$$|u(x,t) - u(x',t')| = |\sum_{j,k,l}(a^l_{jk}\Big(\sin jxe^{ikt}e_l - \sin jx'e^{ikt'}e_l\Big)|$$
$$\leq \sum_{j,k}|\sin jxe^{ikt} - \sin jx'e^{ikt'}||a_{jk}| \leq \sum_{j,k}(c_\mu|x-x'|^\mu + c_\nu|t-t'|^\nu)|a_{jk}|$$
$$\leq C\Big[|x-x'|^\mu\Big(\sum_{j,k}\frac{j^{2\mu}}{|j^2-k^2|}\Big)^{1/2} + |t-t'|^\nu\Big(\sum_{j,k}\frac{j^{2\nu}}{|j^2-k^2|}\Big)^{1/2}\Big]\cdot$$
$$\cdot\Big(\sum_{j,k}|j^2-k^2||a_{jk}|^2\Big)^{1/2}$$
$$\leq C\Big[|x-x'|^\mu\Big(\sum_{j,k}\frac{j^{2\mu}}{|j^2-k^2|}\Big)^{1/2} + |t-t'|^\nu\Big(\sum_{j,k}\frac{j^{2\nu}}{|j^2-k^2|}\Big)^{1/2}\Big]\|u\|^2_E.$$

Let $u(x,t) \in W^+$, that is $j < |k|$, and it is similar to W^-. Then we have

$$\sum \frac{j^{2\mu}}{|j^2 - k^2|} = 2 \sum_{0<j<k} \frac{j^{2\mu}}{(k-j)(k+j)}$$

$$\leq 2 \sum_{0\leq j<k} \frac{1}{(k-j)j^{1-2\mu}} = \sum_{k,j>0} \frac{1}{k(k+j)^{1-2\mu}} \leq \sum_k \frac{1}{j^{2-2\mu}} < \infty.$$

Thus we can obtain the equi-continuous and the uniform bound for the sequence $\{u_n\}$ in W^+. It follows the compactness. □

Lemma 2.2. *Let* $F_{\delta,0}(\xi) = \frac{1}{p+1}\sum_{l=1}^N |\xi^{l+1} - \xi^l|^{p+1} + \frac{\delta}{p+1}\sum_{l=1}^N |\xi^l|^{p+1}$ *and* $\delta \in (0,1)$. *Then*
(i) $F_{\delta,0} \in C^2(\mathbb{R}^N \cap \{\mathbb{1}\}^\perp, \mathbb{R})$ *is strictly convex and coercive.*
(ii) There exists $\alpha \in (2, p+1)$ *such that* $(\xi, \nabla F_{\delta,0}(\xi)) \geq \alpha F_{\delta,0}(\xi) > 0$ *for* $\xi \in \mathbb{R}^N \cap \{\mathbb{1}\}^\perp$.
(iii) There is a positive constant C_1 *such that for* $\xi \in \mathbb{R}^N \cap \{\mathbb{1}\}^\perp$

$$F_{\delta,0}(\xi) \geq C_1 |\xi|^\alpha. \tag{2.10}$$

Proof. (i) Since $\frac{d^2 F_{\delta,0}(\xi+s\zeta)}{dt^2}|_{s=0} = \sum_{l=1}^N p(|\xi^l - \xi^{l-1}|^{p-1} + \delta|\xi^l|^{p-1})(\zeta^l - \zeta^{l-1})^2$, we see that F_δ is strictly convex in $\{\mathbb{1}\}^\perp$.
(ii) It is clear that for $\alpha < p+1$,

$$\langle \nabla F_{\delta,0}(\xi), \xi\rangle - \alpha F_{\delta,0}(\xi) = (1 - \frac{\alpha}{p+1})[\sum_{l=1}^N |\xi^l - \xi^{l-1}|^{p+1} + \delta \sum_{l=1}^N |\xi^l|^{p+1}] \geq 0.$$

Hence we have

$$\langle \nabla F_{\delta,0}(\xi), \xi\rangle \geq \alpha F_{\delta,0}(\xi),$$

for $\alpha \in (m_0, p+1)$ and $\xi \in \{\mathbb{1}\}^\perp$.
(iii) Let $|\xi| = s$ and $\frac{\xi}{|\xi|} = \eta$, so $\xi = s\eta$ and

$$\langle \nabla F_{\delta,0}(s\eta), s\eta\rangle \geq \alpha F_{\delta,0}(s\eta).$$

It directly follows

$$F_{\delta,0}(s\eta) \geq C_1 s^\alpha.$$

□

Lemma 2.3. *(i) For all* $\varphi \in L^p$, *there exists a unique* $\psi(\varphi) \in E^0$ *such that*

$$Q(\varphi) = \int_\Omega F_{\delta,g}(\varphi + \psi(\varphi))\, dxdt. \tag{2.11}$$

(ii) $\psi(\varphi) : L^{p+1} \to E^0$ *is continuous.*
(iii) $Q(\varphi)$ *is of class* C^1 *on* E *and*

$$\langle Q'(\varphi), h\rangle = \int_\Omega (\nabla F_{\delta,g}(\varphi + \psi), h)\, dxdt \quad \textit{for all} \;\; \varphi, h \in E. \tag{2.12}$$

In particular, $Q'(\varphi):\ E\to E^*$ *is compact and for all* $\varphi\in E$.

Proof. The above lemma is a slight generalization of a result by Tanaka [17].
(i) From the property (i) of $F_{\delta,0}$ in Lemma 2.2, we can easily deduce assertion (i) from the fact that

$$\psi\to\int_\Omega F_{\delta,g}(\varphi+\psi)\,dxdt \tag{2.13}$$

is a strictly convex and coercive functional on E^0.
(ii) Suppose that $\varphi_j\to\varphi$ in $L^{p+1}(\Omega)$. We will show that $\psi(\varphi_j)\to\psi(\varphi)$ strongly in E^0. By the definition of $\psi(\varphi_j)$, we have

$$\int_\Omega F_{\delta,g}(\varphi_j+\psi(\varphi))\,dxdt\geq\int_\Omega F_{\delta,g}(\varphi_j+\psi(\varphi_j))\,dxdt. \tag{2.14}$$

We find that $\{\psi(\varphi_j)\}_{j=1}^\infty$ is bounded in E^0 (i.e., in L^{p+1}). We extract a subsequence–still denoted by $\psi(\varphi_j)$–converges weakly to $\bar\psi$ in E^0. Letting $j\to\infty$ in (2.14), by Fatou Lemma and the weak continuity we get

$$\int_\Omega F_{\delta,g}(\varphi+\psi(\varphi))dxdt\geq\limsup_{j\to\infty}\int_\Omega F_{\delta,g}(\varphi_j+\psi(\varphi_j))dxdt\geq\int_\Omega F_{\delta,g}(\varphi+\bar\psi)dxdt.$$

By the uniqueness of $\psi(\varphi)$, we observe $\bar\psi=\psi(\varphi)$ and $\limsup F_{\delta,g}(\varphi_j+\psi(\varphi_j))=F_{\delta,g}(\varphi+\psi(\varphi))$. Thus we obtain $\psi(\varphi_j)\to\psi(\varphi)$ in E^0.
(iii) By the convexity of (2.13), we find that for $w\in E^0$,

$$w=\psi(\varphi)\quad\text{iff}\quad\int_\Omega(\nabla F_{\delta,g}(\varphi+w),\zeta)\,dxdt=0\text{ for all }\zeta\in E^0. \tag{2.15}$$

From the convexity of the function $F_{\delta,g}$ and minimality property of $\psi(\varphi)$, we have for all $\varphi,h\in E$ and $s>0$,

$$Q(\varphi+sh)-Q(\varphi)\geq\int_\Omega(\nabla F_{\delta,g}(\varphi+\psi(\varphi)),sh+\psi(\varphi+sh)-\psi(\varphi))\,dxdt. \tag{2.16}$$

Since $\psi(\varphi+sh)-\psi(\varphi)\in E^0$, we get by (2.15)

$$Q(\varphi+sh)-Q(\varphi)\geq\int_\Omega(\nabla F_{\delta,g}(\varphi+\psi(\varphi)),sh)\,dxdt. \tag{2.17}$$

Similarly we have

$$Q(\varphi+sh)-Q(\varphi)\leq\int_\Omega(\nabla F_{\delta,g}(\varphi+sh+\psi(\varphi+sh)),sh)\,dxdt. \tag{2.18}$$

Letting $s\to0$ in (2.17) and (2.18), we obtain (2.12). Thus $Q(\varphi)\in C^1(E,\mathbb{R})$. Moreover from (2.4) and the continuity of $\psi(\varphi):\ L^{p+1}\to E^0$, we deduce that $Q'(\varphi):\ E\to E^*$ is compact.

□

3. Critical points and periodic solutions in the autonomous case

In this section, we will go to prove the existence of infinite many solutions for the autonomous case, i.e. $g \equiv 0$. Therefore,

$$I(\varphi) = \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - Q_0(\varphi) \text{ for } \varphi = \varphi^+ + \varphi^- \in E.$$

We use the notation B_r for the open ball with radius r and $a_i, r_i, i \in \mathbb{N}$ denote nonnegative constants.

Lemma 3.1. [4] *Let E be a real Hilbert space, E_1 a closed subspace of E, and $E_2 = E_1^\perp$. Denote $I(u) = \Phi(u) + b(u)$. Suppose that $I \in C^1(E, \mathbb{R})$ and satisfies*
(I_1) $\Phi(u) = \frac{1}{2}\langle Lu, u\rangle$ where $u = u_1 + u_2 \in E_1 \oplus E_2, Lu = L_1u_1 + L_2u_2$ and $L_i : E_i \to E_i, i = 1, 2$ is a (bounded) lineal self adjoint mapping.
(I_2) b is weakly continuous and uniformly differentiable on bounded subsets of E.
(I_3) If for a sequence $\{u_m\}_{m=1}^\infty$, $I(u_m)$ is bounded from above and $I'(u_m) \to 0$ as $m \to \infty$ then $\{u_m\}_{m=1}^\infty$ is bounded.
(I_4) There are constants $r_1, r_2, \alpha, \rho, \omega$ with $r_1 > \rho, \alpha > \omega$ and $r_1, r_2, \rho > 0$ and there is an $e \in \partial B_1 \cap E_1$ such that (i) $I \geq \alpha$ on $S := \partial B_\rho \cap E_1$, (ii) $I \leq \omega$ on $\partial\Sigma$ where $\Sigma := \{re|\, 0 \leq r \leq r_1\} \oplus (B_{r_2} \cap E_2)$.
Then I possesses a critical value $c \geq \alpha$.

Lemma 3.2. *(see Proposition 3.12 in* [4]*) Let $\Omega \subset \mathbb{R}^i$ be a bounded domain and for some $s, \widehat{s}, g \in C^1(\Omega \times \mathbb{R}^k, \mathbb{R})$ where $g = g(x, z)$ and*

$$|g_z(x, z)| \leq C_1 + C_2|z|^s, 1 \leq s < \widehat{s}.$$

If $\widehat{E}$ is a subspace of $L^1(\Omega)^k$ with $\|z\|_r \leq a_r\|z\|_{\widehat{E}}$ for all $z \in \widehat{E}$ and all $r \in [2, \widehat{s} + 1]$, then $b(z)$ is uniformly differentiable on bounded subsets of $\widehat{E}$.

Proof of Theorem 1.1. We apply Lemma 3.1 to give the proof of Theorem 1.1 as in [4]. Let $E_1 = E^+$ and $E_2 = E^-$. Then $E_2 = E_1^\perp$, and J satisfies (I_1) with $L_i\varphi_i$ be defined for $\varphi_i = \varphi^\pm \in E_i$ by

$$\langle L_i\varphi_i, \zeta_i\rangle = \int_\Omega \partial_t\varphi_i\partial_t\zeta_i - \partial_x\varphi_i\partial_x\zeta_i \, dxdt,$$

for all $\zeta_i \in E_i, i = 1, 2$. Hence for $\varphi \in E$,

$$\Phi(\varphi) = \frac{1}{2}\langle L\varphi, \varphi\rangle = \frac{1}{2}\int_\Omega |\partial_t\varphi|^2 - |\partial_x\varphi|^2 \, dxdt$$

and

$$b(\varphi) = -Q_0(\varphi) = -\int_\Omega F_{\delta,0}(\varphi + \psi(\varphi)) \, dxdt.$$

By Lemma 2.3 (ii) we get $I \in C^1(E, \mathbb{R})$, It remains to prove $I(\varphi)$ satisfies *(I_2)-(I_4)*.

Step 1. We claim that b is weakly continuous. To prove this, let $\{\varphi_j\} \subset E$ and suppose φ_j converges weakly to φ in E. Then by (2.4), $\varphi_j \to \varphi$ in L^β for all $\beta \in [1, \infty)$. Since

$$F_{\delta,0}(\varphi) \leq C|\varphi|^{p+1}, \tag{3.1}$$

choosing $\beta = p+1$, we see $\varphi_j \to \varphi$ in L^{p+1} and $b(\varphi_j) \to b(\varphi)$ as $j \to \infty$.

Step 2. The uniform differentiability of b on bounded subsets of E is immediately from Lemma 3.2, the form of I and (2.4). Thus we complete (I_2).

Step 3. We claim that $I(\varphi)$ satisfies (I_3). Let $\{\varphi_j\}$ be a sequence with $I(\varphi_j) \leq M$ and $I'(\varphi_j) \to 0$ as $j \to \infty$. Then for large j we have

$$\begin{aligned} |\|\varphi_j^+\|^2 - \|\varphi_j^-\|^2 - \langle Q_0'(\varphi_j), \varphi_j\rangle| &\leq \|\varphi_j\|, \\ \frac{1}{2}\|\varphi_j^+\|^2 - \frac{1}{2}\|\varphi_j^-\|^2 - Q_0(\varphi_j) &\leq M, \end{aligned}$$

where M is a positive constant. It follows that

$$\begin{aligned} \frac{1}{2}\langle Q_0'(\varphi_j), \varphi_j\rangle - Q_0(\varphi_j) &\leq M + \|\varphi_j\|, \\ (\frac{p+1}{2} - 1)Q_0(\varphi_j) &\leq M + \|\varphi_j\|. \end{aligned}$$

Hence

$$Q_0(\varphi_j) \leq C(M + \|\varphi_j\|). \tag{3.2}$$

On the other hand, by the assumption we also obtain for large j

$$|\langle I'(\varphi_j), \varphi_j^+ - \varphi_j^-\rangle| \leq \|\varphi_j\|.$$

So

$$\begin{aligned} &\|\varphi_j\|^2 - \langle Q_0'(\varphi_j), \varphi_j^+ - \varphi_j^-\rangle \leq \|\varphi_j\|, \\ &\|\varphi_j\|^2 \leq |\langle Q_0'(\varphi_j), \varphi_j^+\rangle| + |\langle Q_0'(\varphi_j), \varphi_j^-\rangle| + \|\varphi_j\|. \end{aligned} \tag{3.3}$$

In the following, we will estimate $|\langle Q_0'(\varphi_j), \varphi_j^+\rangle|$ and $|\langle Q_0'(\varphi_j), \varphi_j^-\rangle|$. Firstly

$$|\langle Q_0'(\varphi_j), \varphi_j^+\rangle| \leq (4+\delta)\int_\Omega \sum_{l=1}^N |\varphi_j^l + \psi^l(\varphi_j)|^p |\varphi_j^+|\, dxdt$$

$$\leq \quad C\int_\Omega |\varphi_j + \psi(\varphi_j)|^p |\varphi_j^+|\, dxdt$$

$$\leq \quad C(\int_\Omega |\varphi_j + \psi(\varphi_j)|^\alpha\, dxdt)^{p/\alpha}(\int_\Omega |\varphi_j^+|^{\frac{\alpha}{\alpha-p}}\, dxdt)^{(\alpha-p)/\alpha}$$

$$\leq \quad C(\int_\Omega |\varphi_j + \psi(\varphi_j)|^\alpha\, dxdt)^{p/\alpha}\|\varphi_j^+\|, \tag{3.4}$$

where C is independent of δ. Similarly we have

$$|\langle Q_0'(\varphi_j), \varphi_j^-\rangle| \leq C(\int_\Omega |\varphi_j + \psi(\varphi_j)|^\alpha\, dxdt)^{p/\alpha}\|\varphi_j^-\|. \tag{3.5}$$

Since (2.10) implies

$$\int_\Omega |\varphi + \psi(\varphi_j)|^\alpha\, dxdt \leq C_1\int_\Omega F_{\delta,0}(\varphi_j + \psi(\varphi_j))\, dxdt = C_1 Q_0(\varphi_j),$$

we obtain by (3.4)

$$|\langle Q_0'(\varphi_j), \varphi_j^+\rangle| \leq C_1 Q_0(\varphi_j)^{p/\alpha}\|\varphi_j^+\|.$$

Also by (3.5)we get

$$|\langle Q_0'(\varphi_j), \varphi_j^-\rangle| \leq C_1 Q_0(\varphi_j)^{p/\alpha}\|\varphi_j^-\|.$$

Therefore, by (3.3),(3.2), we have

$$\|\varphi_j\|^2 \leq C_1 Q_0(\varphi_j)^{p/\alpha}\|\varphi_j\| + \|\varphi_j\|$$

$$\leq \quad [C_1\|\varphi_j\|^{p/\alpha} + 1]\|\varphi_j\|.$$

So $\|\varphi_j\|$ is bounded because of $\alpha > p$.

Step 4. We will verify that $F_{\delta,0}$ satisfies (I_4) (i). The form of $F_{\delta,0}$ and the definition $Q(\varphi)$ imply

$$Q_0(\varphi) \leq \int_\Omega F_{\delta,0}(\varphi)\, dxdt \leq C\int_\Omega |\varphi|^{p+1} \leq C\|\varphi\|^{p+1},$$

where C is independent of δ. Thus for $\varphi \in E_1$

$$I(\varphi) \geq \|\varphi\|^2 - C\|\varphi\|^{p+1} = \|\varphi\|^2(1 - C\|\varphi\|^{p-1}).$$

Choosing ρ so that $C\rho^{p-1} < \frac{1}{2}$ gives $I(\varphi) \geq \frac{1}{2}\rho^2$ on $S = \partial B_\rho \cap E_1$.

Step 5. Assume that there exists a set Σ with r_1 and r_2 such that I satisfies (I_4)(ii). In fact, let $\varphi = \varphi^- \in B_{r_2} \cap E_2$ where $r_2 > 0$ and consider

$$I(\varphi + re) = r^2 - \|\varphi^-\|^2 - \int_\Omega F_{\delta,0}(\varphi^- + re + \psi(\varphi^- + re))\, dxdt, \tag{3.6}$$

where $e \in \partial B_1 \cap E_1$. Note that for $r = 0$, $I(\varphi) \le 0$ via $F_{\delta,0} \ge 0$. By (2.10),

$$\begin{aligned}\int_\Omega F_{\delta,0}(\varphi + re + \psi(\varphi + re))\,dxdt &\ge C_1 \int_\Omega |\varphi + re + \psi(\varphi + re)|^\alpha\,dxdt \\ &\ge C_1 (\int_\Omega |\varphi + re + \psi(\varphi + re)|^2\,dxdt)^{\frac{\alpha}{2}} \\ &\ge C_1 (\int_\Omega |\varphi|^2 + |re|^2 + |\psi(\varphi + re)|^2\,dxdt)^{\frac{\alpha}{2}} \ge C_1 [r^{2\alpha} \int_\Omega |e|^2\,dxdt]^{\frac{\alpha}{2}}. \end{aligned} \tag{3.7}$$

Hence by (3.6)(3.7)

$$I(\varphi + re) \le r^2 - \|\varphi^-\|^2 - C_1 r^\alpha. \tag{3.8}$$

Choosing r_1 so that

$$h(r) := r^2 - C_1 r^\alpha \le 0 \tag{3.9}$$

for $r \ge r_1$, it then follows from (3.8)-(3.9) and $F_{\delta,0} \ge 0$ that $I \le 0 := \omega$ on $\partial\Sigma$.

Step 6. We go to complete the proof by convexity of $F_{\delta,0}$. We know that for each $\delta \in (0,1)$, there exists a solution $u_\delta = \varphi_\delta^+ + \varphi_\delta^- + \psi_\delta$ for lattice:

$$\Box u + \nabla F_{\delta,0}(u) = 0. \tag{3.10}$$

It follows from the same argument as in step 3 that also $\varphi_\delta^+, \varphi_\delta^-$ are bounded uniformly for δ in E. Hence there is a sequence δ_j tending to 0 such that $\varphi_{\delta_j} \rightharpoonup \bar\varphi$ in E and $\psi_{\delta_j} \to \bar\psi$ in L^{p+1}.

Now we must show that $\bar\varphi$ is nontrivial critical point of $I(\varphi)$. Recall $I(\varphi_j) \ge \alpha > 0$. Since $\varphi \to Q(\varphi)$ is weakly semi-continuous, it follows from (3.12) that

$$\alpha \le \lim_{j\to\infty} I(\varphi_j) \le I|_{\delta=0}(\bar\varphi).$$

Hence $\bar\varphi \ne 0$.

Finally we employ a standard monotonicity argument in order to show that $\bar u = \bar\varphi + \bar\psi$ is a weak solution of (1.1)-(1.3) when $g^l \equiv 0$. Set $u_j = u_{\delta_j}$. Then u_j satisfies (3.10) with $\delta = \delta_j$. The right hand side of (3.10) is bounded in L^{p^*}. Thus $\Box u_j \to \zeta$ in L^{p^*}, possibly after passing to a subsequence. Since $\Box u_j \to \Box \bar u$ in the sense of distributions, we have $\zeta = \Box \bar u$. For each $\tau \in E$,

$$\begin{aligned} &\int_\Omega (\Box u_j + \nabla F_{\delta_j,0}(\tau))(u_j - \tau)\,dxdt \\ = &\int_\Omega (\nabla F_{\delta_j,0}(\tau) - \nabla F_{\delta_j,0}(u_j))(u_j - \tau)\,dxdt \le 0. \end{aligned} \tag{3.11}$$

Furthermore,

$$\int_\Omega \Box u_j u_j\,dxdt \to \int_\Omega \Box \bar u \bar u\,dxdt, \tag{3.12}$$

so we obtain

$$\int_\Omega (\Box \bar{u} + \nabla F_{0,0}(\tau))(\bar{u} - \tau)\, dxdt \le 0,$$

after passing to the limit in (3.11). Let $\tau = \bar{u} + s\chi$, where $s > 0$ and $\chi \in E \cap C^\infty(\Omega)$. Substituting this τ in the inequality above and letting $s \to 0$ give

$$\int_\Omega (\Box \bar{u} + \nabla F_{0,0}(\bar{u}))\chi\, dxdt \le 0.$$

Since χ was chosen arbitrary, $\bar{u}$ is a solution of (1.1)-(1.3).

Therefore there exists at least one solution u for the autonomous lattice. To prove the existence of infinitely many periodic solutions one relies on the following simple argument (see [12]): assume u is the solutions founded by the preceding argument. Firstly we prove that $u = (u^1, \cdots, u^N)$ depends on the time t. Suppose that u is independent of t, then to multiplying the lattice $\Box u + \nabla F_{\delta,0}(u) = 0$ by u, we have $\int_\Omega u_x^2\, dxdt + (p+1)\int_\Omega F_{\delta,0}(u)\, dxdt = 0$. It is impossible. Therefore, assume that the solution u has a minimal period $\le 2\pi$, say $2\pi/j$ for $j \in \mathbb{N}$. Now consider the same problem on the space $H^1((0,\pi)\times(0,2\pi/j))$, which yields a solution with minimal period $\le 2\pi/j$. Repeating this procedure we find infinitely many distinct 2π-periodic solutions. □

4. A modified functional

In the following sections, we will treat the non-autonomous case. $I(\varphi)$ will be replaced by a modified functional $J(\varphi) \in C^1(E,\mathbb{R})$.

Let $\chi \in C^\infty(\mathbb{R},\mathbb{R})$ such that $\chi(\tau) = 1$ for $\tau \le 1$, $\chi(\tau) = 0$ for $\tau \ge 2$ and $\chi'(\tau) \le 0$, $0 \le \chi \le 1$ for $\tau \in \mathbb{R}$. Denote $\tilde{\chi}(\varphi) = \chi(\frac{Q_0(\varphi)}{a[I(\varphi)^2+1]^{1/2}})$. For $\varphi = \varphi^+ + \varphi^- \in E^+ \oplus E^- = E$ we set

$$J(\varphi) = \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - Q_0(\varphi) - \tilde{\chi}(\varphi)(Q(\varphi) - Q_0(\varphi)), \tag{4.1}$$

where $a = \max\{1, 12/(p-1)\}$, $Q(\varphi) = \min_{\psi\in E^0}\int_\Omega F_{\delta,g}(\varphi + \psi(\varphi))\, dxdt$ and $Q_0(\varphi) = \min_{\psi\in E^0}\int_\Omega F_{\delta,0}(\varphi + \psi(\varphi))\, dxdt$ for $\varphi \in E$.

Proposition 4.1. *The functional $J(\varphi) \in C^1(E,\mathbb{R})$ satisfies:*
(i) There is a constant $c_1 = c_1(\|g\|_{\alpha/(\alpha-1)}) > 0$ such that for $\varphi \in E$

$$|J(\varphi) - J(-\varphi)| \le c_1(|J(\varphi)|^{1/\alpha} + 1). \tag{4.2}$$

(ii) If there is a constant $M_0 = M_0(\|g\|_{\alpha/(\alpha-1)}) > 0$ such that $J(\varphi) \ge M_0$ and $\|J'(\varphi)\|_{E^*} \le 1$, then $J(\varphi) = I(\varphi)$.

To prove this proposition, we need some lemmas as follows:

Lemma 4.2. *There is a constant $C = C(\|g\|_{\alpha/(\alpha-1)}) > 0$ such that for $\varphi \in E$,*

$$|Q(\varphi)| \leq C(Q_0(\varphi) + 1), \tag{4.3}$$

$$|Q(\varphi) - Q_0(\varphi)| \leq C(Q_0(\varphi)^{1/\alpha} + 1). \tag{4.4}$$

Proof. By the definition $Q_0(\varphi)$,

$$\begin{aligned} Q(\varphi) - Q_0(\varphi) &= \min_{\psi \in E^0} \int_\Omega F_{\delta,g}(\varphi + \psi)\, dxdt - \int_\Omega F_{\delta,0}(\varphi + \psi_0(\varphi))\, dxdt \\ &\leq |\langle g, \varphi + \psi_0(\varphi)\rangle| \leq \|g\|_{\alpha/(\alpha-1)} \|\varphi + \psi_0(\varphi)\|_\alpha \\ &\leq C(Q_0(\varphi) + C_0)^{1/\alpha} \leq C(Q_0(\varphi)^{1/\alpha} + 1). \end{aligned} \tag{4.5}$$

Similarly we have

$$Q(\varphi) - Q_0(\varphi) \geq -C(|Q(\varphi)|^{1/\alpha} + 1).$$

Obviously (4.5) implies (4.3). By (4.3) we have

$$Q(\varphi) - Q_0(\varphi) \geq -C(Q_0(\varphi)^{1/\alpha} + 1).$$

Thus we get (4.4) from the above inequality and (4.5). □

Lemma 4.3. *If there is a constant $M_1 = M_1(\|g\|_{\alpha/(\alpha-1)}) > 0$ such that $J(\varphi) \geq M_1$ and $\varphi \in supp\chi$, then $I(\varphi) \geq \frac{1}{3}J(\varphi)$.*

Proof. Form the definition of $J(\varphi)$,

$$\begin{aligned} J(\varphi) &= I(\varphi) + (1 - \tilde{\chi}(\varphi))(Q(\varphi) - Q_0(\varphi)) \\ &\leq I(\varphi) + C(Q_0(\varphi)^{1/\alpha} + 1). \end{aligned}$$

By definition of $\tilde{\chi}$, we get for $\varphi \in \mathrm{supp}\tilde{\chi}$ (i.e., $Q_0(\varphi) \leq a(I^2(\varphi)^2 + 1)^{1/2}$),

$$J(\varphi) \leq I(\varphi) + C(|I(\varphi)|^{1/\alpha} + 1) \leq I(\varphi) + \frac{1}{2}|I(\varphi)| + C_1.$$

Choosing $M_1 = 2C_1$, we get the desired result. □

Lemma 4.4. *For all $\varphi = \varphi^+ + \varphi^- \in E = E^+ \oplus E^-$ and $h \in E^*$,*

$$\begin{aligned} \langle J'(\varphi), h\rangle = (1 + T_1(\varphi))\langle \varphi^+ - \varphi^-, h\rangle - (1 + T_2(\varphi))\langle Q_0'(\varphi), h\rangle \\ -(\tilde{\chi}(\varphi) + T_1(\varphi))\langle Q'(\varphi) - Q_0'(\varphi), h\rangle, \end{aligned} \tag{4.6}$$

where $T_1(\varphi), T_2(\varphi) \in C(E, \mathbb{R})$ are functionals satisfying

$$\sup\{|T_i(\varphi)| \,|\, \varphi \in E, J(\varphi) \geq M_2, i = 1, 2\} \to 0 \text{ as } M_2 \to \infty. \tag{4.7}$$

Proof. For all $\varphi = \varphi^+ + \varphi^- \in E$, we have

$$\begin{aligned} \langle J'(\varphi), h\rangle &= \langle \varphi^+ - \varphi^-, h\rangle - \langle Q_0'(\varphi), h\rangle \\ &\quad -\langle \tilde{\chi}'(\varphi), h\rangle (Q(\varphi) - Q_0(\varphi)) - \tilde{\chi}(\varphi)\langle Q'(\varphi) - Q_0'(\varphi), h\rangle, \end{aligned} \tag{4.8}$$

where

$$\langle \tilde{\chi}'(\varphi), h\rangle =$$
$$= \chi'(Q_0(\varphi)a^{-1}[I(\varphi)^2+1]^{-1/2})[-a^{-1}(I(\varphi)^2+1)^{-3/2}Q_0(\varphi)I(\varphi)\langle I'(\varphi), h\rangle$$
$$+a^{-1}(I(\varphi)^2+1)^{-1/2}\langle Q_0'(\varphi), h\rangle]$$

and

$$\langle I'(\varphi), h\rangle = \langle \varphi^+ - \varphi^-, h\rangle - \langle Q_0'(\varphi), h\rangle - \langle Q'(\varphi) - Q_0'(\varphi), h\rangle.$$

By regrouping terms, we get (4.6) for

$$T_1(\varphi) = a^{-1}\chi'(\cdot)(I^2(\varphi)+1)^{-3/2}I(\varphi)Q_0(\varphi)(Q(\varphi) - Q_0(\varphi)),$$
$$T_2(\varphi) = T_1(\varphi) + a^{-1}\chi'(\cdot)(I^2(\varphi)+1)^{-1/2}(Q(\varphi) - Q_0(\varphi)).$$

Let us prove (4.7). Suppose that $\varphi \in E$ satisfies $J(\varphi) \geq M_2$. Using (4.4) we get

$$|T_1(\varphi)| \leq C|\chi'(\cdot)|(I^2(\varphi)+1)^{-1}Q_0(\varphi)(Q_0(\varphi)^{1/\alpha}+1).$$

If $\varphi \notin \mathrm{supp}\tilde{\chi}$, then $T_1(\varphi) = 0$. Otherwise, by the definition of $\tilde{\chi}$, we have

$$Q_0(\varphi) \leq 2a(I^2(\varphi)+1)^{1/2}.$$

On the other hand, we get from Lemma 4.3,

$$a(I^2(\varphi)+1)^{1/2} \geq I(\varphi) \geq \frac{1}{3}J(\varphi) \geq \frac{1}{3}M_2.$$

Hence we obtain

$$|T_1(\varphi)| \leq C(a(I^2(\varphi)+1)^{1/2})^{-(\alpha-1)/\alpha} \leq CM_2^{-(\alpha-1)/\alpha} \to 0 \text{ as } M_2 \to \infty.$$

Similarly we have $T_2(\varphi) \to 0$ as $M_2 \to \infty$. Thus we get (4.7). □

Lemma 4.5. *For all $\varphi \in E$, there exists a constant $C = C(\|g\|_{\alpha/(\alpha-1)})$ such that*

(4.9) *(i)* $|\langle Q'(\varphi), \varphi\rangle - (p+1)Q(\varphi)| \leq C(|Q(\varphi)|^{1/\alpha}+1).$

(4.10) *(ii)* $|\langle Q_0'(\varphi), \varphi^+ - \varphi^-\rangle| \leq C|Q_0(\varphi)|^{p/\alpha}\|\varphi\|,$

(4.11) *(iii)* $|\langle Q'(\varphi), \varphi^+ - \varphi^-\rangle| \leq C(|Q(\varphi)|^{p/\alpha}+1)\|\varphi\|,$

(4.12) *(iv)* $|\langle Q'(\varphi) - Q_0'(\varphi), \varphi\rangle| \leq C(|Q_0(\varphi)|^{p/\alpha}+1)\|\varphi\|.$

Proof. (i) By (2.11)(2.12)(2.15),

$$|(p+1)Q(\varphi) - \langle Q'(\varphi), \varphi\rangle| \leq p|\int_\Omega (g, \varphi + \psi(\varphi))\, dxdt|$$
$$\leq \quad C\|g\|_{\alpha/(\alpha-1)}\|\varphi + \psi(\varphi)\|_\alpha.$$

Since

$$\int_\Omega |\varphi+\psi(\varphi)|^\alpha\,dxdt \le C_1\int_\Omega F_{\delta,0}(\varphi+\psi(\varphi))\,dxdt$$
$$\le \quad C_1|\int_\Omega F_{\delta,g}(\varphi+\psi(\varphi))\,dxdt|+|\int_\Omega (g,\varphi+\psi(\varphi))\,dxdt|$$
$$\le \quad C_1|Q(\varphi)|+\int_\Omega (\epsilon|\varphi+\psi(\varphi)|^\alpha+C_\epsilon|g|^{\alpha/(\alpha-1)})\,dxdt,$$

we have

$$\|\varphi+\psi(\varphi)\|_\alpha \le C(|Q(\varphi)|^{1/\alpha}+1).$$

(ii) Note that

$$\langle Q_0'(\varphi),\varphi^+\rangle \le C\int_\Omega \sum_{l=1}^N |\varphi^l+\psi_0^l(\varphi)|^p|\varphi^+|\,dxdt$$
$$\le C(\int_\Omega |\varphi+\psi_0(\varphi)|^\alpha\,dxdt)^{p/\alpha}(\int_\Omega |\varphi^+|^{\frac{\alpha}{\alpha-p}}\,dxdt)^{(\alpha-p)/\alpha}$$
$$\le C(\int_\Omega |\varphi+\psi_0(\varphi)|^\alpha\,dxdt)^{p/\alpha}\|\varphi^+\|.$$

Similarly, we have

$$\langle Q_0'(\varphi),\varphi^-\rangle \le C(\int_\Omega |\varphi+\psi_0(\varphi)|^\alpha\,dxdt)^{p/\alpha}\|\varphi^-\|.$$

Since

$$\int_\Omega |\varphi+\psi_0(\varphi)|^\alpha\,dxdt \le C_1\int_\Omega F_{\delta,0}(\varphi+\psi_0(\varphi))\,dxdt,$$

we get $|\langle Q_0'(\varphi),\varphi^+-\varphi^-\rangle| \le CQ_0(\varphi)^{p/\alpha}\|\varphi\|$.
(iii) To prove the other one in similar way,

$$\langle Q'(\varphi),\varphi^+\rangle \le C\int_\Omega\sum_{l=1}^N|\varphi^l+\psi^l(\varphi)|^p|\varphi^+|dxdt+\int_\Omega(g,\varphi+\psi(\varphi))dxdt$$
$$\le C(\int_\Omega|\varphi+\psi(\varphi)|^\alpha dxdt)^{p/\alpha}(\int_\Omega|\varphi^+|^{\frac{\alpha}{\alpha-p}}dxdt)^{(\alpha-p)/\alpha}+\|g\|_{\alpha/(\alpha-1)}\|\varphi^+\|$$
$$\le C[(\int_\Omega |\varphi+\psi(\varphi)|^\alpha\,dxdt)^{p/\alpha}+1]\|\varphi^+\|.$$

Similarly, we have

$$\langle Q'(\varphi),\varphi^-\rangle \le C[(\int_\Omega |\varphi+\psi(\varphi)|^\alpha\,dxdt)^{p/\alpha}+1]\|\varphi^-\|.$$

Since

$$\int_\Omega |\varphi+\psi(\varphi)|^\alpha dxdt \le C_1\int_\Omega F_{\delta,g}(\varphi+\psi(\varphi))dxdt+C_1\int_\Omega(g,\varphi+\psi(\varphi))dxdt$$
$$\le C_1|Q(\varphi)|+C_1\|g\|_{\alpha/(\alpha-1)}\|\varphi+\psi(\varphi)\|_\alpha,$$

by Young's inequality, we have

$$\int_\Omega |\varphi+\psi(\varphi)|^\alpha \, dxdt \le C_1|Q(\varphi)| + \epsilon\|\varphi+\psi(\varphi)\|_\alpha^\alpha + C_\epsilon(C_1\|g\|_{\alpha/(\alpha-1)})^{\alpha/(\alpha-1)}.$$

Therefore

$$\int_\Omega |\varphi + \psi(\varphi)|^\alpha \, dxdt \le C_1|Q(\varphi)|,$$

we get $|\langle Q'(\varphi), \varphi^+ - \varphi^-\rangle| \le C(|Q(\varphi)|^{p/\alpha} + 1)\|\varphi\|$.

(iv) The proof is similar to (ii)(iii). It is easy to get

$$\langle Q'(\varphi), \varphi\rangle \le C(Q_0(\varphi)^{p/\alpha} + 1)\|\varphi\|,$$
$$\langle Q_0'(\varphi), \varphi\rangle \le CQ_0(\varphi)^{p/\alpha}\|\varphi\|,$$

which implies (iv). □

Proof of Proposition 4.1 (i) From the definition of $J(\varphi)$, we have

$$|J(\varphi) - J(-\varphi)| \le \tilde\chi(\varphi)|Q(\varphi) - Q_0(\varphi)| + \tilde\chi(-\varphi)|Q(-\varphi) - Q_0(-\varphi)|.$$

Suppose that $-\varphi \in \mathrm{supp}\tilde\chi$, i.e., $Q_0(\varphi) \le 2a(I(-\varphi)^2 + 1)^{1/2}$. From the definition of $J(\varphi)$,

$$I(-\varphi) = J(\varphi) + Q_0(\varphi) - Q(-\varphi) - \tilde\chi(\varphi)(Q(\varphi) - Q_0(\varphi)).$$

By Lemma 4.2, we get that

$$|I(-\varphi)| \le |J(\varphi)| + C(Q_0(\varphi)^{1/\alpha} + 1) \le |J(\varphi)| + C(I(-\varphi)^2 + 1)^{1/2\alpha}.$$

Using Young's inequality, we deduce that

$$|I(-\varphi)| \le 2|J(\varphi)| + C.$$

Hence we get for $-\varphi \in \mathrm{supp}\tilde\chi$,

$$Q_0(-\varphi) \le 2a(I(-\varphi)^2 + 1)^{1/2} \le C|J(\varphi)| + C.$$

Similarly we have for $\varphi \in \mathrm{supp}\tilde\chi$,

$$Q_0(\varphi) \le C|J(\varphi)| + C.$$

Combining the above inequalities, we obtain for $\varphi \in E$,

$$|J(\varphi) - J(-\varphi)| \le C(\tilde\chi(\varphi) + \tilde\chi(-\varphi))(Q_0(\varphi)^{1/\alpha}).$$

This is the desired result.

(ii) It suffices to show that $\tilde\chi = 1$, that is, by the definition of $\tilde\chi$, to show that

$$Q_0(\varphi) \le a(I(\varphi)^2 + 1)^{1/2} \tag{4.13}$$

for $\varphi \in E$ such that $J(\varphi) \geq M_0$ and $\|J'(\varphi)\| \leq 1$. For sufficiently large $M_0 > 0$ and we can assume by (4.7) that $|T_1(\varphi)| \leq \frac{1}{2}, |T_2(\varphi)| \leq 1$ and

$$\frac{(p+1)(1+T_2(\varphi))}{2(1+T_1(\varphi))} - 1 > \frac{p-1}{4} = b.$$

From (4.6) we obtain

$$\begin{aligned}
I(\varphi) - &\frac{1}{2(1+T_1(\varphi))}\langle J'(\varphi), \varphi\rangle = \\
&= -Q(\varphi) + \frac{1+T_2(\varphi)}{2(1+T_1(\varphi))}\langle Q_0'(\varphi), \varphi\rangle + \frac{\tilde{\chi}(u)+T_1(\varphi)}{2(1+T_1(\varphi))}\langle Q'(\varphi) - Q_0'(\varphi), \varphi\rangle \\
&= -Q(\varphi) + \frac{1+T_2(\varphi)}{2(1+T_1(\varphi))}\langle Q_0'(\varphi), \varphi\rangle + (III) \\
&= \Big(\frac{(p+1)(1+T_2(\varphi))}{2(1+T_1(\varphi))} - 1\Big)Q_0(\varphi) - (Q(\varphi) - Q_0(\varphi)) + (III) \\
&= (I) + (II) + (III).
\end{aligned}$$

By (4.4)

$$|(II)| \leq C(Q_0(\varphi)^{1/\alpha} + 1). \tag{4.14}$$

Using (4.3)(4.4) we get

$$\begin{aligned}
|(III)| &\leq C|\langle Q'(\varphi) - Q_0'(\varphi), \varphi\rangle| \\
&\leq |(p+1)Q(\varphi) - \langle Q'(\varphi), \varphi\rangle| + (p+1)|Q(\varphi) - Q_0(\varphi)| \\
&\leq C(Q_0(\varphi)^{1/\alpha} + 1). \qquad (4.15)
\end{aligned}$$

On the other hand, letting $h = \varphi^+ - \varphi^-$ in (4.6) we get

$$\begin{aligned}
\langle J'(\varphi), \varphi^+ - \varphi^-\rangle &= (1+T_1(\varphi))\|\varphi\|^2 - (1+T_2(\varphi))\langle Q_0'(\varphi), \varphi^+ - \varphi^-\rangle \\
&\cdot(\tilde{\chi}(\varphi) + T_1(\varphi))\langle Q'(\varphi) - Q_0'(\varphi), \varphi^+ - \varphi^-\rangle. \qquad (4.16)
\end{aligned}$$

Therefore we get from (4.16) and Lemma 4.4, $T_1(\varphi) \leq \frac{1}{2}$ and the assumption: $\|J'(\varphi)\|_{E^*} \leq 1$,

$$\frac{1}{2}\|\varphi\|^2 \leq \|J'(\varphi)\|_{E^*}\|\varphi\| + C(Q_0(\varphi)^{p/\alpha} + 1)\|\varphi\| \leq C(Q_0(\varphi)^{p/\alpha} + 1)\|\varphi\|,$$

that is

$$\|\varphi\| \leq C(Q_0(\varphi)^{p/\alpha} + 1). \tag{4.17}$$

We obtain by (4.14)(4.15),

$$I(\varphi) - \frac{1}{2(1+T_1(\varphi))}\langle J'(\varphi), \varphi\rangle \geq (I) - C(Q_0(\varphi)^{1/(p+1)} + 1).$$

It follows by (4.17) and $\|J'(\varphi)\|_{E^*} \le 1$,

(4.18)
$$I(\varphi) \ge -C\|J'(\varphi)\|_{E^*}\|\varphi\| + (I) - C(Q_0(\varphi)^{1/(p+1)} + 1) \ge bQ_0(\varphi)/2 - C_0.$$

We remark that

$$\inf\{Q_0(\varphi) | \, \|J'(\varphi)\|_{E^*} \le 1 \text{ and } J(\varphi) \ge M\} \to \infty \text{ as } M \to \infty.$$

This follows from (4.17). In fact, $J(\varphi) \to \infty$ implies $\|\varphi\| \to \infty$. So it follows $Q_0(\varphi) \to \infty$, by (4.17). Now we may assume that $J(\varphi) \ge M_0$ implies $bQ_0(\varphi)/6 - C_0 \ge 0$, i.e., $I(\varphi) \ge bQ_0(\varphi)/3$ by (4.18). Thus

$$Q_0(\varphi) \le aI(\varphi) \le a(I(\varphi)^2 + 1)^{1/2}.$$

□

Proposition 4.6. *$J(\varphi) \in C^1(E, \mathbb{R})$ satisfies the following Palias-Smale compactness condition (P.S.): Whenever a sequence $\{\varphi_j\}_{j=1}^{\infty}$ in E satisfies for a large M_2 and some $M_3 > 0$,*

$$M_2 \le J(\varphi_j) \le M_3 \quad \textit{for all } j,$$
$$J'(\varphi_j) \to 0 \quad \textit{in } E^* \textit{ as } j \to \infty,$$

there is a subsequence of $\{\varphi_j\}$ which is convergent in E.

Proof. Setting $h = \varphi_j$ and $h = \varphi_j^+ - \varphi_j^-$ in (4.6)

$$\begin{aligned}&\Big|(1 + T_1(\varphi_j))\Big(\|\varphi_j^+\|^2 - \|\varphi_j^-\|^2\Big) - (1 + T_2(\varphi_j))\langle Q_0'(\varphi_j), \varphi_j\rangle \\ &-\Big(\tilde\chi(\varphi_j) + T_1(\varphi_j)\Big)\langle Q'(\varphi_j) - Q_0'(\varphi_j), \varphi_j\rangle\Big| \le m\|\varphi_j\|,\end{aligned} \tag{4.19}$$

$$\begin{aligned}&\Big|(1 + T_1(\varphi_j))\|\varphi_j\|^2 - (1 + T_2(\varphi_j))\langle Q_0'(\varphi_j), \varphi_j^+ - \varphi_j^-\rangle \\ &-\Big(\tilde\chi(\varphi_j) + T_1(\varphi_j)\Big)\langle Q'(\varphi_j) - Q_0'(\varphi_j), \varphi_j^+ - \varphi_j^-\rangle\Big| \le m\|\varphi_j\|,\end{aligned} \tag{4.20}$$

where $m = \sup \|J'(\varphi_j)\|_{E^*}$. Since

$$J(\varphi_j) = \frac{1}{2}\|\varphi_j^+\|^2 - \frac{1}{2}\|\varphi_j^-\|^2 - Q_0(\varphi_j) - \tilde\chi(\varphi_j)(Q(\varphi_j) - Q_0(\varphi_j)) \le M_3,$$

it follows from (4.19)

$$\begin{aligned}&\frac{1 + T_2(\varphi_j)}{2(1 + T_1(\varphi_j))}\langle Q_0'(\varphi_j), \varphi_j\rangle - Q_0(\varphi_j) - \frac{\tilde\chi(\varphi_j) + T_1(\varphi_j)}{2(1 + T_1(\varphi_j))}\langle Q'(\varphi_j) \\ &\qquad -Q_0'(\varphi_j), \varphi_j\rangle - \tilde\chi(\varphi_j)(Q(\varphi_j) - Q_0(\varphi_j)) \le M_3 + m\|\varphi_j\|.\end{aligned}$$

Hence for large M_2, $T_1(\varphi_j), T_2(\varphi_j)$ are small. One can see that there is C_0 such that

$$\frac{1 + T_2(\varphi_j)}{2(1 + T_1(\varphi_j))}\langle Q_0'(\varphi_j), \varphi_j\rangle - Q_0(\varphi_j) \le C_0 Q_0(\varphi_j).$$

It follows by Lemma 4.2 and Lemma 4.5 that

$$C_0Q_0(\varphi_j) - C_1(|Q_0(\varphi_j)|+1)^{p/\alpha} \le M_3 + m\|\varphi_j\|.$$

Hence by Young's inequality,

$$|Q_0(\varphi_j)| \le C_1(1+\|\varphi_j\|) \quad \text{for all } \ j. \tag{4.21}$$

Then, by (4.20) we obtain for large j,

$$\begin{aligned}
\|\varphi_j\|^2 \quad &\le \quad \Big|\frac{1+T_2(\varphi_j)}{1+T_1(\varphi_j)}\langle Q_0'(\varphi_j), \varphi_j^+ - \varphi_j^-\rangle \\
&\quad - \frac{\tilde{\chi}(\varphi_j)+T_1(\varphi_j)}{1+T_1(\varphi_j)}\langle Q'(\varphi_j) - Q_0'(\varphi_j), \varphi_j^+ - \varphi_j^-\rangle\Big| + m\|\varphi_j\| \\
&\le \quad C_2[(|Q(\varphi_j)|^{p/\alpha} + \|g\|_{\alpha/(\alpha-1)}\|\varphi_j\|_\alpha)\|\varphi_j\| + \|\varphi_j\|] \\
&\le \quad C_2(\|\varphi_j\|^{p/\alpha}+1)\|\varphi_j\|,
\end{aligned}$$

where C_2 is independent of δ. So $\|\varphi_j\|$ is bounded, which is independent of δ. Observe that $J'(\varphi_j) = \varphi_i^+ - \varphi_j^- + P(\varphi_j)$ where $P: E \to E^*$ is compact operator and $J'(\varphi_j) \to 0$ as $j \to \infty$. Hence $\varphi_j^+ - \varphi_j^-$ is precompact in E. That is, φ_j is precompact in E. Thus the proof is completed. □

5. Minimax methods

In this section, we construct critical points of $J(\varphi)$ via minimax methods. For convenience, we define the usual lexicographical order for 2-tuples $(k,i) \in \mathcal{D}$ as follows, where $\mathcal{D} = \mathbb{N} \times \{1, 2, \cdots, N\}$,

$$\begin{aligned}
(j,m) &= (k,i), \quad \text{if } \ j=k \text{ and } m=i, \\
(j,m) &< (k,i), \quad \text{if } \ j<k \text{ or } j=k \text{ and } m<i.
\end{aligned}$$

Moreover, we write $(k,i) \equiv (k+[\frac{i}{N}], i-[\frac{i}{N}])$ for any $i \in \mathbb{N}$, where $[a]$ is the integer part of a, $(k,0) \equiv (k-1, N)$ for $k \in \mathbb{N}$.

We observe that the eigenvalues of the wave operator under periodic-Dirichlet conditions are $\{j^2 - k^2 |\ j \in \mathbb{N}, k \in \mathbb{Z}\}$ and corresponding eigenfunctions are $\sin jx \cos kt$ and $\sin jx \sin kt$. We arrange the negative eigenvalues in the following order, denoted by $0 > -\mu_1 \ge -\mu_2 \ge -\mu_3 \ge \cdots$ with repetitions according to the multiplicity of each eigenvalue and denote by v_j the eigenfunctions which correspond to μ_j. We assume $\langle v_i, v_j\rangle = \delta_{ij}$ for $i, j \in \mathbb{N}$. Let $e_1, \cdots, e_N$ denote the usual orthogonal basis in $\mathbb{R}^N$. Define $v_{jk} = v_j e_k$ for $j \in \mathbb{N}$ and $1 \le k \le N$. Let

$$E_q^+ = E_{mi}^+ = \text{span}\{v_{jk} |\, (1,1) \le (j,k) \le (m,i)\},$$

where $1 \le i \le N$ and $q = mN + i$.

Lemma 5.1. *For all* $\theta \in (0, 1/\alpha)$*, there is a constant* $C > 0$ *independent of* $m \in \mathbb{N}$ *such that*

$$\|\varphi\|_\alpha \leq C\mu_m^{-\theta}\|\varphi\| \quad \textit{for } \varphi \in (E_q^+)^\perp,$$

where $(E_q^+)^\perp = E^+ \setminus E_q^+$ *and* $m = [q/N]$ *is the integer part of* q.

Proof. We have $m = [q/N]$ and by the definition of μ_m, it follows for all $l \in \mathfrak{N}$

$$\int_\Omega |\varphi^l|^2 \, dxdt \leq \mu_m^{-1} \int_\Omega |\varphi_t^l|^2 - |\varphi_x^l|^2 \, dxdt \quad \text{for } \varphi \in (E_q^+)^\perp.$$

Summing the inequalities from 1 to N, we get

$$\|\varphi\|_2 \leq \mu_m^{-1/2}\|\varphi\| \quad \text{for } \varphi \in (E_q^+)^\perp.$$

On the other hand, by the embedding property,

$$\|\varphi\|_s \leq C_s\|\varphi\| \quad \text{for all } \varphi \in E^+ \text{ and } s \in [1, \infty).$$

Using Hölder's inequality, we obtain for $s \in (\alpha, \infty)$

$$\|\varphi\|_\alpha \leq \|\varphi\|_2^\tau \|\varphi\|_s^{1-\tau} \quad \text{for } \varphi \in E^+,$$

where

$$\tau = \frac{2(s-\alpha)}{\alpha(s-2)} \in (0, \frac{2}{\alpha}).$$

Combining the above inequalities, we have

$$\|\varphi\|_\alpha \leq C_s^{1-\tau}\mu_m^{-\tau/2}\|\varphi\| \quad \text{for } \varphi \in (E_q^+)^\perp.$$

□

Note that

$$\|\varphi\| \leq \mu_m^{1/2}\|\varphi\|_2 \quad \text{for } \varphi \in E_q^+, m = [q/N].$$

For $\varphi = \varphi^+ + \varphi^- \in E_q^+ \oplus E^-$, we have

$$\begin{aligned}
J(\varphi) &= \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - Q_0 - \tilde{\chi}(u)(Q - Q_0) \\
&\leq \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - Q_0 + C(Q_0(\varphi)^{1/\alpha} + 1) \\
&\leq \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - \frac{1}{2}Q_0 + C \\
&\leq \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}F_{\delta,0}(\varphi^+ - \varphi^- + \psi) - \frac{1}{2}\|\varphi^-\|^2 + C \\
&\leq \frac{1}{2}\|\varphi^+\|^2 - c\|\varphi^+ + \varphi^- + \psi\|_\alpha^\alpha - \frac{1}{2}\|\varphi^-\|^2 + C \\
&\leq \frac{1}{2}\|\varphi^+\|^2 - c\|\varphi^+ + \varphi^- + \psi\|_2^\alpha - \frac{1}{2}\|\varphi^-\|^2 + C \\
&\leq \frac{1}{2}\|\varphi^+\|^2 - c\|\varphi^+\|_2^\alpha - \frac{1}{2}\|\varphi^-\|^2 + C \\
&\leq \frac{1}{2}\|\varphi^+\|^2 - c\mu_m^{-\alpha/2}\|\varphi^+\|^\alpha - \frac{1}{2}\|\varphi^-\|^2 + C.
\end{aligned}$$

Hence there is a constant R_q such that

$$J(\varphi) \leq 0 \quad \text{for all } \varphi \in E_q^+ \oplus E^- \text{with } \|\varphi\| \geq R_q. \tag{5.1}$$

We may assume that $R_q < R_{q+1}$ for all $i \in \mathbb{N}$. Let

$$\begin{aligned}
&B_R = \{\varphi \in E | \, \|\varphi\|_E \leq R\} \text{ for } R \geq 0, \\
&D_q = B_{R_q} \cap (E_q^+ \oplus E^-), \\
&\Gamma_q = \{\gamma \in C(D_q, E) | \, \gamma \text{ satisfies } (\gamma_1) - (\gamma_3)\},
\end{aligned}$$

where
(γ_1) γ is odd, i.e., $\gamma(-\varphi) = -\gamma(\varphi)$ for all $\varphi \in D_q$,
(γ_2) $\gamma(\varphi) = \varphi$ for all $\varphi \in \partial D_q$,
(γ_3) for $\varphi = \varphi^+ + \varphi^- \in D_q, \gamma(\varphi) = \alpha(\varphi)\varphi + \kappa(\varphi)$ where $\alpha \in C(D_q, [1, \bar{\alpha}])$ is a even functional ($\bar{\alpha}$ depends on γ) and κ is a compact operator such that $\alpha(\varphi) = 1$ and $\kappa = 0$ on ∂D_q.
Moreover, set

$$U_q = \{\varphi = w + \tau v_{q+1} | \, w \in B_{R_{q+1}} \cap E_q, \tau \in [0, R_{q+1}], \|\varphi\| \leq R_{q+1}\}.$$

Let

$$\Lambda_q = \{\lambda \in C(U_q, E) | \, \lambda \text{ satisfies } (\lambda_1) - (\lambda_3)\},$$

where
(λ_1) $\lambda|_{D_q} \in \Gamma_q$
(λ_2) $\lambda(\varphi) = \varphi$ on $\partial U_q \setminus D_q$,
(λ_3) for $\varphi = \varphi^+ + \varphi^- \in U_q, \lambda(\varphi) = \widetilde{\alpha}(\varphi)\varphi + \widetilde{\kappa}(\varphi)$ where $\widetilde{\alpha} \in C(U_q, [1, \bar{\alpha}])$ is

a even functional ($\bar{\alpha}$ depends on λ) and $\widetilde{\kappa}$ is a compact operator such that $\widetilde{\alpha}(\varphi) = 1$ and $\widetilde{\kappa} = 0$ on $\partial U_q \setminus D_q$.

Define for $q \in \mathbb{N}$,

$$b_q = \inf_{\gamma \in \Gamma_q} \sup_{\varphi \in D_q} J(\gamma(\varphi)), \quad c_q = \inf_{\lambda \in \Lambda_q} \sup_{\varphi \in U_q} J(\lambda(\varphi)). \tag{5.2}$$

Lemma 5.2. *Suppose that $c > M_0$ is a regular value of $J(\varphi)$, that is, $J'(\varphi) \neq 0$ when $J(\varphi) = c$. Then for any $\bar{\epsilon}$ there exists an $\epsilon \in (0, \bar{\epsilon})$ and $\eta \in C([0,1] \times E, E)$ such that*
(i) $\eta(t, \cdot)$ is odd, for $t \in [0,1]$ if $g(x,t) = 0$;
(ii) $\eta(t, \cdot)$ is homeomorphism of E onto E for all t;
(iii) $\eta(0, \varphi) = \varphi$ for all $\varphi \in E$;
(iv) $\eta(t, \varphi) = \varphi$ if $J(\varphi) \notin [c - \bar{\epsilon}, c + \bar{\epsilon}]$;
(v) $J(\eta(1, \varphi)) \leq c - \epsilon$ if $J(\varphi) \leq c + \epsilon$;
(vi) for $\varphi = \varphi^+ + \varphi^- \in E^+ \oplus E^-, \eta(1, \varphi) = \alpha^+(\varphi)\varphi^+ + \alpha^-(\varphi)\varphi^- + \kappa(\varphi)$ where $\alpha^+ \in C(E, [0,1])$, $\alpha^- \in C(E, [1, \bar{\alpha}])$ is a even functional ($\bar{\alpha} \geq 1$ is constant) and κ is a compact operator.

Proof. Since $J(\varphi) \in C^1(E, \mathbb{R})$ and satisfies (P.S.) condition via Proposition 4.6, the assertions are standard. As [15] Proposition A.18, we know that η is the solution of the initial value problem

$$\frac{d\eta}{dt} = -\omega(\eta) V(\eta), \quad \eta(0, \varphi) = \varphi,$$

where $\omega \in C^{0,1}(E, \mathbb{R})$ satisfying $0 \leq \omega \leq 1$ and V is pseudogradient vector field for J' on E^*. By Lemma 4.4, it follows that

$$\begin{cases} \frac{d\eta}{dt} = -\omega(\eta)[(1 + T_1(\eta))(\eta^+ - \eta^-) + \mathcal{P}(\eta)], \\ \eta(0) = \varphi, \end{cases}$$

where $\mathcal{P}(\eta)$ is compact. It yields

$$\begin{cases} \frac{d\eta^\pm}{dt} = -\omega(\eta)[\pm(1 + T_1(\eta))\eta^\pm + P^\pm \mathcal{P}(\eta)], \\ \eta^\pm(0) = \varphi^\pm, \end{cases}$$

where $P^\pm : E \to E^\pm$ are the orthogonal projection. Integrating these shows

$$\eta^\pm(t, \varphi) = e^{A(t)}[\varphi^\pm - \int_0^t e^{-A(\tau)} \omega(\eta(s, \varphi)) P^\pm \mathcal{P}(\eta(\tau, \varphi))\, d\tau],$$

where $A(t) = \mp \int_0^t \omega(\eta(s, \varphi))(1 + T_1(\eta(s, \varphi))))\, ds$. Thus η is the form asserted. □

Proposition 5.3. *Suppose that $c_q > b_q \geq M_0$. Let $d \in (0, c_q - b_q)$ and*

$$\Lambda_q(d) = \{\lambda \in \Lambda_q |\ J(\lambda) \leq b_q + d \text{ on } D_q\}.$$

Define

$$c_q(d) = \inf_{\lambda\in\Lambda_q(d)} \sup_{\varphi\in U_q} J(\lambda(\varphi)) \quad (\geq c_q). \tag{5.3}$$

Then $c_q(d)$ is a critical value of $J(\varphi)$.

Proof. From Proposition 4.1 and 4.6, we obtain that whenever $\varphi \in E$ satisfies $J'(\varphi) = 0$ and $J(\varphi) \geq M_0$, then $I(\varphi) = J(\varphi)$ and $I'(\varphi) = 0$ and that $J(\varphi)$ satisfies the Palais-Smale condition on $A_{M_0} = \{\varphi \in E|\, J(\varphi) \geq M_0\}$. Note that

$$J'(\varphi) = (1 + T_1(\varphi))(\varphi^+ - \varphi^-) + (\text{compact})$$

where $|T_1(\varphi)| \leq \frac{1}{2}$ on $\{\varphi \in E\,|J(\varphi) \geq M_0\}$, see Lemma 4.4. Therefore we can show that c_q is a critical value of $J(\varphi)$ as in [14]. In fact, note that by the definition of b_q and Λ_q, $\Lambda_q(\delta) \neq \emptyset$. Choose $\bar{\varepsilon} = \frac{1}{2}(c_q - b_q - d) > 0$. If $c_q(d)$ is not a critical value of J, there are an ε and an η as Lemma 5.2. Choose $\lambda \in \Lambda_q(d)$ such that

$$\max J(\lambda(\varphi)) \leq c_q(d) + \epsilon. \tag{5.4}$$

Consider $\eta(1, \lambda(\varphi)) \in C(U_q, E)$. Note that if $\|\varphi\| = R_{q+1}$ or $\varphi \in (B_{R_{q+1}} \setminus B_{R_q}) \cap E_q$, $J(\lambda(\varphi)) = J(\varphi) \leq 0$, so $\eta(1, \lambda(\varphi)) = \varphi$ by Lemma 5.2. Therefore $\eta(1, \lambda) \in \Lambda_q$. Moreover on D_q, $J(\lambda(\varphi)) \leq b_q + d \leq c_q - \bar{\varepsilon} \leq c_q(d) - \bar{\varepsilon}$ via our choice of d and $\bar{\varepsilon}$. Then $\eta(1, \lambda) = \lambda$, $J(\eta(1, \lambda)) \leq b_q + d$ on D_q, again by Lemma 5.2. Thus $\eta(1, \lambda) \in \Lambda_q(d)$ and by (5.4) and Lemma 5.2,

$$\max J(\lambda(\varphi)) \leq c_q(d) - \epsilon.$$

contrary to the definition of $c_q(d)$. Hence $c_q(d)$ is a critical value of J. □

Proposition 5.4. *If $c_q = b_q$ for all $q \geq q_0$, then there is a constant $C > 0$ such that*

$$b_q \leq Cq^{(p+1)/p} \quad \textit{for all } q \in \mathbb{N}.$$

Proof. Let $q \geq q_0$ and $\epsilon > 0$ and $\gamma \in \Gamma_q$ such that

$$\max_{\gamma\in\Gamma_q} J(\gamma(\varphi)) \leq b_q + \epsilon.$$

Let $\widetilde{\gamma}(\varphi) = \gamma(\varphi)$ for $\varphi \in \Gamma_q$ and $\widetilde{\gamma}(-\varphi) = -\gamma(\varphi)$ for $\varphi \in -\Gamma_q$. Note that $\Gamma_q \cup (-\Gamma_q) = \Gamma_{q+1}$. Moreover, since $\gamma \in \Gamma_q$ implies $\widetilde{\gamma} \in \Gamma_{q+1}$. Therefore,

$$b_{q+1} \leq b_q + C(|b_q|^{1/\alpha} + 1).$$

Hence

$$b_q \leq Cq^{\frac{\alpha}{\alpha-1}} \leq Cq^{\frac{p+1}{p}}.$$

□

Therefore, the existence of subsequence of c_q which satisfy $c_q > b_q \leq M_0$ guarantees the existence of critical values. In other words, we should show the existence of subsequence $\{q_j\}$ such that

$$c_{q_j} > b_{q_j} \geq M_0 \quad \text{for large } \ q_j \in \mathbb{N},$$
$$b_{q_j} \to \infty \quad \text{as} \ \ q_j \to \infty.$$

To show the above properties, we will prove the existence of a sequence $\{q_j\}$ such that for any $\epsilon > 0$, there is a $C_\epsilon > 0$ satisfying

$$b_{q_j} \geq C_\epsilon q_j^{(p+1)/(p-1)-\epsilon} \quad \text{for large } \ q_j \in \mathbb{N}, \tag{5.5}$$

which make sure that the case in Proposition 5.4 does not happen.

Let us look for the comparison functional for $J(\varphi)$.

$$\begin{aligned} J(\varphi) &= \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - Q_0 - \tilde{\chi}(\varphi)(Q - Q_0) \\ &\geq \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - 2Q_0 - a_1 \\ &\geq \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - F_{\delta,0}(\varphi^+ + \varphi^-) - a_1 \\ &\geq \frac{1}{2}\|\varphi^+\|^2 - \frac{1}{2}\|\varphi^-\|^2 - \frac{a_0}{p+1}\|\varphi^+\|_{p+1}^{p+1} - \frac{a_0}{p+1}\|\varphi^-\|_{p+1}^{p+1} - a_1. \qquad (5.6) \end{aligned}$$

where $a_0, a_1 > 0$ are constants independent of φ. We set

$$K(\varphi^+) = \frac{1}{2}\|\varphi^+\|^2 - \frac{a_0}{p+1}\|\varphi^+\|_{p+1}^{p+1} \in C^2(E^+, \mathbb{R}). \tag{5.7}$$

Here we recall the definitions of (P.S.)$_*$ and (P.S.)$_n$ conditions:
(P.S.)$_*$: If $\{\varphi_n\} \subset E^+$ satisfies $\varphi_n \in E_n$, $K(\varphi_n) \leq C$ and $\|(K|_{E_n^+})'(\varphi_n)\|_{(E_n^+)^*} \to 0$ as $n \to \infty$, then $\{\varphi_n\}$ is relative compact in E^+;
(P.S.)$_n$: If $\{\varphi_j\} \subset E_n^+$ satisfies $K(\varphi_j) \leq C$ and $\|(K|_{E_n^+})'(\varphi_j)\|_{(E_n^+)^*} \to 0$ as $j \to \infty$, then $\{\varphi_j\}$ is relative compact in E_n^+.

Then we have

Lemma 5.5. *(i) $J(\varphi^+) \geq K(\varphi^+) - a_1$ for all $\varphi^+ \in E^+$.*
(ii) $K(\varphi^+)$ satisfied the (PS), (PS)$_*$ and (PS)$_n$ conditions on E^+.

Proof. The arguments to show (PS), (PS)$_*$ and (PS)$_n$ are very similar. Therefore we just give the proof of (PS)$_*$. Let $\varphi_n \subset E^+$ be a sequence such that $\varphi_n \subset E_n^+$, $K(\varphi_n) \leq C$ and

$$\|(K|_{E_n^+})'(\varphi_n)\|_{(E_n^+)^*} \to 0,$$

that is, for all $h \in E^+$

$$\langle \varphi_n, h \rangle - a_0 \int_\Omega P_n^+(|\varphi_n|^{p-1}\varphi_n) h \, dxdt = \epsilon_n \to 0,$$

which lead to a priori estimate of type: $\|\varphi_n\|_{E^+} = \|\varphi\|_{E_n^+} \le C$. Hence, for a subsequence denote again by φ_n, one has $\varphi_n \to \varphi$ weakly in E^+, strongly in L^{p+1}. Consequently,

$$\int_\Omega P_n^+(|\varphi_n|^{p-1}\varphi_n)\varphi_n\, dxdt \to \int_\Omega (|\varphi|^{p-1}\varphi)\varphi\, dxdt.$$

Therefore,

$$\|\varphi_n\|^2 = a_0 \int_\Omega P_n^+(|\varphi_n|^{p-1}\varphi_n)\varphi_n\, dxdt + \epsilon_n \to a_0 \int_\Omega (|\varphi|^{p-1}\varphi)\varphi\, dxdt = \|\varphi\|^2.$$

It follows that $\|\varphi_n - P_n^+\varphi\|_{E^+} = \|\varphi_n - P_n^+\varphi\|_{E_n^+}$ converges to 0 as $n \to \infty$. This shows $\varphi_n \to \varphi$ in E^+. □

Now we are concerned with the functional $K(\varphi^+)$ and state index property of Bahri-Berestycki's max-min value σ_q. For $n > q, n, q \in \mathbb{N}$ set

$$A_q^n = \{\sigma \in C(S^{n-q}, E_n^+) |\, \sigma(-y) = -\sigma(y) \quad \text{for all } y\}, \tag{5.8}$$

$$\sigma_q^n = \sup_{\sigma \in A_q^n} \min_{y \in S^{n-q}} K(\sigma(y)), \tag{5.9}$$

where $q = mN + i$.

Lemma 5.6. *([17]) Let $a, b \in \mathbb{N}$. Suppose that $h_1 \in C(S^a, \mathbb{R}^{a+b})$ and $h_2 \in C(\mathbb{R}^b, \mathbb{R}^{a+b})$ are continuous such that*

$$h_1(-y) = -h_1(y) \quad \textit{for all } y \in S^a, \qquad h_2(-y) = -h_2(y) \quad \textit{for all } y \in \mathbb{R}^b,$$

and there is a $r_0 > 0$ such that $h_2(y) = y$ for $|y| \ge r_0$. Then $h_1(S^a) \cap h_2(\mathbb{R}^b) \ne \emptyset$.

Lemma 5.7. *For all $\sigma \in A_q^n$,*

$$\sigma(S^{n-q}) \cap E_q^+ \ne \emptyset. \tag{5.10}$$

Proof. Apply Lemma 5.6 to $h_1 = \sigma : S^{n-q} \to E_n^+$ and $h_2 = id : E_q^+ \to E_n^+$. Then we get the result. □

Proposition 5.8. *(i) $0 \le \sigma_q^n \le \sigma_{q+1}^n$ for all q, n;*
(ii) For all $n \in \mathbb{N}$ there exist $\nu(q)$ and $\widetilde{\nu}(q)$ such that

$$0 \le \nu(q) \le \sigma_q^n \le \widetilde{\nu}(q) \le \infty \textit{ for all } n \ge q+1; \tag{5.11}$$

(iii) Moreover, $\nu(q) \to \infty$ as $q \to \infty$.

Proof. (i) For any $\sigma \in A_q^n$, it is clear that there is a $\bar{\sigma} \in A_{q+1}^n$ with $\bar{\sigma}(S^{n-q-1}) \subset \sigma(S^{n-q})$. Hence we have $\sigma_q^n \le \sigma_{q+1}^n$.
(ii) We now prove the existence of $\widetilde{\nu}(q)$. By Lemma 5.7 we have for all $\sigma \in A_q^n$,

$$\min_{y \in S^{n-q}} K(\sigma(y)) \le \sup_{\varphi \in E_q^+} K(\varphi). \tag{5.12}$$

For all $\varphi \in E_q^+$, we have
(5.13)
$$K(\varphi) \leq \frac{1}{2}\|\varphi\|^2 - C\|\varphi\|_{p+1}^{p+1} \leq \frac{1}{2}\|\varphi\|^2 - C\|\varphi\|_2^{p+1} \leq \frac{1}{2}\|\varphi\|^2 - C\mu_m^{-(p+1)}\|\varphi\|^{p+1}.$$

Thus the right-hand of (5.13) is finite and independent of σ and n. Set
$$\widetilde{\nu}(q) = \sup_{\varphi \in E_q^+} K(\varphi) \leq \infty,$$
then we obtain
$$\sigma_q^n = \sup_{\sigma \in A_q^n} \min_{y \in S^{n-q}} K(\sigma(y)) \leq \widetilde{\nu}(q).$$

(iii) We claim that the existence of $\nu(n)$. We construct a special $\sigma \in A_q^n$ as follows: write
$$S^{n-q} = \{y = (y_q, \ldots, y_n) \in \mathbb{R}^{n-q+1} | \sum_{i=q}^{n} y_i^2 = 1\}$$
and set $\sigma : S^{n-q} \to E_q^+ \setminus \{0\}$ by
$$\sigma(y) = a_0^{-1/(p+1)}\|w(y)\|_{p+1}^{-(p+1)/(p-1)} w(y),$$
where $w(y)$ is defined by
$$w(y) = \sum_{i=q}^{n} y_i v_i,$$
and v_i are eigenfunctions corresponding to μ_i. Obviously we have $\sigma \in A_q^n$. Since $\|w(y)\| = 1$ on S^{n-q}, we have
$$K(\sigma) \geq (\frac{1}{2} - \frac{1}{p+1}) a_0^{-2/(p-1)} \|w\|_{p+1}^{-2(p+1)/(p-1)}.$$
Since
$$w(y) \in (E_q^+)^{\perp}, \ \ \|w(y)\| = 1 \ \text{ for all } \ y \in S^{n-q},$$
we get that
$$\|w(y)\|_{p+1} \leq C_\theta \mu_{n-1}^{-\theta},$$
where $\theta \in (0, 1/(p+1))$ and C_θ is a constant independent of n and y. Therefore,
$$K(\varphi) \geq C'_\theta \mu_{q-1}^{\theta(p+1)/(p-1)} := \nu(q).$$
Then we have
$$\sigma_q^n \geq \min_{y \in S^{n-q}} K(\sigma(y)) \geq \nu(q) \quad \text{for} \ \ n > q.$$

Since μ_{q-1} as $q \to \infty$, we obtain $\nu(q) \to \infty$ as $n \to \infty$. □

Proposition 5.9. *Suppose that $\nu(q) > 0$. Then σ_q^n is a critical point of the restriction of K to E_n^+. Furthermore, the limit σ_q of any convergent subsequence of σ_q^n is a critical value of $K \in C^2(E_n^+, \mathbb{R})$, $\sigma_q \leq \sigma_{q+1}$ for all $q \in \mathbb{N}$ and $\sigma_q \to \infty$ as $q \to \infty$.*

Proof. Since K satisfies (PS), (PS)$_*$ and (PS)$_n$ by Lemma 5.5, we have σ_q^n is a critical value of $K_{E_n^+} \in C^2(E_n^+, \mathbb{R})$. By Proposition 5.8, we can choose a sequence n_j such that $n_j \to \infty$ as $j \to \infty$,

$$\sigma_q = \lim_{j\to\infty} \sigma_q^{n_j} \quad \text{exist for all } q \in \mathbb{N}. \tag{5.14}$$

Using the (PS)$_*$ condition, we can extract a convergent subsequence $\varphi_q^{n_j} \to \varphi_q$, then observe easily that $K(\varphi_q) = \lim \sigma_q^{n_j}$ and $K'(\varphi_q) = 0$. Therefore σ_q is a critical value of $K \in C^2(E^+, \mathbb{R})$, the other properties follows directly from Proposition 5.8. □

Next we state the relation between b_q and σ_q.

Proposition 5.10. *For all $q \in \mathbb{N}$,*

$$b_q \geq \sigma_q - a_1, \tag{5.15}$$

where a_1 is the constant appeared in (5.6).

To prove this proposition, we need the lemma:

Lemma 5.11. *For all $\gamma \in \Gamma_q$ and $\sigma \in A_q^n$,*

$$\Big((P_n\gamma)(D_q) \cup \{\varphi \in E_q^+ \oplus E^- \,|\, \|\varphi\| \geq R_q\}\Big) \cap \sigma(S^{n-q}) \neq \emptyset,$$

where $P_n : E = E^+ \oplus E^- \to E_n^+ \oplus E^-$ is orthogonal projection.

Proof. We extend γ to $\widetilde{\gamma} \in C(E_q^+ \oplus E^-, E)$ by

$$\widetilde{\gamma}(\varphi) = \gamma(\varphi) \text{ if } \|\varphi\| \leq R_q \quad \widetilde{\gamma}(\varphi) = \varphi \text{ if } \|\varphi\| \geq R_q.$$

Obviously, $\widetilde{\gamma}(\varphi)$ is well defined and odd in $E_q^+ \oplus E^-$ and

$$P_n\widetilde{\gamma}(E_q^+ \oplus E^-) = P_n\gamma(D_q) \cup \{\varphi \in E_q \oplus E^- \,|\, \|\varphi\|_E \geq R_q\}.$$

Therefore, it suffices to prove $P_n\widetilde{\gamma}(E_q^+ \oplus E^-) \cap \sigma(S^{n-q}) \neq \emptyset$. We set

$$E_s^- = \{sinjxe^{ikt} \,|\, 0 \leq k, j \leq s, j > |k|\}$$

and let $P_{q,s} : E = E^+ \oplus E^- \to E_q^+ \oplus E_s^-$ be the orthogonal projection. Consider the operators

$$\sigma : S^{n-q} \to E_n^+ \subset E_n^+ \oplus E_s^-, \quad P_{n,s}\widetilde{\gamma} : E_q^+ \oplus E_s^- \to E_n^+ \oplus E_s^-.$$

Apply Lemma 5.6 for $h_1 = \sigma$ and $h_2 = P_{q,s}\widetilde{\gamma}$, we get for some $y_s \in S^{n-q}$ and $\varphi_s \in E_q^+ \oplus E_s^-$,

$$\sigma(y_s) = P_{n,s}\widetilde{\gamma}(\varphi_s).$$

Since S^{n-q} is compact, there is a subsequence y_{s_i} such that

$$y_{s_i} \to y \text{ in } S^{n-q}, \tag{5.16}$$

$$\sigma(y_{s_i}) \to \sigma(y) \text{ in } E_q^+. \tag{5.17}$$

On the other hand, by (γ_3),

$$P_{n,s}\widetilde{\gamma}(\varphi_s) = P_{n,s}[\alpha(\varphi_s)\varphi_s + \kappa(\varphi_s)] = \alpha(\varphi_s)\varphi_s + P_{n,s}\kappa(\varphi_s),$$

where $\alpha(\varphi) \geq 1$ on $E_q^+ \oplus E^-$ and $\overline{\kappa(E_q^+ \oplus E^-)} = \overline{\kappa(D_q)}$ is compact. Hence we have

$$\varphi_s = \frac{1}{\alpha(\varphi_s)} P_{n,s}[\widetilde{\gamma}(\varphi_s) - \kappa(\varphi_s)] = \frac{1}{\alpha(\varphi_s)} P_{n,s}[\sigma(\varphi_s) - \kappa(\varphi_s))].$$

By (5.17) (φ_s) has a convergent subsequence (φ_{s_i}), that is,

$$\varphi_{s_i} \to \varphi \text{ in } E_q^+ \oplus E^-.$$

Passing to the limit we obtain

$$P_n\widetilde{\gamma}(\varphi) = \sigma(y), \text{ i.e., } P_n\widetilde{\gamma}(E_q^+ \oplus E^-) \cap \sigma(S^{n-q}) \neq \emptyset.$$

□

Proof of Proposition 5.10. Since $J(\varphi) \leq 0$ on $\{\varphi \in E_n^+ \oplus E^- \mid \|\varphi\| \geq R_q\}$, we have from Lemma 5.11

$$\min_{y \in S^{n-q}} J(\sigma(y)) \leq \sup_{\varphi \in D_q} J(P_n\gamma(\varphi))$$

for all $\gamma \in \Gamma_q$ and $\sigma \in A_q^n$. By Lemma 5.5

$$\min_{y \in S^{n-q}} J(\sigma(y)) - a_1 \leq \sup_{\varphi \in D_q} J(P_n\gamma(\varphi)).$$

Hence we obtain

$$\sup_{\sigma \in A_q^n} \min_{y \in S^{n-q}} J(\sigma(y)) - a_1 \leq \inf_{\gamma \in \Gamma_q} \sup_{\varphi \in D_q} J(P_n\gamma(\varphi)) =: b_q^n.$$

Letting $n = n_i \to \infty$, we get

$$\sigma_q - a_1 \leq \limsup_{n \to \infty} b_q^n.$$

Thus it suffices to show the following lemma.

Lemma 5.12. *For $q \in \mathbb{N}$, $b_q = \limsup_{n\to\infty} b_q^n$.*

Proof. Since $P_n\Gamma_q = \{P_n\gamma \mid \gamma \in \Gamma_q\} \subset \Gamma_q$, it is clear that $b_q \leq b_q^n$ for $n > q$. Let us prove $b_q \geq \limsup_{n\to\infty} b_q^n$ for $q \in \mathbb{N}$. From the definition of b_q, for any $\epsilon > 0$, there is a $\gamma \in \Gamma_q$ such that

$$\sup_{\varphi \in D_q} J(\gamma(\varphi)) \leq b_q + \epsilon.$$

By (γ_3), $\gamma(\varphi)$ takes a form $\gamma(\varphi) = \alpha(\varphi)\varphi + \kappa(\varphi)$ where $\alpha(\varphi) \in C(D_q, [1, \bar{\alpha}])$ and $\overline{\kappa(D_q)}$ is compact. Since $P_n\kappa(\varphi) \to \kappa(\varphi)$ as $n \to \infty$ uniformly in D_q, we have

$$P_n\gamma(\varphi) = \alpha(\varphi)\varphi + P_n\kappa(\varphi) \to \alpha(\varphi)\varphi + \kappa(\varphi) = \gamma(\varphi)$$

uniformly in D_q. Hence

$$\sup_{\varphi\in D_q} J(P_n\gamma(\varphi)) \to \sup_{\varphi\in D_q} J(\gamma(\varphi))$$

as $n \to \infty$. By the above inequality, we obtain

$$\begin{aligned}\limsup_{n\to\infty} b_q^n &\leq \limsup_{n\to\infty} \sup_{\varphi\in D_q} J(P_n\gamma(\varphi))\\ &= \sup_{\varphi\in D_q} J(\gamma(\varphi)) \leq b_q + \epsilon.\end{aligned}$$

Since ϵ is arbitrary, we get the desired result. □

6. Morse index and spectral estimate

In this section, we go to get the lower and upper bound for Morse index of K''. For $\varphi \in E^+$, we define a index of $K''(\varphi)$ by

index $K''(\varphi) =$ the number of eigenvalues of $K''(\varphi)$ which are non-positive.

That is,

index $K''(\varphi) =$

$$\max\{\dim H |\, H \text{ is subspace such that } \langle K''(\varphi)h, h\rangle \leq 0 \text{ for } h \in H\}.$$

Firstly we have lower bound:

Proposition 6.1. *Suppose that $\sigma_q < \sigma_{q+1}$. Then there exists a $\varphi_q \in E^+$ such that*

$$K(\varphi_q) \leq \sigma_q, \tag{6.1}$$

$$K'(\varphi_q) = 0, \tag{6.2}$$

$$\textit{index}\, K''(\varphi_q) \geq q. \tag{6.3}$$

Proposition 6.2. *Suppose that $\sigma_q^n < \sigma_{q+1}^n, n > q + 1$. Then there exists a $\varphi_q^n \in E_n^+$ such that*

$$K(\varphi_q^n) \leq \sigma_q^n, \tag{6.4}$$

$$(K|_{E_n^+})'(\varphi_q^n) = 0, \tag{6.5}$$

$$\textit{index}\, (K|_{E_n^+})''(\varphi_q^n) \geq q. \tag{6.6}$$

To get those, we need several lemmas.

Lemma 6.3. *([3][17]) Let U be a C^2 open subset of some Hilbert space H and let $\phi_1 \in C^2(U,\mathbb{R})$. Assume that ϕ'' is a Fredholm operator (of null index) on the critical set $Z(\phi_1) = \{x \in U |\, \phi_1'(x) = 0\}$. Lastly suppose that ϕ_1 satisfies the condition (P.S.) and that $Z(\phi_1)$ is compact. Then for any $\epsilon > 0$, there exists $\phi_2 \in C^2(U,\mathbb{R})$ satisfying (P.S.) with the following properties:*
(i) $\phi_2 = \phi_1(x)$ if distance $\{x, Z(\phi_1)\} \geq \epsilon$;
(ii) $|\phi_1(x) - \phi_2(x)|, \|\phi_1'(x) - \phi_2'(x)\|, \|\phi_1''(x) - \phi_2''(x)\| \leq \epsilon$ for all $x \in U$;
(iii) the critical points of ϕ_2 are finite in number and non-degenerate.

We remark that $K|_{E_n^+} \in C^2(E_n^+,\mathbb{R})$ satisfies (P.S.) and that all critical values of $K|_{E_n^+}$ are nonnegative, in fact, suppose that $\varphi \in E_n^+$ is critical point of $K|_{E_n^+}$, then we have

$$K(\varphi) = K(\varphi) - \frac{1}{2}\langle (K|_{E_n^+})'(\varphi), \varphi\rangle = (\frac{1}{2} - \frac{1}{p+1})a_0\|\varphi\|_{p+1}^{p+1} \geq 0.$$

On the other hand, there is a constant $\bar{R}_n$ such that $K(\varphi) < 0$ for $\varphi \in E_n^+$ with $\|\varphi\| \geq \bar{R}_n$. Therefore $Z(K|_{E_n^+})$ is compact. Applying Lemma 6.3, for all $\epsilon > 0$ there exists a $\phi_\epsilon \in C^2(E_n^+,\mathbb{R})$ satisfying (P.S.) with the following properties:

$$(6.7)\quad |\phi_\epsilon(\varphi) - K(\varphi)|, \|\phi_\epsilon'(\varphi) - (K|_{E_n^+})'(\varphi)\|, \|\phi_\epsilon''(\varphi) - (K|_{E_n^+})''(\varphi)\| \leq \epsilon$$

for all $\varphi \in E_n^+$; the critical points of ϕ_ϵ are finite in number and nondegenerate. We set for $n > q$ and $\epsilon > 0$

$$\sigma_q^n(\epsilon) = \sup_{\sigma \in A_q^n} \min_{y \in S^{n-q}} \phi_\epsilon(\sigma(y)).$$

By (6.7),

$$\sigma_q^n - \epsilon \leq \sigma_q^n(\epsilon) \leq \sigma_q^n + \epsilon.$$

Moreover we have the following lemmas as Tanaka in [17],

Lemma 6.4. [17] *Suppose that $a_\epsilon \in \mathbb{R}$ satisfies*

$$\sigma_q^n(\epsilon) < a_\epsilon - 2\epsilon < a_\epsilon < \sigma_{q+1}^n(\epsilon).$$

Then

$$(6.8)\qquad \pi_{n-q-1}([\phi_\epsilon \geq a_\epsilon]_n, p) \neq 0 \text{ for some } p \in [\phi_\epsilon \geq a_\epsilon]_n,$$

where $[\phi_\epsilon \geq a_\epsilon]_n = \{\varphi \in E_n^+ |\, \phi_\epsilon(\varphi) \geq a_\epsilon\}$.

Lemma 6.5. [17] *For a regular value $a \in \mathbb{R}$ of ϕ_ϵ, set*

$$L(\epsilon; a) = \max\{\mathit{index}\, \phi_\epsilon''(x) |\, \phi_\epsilon(x) \leq a, \phi_\epsilon'(x) = 0\}.$$

Then

$$\pi_s([\phi_\epsilon(x) \geq a]_n, p) = 0 \text{ for all } p \in [\phi_\epsilon \geq a]_n \text{ and } s \leq n - L(\epsilon; a) - 2.$$

Proof of Proposition 6.2. Since $\sigma_q^n < \sigma_{q+1}^n$ and the critical points of ϕ_ϵ are finite in the number and nondegenerate, there is a sequence $a_\epsilon \in \mathbb{R}(0 < \epsilon \le \epsilon_0)$ such that

$$a_\epsilon \text{ is a regular value of } \phi_\epsilon, \tag{6.9}$$

$$\sigma_q^n(\epsilon) < a_\epsilon - 2\epsilon < a_\epsilon < \sigma_{q+1}^n(\epsilon), \tag{6.10}$$

$$a_\epsilon \to \sigma_q^n \text{ as } \epsilon \to 0. \tag{6.11}$$

Applying Lemma 6.4 and Lemma 6.5, we observe

$$L(\epsilon; a) \ge q \quad \text{for} \quad 0 < \epsilon < \epsilon_0.$$

Therefore there is a $\varphi_\epsilon \in E_n^+$ such that

$$\phi_\epsilon(\varphi_\epsilon) \le a_\epsilon, \tag{6.12}$$

$$\phi_\epsilon'(\varphi_\epsilon) = 0, \tag{6.13}$$

$$\text{index}\phi_\epsilon''(\varphi_\epsilon) \ge q. \tag{6.14}$$

It follows from (6.7) that satisfies

$$K(\varphi_\epsilon) \text{ is bounded as } \epsilon \to 0,$$
$$(K|_{E_n^+})'(\varphi_\epsilon) \to 0 \text{ as } \epsilon \to 0.$$

Since $K|_{E_n^+}$ satisfies (P.S) on E_n^+, we can choose a convergent subsequence $\varphi_{\epsilon_j} \to \varphi_q^n$ $(\epsilon_j \to 0)$.

Proof of proposition 6.1. Since $\sigma_q < \sigma_{q+1}$, we have $\sigma_q^{n_j} < \sigma_{q+1}^{n_j}$ for sufficiently large j. Hence there is a $\varphi_q^{n_j} \in E_{n_j}^+$ satisfying (6.4)-(6.6) by Proposition 6.2. Since $K \in C^2(E^+, \mathbb{R})$ satisfies (P.S.)$_*$, $\varphi_q^{n_j}$ has a convergent subsequence $\varphi_q^{n_{j'}}$. Let $\varphi_q = \lim_{j' \to \infty} \varphi_q^{n_{j'}}$. Then (6.1)(6.2) follow from (6.4)(6.5) easily. Let us prove (6.3).

First we have

$$\text{index}\, K''(\varphi_q^n) \ge \text{index}\, (K|_{E_n^+})''(\varphi_q^n)$$

for all $n \in \mathbb{N}$.

On the other hand, we observe that $K''(\varphi_q)$ is an operator of type $K'' = id + \text{(compact)}$. Hence there is an $\epsilon > 0$ such that for $h \in E^+$,

$$\langle K''(\varphi_q)h, h\rangle \le 0 \quad \text{iff} \quad \langle K''(\varphi_q)h, h\rangle \le \epsilon \|h\|^2.$$

i.e.,

$$\text{index}\, K''(\varphi_q) = \text{index}\, (K''(\varphi_q) - \epsilon).$$

Since $K \in C^2(E^+, \mathbb{R})$, we have for some j_0',

$$\|K''(\varphi_q^{n_{j'}}) - K''(\varphi_q)\| \le \epsilon \quad \text{for} \quad j > j_0'.$$

Thus for $j' \ge j_0'$ and $h \in E^+$

$$\langle K''(\varphi_q)h, h\rangle - \epsilon\|h\|^2 \le \langle K''(\varphi_q^{n_{j'}})h, h\rangle.$$

That is

$$\operatorname{index} K''(\varphi_q^{n_{j'}}) \leq \operatorname{index}(K''(\varphi_q) - \epsilon).$$

Therefore by the above inequality, we complete the proof. □

Now we go to prove the upper bound.

Proposition 6.6. *For any $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that for $\varphi \in E^+$,*

$$\text{(6.15)} \qquad \textit{index}\, K''(\varphi) \leq C_\epsilon \|\varphi\|_{(p-1)(1+\epsilon)}^{(p-1)(1+\epsilon)}.$$

Note that for $\varphi, h \in E^+$,

$$\text{(6.16)} \qquad \langle K''(\varphi)h, h\rangle = \|h\|^2 - pa_0\langle|\varphi|^{p-1}h, h\rangle.$$

From the definition of index $K''(\varphi)$, it is clear that

$$\text{(6.17)} \quad \operatorname{index} K''(\varphi) =$$

$$\max\{\dim H |\, H \subset E^+ \text{ subspace such that } \|h\|^2 \leq pa_0\langle|\varphi|^{p-1}h, h\rangle, \text{for } h \in H\}$$

We define an operator $D : L^2 \to E^+$ by

$$\text{(6.18)} \qquad (Dv) = \sum_{j<|k|}(k^2 - j^2)^{-1/2}\sum_l a_{jk}^l sinjxe^{ikt}e_l,$$

for $v = \sum_{l,j,k} a_{jk}^n sinjxe^{ikt}e_l$. It is easily seen that D is an isometry from $L^2_+ = L^2-$closure of span $\{sinjxe^{ikt}e_l |\, j < |k|\}$ to E^+ and $D = 0$ on $(L^2_+)^\perp$. Setting $h = Dv$ in (6.18), we get

$$\text{(6.19)} \quad \operatorname{index} K''(\varphi)$$

$$= \max\{\dim H |\, H \subset L^2 \text{ such that } \|v\|_2^2 \leq \langle pa_0|\varphi|^{p-1}Dv, Dv\rangle, v \in L^2\},$$

which means that index $K''(\varphi)$ is the number of the eigenvalues of $D^*(pa_0|\varphi|^{p-1})D$ that are greater than or equal to 1.

For the above reason, we are concerned with an operator $T_{V,\theta} : L^2 \to L^2$ defined by

$$\text{(6.20)} \quad T_{V,\theta} = V(x,t)\sum_{j,k}\theta_{jk}\sum_l a_{jk}^l sinjxe^{ikt}e_l \text{ for } v = \sum_{l,j,k} a_{jk}^l sinjxe^{ikt}e_l,$$

where $V(x,t)$ is a function on Ω and $\theta = (\theta_{jk})$ is a sequence on $\mathbb{N} \times \mathbb{Z}$. If we set

$$\text{(6.21)} \qquad \widetilde{V}(x,t) = \sqrt{pa_0}|\varphi|^{(p-1)/2},$$

$$\text{(6.22)} \qquad \widetilde{\theta}_{ij} = \begin{cases} (k^2 - j^2)^{-1/2} & \text{if } j < |k|, \\ 0 & \text{if } j \geq |k|, \end{cases}$$

then letting $\widetilde{\theta} = (\widetilde{\theta}_{ij})$, we have

$$\text{(6.23)} \qquad D^*(pa_0|\varphi|^{(p-1)})D = T^*_{\widetilde{V},\widetilde{\theta}}T_{\widetilde{V},\widetilde{\theta}}.$$

In order to analyze the operator $T_{V,\Theta}$, let us recall the definition of the singular values of a compact operator. Let $A : L^2 \to L^2$ be a compact operator. The singular values of A, $s_n(A)$ are the eigenvalues of $|A| = \sqrt{A^*A}$ listed according to $s_1(A) \geq s_2(A) \geq \dots$. For $1 \leq q < \infty$, A is said to lie in trace ideal I_q if and only if

$$\|A\|_{I_q} = (\sum_{n=1}^{\infty} s_n(A)^q)^{1/q} < \infty \text{ for } 1 \leq q < \infty.$$

For $q = \infty$, we set $I_\infty =$ the set of bounded linear operators $L^2 \to L^2$ and

$$\|A\|_{I_\infty} = \sup\{\|Av\|_2 | \, \|v\|_2 \leq 1\} < \infty.$$

The following properties of trace ideals are known :
(i) I_2 is the Hilbert-Schmidt class on L^2. (ii) Let $\mathfrak{B}$ denote the family of orthogonal sequences in L^2, then

$$\|A\|_{I_q} = \sup_{\{u_n\},\{v_n\} \in \mathfrak{B}} (\sum_{n=1}^{\infty} |\langle u_n, Av_n\rangle|^q)^{1/q}.$$

When $q = 2$, for any complete orthogonal sequence $\{v_n\}$ in L^2,

$$\|A\|_{I_2} = (\sum_{n=1}^{\infty} \|Av_n\|_2^2)^{1/2}$$

(iii) For $q \geq 2$, $A \in I_q$ if and only if $A^*A \in I_{q/2}$ and $\|A\|_{I_q}^2 = \|A^*A\|_{I_{q/2}}$. We denote by $l^q = l^q(\mathbb{N} \times \mathbb{Z})$ the space of sequences $\theta = (\theta_{jk})$ which satisfy

$$\|\theta\|_{l^q} = (\sum_{jk} |\theta_{jk}|^q)^{1/q} < \infty \quad \text{for } q \in q \in [1, \infty),$$
$$\|\theta\|_{l^\infty} = \sup_{jk} |\theta_{jk}| < \infty \quad \text{for } q \in q \in [1, \infty).$$

Lemma 6.7. *([17]) Suppose that $V \in L^q$ and $\theta = (\theta_{jk}) \in l^q$ for $q \in [2, \infty]$. Then $T_{V,\theta} \in L^q$ and there is a constant $C_q > 0$, which is independent of V and θ, such that*

$$\|T_{V,\theta}\|_{I_q} \leq C_q N \|V\|_q \|\theta\|_{l^q} \quad \textit{for all } V \textit{ and } \theta. \tag{6.24}$$

Proof. It follows directly as [17]. Firstly we deal with the case $q = 2$. Setting $\{v\} = \{\frac{1}{\pi}\} \sin jx e^{ikt} e_l$, we get

$$\begin{aligned} \|T_{V,\theta}\|_{I_2} &= \sum_{j,k,l} \frac{1}{\pi^2} \|T_{V,\theta}(\sin jx e^{ikt} e_l)\|_2^2 \\ &= \sum_{j,k,l} \frac{1}{\pi^2} \|V(x,t)\theta_{jk} \sin jx e^{ikt} e_l\|_2^2 \\ &\le \sum_{j,k,l} \frac{1}{\pi^2} \|V(x,t)\|_2^2 |\theta_{jk}|^2 = \frac{N}{\pi^2} \|\theta\|_{l^2}^2 \|V\|_2^2. \end{aligned}$$

Next we deal with the case $q = \infty$. For $v = \sum_{l,j,k} a_{j,k}^l \sin jx e^{ikt} e_l$,

$$\|T_{V,\theta} v\|_2^2 = \|V \sum_{l,j,k} a_{j,k}^l \sin jx e^{ikt} e_l\|_2^2 \le \|V\|_\infty^2 \|\theta\|_{l^\infty}^2 \|v\|_2^2.$$

That is,

$$\|T_{V,\theta}\| = \sup_{\|v\|_2 = 1} \|T_{V,\theta} v\|_2 \le \|V\|_\infty \|\theta\|_{l^\infty}.$$

Lastly for $2 \le q \le \infty$, fix $\{u_n\}, \{v_n\} \in \mathcal{B}$ and consider the operator $L^q \times l^q \to l^q$ defined by $(V, \theta) \to \{(u_n, T_{V,\theta} v_n)\}_{n \in \mathbb{N}}$. By the case $q = 2, \infty$, we get

$$\begin{aligned} \|(u_{n,T_{V,\theta}} v_n)\|_{l^2} &\le \|T_{V,\theta}\|_{I_2} \le \frac{1}{\pi^2} \|\theta\|_{l^2} \|V\|_2, \\ \|(u_{n,T_{V,\theta}} v_n)\|_{l^\infty} &\le \|T_{V,\theta}\|_{I_\infty} \le \|\theta\|_{l^\infty} \|V\|_\infty. \end{aligned}$$

By the complex interpolation, we get for $q \in (2, \infty)$,

$$\|(u_{n,T_{V,\theta}} v_n)\|_{l^q} \le C_q \|\theta\|_{l^q} \|V\|_q$$

where C_q is constant independent of $\{u_n\}, \{v_n\} \in \mathcal{B}$. By the definition of $\|T_{V,\theta}\|_{I_q}$, we get the desired result. □

Proof of Proposition 6.6. Since $T_{V,\theta}^* T_{V,\theta}$ is a positive self-adjoint operator,

$$\|T_{V,\theta}^* T_{V,\theta}\|_{I_{q/2}} = (\sum_n s_n^{q/2})^{2/q} \quad \text{for } q \ge 2,$$

where s_n are the eigenvalues of $T_{V,\theta}^* T_{V,\theta}$. Hence we have from the definition of I_q and (6.19)

$$\operatorname{index} K''(\varphi) \le \|T_{V,\theta}^* T_{V,\theta}\|_{I_{q/2}}^{q/2} \le \|T_{V,\theta}\|_{I_q}^q \quad \text{for } q \ge 2.$$

Set $\widetilde{V}$ and $\widetilde{\theta}$ as in (6.21),(6.22). Then we have from (6.23)

$$\operatorname{index} K''(\varphi) \le \|T_{\widetilde{V},\widetilde{\theta}}\|_{I_q}^q \quad \text{for } q \in (2, \infty].$$

Note that for any $q \in (2, \infty]$ as in [17]

$$\begin{aligned} \|\widetilde{\theta}\|_{l^q}^q &= \sum_{j<|k|} (k^2 - j^2)^{-q/2} = 2 \sum_{j,s \in \mathbb{N}} ((j+s)^2 - j^2)^{-q/2} \\ &= 2 \sum_{j,s} [s(2j+s)])^{-q/2} \le \sum_{j,s} s^{-q/2} j^{-q/2} < \infty. \end{aligned}$$

Then we deduce from Lemma 6.7 that

$$\text{index} K''(\varphi) \le N \|T_{\widetilde{V}, \widetilde{\theta}}\|_{I_q}^q \le C_q N \|\widetilde{\theta}\|_{l^q}^q \|\widetilde{V}\|_q^q \le CN \|\varphi\|_{(p-1)q/2}^{(p-1)q/2}.$$

□

7. Proof of Theorem 1.2

Step 1. By Proposition 5.3 and Proposition 5.4, we see that

$$b_{q_j} \ge C_\epsilon q_j^{(p+1)/(p-1)-\epsilon} \quad \text{for large } \ q_j \in \mathbb{N},$$

ensures the existence of an unbounded sequence $\{\varphi_j\}$ of critical points of J. Then by Proposition 4.1, we know that the unbounded critical points $\{\varphi_j\}$ satisfy $I(\varphi_j) = J(\varphi_j)$.

By Proposition 5.10, it suffices to show the existence of a sequence $q_j \to \infty$, as $j \to \infty$, with the following property: for any $\epsilon > 0$ there is a $C_\epsilon > 0$ such that

$$\sigma_{q_j} \ge C_\epsilon q_j^{(p+1)/(p-1)-\epsilon} \quad \text{for large} \quad j \in \mathbb{N}.$$

Since $\sigma_q \to \infty$ as $q \to \infty$, there is a sequence q_j such that $\sigma_{q_j} < \sigma_{q_j+1}$. Applying Proposition 6.1, there are $\{\varphi_j\} \in E^+$ such that

$$K(\varphi_j) \le \sigma_{q_j}, \tag{7.1}$$

$$K'(\varphi_j) = 0 \tag{7.2}$$

$$\text{index}\, K''(\varphi_j) \ge q_j \quad \text{for large} \quad j \in \mathbb{N}. \tag{7.3}$$

Next applying Proposition 6.6, we get

$$C_\epsilon \|\varphi_j\|_{(p-1)(1+\epsilon)}^{(p-1)(1+\epsilon)} \ge q_j.$$

Choosing $\epsilon \in (0, 2/(p-1))$, we obtain

$$\|\varphi_j\|_{p+1}^{p+1} \ge C \|\varphi_j\|_{(p-1)(1+\epsilon)}^{p+1} \ge C'_\epsilon q_j^{(p+1)/[(p-1)(1+\epsilon)]} \quad \text{for } j \in \mathbb{N}. \tag{7.4}$$

On the other hand, we have by (7.2)

$$\langle K'(\varphi_j), \varphi_j \rangle = \|\varphi_j\|^2 - a_0 \|\varphi_j\|_{p+1}^{p+1} = 0.$$

By (7.1), we obtain

$$\sigma_{q_j} \ge K(\varphi_j) = \frac{1}{2} \|\varphi_j\|^2 - \frac{a_0}{p+1} \|\varphi_j\|_{p+1}^{p+1} = (\frac{1}{2} - \frac{a_0}{p+1}) \|\varphi_j\|_{p+1}^{p+1}.$$

Therefore by (7.4), we conclude that for a unbounded sequence q_j (as $j \to \infty$)

$$\sigma_{q_j} \geq C_\epsilon q_j^{(p+1)/(p-1)-\epsilon}.$$

Step 2. Passing limit: We complete the proof by convexity of $F_{\delta,g}$. We may assume that for each $\delta \in (0,1)$, $u_\delta = \varphi_\delta^+ + \varphi_\delta^- + \psi_\delta$ is a solution of the equation

$$\Box u + \nabla F_{\delta,g}(u) = 0. \tag{7.5}$$

It follows from the same argument as in section 3 that also $\varphi_\delta^+, \varphi_\delta^-$ are bounded uniformly for δ in E. Hence there is a sequence δ_j tending to 0 such that $\varphi_{\delta_j} \rightharpoonup \bar\varphi$ in E and $\psi_{\delta_j} \to \bar\psi$ in L^{p+1}.

Now we employ a standard monotonicity argument in order to show that $\bar u = \bar\varphi + \bar\psi$ is a weak solution of (1.1)-(1.3). Set $u_j = u_{\delta_j}$. Then u_j satisfies (7.5) with $\delta = \delta_j$. The right hand side of (7.5) is integrable in Ω. Thus $\Box u_j \to \zeta$ in $L^{(p-1)/p}$, possibly after passing to a subsequence. Since $\Box u_j \to \Box \bar u$ in the sense of distributions, we have $\zeta = \Box \bar u$. For each $\tau \in E$,

$$\begin{aligned} &\int_\Omega (\Box u_j + \nabla F_{\delta_j,g}(\tau))(u_j - \tau)\, dxdt \\ = &\int_\Omega (\nabla F_{\delta_j,g}(\tau) - \nabla F_{\delta_j,g}(u_j))(u_j - \tau)\, dxdt \leq 0. \end{aligned} \tag{7.6}$$

Furthermore,

$$\int_\Omega \Box u_j u_j \, dxdt \to \int_\Omega \Box \bar u \bar u \, dxdt, \tag{7.7}$$

so we obtain

$$\int_\Omega (\Box \bar u + \nabla F_{0,g}(\tau))(\bar u - \tau)\, dxdt \leq 0,$$

after passing to the limit in (7.6). Let $\tau = \bar u + s\chi$, where $s > 0$ and $\chi \in E \cap C^\infty(\Omega)$. Substituting this τ in the inequality above and letting $s \to 0$ give

$$\int_\Omega (\Box \bar u + \nabla F_{0,g}(\bar u))\chi \, dxdt \leq 0.$$

Since χ was chosen arbitrary, $\bar u$ is a solution of (1.1)-(1.3). □

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